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ABSTRACT

We use Hamiltonian Floer theory to recover and generalize a classic rigidity theorem of Ekeland and Lasry. That theorem can be rephrased as an assertion about the existence of multiple closed Reeb orbits for certain tight contact forms on the sphere that are close, in a suitable sense, to the standard contact form. We first generalize this result to Reeb flows of contact forms on prequantization spaces that are suitably close to Boothby–Wang forms. We then establish, under an additional nondegeneracy assumption, the same rigidity phenomenon for Reeb flows on any closed contact manifold. A natural obstruction to obtaining sharp multiplicity results for closed Reeb orbits is the possible existence of fast closed orbits. To complement the existence results established here, we also show that the existence of such fast orbits cannot be precluded by any condition which is invariant under contactomorphisms, even for nearby contact forms.

1. Introduction

The following theorem of Ekeland and Lasry appeared in 1980.

THEOREM 1.1 [EL80]. *Let $\Sigma \subset \mathbb{R}^{2n}$ be a C^2 -smooth hypersurface which forms the boundary of a compact convex neighborhood of the origin. If there are positive numbers $r \leq R$ such that $R < \sqrt{2}r$ and*

$$r \leq \|x\| \leq R$$

for all $x \in \Sigma$, then Σ carries at least n geometrically distinct closed characteristics with actions in $[\pi r^2, \pi R^2]$.

This result and the ideas developed in its proof have been highly influential. Much of the subsequent progress on the problem of detecting closed characteristics on convex hypersurfaces, such as the remarkable results of Long and Zhu in [LZ02], is built on the foundation laid down in [EL80]. Indeed, Theorem 1.1 is still one of the most compelling facts supporting the following well-known conjecture.

CONJECTURE 1.2. Every compact convex hypersurface in \mathbb{R}^n carries at least n geometrically distinct closed characteristics.

The basic idea underlying the proof of Theorem 1.1 is the following: *Closed characteristics on a convex hypersurface Σ are critical points of Clarke’s dual action principle which is invariant under the natural S^1 -action on loops. If Σ satisfies the stated pinching condition, then the*

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S^{2n-1} -worth of closed characteristics (critical points) on the sphere of radius r influences the topology of the negative sublevels of the dual action principle for Σ . In particular, the S^1 -action is free on these sublevels and they must also contain an invariant copy of S^{2n-1} . This forces the restriction of the dual action to these sublevels to have at least $\text{cuplength}(\mathbb{C}\mathbb{P}^{n-1}) + 1 = n$ critical points.

In the present work we detect such influences using tools from Hamiltonian Floer theory. With these tools we recover Theorem 1.1 and generalize the rigidity phenomenon underlying it to Reeb flows on any closed contact manifold.

1.1 Recovery

We begin by recovering Theorem 1.1 in a different but equivalent setting. Let λ be a contact form on M^{2n-1} . It defines a unique Reeb vector field R_λ on M via the equations

$$i_{R_\lambda}d\lambda = 0 \quad \text{and} \quad \lambda(R_\lambda) = 1.$$

It also defines a contact structure $\xi = \ker(\lambda)$. Any other contact form defining the same contact structure is of the form $f\lambda$ for some nonvanishing smooth function $f : M \rightarrow \mathbb{R}$.

Let λ_0 be the standard contact form on the unit sphere $S^{2n-1} \subset \mathbb{R}^{2n}$ obtained by restricting the form $\frac{1}{2} \sum_{j=1}^n (p_j dq_j - q_j dp_j)$. The following result is equivalent to Theorem 1.1 for the case of smooth hypersurfaces.

THEOREM 1.3. *Let $\lambda = f\lambda_0$ for some positive function f . If*

$$\frac{\max(f)}{\min(f)} < 2, \tag{1}$$

then there are at least n distinct closed orbits of R_λ with periods in the interval $[\pi \min(f), \pi \max(f)]$.

Recall that a closed orbit γ of R_λ (closed Reeb orbit of λ) is said to be *distinct* from another such orbit $\tilde{\gamma}$ if for all $k \in \mathbb{N}$ and $c \in \mathbb{R}$ there is a $t \in \mathbb{R}$ such that

$$\gamma(t) \neq \tilde{\gamma}(kt + c).$$

Two closed orbits are said to be *geometrically distinct* if they have disjoint images. While geometrically distinct orbits are distinct, the converse does not hold since two distinct orbits can both be relatively prime multiples of a third closed orbit.

To see that Theorem 1.3 implies Theorem 1.1, consider a smooth hypersurface Σ as in the statement of Theorem 1.1. It can be described in the form

$$\Sigma = \{z\sqrt{f(z)} \mid z \in S^{2n-1}\},$$

where f is a positive smooth function on S^{2n-1} . There is a bijective correspondence between the simple closed Reeb orbits of $\lambda = f\lambda_0$ with period T (modulo translation) and the closed characteristics on Σ with action equal to T . To show that Theorem 1.1 follows from Theorem 1.3, it therefore suffices to show that the convexity of Σ implies that the closed Reeb orbits of λ with period in $[\pi \min(f), \pi \max(f)]$ are simple and hence geometrically distinct. Since $\max(f) < 2 \min(f)$, it suffices to show that the convexity of Σ implies that R_λ has no closed orbits of period less than $\pi \min(f)$ or, equivalently, that Σ has no closed characteristics with action less than $\pi \min(f)$. This last condition follows immediately from the main result of Croke and Weinstein in [CW81], which asserts that the closed characteristics of convex hypersurfaces containing the ball of radius r have action at least πr^2 .

Remark 1.4. Note that the convexity assumption for hypersurfaces is not invariant under symplectomorphisms and that no assumption like convexity appears in the statement of Theorem 1.3.

Remark 1.5. Given a contact structure ξ on M , there is a natural coarse pseudo-metric on the space of contact forms defining ξ . For any two such contact forms, say λ and $\tilde{\lambda}$, there is a smooth nonvanishing function $f_{\lambda/\tilde{\lambda}}$ on M such that

$$\lambda = f_{\lambda/\tilde{\lambda}}\tilde{\lambda}.$$

The *distance* between them can then be defined as

$$d(\lambda, \tilde{\lambda}) = \ln\left(\frac{\max(f_{\lambda/\tilde{\lambda}})}{\min(f_{\lambda/\tilde{\lambda}})}\right). \tag{2}$$

It is easy to verify that d is a pseudo-metric and that its degeneracy only reflects the fact that it is invariant under scalar multiplication of the contact forms. In these terms, condition (1) can be rephrased geometrically as $d(\lambda, \lambda_0) < \ln 2$.

1.2 Prequantization spaces

Our first generalization of Theorem 1.3 establishes the same rigidity phenomenon for all prequantization spaces. Consider a closed symplectic manifold (Q, ω) such that the de Rham cohomology class $-\omega/2\pi$ is the image of an integral class $\mathbf{e} \in H^2(Q; \mathbb{Z})$. Let $p : M \rightarrow Q$ be an S^1 -bundle over Q with first Chern class equal to \mathbf{e} . Denote the corresponding Boothby–Wang contact form on M by λ_Q and the corresponding contact structure by ξ_Q . We then have $d\lambda_Q = p^*\omega$ and the Reeb vector field of λ_Q generates the circle action on M with period 2π .

Let $\alpha_{\mathbf{f}} \in [S^1, M]$ be the free homotopy class corresponding to the fibres of the bundle $p : M \rightarrow Q$ and denote its order by $|\alpha_{\mathbf{f}}|$.

THEOREM 1.6. *Let $\lambda = f\lambda_Q$ for some positive function f . If*

$$\frac{\max(f)}{\min(f)} < 2,$$

then there are at least $\frac{1}{2} \dim(Q) + 1$ distinct closed Reeb orbits of λ which represent the class $\alpha_{\mathbf{f}}$ and have period in the interval

$$[2\pi \min(f), 2\pi \max(f)].$$

These orbits are geometrically distinct from one another if the class $\alpha_{\mathbf{f}}$ is either primitive or of infinite order. Otherwise, they are geometrically distinct if there are no closed Reeb orbits of λ which have period less than or equal to

$$\frac{2\pi}{|\alpha_{\mathbf{f}}|}(\max(f) - \min(f))$$

and which represent a class β such that $\beta^k = \alpha_{\mathbf{f}}$ for some integer $k > 1$.

Note that we can detect geometrically distinct orbits in many cases without a geometric assumption like convexity or an equivalent substitute like dynamical convexity.

Remark 1.7. It is shown in [GGM15] that the class α_f is of infinite order if (Q, ω) is symplectically aspherical and is primitive if $\pi_1(Q)$ is torsion free. Under the assumption that both these conditions hold, it is also shown in [GGM15] that if all the closed Reeb orbits of $\lambda = f\lambda_Q$ are nondegenerate and λ has no contractible closed Reeb orbits with Conley–Zehnder index within one of $2 - \frac{1}{2} \dim Q$, then λ has infinitely many closed Reeb orbits with contractible projections to Q .

1.3 Rigid constellations

We now present a more extensive generalization of the Ekeland–Lasry rigidity phenomenon. Let (M, λ_0) be any closed contact manifold and let $\alpha \in [\mathbb{S}^1, M]$ be any free homotopy class. (The trivial class will be denoted here by e .) We first identify collections of closed orbits of λ_0 in class α which can meaningfully influence the Reeb flows of contact forms which are nearby in the sense of Remark 1.5. As the original proof of Theorem 1.1 suggests, these collections should consist of simple orbits (so that the natural \mathbb{R}/\mathbb{Z} -action is free) and their periods should be isolated in the period spectrum.

Let $\mathcal{T}(\lambda_0)$ be the set of periods of all closed Reeb orbits of λ_0 and, assuming that it is nonempty, let $T_{\min}(\lambda_0)$ be the smallest such period. Restricting to orbits in class α , we get the similarly defined set $\mathcal{T}(\lambda_0, \alpha)$ and minimal α -period, $T_{\min}(\lambda_0, \alpha)$.

Given a T in $\mathcal{T}(\lambda_0, \alpha)$, let $\mathcal{C}_{\lambda_0, \alpha}(T)$ be the collection of closed orbits of R_{λ_0} which represent the class α and have period in the interval $[T_{\min}(\lambda_0, \alpha), T]$.

DEFINITION 1.8. The collection of closed orbits $\mathcal{C}_{\lambda_0, \alpha}(T)$ is a *rigid constellation* if every orbit in $\mathcal{C}_{\lambda_0, \alpha}(T)$ is simple, no decreasing sequence in $\mathcal{T}(\lambda_0, \alpha)$ converges to T and

$$T < T_{\min}(\lambda_0, \alpha) + T_{\min}(\lambda_0). \tag{3}$$

Given a rigid constellation $\mathcal{C}_{\lambda_0, \alpha}(T)$, we set

$$T^+ = \min\{T' \in \mathcal{T}(\lambda_0, \alpha) : T' > T\}.$$

This number is strictly greater than T by the second condition of the definition above.

Since Reeb vector fields are autonomous, the elements of a rigid constellation $\mathcal{C}_{\lambda_0, \alpha}(T)$ can be divided into separate \mathbb{R}/\mathbb{Z} -families of closed Reeb orbits, the elements of which differ only by simple translation reparameterizations. It follows from the simplicity condition in Definition 1.8 that closed orbits belonging to different \mathbb{R}/\mathbb{Z} -families of a rigid constellation are geometrically distinct from one another.

Example 1.9. Let λ_0 be the standard contact form on the unit sphere $M = \mathbb{S}^{2n-1} \subset \mathbb{R}^{2n}$. The time- t Reeb flow of λ_0 is

$$(z_1, \dots, z_n) \mapsto (e^{i2t} z_1, \dots, e^{i2t} z_n)$$

and so every Reeb orbit is closed with (minimal) period π . For the choice $T = \pi$, the set $\mathcal{C}_{\lambda_0, e}(\pi)$ is then a rigid constellation (diffeomorphic to \mathbb{S}^{2n-1}) with $T^+ = 2\pi$.

Example 1.10. Consider an n -tuple $\mathbf{r} = (r_1, \dots, r_n)$ in \mathbb{R}^n such that

$$0 < r_1 < r_2 < \dots < r_n < \sqrt{2}r_1.$$

Equip the unit sphere \mathbb{S}^{2n-1} with the contact form

$$\lambda_{\mathbf{r}}(z) = \left(\frac{|z_1|^2}{r_1^2} + \dots + \frac{|z_n|^2}{r_n^2} \right)^{-1} \lambda_0(z),$$

where λ_0 is the form from Example 1.9. The time- t Reeb flow of λ_r is

$$(z_1, \dots, z_n) \mapsto (e^{i2t/r_1^2} z_1, \dots, e^{i2t/r_n^2} z_n).$$

We then have $T_{\min} = \pi r_1^2$ and are free to choose a T of the form $T_k = \pi r_k^2$ for some $k = 1, \dots, n$. For these choices, we have

$$\mathcal{C}_{\lambda_r, e}(T_k) = \{\Gamma_1, \dots, \Gamma_k\},$$

where each Γ_i is the \mathbb{R}/\mathbb{Z} -family of closed Reeb orbits with image equal to the intersection of \mathbb{S}^{2n-1} with the z_i -plane. Each $\mathcal{C}_{\lambda_r, e}(T_k)$ is a rigid constellation with

$$T_k^+ = \begin{cases} \pi r_{k+1}^2 & \text{if } k < n, \\ 2\pi r_1^2 & \text{if } k = n. \end{cases}$$

Example 1.11. Let (B, g) be a Riemannian manifold with negative sectional curvature. Let $\Sigma_{g^*} \subset T^*B$ be the unit cosphere bundle and let λ_{g^*} be the restriction to Σ_{g^*} of the tautological one-form on T^*B . Then λ_{g^*} is a contact form whose Reeb vector field generates the cogeodesic flow on Σ_{g^*} . Let $\tilde{\alpha}$ be a nontrivial primitive element of $[\mathbb{S}^1, B]$ and let α be its lift to $[\mathbb{S}^1, \Sigma_{g^*}]$. The assumption of negative sectional curvature implies that there is a unique \mathbb{R}/\mathbb{Z} -family of closed Reeb orbits of λ_{g^*} in the class α . Thus, for $T = T_{\min}(\lambda_{g^*}, \alpha)$, the set $\mathcal{C}_{\lambda_{g^*}, \alpha}(T)$ is a rigid constellation with $T^+ = +\infty$.

A collection of closed Reeb orbits of a contact form λ is said to be *nondegenerate* if each of its elements $\gamma(t)$ is nondegenerate (and hence isolated) in the usual sense. The rigid constellation in Example 1.9 is degenerate (Morse–Bott nondegenerate) while those described in Examples 1.10 and 1.11 are nondegenerate.

In §2, we will associate to each nondegenerate rigid constellation $\mathcal{C}_{\lambda_0, \alpha}(T)$ a version of \mathbb{S}^1 -equivariant Floer homology. With this we will define the rank of $\mathcal{C}_{\lambda_0, \alpha}(T)$. At this point we can now state our broadest generalization of Theorem 1.1.

THEOREM 1.12. *Let (M, λ_0) be a closed contact manifold and let $\mathcal{C}_{\lambda_0, \alpha}(T)$ be a nondegenerate rigid constellation. Let $\lambda = f\lambda_0$ for some positive function f . If*

$$\frac{\max(f)}{\min(f)} < \min \left\{ \frac{T^+}{T}, \frac{T_{\min}(\lambda_0) + T_{\min}(\lambda_0, \alpha)}{T} \right\} \tag{4}$$

and every closed Reeb orbit of λ in class α and with period in the interval

$$[\min(f)T_{\min}, \max(f)T]$$

is nondegenerate, then there are at least $\text{rank}(\mathcal{C}_{\lambda_0, \alpha}(T))$ such orbits which are distinct from one another. These orbits are geometrically distinct from one another if the class α is either primitive or of infinite order. Otherwise, these orbits are geometrically distinct if there are no (fast) closed Reeb orbits of λ with period less than or equal to

$$\frac{1}{|\alpha|} (T \max(f) - T_{\min} \min(f))$$

that represent a class β such that $\beta^k = \alpha$ for some integer $k > 1$.

The primary point of this result is that it establishes the rigidity of closed Reeb orbits without assumptions on the ambient contact manifold. In particular, it does not assume the existence of strong symplectic fillings. The price of this generality is the presence of the term

$$\frac{T_{\min}(\lambda_0) + T_{\min}(\lambda_0, \alpha)}{T}.$$

While this term constrains the range of the rigidity phenomenon, it is needed to achieve compactness for the relevant moduli spaces of Floer trajectories used here to detect it. Despite the fact that the term is expressed in the language of dynamics, we do not know whether it represents an actual boundary to the generalized Ekeland–Lasry rigidity phenomenon. When (M, λ_0) does admit strong symplectic fillings the proof of Theorem 1.12 simplifies greatly and yields rigidity results with larger, sometimes infinite, range. These results are described below in § 1.5.

In Theorem 1.12 we have imposed nondegeneracy assumptions on certain closed Reeb orbits of λ while in Theorem 1.6 no such assumptions are made. The lower bounds of Theorem 1.6 correspond to cup-length estimates in the spirit of those predicted in the strong form of Arnold’s conjecture for Hamiltonian diffeomorphisms. Theorem 1.12 is more in the spirit of the nondegenerate form of Arnold’s conjecture. Note that rather than assuming the nondegeneracy of all closed Reeb orbits of λ (strong nondegeneracy) we have only assumed nondegeneracy for the closed Reeb orbits of λ in a fixed range of periods. For the Ekeland–Lasry rigidity phenomenon, this seems to be the appropriate assumption to obtain Morse-type inequalities. The difference between this nondegeneracy assumption and the strong form is analogous, in the setting of Arnold’s conjecture, to the difference between the assumption that the fixed points of a Hamiltonian diffeomorphism are nondegenerate and the assumption (clearly irrelevant in that case) that all its periodic points are nondegenerate. This point is captured nicely by the contact forms $\lambda_{\mathbf{r}}$ from Example 1.10. Given $\mathbf{r} = (r_1, r_2)$ with $r_1 \leq r_2 < \sqrt{2}r_1$, we know that all contact forms $\lambda_{\mathbf{r}}$ have at least two closed Reeb orbits with periods in $[\pi r_1^2, \pi r_2^2]$ and of course we would like our theorems to see this too. For $r_1 = r_2$, these orbits are detected by Theorem 1.3 and, for $r_1 < r_2$, they are detected by Theorem 1.12. In contrast, contact forms $\lambda_{\mathbf{r}}$ satisfy the strong nondegeneracy assumption if and only if r_1 and r_2 are rationally independent.

Remark 1.13. Under the strong nondegeneracy assumption one can, in certain cases, detect more dramatic rigidity phenomena. This is because tools such as the index iteration formulas for Maslov indices become much sharper, and the rich machinery of symplectic and contact homology may also become available. For example, Conjecture 1.2 was proven for the case of convex hypersurfaces all of whose closed characteristics are strongly nondegenerate by Long and Zhu in [LZ02] (starting from the analytic framework of [EL80]). More recently, this was reproved under weaker *dynamical* convexity assumptions by Gutt and Kang in [GK16] using S^1 -equivariant symplectic homology, and Abreu and Macarini in [AM17] using contact homology.

1.4 Applications (with no strong symplectic fillings)

Theorem 1.12 can be applied to any Reeb flow on a closed manifold for which one has a basic understanding of its fastest closed orbits. One does not need the contact manifold to admit a strong symplectic filling, nor must one preclude the existence of certain closed Reeb orbits. These points are illustrated by the following three applications.

(A) Consider the three-dimensional torus $T^3 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ with angular coordinates (x, y, θ) and the familiar family of contact forms

$$\lambda_k = \cos(k\theta) dx + \sin(k\theta) dy \tag{5}$$

for $k \in \mathbb{N}$. The underlying contact structures $\xi_k = \ker \lambda_k$ are all weakly symplectically fillable (and hence tight). However, Eliashberg proved in [Eli96] that for $k \geq 2$ the contact structures ξ_k are not strongly symplectically fillable. Hence, one cannot define symplectic cohomology or linearized contact homology for the contact forms they support. Nevertheless, Theorem 1.12 can be applied to all such forms.

The Reeb vector field of λ_k is

$$\cos(k\theta) \frac{\partial}{\partial x} + \sin(k\theta) \frac{\partial}{\partial y}.$$

Hence, the Reeb flow is linear on each xy -torus. The closed Reeb orbits represent nonzero classes of the form

$$(m, n, 0) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} = H_1(\mathbb{T}^3; \mathbb{Z}).$$

For a fixed pair of integers (m, n) these orbits all have period $\sqrt{m^2 + n^2}$. They foliate k subtori of the form

$$\mathbb{T}^2 \times \{\theta_{m,n}^j\} \subset \mathbb{T}^3,$$

where $\theta_{m,n}^1, \dots, \theta_{m,n}^k$ are the solutions of the two equations

$$\cos(k\theta) = \frac{m}{\sqrt{m^2 + n^2}}$$

and

$$\sin(k\theta) = \frac{n}{\sqrt{m^2 + n^2}}.$$

The orbits on these tori are simple if and only if m and n are relatively prime. In this case, for $T = \sqrt{m^2 + n^2}$ the collection $\mathcal{C}_{\lambda_k, (m,n,0)}(T)$ is a rigid constellation with $T^+ = +\infty$. (Here we have identified $H_1(\mathbb{T}^3; \mathbb{Z})$ with $[S^1, \mathbb{T}^3]$ in the obvious way.) A simple perturbation argument, see for example [Bou03], shows that each torus contributes 2 to the rank of $\mathcal{C}_{\lambda_k, (m,n,0)}(T)$. Hence, $\text{rank}(\mathcal{C}_{\lambda_k, (m,n,0)}(T)) = 2k$. With this, Theorem 1.12 implies the following result.

THEOREM 1.14. *Let $\lambda = f\lambda_k$, where $f : \mathbb{T}^3 \rightarrow \mathbb{R}$ is a positive function. Let m and n be relatively prime integers. If*

$$\frac{\max(f)}{\min(f)} < \frac{\sqrt{m^2 + n^2} + 1}{\sqrt{m^2 + n^2}}$$

and the closed Reeb orbits of λ in class $(m, n, 0)$ and with period in the interval

$$[\min(f)\sqrt{m^2 + n^2}, \max(f)\sqrt{m^2 + n^2}]$$

are nondegenerate, then there are at least $2k$ such orbits which are geometrically distinct from one another.

(B) Next we apply Theorem 1.12 to an overtwisted contact three-manifold. Consider the manifold $M = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \times [0, 2\pi]$ with coordinates (x, y, t) and the family of contact forms

$$\eta_k = \cos(kt) dx + \sin(kt) dy$$

for $k \in \mathbb{N} \cup \{0\}$. Each η_k is invariant under the free \mathbb{R}/\mathbb{Z} -action generated by the vector field $\partial/\partial y$. Thus, we can perform the contact cut operation, as defined by Lerman in [Ler01], with respect to the restriction of this action to the boundary of M (see [Ler01, Example 2.12]). We obtain in

this way contact forms $\tilde{\eta}_k$ on $\mathbb{R}/\mathbb{Z} \times \mathbb{S}^2$. Here x can still be viewed as a coordinate parameterizing the \mathbb{R}/\mathbb{Z} -fibres of the product $\mathbb{R}/\mathbb{Z} \times \mathbb{S}^2$ and we can identify $\tilde{\eta}_k$ with η_k away from the fibres over the poles of \mathbb{S}^2 (which correspond to $t = 0$ and $t = 2\pi$). As shown in [Ler01], the contact structures $\ker(\tilde{\eta}_k)$ are overtwisted and contactomorphic to one another for all $k > 0$. Hence, the contact manifolds $(\mathbb{R}/\mathbb{Z} \times \mathbb{S}^2, \tilde{\eta}_k)$ have no strong symplectic fillings for $k > 0$.

The Reeb vector field of $\tilde{\eta}_k$ (η_k) is

$$\cos(kt) \frac{\partial}{\partial x} + \sin(kt) \frac{\partial}{\partial y}$$

and so the closed Reeb orbits in class

$$m \in \mathbb{Z} = H_1(\mathbb{R}/\mathbb{Z} \times \mathbb{S}^2; \mathbb{Z})$$

occur for values of t such that

$$\cos(kt) = \frac{m}{\sqrt{m^2 + n^2}} \tag{6}$$

and

$$\sin(kt) = \frac{n}{\sqrt{m^2 + n^2}} \tag{7}$$

for some $n \in \mathbb{Z}$. For a solution t of these equations located in $(0, 2\pi)$, the corresponding orbits occur in an \mathbb{S}^1 -family that foliates the two-dimensional xy -tori, $\mathbb{T}^2 \times \{t\}$, as in the previous example. For $t = 0$ and $t = 2\pi$, the corresponding orbits are isolated and correspond to the fibres of $\mathbb{R}/\mathbb{Z} \times \mathbb{S}^2$ over the poles of \mathbb{S}^2 .

The fastest simple closed orbits of $\tilde{\eta}_k$ all have period 2π and correspond to the four cases $(m, n) = (\pm 1, 0)$ and $(m, n) = (0, \pm 1)$. For the case $(m, n) = (1, 0)$, there are $k + 1$ solutions to (6) and (7),

$$0, \frac{2\pi}{k}, \dots, \frac{2\pi(k-1)}{k}, 2\pi.$$

As described above, the values $t = 0$ and $t = 1$ correspond to isolated simple closed Reeb orbits, and the other $k - 1$ values of t correspond to subtori foliated by simple closed Reeb orbits (in class $(1, 0)$). The collection of all these orbits forms a rigid constellation of rank $2k$ with $T = 2\pi$ and $T^+ = 2\sqrt{2}\pi$.

For the case $(m, n) = (-1, 0)$, there are k solutions of (6) and (7),

$$\frac{\pi}{k}, \frac{3\pi}{k}, \dots, 2\pi - \frac{\pi}{k}.$$

Each of these corresponds to unique subtori foliated by simple closed Reeb orbits (in class $(-1, 0)$). The collection of these tori again forms a rigid constellation of rank $2k$ with $T = 2\pi$ and $T^+ = 2\sqrt{2}\pi$.

Finally, for the cases $(m, n) = (0, \pm 1)$, there are again k solutions of (6) and (7) each of which corresponds to a subtorus foliated by simple closed Reeb orbits which are contractible in $\mathbb{R}/\mathbb{Z} \times \mathbb{S}^2$. These also form a rigid constellation of rank $2k$, this time with $T = 2\pi$ and $T^+ = 4\pi$.

Applying Theorem 1.12 to these three rigid constellations, we get the following rigidity theorem.

THEOREM 1.15. *Consider a contact form on $\mathbb{R}/\mathbb{Z} \times \mathbb{S}^2$ of the form $\lambda = f\tilde{\eta}_k$, where f is a positive function on $\mathbb{R}/\mathbb{Z} \times \mathbb{S}^2$.*

(i) *If $\max(f) < \sqrt{2} \min(f)$ and every closed Reeb orbit of λ in class*

$$\pm 1 \in \mathbb{Z} = H_1(\mathbb{R}/\mathbb{Z} \times \mathbb{S}^2; \mathbb{Z})$$

and with period in the interval $[\min(f)2\pi, \max(f)2\pi]$ is nondegenerate, then there are at least $2k$ such orbits which are geometrically distinct from one another.

(ii) *If $\max(f) < 2 \min(f)$ and every contractible closed Reeb orbit of λ with period in the interval $[\min(f)2\pi, \max(f)2\pi]$ is nondegenerate, then there are at least $4k$ such orbits which are distinct from one another. They are geometrically distinct if there are no contractible closed Reeb orbits of λ with period less than or equal to $2\pi(\max(f) - \min(f))$.*

(C) Now we apply Theorem 1.12 to a family of overtwisted contact forms on \mathbb{S}^3 . We start again with the manifold $M = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \times [0, 2\pi]$ with coordinates (x, y, t) . For $k \in \mathbb{N} \cup \{0\}$, we now consider the contact forms

$$\zeta_k = \cos\left(\left(k + \frac{1}{4}\right)t\right) dx + \sin\left(\left(k + \frac{1}{4}\right)t\right) dy.$$

Each ζ_k is invariant under the free \mathbb{R}/\mathbb{Z} -actions generated by the vector fields $\partial/\partial x$ and $\partial/\partial y$. This time we perform a different contact cut operation by choosing the \mathbb{R}/\mathbb{Z} -action generated by $\partial/\partial y$ at the boundary component $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \times \{0\}$, and the \mathbb{R}/\mathbb{Z} -action generated by $\partial/\partial x$ at the boundary component $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \times \{2\pi\}$. The resulting manifold is \mathbb{S}^3 and we denote the resulting contact form by $\tilde{\zeta}_k$. For all $k > 0$, the contact structures $\ker(\tilde{\zeta}_k)$ are overtwisted and contactomorphic to one another (see the proof of [Ler01, Theorem 3.1]).

Away from the fibres of the Hopf fibration that lie over the two poles in \mathbb{S}^2 , we can identify $\tilde{\zeta}_k$ with ζ_k . Thus, we can analyze the closed Reeb orbits as before. Here the set of the fastest simple closed orbits of $\tilde{\zeta}_k$ is a rigid constellation with $T = 2\pi$, $T^+ = 2\sqrt{2}\pi$. It comprises two isolated orbits and $4k$ \mathbb{S}^1 -families of closed orbits and so has rank $8k + 1$. In this setting Theorem 1.12 implies the following result.

THEOREM 1.16. *Consider a contact form on \mathbb{S}^3 of the form $\lambda = f\tilde{\zeta}_k$, where f is a positive function. If $\max(f) < \sqrt{2} \min(f)$ and every contractible closed Reeb orbit of λ with period in the interval $[\min(f)2\pi, \max(f)2\pi]$ is nondegenerate, then there are at least $8k + 2$ such orbits which are distinct from one another. They are geometrically distinct if there are no closed Reeb orbits of λ with period less than or equal to $2\pi(\max(f) - \min(f))$.*

1.5 Applications (with strong symplectic fillings)

For contact manifolds which admit a symplectic filling, Theorems 1.6 and 1.12 both hold as stated but with improved ranges (if α is interpreted as a free homotopy class of the filling). For example, if we assume that the prequantization space (M, λ_Q) admits an exact symplectic filling, then Theorem 1.6 holds, as stated, with the larger upper bound

$$\frac{\max(f)}{\min(f)} < |\alpha_{\mathbf{f}}| + 1.$$

Similarly, assuming that there is an exact symplectic filling of (M, λ_0) , Theorem 1.12 holds, as stated, with the upper bound

$$\frac{\max(f)}{\min(f)} < \frac{T^+}{T}.$$

As the next set of applications demonstrates, this version of Theorem 1.12 allows one to view a variety of previously known rigidity phenomena from a single perspective.

(D) Applying Theorem 1.12 to the contact forms in Example 1.10, one can recover the various pinching theorems of Berestycki *et al.* established in [BLMR85] up to some extra nondegeneracy assumptions.

(E) Returning to the setting of Example 1.11, consider a closed Riemannian manifold (B, g) with negative sectional curvature and the corresponding unit cosphere bundle $\Sigma_{g^*} \subset T^*B$. Let $\tilde{\alpha}$ be a nontrivial primitive element of $[\mathbb{R}/\mathbb{Z}, B]$ and denote its lift to $[\mathbb{R}/\mathbb{Z}, \Sigma_{g^*}]$ by α . As described in Example 1.11, the rigid constellation $\mathcal{C}_{\lambda_{g^*}, \alpha}(T_{\min}(\lambda_{g^*}, \alpha))$ has $T^+ = +\infty$. It also has rank 1 and so Theorem 1.12 implies that every hypersurface Σ in T^*B which is fibrewise star-shaped about the zero section must carry at least one closed characteristic in class α . A similar but stronger result in this direction is proved in [BPS03] as Corollary 3.4.2 (see also [HV88]).

(F) Let g_n be the standard flat metric on the n -dimensional torus \mathbb{T}^n . Denote its cosphere bundle by $\Sigma_{g_n^*} \subset T^*\mathbb{T}^n$ and the restriction of the tautological one-form to $\Sigma_{g_n^*}$ by $\lambda_{g_n^*}$. Every nontrivial primitive class $\tilde{\alpha}$ in $[\mathbb{R}/\mathbb{Z}, \mathbb{T}^n]$ lifts to a class α in $[\mathbb{R}/\mathbb{Z}, \Sigma_{g_n^*}]$, which is again nontrivial and primitive. There is a unique \mathbb{T}^n -family of closed Reeb orbits in class α . Denoting their period by $T(\alpha)$, the constellation $\mathcal{C}_{\lambda_{g_n^*}, \alpha}(T(\alpha))$ is rigid with $T^+ = +\infty$ and $\text{rank}(\mathcal{C}_{\lambda_{g_n^*}, \alpha}(T)) = 2^n$. With this, Theorem 1.12 implies the following.

THEOREM 1.17. *Let Σ be a smooth hypersurface in $T^*\mathbb{T}^n$ which is fibrewise star-shaped about the zero section. Let λ_Σ be the restriction of the tautological one-form to Σ . If the closed Reeb orbits of λ_Σ in class α are all nondegenerate, then there must be at least 2^n such orbits which are geometrically distinct.*

This complements a theorem of Cielebak from [Cie94], which establishes the existence of at least

$$\left\lfloor \frac{n}{2} \right\rfloor + 1$$

geometrically distinct closed Reeb orbits with no nondegeneracy assumption.

(G) The finite collection of prime closed geodesics in Katok’s famous examples of Finsler metrics from [Kat73] also yields a rich source of rigid constellations. Consider for example a Katok metric g_K on the sphere $P = \mathbb{S}^{2n}$ or \mathbb{S}^{2n-1} . It has precisely $2n$ prime closed geodesics and can be constructed so that for any $\epsilon \in (0, 1)$ the longest prime geodesic has length $1 + \epsilon$ and the shortest has length $1 - \epsilon$. For $\epsilon < \frac{1}{2}$ and $T = 1 + \epsilon$, the collection of prime closed geodesics is then a rigid constellation with $T^+ = 2(1 - \epsilon)$ and rank $2n$. If, as above, we let $\Sigma_{g_K^*} \subset T^*P$ be the cosphere bundle of g_K , then for any other hypersurface Σ of $T^*\mathbb{S}^{2n}$ which is fibrewise star-shaped about the zero section there is a unique smooth function $f_\Sigma : \Sigma_{g_K^*} \rightarrow (0, \infty)$ such that

$$\Sigma = \{(q, f_\Sigma(q, p)p) : (q, p) \in \Sigma_{g_K^*}\}.$$

Let λ_Σ be the restriction of the tautological one-form to Σ . In this setting Theorem 1.12 implies the following result.

THEOREM 1.18. *If*

$$\frac{\max(f_\Sigma)}{\min(f_\Sigma)} < \frac{2(1 - \epsilon)}{1 + \epsilon}$$

and all the closed Reeb orbits of λ_Σ with period in the interval

$$\left[\frac{2\pi \min(f_\Sigma)}{1 + \epsilon}, \frac{2\pi \max(f_\Sigma)}{1 - \epsilon} \right]$$

are nondegenerate, then λ_Σ has at least $2n$ distinct closed Reeb orbits with periods in this interval. These orbits are geometrically distinct if there are no closed Reeb orbits of λ_Σ with period less than or equal to the length of this interval.

If one restricts Theorem 1.18 to hypersurfaces that are cosphere bundles of Finsler metrics, it then yields multiplicity results for closed geodesics which complement recent work of Rademacher in [Rad07] and of Wang in [Wan15].

1.6 Fast orbits

The possible presence of periodic orbits with small periods sometimes obstructs our ability to conclude that the distinct orbits detected by Theorems 1.6 and 1.12 are in fact geometrically distinct. This is a fundamental, and somewhat notorious, difficulty common to such multiplicity problems. In certain settings one can preclude the existence of such fast orbits by imposing geometric restrictions, like the convexity of the hypersurfaces in [EL80]. It is natural then to ask if one can find conditions on the contact structure which preclude fast orbits and are invariant under contactomorphisms. We prove here that this is not possible. This is one implication of our final theorem, which can also be viewed as a *soft* compliment to Theorems 1.6 and 1.12.

THEOREM 1.19. *Let (M, λ_0) be a contact manifold. For any free homotopy class $\alpha \in [\mathbb{S}^1, M]$ and any positive constants $c_1, c_2 > 0$, there is a contact form $\lambda = f\lambda_0$ on M such that $\min(f) = 1$, $\max(f) < 1 + c_1$ and λ has a closed Reeb orbit in class α of period less than c_2 .*

There are some intriguing phenomena hidden in the gap between the construction underlying Theorem 1.19 and, say, Theorem 1.12. For example, in creating the nearby form λ in Theorem 1.19 with an arbitrarily fast closed orbit we are forced, by the Reeb condition, to cede all control over the number and basic properties of any additional closed Reeb orbits we might also create in the process. This follows from work of Rechtman in [Rec09] and is discussed further in Remark 5.5.

1.7 Related works

Our proof of Theorem 1.6 is motivated by the method to obtain cup-length estimates in Floer theory that was introduced by Albers and Momin in [AM10] and further developed by Albers and Hein in [AH16].

The proof of Theorem 1.12 has two main parts. The first involves the construction of a version of Hamiltonian Floer theory for rigid constellations. In this construction we use several results concerning the Hamiltonian Floer theory of autonomous Hamiltonians established by Bourgeois and Oancea in [BO09]. To obtain the C^0 -bounds for our Floer trajectories (which allows us to do away with fillings), we also adapt an argument of Albers *et al.* from [AFM15]. In the second part of the proof we use the Floer-theoretic tools developed in the first to adapt an argument of Chekanov from [Che98] to detect the desired closed orbits. This is based on previous work of the author from [Ker08].

Similar ideas to those underlying Theorem 1.12 were developed by Gutt in his thesis [Gut14] and subsequent paper [Gut15].¹ In these works, Gutt shows that positive \mathbb{S}^1 -equivariant symplectic homology can be used as a contact invariant for a certain class of fillable contact manifolds that can be realized as the boundary of Liouville domains. Among the many interesting applications of his theory, Gutt reproves Theorem 1.1 under the additional strong nondegeneracy assumption, and also proves a result [Gut15, Theorem 1.6] very similar in content to Theorem 1.6 here. Happily, besides a shared debt owed to the technical foundations for \mathbb{S}^1 -equivariant Hamiltonian Floer theory laid down by Bourgeois and Oancea, this is essentially the extent of the overlap between the two projects.

The construction of a Reeb semi-plug described in the proof of Theorem 1.19 is similar in several details to Cieliebak’s construction of a confoliation-type plug from [Cie97]. The goals of the two constructions diverge at an early stage, however. In keeping within the class of Reeb flows we sacrifice here the possibility, achieved in [Cie97], of realizing the insertion of our plugs on fillable contact manifolds, as the result of a symplectomorphism acting on the interior of a filling.

2. Hamiltonian Floer theory and rigid constellations

Let $\mathcal{C}_{\lambda_0,\alpha}(T)$ be a rigid constellation for the contact form λ_0 on M . In this section we develop a version of Hamiltonian Floer theory adapted to $\mathcal{C}_{\lambda_0,\alpha}(T)$.

2.1 Admissible families of functions

Denoting the \mathbb{R} -coordinate of $\mathbb{R} \times M$ by τ , we consider the symplectization

$$(\mathbb{R} \times M, d(e^\tau \lambda_0)).$$

DEFINITION 2.1. A smooth function $H : \mathbb{R} \times M \rightarrow \mathbb{R}$ is an *admissible* Hamiltonian if it is nonnegative, satisfies

$$H(\tau, p) = 0 \quad \text{for all } \tau \leq 0 \tag{8}$$

and if there is a $\mathbf{T} > 0$ such that

$$dH_{(\tau,p)} = 0 \quad \text{for all } \tau \geq \mathbf{T}. \tag{9}$$

The Hamiltonian vector field, V_H , of H is defined by the equation

$$i_{V_H} \omega = -dH.$$

Conditions (8) and (9) imply that the support of dH is compact and so the flow of V_H is defined for all times.

Let $x(t)$ be a 1-periodic orbit of H (i.e. V_H). We define the action of $x(t)$ by

$$\mathcal{A}_H(x) = - \int_{\mathbb{R}/\mathbb{Z}} x^*(e^\tau \lambda_0) + \int_{\mathbb{R}/\mathbb{Z}} H(x(t)) dt.$$

Let $\mathcal{P}^-(H)$ be the set of all 1-periodic orbits of H which have negative action. Since H is nonnegative, every x in $\mathcal{P}^-(H)$ is nonconstant. We say that H is *nondegenerate* if every x in $\mathcal{P}^-(H)$ is also transversally nondegenerate. Strict nondegeneracy is impossible since H is

¹ The author is grateful to Peter Albers for notifying him of Gutt’s thesis when the author spoke of the results presented here at the Lorentz Centre in July 2014.

autonomous. In particular, every nonconstant 1-periodic orbit $x(t)$ of H belongs to an \mathbb{R}/\mathbb{Z} -family of such orbits, which we will denote by X . The elements of X all have the same action and so the notation $\mathcal{A}_H(X)$ can and will be used unambiguously. The collection of all \mathbb{R}/\mathbb{Z} -families of 1-periodic orbits of H with negative actions will be denoted by $\mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H)$.

The *action gap* of a nondegenerate admissible Hamiltonian H is defined to be

$$\Delta(H) = \sup_{X, Y \in \mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H)} \{ \mathcal{A}_H(X) - \mathcal{A}_H(Y) : \mathcal{A}_H(X) \neq \mathcal{A}_H(Y) \},$$

where we set $\Delta(H) = 0$ if there are no families X and Y meeting the required conditions (for example if dH is sufficiently C^1 -small). Given two nondegenerate admissible Hamiltonians H^0 and H^1 , we set

$$\Delta(H^0, H^1) = \sup \{ \mathcal{A}_{H^0}(X^0) - \mathcal{A}_{H^1}(X^1) : \mathcal{A}_{H^0}(X^0) \neq \mathcal{A}_{H^1}(X^1) \},$$

where here the supremum is over pairs $(X^0, X^1) \in \mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H^0) \times \mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H^1)$ and we again set $\Delta(H^0, H^1) = 0$ if no relevant pairs exist.

An *admissible homotopy* from H^0 to H^1 is a smooth family of functions H^s for $s \in \mathbb{R}$ such that for some $\mathbf{S} > 0$ and $\mathbf{T} > 0$ we have:

- (H^s1) $H^s = H^0$ for all $s \leq -\mathbf{S}$;
- (H^s2) $H^s = H^1$ for all $s \geq \mathbf{S}$;
- (H^s3) for all $s \in \mathbb{R}$, the support of $d(H^s)$ is contained in $[0, \mathbf{T}] \times M$.

The *cost* of the homotopy H^s is defined to be

$$\text{cost}(H^s) = \int_{\mathbb{R}} \max_{(\tau, p)} (\partial_s(H^s(\tau, p))) ds.$$

Finally, an *admissible homotopy of homotopies* between H^0 and H^1 is a smooth \mathbb{R}^2 -family of functions $H^{r,s}$ such that for each $r \in \mathbb{R}$, $H^{r,s}$ is an admissible homotopy from H^0 to H^1 and the supports of all the $d(H^{r,s})$ are contained in a single neighborhood of the form $[0, \mathbf{T}] \times M$. The cost of $H^{r,s}$ is defined as

$$\text{cost}(H^{r,s}) = \int_{\mathbb{R}} \max_{(\tau, p, r)} (\partial_s(H^{r,s}(\tau, p))) ds.$$

2.2 Dividing and tuned Hamiltonians

Let H be an admissible Hamiltonian of the form

$$H(\tau, p) = h(e^\tau).$$

We will refer to h as the *profile* of H . The Hamiltonian vector field of H is

$$V_H(\tau, p) = h'(e^\tau)R_{\lambda_0}(p).$$

Thus, every nonconstant 1-periodic orbit $x(t)$ of V_H corresponds to a unique closed Reeb orbit $\gamma_x(t)$ of λ_0 such that

$$x(t) = (\tau_x, \gamma_x(h'(e^{\tau_x})t)). \tag{10}$$

In particular, $h'(e^{\tau_x})$ is the period of the orbit γ_x , which we will also denote by T_{γ_x} . The action of $x(t)$ is

$$\mathcal{A}_H(x) = -e^{\tau_x}h'(e^{\tau_x}) + h(e^{\tau_x}). \tag{11}$$

Note also that the \mathbb{R}/\mathbb{Z} -family of 1-periodic orbits of V_H containing x , X corresponds to the unique \mathbb{R}/\mathbb{Z} -family of closed Reeb orbits of λ_0 containing γ_x . Denoting this family of Reeb orbits by Γ_X , we have

$$\Gamma_X = \bigcup_{x \in X} \gamma_x.$$

We now define a class of Hamiltonians which have useful collections of 1-periodic orbits (related to useful collections of closed Reeb orbits) that can be identified simply by the fact that their actions are negative. For positive constants $a, b > 0$ and $c > 1 + a$, let $\mathfrak{h}_{a,b,c}$ be the space of smooth profile functions $h : \mathbb{R} \rightarrow \mathbb{R}$ with the following properties:

- (h1) $h(s) = 0$ for $s \leq 1$;
- (h2) $h''(s) > 0$ for $s \in (1, 1 + a)$;
- (h3) $h(1 + a) = a^2$;
- (h4) $h'(s) = b$ for $s \in [1 + a, c]$;
- (h5) $h''(s) < 0$ for $s \in (c, c + a)$;
- (h6) $h(s) = b(c - 1 - a) + a^2$ for $s \geq c + a$.

Let $\mathcal{R}^b(\lambda_0)$ be the set of closed Reeb orbits of λ_0 with period less than b .

LEMMA 2.2. *Suppose that $H(\tau, p) = h(e^\tau)$ for some profile h in $\mathfrak{h}_{a,b,c}$. If b is not the period of a closed Reeb orbit of λ_0 and c is sufficiently large, then every $x \in \mathcal{P}^-(H)$ is nonconstant and of the form*

$$x = (\tau_x, \gamma_x(h'(e^{\tau_x})t))$$

for some e^{τ_x} in $(1, 1 + a)$ and some γ_x in $\mathcal{R}^b(\lambda_0)$. Moreover, the correspondence $x \rightarrow \gamma_x$ defines a bijection between $\mathcal{P}^-(H)$ and $\mathcal{R}^b(\lambda_0)$.

Proof. Every point (τ, p) in $\mathbb{R} \times M$ with $e^\tau \leq 1$ or $e^\tau \geq c + a$ corresponds to a constant periodic orbit of H . The constant orbits in the first region have action (H -value) equal to zero by (h1). The orbits in the second region have strictly positive action (H -value) by conditions (h3)–(h6). Thus, neither set contributes to $\mathcal{P}^-(H)$.

It follows from our choice of b that the nonconstant periodic orbits of H are of the form

$$x(t) = (\tau_x, \gamma_x(h'(e^{\tau_x})t))$$

for some e^{τ_x} in either $(1, 1 + a)$ or $(c, c + a)$. As mentioned above, the action of such an orbit is

$$\mathcal{A}_H(x) = -e^{\tau_x} h'(e^{\tau_x}) + h(e^{\tau_x}). \tag{12}$$

These action values correspond to values of the function

$$F(s) = -sh'(s) + h(s).$$

By (h1), we have $F(1) = 0$ and by (h2) we have $F'(s) = -sh''(s) < 0$ in $(1, 1 + a)$. Thus, all the nonconstant periodic orbits x with e^{τ_x} in $(1, 1 + a)$ have negative action and so appear in $\mathcal{P}^-(H)$.

On the other hand, given a closed Reeb orbit γ of λ_0 with period $T_\gamma < b$, it follows from (h2) and (h4) that there is a unique solution τ_γ of

$$h'(e^\tau) = T_\gamma$$

contained in the interval $(1, 1 + a)$. Then

$$x(t) = (\tau_\gamma, \gamma(T_\gamma t))$$

belongs to $\mathcal{P}^-(H)$.

To complete the proof it just remains to show that a nonconstant periodic orbit x as above with $e^{\tau x}$ in $(c, c + a)$ has positive action for all large enough values of c . For such an x , we have

$$\mathcal{A}_H(x) > -(c + a)h'(e^{\tau x}) + (c - 1 - a)b$$

and thus

$$\mathcal{A}_H(x) > c(b - T_{\gamma_x}) - b(2a + 1). \tag{13}$$

Since the period spectrum, $\mathcal{T}(\lambda_0)$, of λ_0 is closed and b lies outside it, the quantity

$$b - \max\{t \in \mathcal{T}(\lambda_0) : t < b\}$$

is positive. From (13), we then get

$$\mathcal{A}_H(x) > c(b - \max\{t \in \mathcal{T}(\lambda_0) : t < b\}) - b(2a + 1).$$

Hence, for all sufficiently large $c > 0$, we have $\mathcal{A}_H(x) > 0$ for all $x \in \text{Crit}(\mathcal{A}_H)$ with $e^{\tau x} \in (c, c + a)$, as desired. \square

Let $\mathcal{H}_{a,b,c,\kappa}$ be the space of smooth functions of the form

$$H(\tau, p) = h(e^{\tau - \kappa}),$$

where $\kappa \geq 0$ and h is in $\mathfrak{h}_{a,b,c}$. For an H in $\mathcal{H}_{a,b,c,\kappa}$, we have

$$V_H(\tau, p) = h'(e^{\tau - \kappa})e^{-\kappa}R_{\lambda_0}(p)$$

and for every nonconstant 1-periodic orbit $x(t)$ of V_H there is a unique closed Reeb orbit $\gamma_x(t)$ of λ_0 such that

$$x(t) = (\tau_x, \gamma_x(h'(e^{\tau_x - \kappa})e^{-\kappa}t)) \tag{14}$$

and

$$\mathcal{A}_H(x) = -e^{\tau_x}T_{\gamma_x} + h(e^{\tau_x - \kappa}). \tag{15}$$

Arguing as above, we then get the following.

LEMMA 2.3. *Let H be in $\mathcal{H}_{a,b,c,\kappa}$. If $be^{-\kappa}$ is not in $\mathcal{T}(\lambda_0)$ and c is sufficiently large, then every $x \in \mathcal{P}^-(H)$ is nonconstant and of the form*

$$x(t) = (\tau_x, \gamma_x(h'(e^{\tau_x - \kappa})e^{-\kappa}t))$$

for some $e^{\tau_x} \in (e^\kappa, (1 + a)e^\kappa)$ and some $\gamma_x \in \mathcal{R}^{be^{-\kappa}}(\lambda_0)$. Moreover, the correspondence $x \rightarrow \gamma_x$ defines a bijection between $\mathcal{P}^-(H)$ and $\mathcal{R}^{be^{-\kappa}}(\lambda_0)$.

DEFINITION 2.4. A function H as in Lemma 2.3 will be called a *dividing* Hamiltonian.

Remark 2.5. Note that, changing all the forms λ_0 above to forms λ , we can use the same definition for the notion of dividing functions which detect collections of closed Reeb orbits of λ .

DEFINITION 2.6. A Hamiltonian H is *tuned* to the rigid constellation $\mathcal{C}_{\lambda_0,\alpha}(T)$ if it belongs to some $\mathcal{H}_{a,b,c,\kappa}$ for which:

- (t1) $e^\kappa < \min\{T^+/T, (T_{\min}(\lambda_0) + T_{\min}(\lambda_0, \alpha))/T\}$;
- (t2) $Te^\kappa < b < T^+$;
- (t3) $c > 2b/(b - Te^\kappa)$.

Every H tuned to $\mathcal{C}_{\lambda_0,\alpha}(T)$ is dividing. So, the sets $\mathcal{P}^-(H)$ and $\mathcal{R}^{be^{-\kappa}}(\lambda_0)$ are in bijection. The class $\alpha \in [\mathbb{S}^1, M]$ determines a unique class in $[\mathbb{R}/\mathbb{Z}, \mathbb{R} \times M]$, which we will also denote by α . Let $\mathcal{P}_\alpha^-(H)$ be the subset $\mathcal{P}^-(H)$ consisting of 1-periodic orbits of H which have negative action and which represent the class α . The point of Definition 2.6 is that, if H is tuned to $\mathcal{C}_{\lambda_0,\alpha}(T)$, then the elements of $\mathcal{P}_\alpha^-(H)$ correspond precisely to the elements of $\mathcal{C}_{\lambda_0,\alpha}(T)$, i.e.

$$x \in \mathcal{P}_\alpha^-(H) \iff \gamma_x \in \mathcal{C}_{\lambda_0,\alpha}(T). \tag{16}$$

To identify tuned Hamiltonians that are suitable for defining Floer theory on the symplectization of (M, λ_0) , we must refine this notion further. It is clear from (15) that for a tuned H in $\mathcal{H}_{a,b,c,\kappa}$ with a small value of a , the actions of the orbits in $\mathcal{P}_\alpha^-(H)$ are close to $-e^\kappa$ times the period of an element of $\mathcal{C}_{\lambda_0,\alpha}(T)$. Developing this further, we get the following useful set of inequalities.

LEMMA 2.7. *For a rigid constellation $\mathcal{C}_{\lambda_0,\alpha}(T)$, there is a positive constant $\bar{a} > 0$ such that the following hold.*

- (i) *If H in $\mathcal{H}_{a,b,c,\kappa}$ is tuned to $\mathcal{C}_{\lambda_0,\alpha}(T)$ and $a < \bar{a}$, then*

$$\mathcal{A}_H(x) < -T_{\min}(\lambda_0, \alpha)/2 \tag{17}$$

for every $x \in \mathcal{P}_\alpha^-(H)$.

- (ii) *If H in $\mathcal{H}_{a,b,c,\kappa}$ is tuned to $\mathcal{C}_{\lambda_0,\alpha}(T)$ and $a < \bar{a}$, then*

$$\Delta(H) < T_{\min}(\lambda_0). \tag{18}$$

- (iii) *If $H^0 \in \mathcal{H}_{a^0,b^0,c^0,\kappa^0}$ and $H^1 \in \mathcal{H}_{a^1,b^1,c^1,\kappa^1}$ are tuned to $\mathcal{C}_{\lambda_0,\alpha}(T)$ and $a^0, a^1 < \bar{a}$, then*

$$\Delta(H^0, H^1) < T_{\min}(\lambda_0). \tag{19}$$

Proof. By (15) and the properties of h , we have

$$-e^\kappa(1+a)T < \mathcal{A}_H(x) < -e^\kappa T_{\min}(\lambda_0, \alpha) + a^2 \tag{20}$$

for every $x \in \mathcal{P}_\alpha^-(H)$. The fact that inequality (17) holds for sufficiently small a follows immediately from this and the fact that $\kappa \geq 0$. The inequalities of (20) also imply that

$$\Delta(H) < e^\kappa(T - T_{\min}(\lambda_0, \alpha)) + ae^\kappa(T - a). \tag{21}$$

Since $\mathcal{C}_{\lambda_0,\alpha}(T)$ is rigid, we have $T < T_{\min}(\lambda_0, \alpha) + T_{\min}(\lambda_0)$. Together with condition (t1), this yields

$$e^\kappa < \frac{T_{\min}(\lambda_0)}{T - T_{\min}(\lambda_0, \alpha)}. \tag{22}$$

The fact that inequality (18) holds for sufficiently small a now follows immediately from (21) and (22). Finally, (20) also implies that

$$\Delta(H^0, H^1) \leq -e^{\kappa^0}T_{\min}(\lambda_0, \alpha) + e^{\kappa^1}T + (a^0)^2 + e^{\kappa^1}a^1T \tag{23}$$

$$\leq e^{\kappa^1}T - T_{\min}(\lambda_0, \alpha) + (a^0)^2 + e^{\kappa^1}a^1T. \tag{24}$$

This, together with (t1), implies that (19) holds when both a^0 and a^1 are sufficiently small. \square

DEFINITION 2.8. A function H in $\mathcal{H}_{a,b,c,\kappa}$ tuned to the rigid constellation $\mathcal{C}_{\lambda_0,\alpha}(T)$ is said to be *finely tuned* to $\mathcal{C}_{\lambda_0,\alpha}(T)$ if $a < \bar{a}$ and hence inequalities (17), (18) and (19) hold.

2.3 Almost complex structures

For the next four subsections we will assume that H is an admissible Hamiltonian and that each element of $\mathcal{P}^-(H)$, that is, each 1-periodic orbit of H with negative action, is transversally nondegenerate. Given such an H , we now define a useful class of almost complex structures on $\mathbb{R} \times M$. Recall first that an almost complex structure J on the symplectization $(\mathbb{R} \times M, d(e^\tau \lambda))$ is said to be *cylindrical* if it is invariant under τ -translations and satisfies $J(\partial/\partial\tau) = R_\lambda$. The related notion of being cylindrical on subsets of the form $\{\tau \leq \mathbf{T}\}$ or $\{\tau \geq \mathbf{T}\}$ is defined in the obvious way.

Denote by $\mathcal{J}(H)$ the set of smooth almost complex structures on $\mathbb{R} \times M$ with the following properties:

- (J1) J is compatible with $d(e^\tau \lambda_0)$;
- (J2) $J = J_0$ on $\{\tau \leq 0\}$, where J_0 is fixed and cylindrical;
- (J3) J is cylindrical on $\{\tau \geq \mathbf{T}\}$ for some $\mathbf{T} > 0$;
- (J4) for any point $z = (\tau, p)$ on the image of a family X in $\mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H)$, we have

$$[V_H, JV_H](z) \neq 0 \quad \text{and} \quad [V_H, JV_H](z) \notin \text{Span}\{V_H(z), JV_H(z)\}.$$

LEMMA 2.9 [BO09, see proof of Proposition 3.5(i) in §4]. *The set $\mathcal{J}(H)$ is a nonempty open subset of the set of all smooth almost complex structures with properties (J1)–(J3).*

2.4 Floer trajectories: transversality

Consider a pair

$$F = (H, J)$$

consisting of an admissible Hamiltonian H and an almost complex structure J in $\mathcal{J}(H)$. We will refer to F as a Floer data set and will denote the set of all Floer data sets by \mathbf{F} .

Given two (nonconstant) \mathbb{R}/\mathbb{Z} -families X and Y in $\mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H)$ and an $F = (H, J)$ in \mathbf{F} , we define

$$\widehat{\mathcal{M}}(X, Y; F)$$

to be the space of solutions $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R} \times M$ of

$$\partial_s u + J(u)(\partial_t u - V_H(u)) = 0$$

which satisfy the asymptotic conditions

$$\lim_{s \rightarrow -\infty} u(s, t) \in X, \quad \lim_{s \rightarrow +\infty} u(s, t) \in Y \quad \text{and} \quad \lim_{s \rightarrow \pm\infty} \partial_s u(s, t) = 0,$$

where the convergences are all uniform in t . The following transversality statement for these spaces is established by Bourgeois and Oancea in [BO09].

PROPOSITION 2.10 [BO09, Proposition 3.5(i)]. *There is a subset $\mathcal{J}_{\text{reg}}(H)$ of $\mathcal{J}(H)$ of second category such that for any $J \in \mathcal{J}_{\text{reg}}(H)$ and any pair of \mathbb{R}/\mathbb{Z} -families $X, Y \in \mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H)$ such that the orbits of either X or Y are simple, each set $\widehat{\mathcal{M}}(X, Y; F)$ for $F = (H, J)$ is a smooth finite-dimensional manifold.*

The assumption that either X or Y is simple implies that the elements of $\widehat{\mathcal{M}}(X, Y; F)$ are all somewhere injective. Having thus avoided the fundamental difficulty of dealing with multiply covered maps, transversality can then be established in the manner of [HS95]. The subtle point, observed and overcome in [BO09], is that condition (J4) can be used to prove that the set of injective points in the domain of an element of $\widehat{\mathcal{M}}(X, Y; F)$ constitutes an open and dense subset of some neighborhood of an end asymptotic to a simple orbit. Let \mathbf{F}_{reg} be the subset of \mathbf{F} consisting of Floer data sets $F = (H, J)$ with $J \in \mathcal{J}_{\text{reg}}(H)$.

Next we consider spaces of Floer continuation trajectories. Let H^s be an admissible homotopy between admissible Hamiltonians H^0 and H^1 whose nonconstant 1-periodic orbits with negative action are transversally nondegenerate. Let J^s be a smooth family of $d(e^\tau \lambda_0)$ -compatible almost complex structures such that for some $\mathbf{S} > 0$ and $\mathbf{T} > 0$, we have:

- (J^s1) $J^s = J^0 \in \mathcal{J}(H^0)$ for all $s \leq -\mathbf{S}$;
- (J^s2) $J^s = J^1 \in \mathcal{J}(H^1)$ for all $s \geq \mathbf{S}$;
- (J^s3) $J^s = J_0$ on $(-\infty, 0] \times M$ for all $s \in \mathbb{R}$;
- (J^s4) J^s is cylindrical (for λ_0) on $[\mathbf{T}, +\infty) \times M$ for all $s \in \mathbb{R}$.

We refer to the pair $F^s = (H^s, J^s)$ as Floer continuation data (connecting $F^0 = (H^0, J^0)$ to $F^1 = (H^1, J^1)$) and will denote the set of all such triples as $\mathbf{F}^s = \mathbf{F}^s(F^0, F^1)$.

For an $F^s = (H^s, J^s)$ in \mathbf{F}^s and families X^0 in $\mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H^0)$ and X^1 in $\mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H^1)$, let

$$\widehat{\mathcal{M}}_s(X^0, X^1; F^s)$$

be the space of solutions $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R} \times M$ of

$$\partial_s u + J^s(u)(\partial_t u - V_{H^s}(u)) = 0$$

which satisfy the asymptotic conditions

$$\lim_{s \rightarrow -\infty} u(s, t) \in X^0, \quad \lim_{s \rightarrow +\infty} u(s, t) \in X^1 \quad \text{and} \quad \lim_{s \rightarrow \pm\infty} \partial_s u(s, t) = 0.$$

Arguing as above, one gets the following basic transversality statement.

PROPOSITION 2.11. *Suppose that F^0 and F^1 are in \mathbf{F}_{reg} . Then there is a subset $\mathbf{F}_{\text{reg}}^s$ of $\mathbf{F}^s(F^0, F^1)$ of second category such that for any $F^s \in \mathbf{F}_{\text{reg}}^s$ and any families $X^0 \in \mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H^0)$ and $X^1 \in \mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H^1)$ such that the orbits of either X^0 or X^1 are simple, each $\widehat{\mathcal{M}}_s(X^0, X^1; F^s)$ is a smooth finite-dimensional manifold.*

More generally, we have the moduli space of Floer trajectories corresponding to an admissible homotopy of homotopies, $H^{r,s}$, from H^0 to H^1 . Let $J^{r,s}$ be a smooth \mathbb{R}^2 -family of $d(e^\tau \lambda_0)$ -compatible almost complex structures such that for some smooth positive function $S(r)$ and for some $\mathbf{T} > 0$, the following condition holds:

- (J^{r,s}1) for each r , the \mathbb{R} -family $J^{r,s}$ satisfies (J^s1)–(J^s4) for $S = S(r)$.

Following the pattern above, we refer to the triple $F^{r,s} = (H^{r,s}, J^{r,s})$ as Floer homotopy data and will denote the set of all such triples by $\mathbf{F}^{r,s}$. For an $F^{r,s} = (H^{r,s}, J^{r,s})$ in $\mathbf{F}^{r,s}$ and two families X^0 in $\mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H)$, X^1 in $\mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H^1)$, let

$$\widehat{\mathcal{M}}_{r,s}(X^0, X^1; F^{r,s}) = \{(r, u) : r \in \mathbb{R}, u \in \widehat{\mathcal{M}}_s(X^0, X^1; F^{r,s})\}.$$

PROPOSITION 2.12. *There is a subset $\mathbf{F}_{\text{reg}}^{r,s}$ of $\mathbf{F}^{r,s}$ of second category such that for any $F^{r,s} \in \mathbf{F}_{\text{reg}}^{r,s}$ and any families $X^0 \in \mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H)$ and $X^1 \in \mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H^1)$ such that the orbits of either X^0 or X^1 are simple, each $\widehat{\mathcal{M}}_{r,s}(X^0, X^1; F^{r,s})$ is a smooth finite-dimensional manifold.*

2.5 Floer trajectories: C^0 -bounds

With transversality in hand, we now turn to compactness. Since the manifold $\mathbb{R} \times M$ is open, we must first establish C^0 -bounds. The positive end of this manifold ($\tau \rightarrow +\infty$) is never a possible source of noncompactness. Since every (family of) Hamiltonian(s) we consider is constant for $\tau \gg 0$ and every (family of) almost complex structure(s) we consider is cylindrical for $\tau \gg 0$, the maximal principle forbids our curves from entering these regions. It remains for us to deal with the negative end of $\mathbb{R} \times M$.

Before proceeding, we recall the relevant notions of energy in this context and some useful equalities and inequalities involving them. For $F = (H, J)$, the L^2 -energy of each $u \in \widehat{\mathcal{M}}(X, Y; F)$ is defined to be

$$E(u) = \int_{\mathbb{R} \times \mathbb{R}/\mathbb{Z}} d(e^\tau \lambda_0)(\partial_s u, J(u)\partial_s u) ds dt. \tag{25}$$

The following well-known identity then follows from Stokes' theorem and the definition of $\widehat{\mathcal{M}}(X, Y; F)$:

$$E(u) = \mathcal{A}_H(X) - \mathcal{A}_H(Y). \tag{26}$$

Hence, we have

$$E(u) \leq \Delta(H) \tag{27}$$

for any u in any $\widehat{\mathcal{M}}(X, Y; F)$.

Similarly, for $F^s = (H^s, J^s)$, the L^2 -energy of u in $\widehat{\mathcal{M}}_s(X^0, X^1; F^s)$ is defined to be

$$E_s(u) = \int_{\mathbb{R} \times \mathbb{R}/\mathbb{Z}} d(e^\tau \lambda_0)(\partial_s u, J^s(u)\partial_s u) ds dt. \tag{28}$$

In this case, Stokes' theorem yields

$$E_s(u) = \mathcal{A}_{H^0}(X^0) - \mathcal{A}_{H^1}(X^1) + \int_{\mathbb{R} \times \mathbb{R}/\mathbb{Z}} (\partial_s H^s)(u(s, t)) ds dt \tag{29}$$

and

$$E_s(u) \leq \Delta(H^0, H^1) + \text{cost}(H^s). \tag{30}$$

Finally, for $F^{r,s} = (H^{r,s}, J^{r,s})$ and (r, u) in $\widehat{\mathcal{M}}_{r,s}(X^0, X^1; F^{r,s})$, we have

$$E_{r,s}((r, u)) = \int_{\mathbb{R} \times \mathbb{R}/\mathbb{Z}} d(e^\tau \lambda_0)(\partial_s u, J^{r,s}(u)\partial_s u) ds dt \leq \Delta(H^0, H^1) + \text{cost}(H^{r,s}). \tag{31}$$

We now prove that for a fixed H we have uniform C^0 -bounds for the elements of the spaces $\widehat{\mathcal{M}}(X, Y; F)$. Recall that $T_{\min}(\lambda_0)$ is the smallest period of any closed Reeb orbit of λ_0 .

PROPOSITION 2.13. *Suppose that H is an admissible Hamiltonian and that X and Y are transversally nondegenerate families in $\mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H)$. If*

$$\mathcal{A}_H(X) - \mathcal{A}_H(Y) < T_{\min}(\lambda_0),$$

then there is a $K > 0$ such that for any choice of Floer data of the form $F = (H, J)$ the image of every $u \in \widehat{\mathcal{M}}(X, Y; F)$ is contained in $[-K, +\infty) \times M \subset \mathbb{R} \times M$.

Proof. In terms of the product structure of $\mathbb{R} \times M$, any $u \in \widehat{\mathcal{M}}(X, Y; F)$ can be written in the form

$$u(s, t) = (\rho(s, t), \xi(s, t)).$$

Arguing by contradiction, we assume that there are a sequence of almost complex structures J_k in $\mathcal{J}(H)$ and a sequence of curves

$$u_k(s, t) = (\rho_k(s, t), \xi_k(s, t))$$

in $\widehat{\mathcal{M}}(X, Y; F_k)$ for $F_k = (H, J_k)$ such that

$$\lim_{k \rightarrow \infty} \left(\min_{(s,t) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}} \rho_k(s, t) \right) = -\infty. \tag{32}$$

To obtain the desired contradiction, we will argue as in [AFM15] (see also [EHS95]). We begin by isolating purely J_0 -holomorphic portions of the u_k . Fix a decreasing sequence $\epsilon_k \searrow 0$ such that u_k is transverse to $\{-\epsilon_k\} \times M$ for all $k \in \mathbb{N}$ and set

$$V_k = u_k^{-1}((-\infty, -\epsilon_k] \times M).$$

By (32), we may assume, by passing to a subsequence if necessary, that each V_k is nonempty. Let $v_k = u_k|_{V_k}$. Since H is admissible and J_k belongs to $\mathcal{J}(H)$, each v_k is a J_0 -holomorphic curve with a possibly disconnected domain and image in $\{\tau \leq 0\}$. For these curves, we have

$$\int_{V_k} v_k^* d(e^\tau \lambda_0) = \int_{V_k} d(e^\tau \lambda_0)(\partial_s u_k, J_k(u_k)\partial_s u_k) ds dt < E(u_k)$$

and so, by (26), we have

$$\int_{V_k} v_k^* d(e^\tau \lambda_0) < \mathcal{A}_H(X) - \mathcal{A}_H(Y). \tag{33}$$

The Hofer energy of v_k is

$$E_{\text{Hofer}}(v_k) = \sup_{\phi} \int_{V_k} v_k^* d(\phi \lambda_0),$$

where the supremum is over all functions ϕ in $C^\infty(\mathbb{R}, [0, 1])$ that are nondecreasing.

Applying Stokes' theorem twice, we get

$$\begin{aligned} E_{\text{Hofer}}(v_k) &= \sup_{\phi} \int_{V_k} v_k^* d(\phi \lambda_0) \\ &= \sup_{\phi} \int_{\partial V_k} v_k^*(\phi \lambda_0) \\ &= \int_{\partial V_k} v_k^* \lambda_0 \\ &= e^{\epsilon_k} \int_{\partial V_k} v_k^*(e^\tau \lambda_0) \\ &= e^{\epsilon_k} \int_{V_k} v_k^* d(e^\tau \lambda_0) \\ &= e^{\epsilon_k} \int_{V_k} v_k^* d(e^\tau \lambda_0). \end{aligned}$$

Hence, by (33), we have

$$E_{\text{Hofer}}(v_k) < e^{\epsilon_k}(\mathcal{A}_H(X) - \mathcal{A}_H(Y)).$$

So, we have a sequence, v_k , of J_0 -holomorphic curves in the symplectization $(\mathbb{R} \times M, d(e^\tau \lambda_0))$ whose Hofer energies are uniformly bounded from above, which all intersect the hypersurface $\{-\epsilon_1\} \times M$ and whose \mathbb{R} -components have minima that converge to negative infinity. This suggests that the curves break at $\tau = -\infty$ along closed Reeb orbits of λ_0 with period less than the limiting energy bound. Indeed, this is the case. This is a consequence of [AFM15, Theorem 5.3], which utilizes the compactness argument from [CM05] and yields the following precise statement in the present setting.

PROPOSITION 2.14 [AFM15, Theorem 5.3]. *There are a subsequence k_n and cylinders $C_n \subset U_{k_n}$ that are biholomorphically equivalent to the standard cylinders $[-L_n, L_n] \times \mathbb{R}/\mathbb{Z}$ such that the lengths $L_n \rightarrow \infty$ and the curves $v_{k_n}|_{C_n}$ converge in $C_{\text{loc}}^\infty(\mathbb{R} \times \mathbb{R}/\mathbb{Z}, \mathbb{R} \times M)$ to a trivial cylinder over a closed Reeb orbit of λ_0 of period at most $\mathcal{A}_H(X) - \mathcal{A}_H(Y)$.*

The existence of this closed Reeb orbit of λ_0 implies that $T_{\min}(\lambda_0) \leq \mathcal{A}_H(X) - \mathcal{A}_H(Y)$ and we have arrived at the desired contradiction. □

Starting from the uniform bounds (30) and (31) and arguing as above, one also obtains C^0 -bounds for moduli spaces of the form $\widehat{\mathcal{M}}_s(X^0, X^1; F^s)$ and $\widehat{\mathcal{M}}_{r,s}(X^0, X^1; F^{r,s})$ whenever $\mathcal{A}_{H^0}(X^0) - \mathcal{A}_{H^1}(X^1) + \text{cost}(H^s) < T_{\min}(\lambda_0)$ and $\mathcal{A}_{H^0}(X^0) - \mathcal{A}_{H^1}(X^1) + \text{cost}(H^{r,s}) < T_{\min}(\lambda_0)$, respectively.

2.6 Floer trajectories: quotients and compactifications

Consider a regular Floer data set $F = (H, J)$ in \mathbf{F}_{reg} and two distinct families X and Y in $\mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H)$ at least one of which is simple. Since both H and J do not depend on t , it follows that the $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$ -action on $\widehat{\mathcal{M}}(X, Y; F)$ given by

$$(s, t) * u(\cdot, \cdot) = u(\cdot + s, \cdot + t)$$

is free. The quotient

$$\mathcal{M}(X, Y; F) = \widehat{\mathcal{M}}(X, Y; F)/(\mathbb{R} \times \mathbb{R}/\mathbb{Z})$$

is then a smooth manifold. Let $\mathcal{M}^k(X, Y; F)$ be the submanifold of $\mathcal{M}(X, Y; F)$ which consists of all its components which have dimension k . Given Proposition 2.13, the follow compactness statements are then standard.

PROPOSITION 2.15. *Suppose that $\mathcal{A}_H(X) - \mathcal{A}_H(Y) < T_{\min}(\lambda_0)$. Then $\mathcal{M}^0(X, Y; F)$ is a compact manifold of dimension zero. If, in addition, both X and Y are simple, then $\mathcal{M}^1(X, Y; F)$ admits a compactification $\overline{\mathcal{M}}^1(X, Y; F)$ which is a one-dimensional manifold with boundary equal to*

$$\bigcup_{Z \in \mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H)} \mathcal{M}^0(X, Z; F) \times \mathcal{M}^0(Z, Y; F).$$

Now consider two admissible and nondegenerate Hamiltonians H^0 and H^1 , regular Floer data $F^0 = (H^0, J^0)$ and $F^1 = (H^1, J^1)$ and regular Floer continuation data $F^s \in \mathbf{F}^s(F^0, F^1)$

in $\mathbf{F}_{\text{reg}}^s$. For families $X^0 \in \mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H^0)$ and $X^1 \in \mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H^1)$, at least one of which is simple, the manifold $\widehat{\mathcal{M}}_s(X^0, X^1; F^s)$ admits a free \mathbb{R}/\mathbb{Z} -action,

$$t * u(\cdot, \cdot) = u(\cdot, \cdot + t).$$

The quotient

$$\mathcal{M}_s(X^0, X^1; F^s) = \widehat{\mathcal{M}}_s(X^0, X^1; F^s)/(\mathbb{R}/\mathbb{Z})$$

is then a smooth manifold and we let $\mathcal{M}_s^k(X^0, X^1; F^s)$ be the collection of its k -dimensional components. In this case we get the following compactness result.

PROPOSITION 2.16. *Suppose that $\mathcal{A}_{H^0}(X^0) - \mathcal{A}_{H^1}(X^1) + \text{cost}(H^s) < T_{\min}(\lambda_0)$. Then $\mathcal{M}_s^0(X^0, X^1; F^s)$ is compact. If, in addition, both X^0 and X^1 are simple and $\text{cost}(H^s) < -\mathcal{A}_H(X^0)$, then $\mathcal{M}_s^1(X^0, X^1; F^s)$ admits a compactification $\overline{\mathcal{M}}_s^1(X^0, X^1; F^s)$ which is a one-dimensional manifold whose boundary is*

$$\begin{aligned} & \bigcup_{Y^0 \in \mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H^0)} \mathcal{M}^0(X^0, Y^0; F^0) \times \mathcal{M}_s^0(Y^0, X^1; F^s) \\ & \cup \bigcup_{Y^1 \in \mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H^1)} \mathcal{M}_s^0(X^0, Y^1; F^s) \times \mathcal{M}^0(Y^1, X^1; F^1). \end{aligned}$$

Note that the condition $\text{cost}(H^s) < -\mathcal{A}_H(X^0)$ is needed to ensure that the orbits Y_0 that appear in the expression above have negative action.

Finally, for regular Floer homotopy data $F^{r,s} = (H^{r,s}, J^{r,s}) \in \mathbf{F}^{r,s}(F^0, F^1)$, we set

$$\mathcal{M}_{r,s}(X^0, X^1; F^{r,s}) = \widehat{\mathcal{M}}_{r,s}(X^0, X^1; F^{r,s})/(\mathbb{R}/\mathbb{Z})$$

and define $\mathcal{M}_{r,s}^k(X^0, X^1; F^{r,s})$ as above. For $k = 0$, we get the following.

PROPOSITION 2.17. *If $\mathcal{A}_{H^0}(X^0) - \mathcal{A}_{H^1}(X^1) + \text{cost}(H^{r,s}) < T_{\min}(\lambda_0)$, then $\mathcal{M}_{r,s}^0(X^0, X^1; F^{r,s})$ is a compact manifold of dimension zero.*

For $k = 1$, there are two important versions of the relevant compactness statement to state.

Version 1: A closed homotopy of homotopies. We say that the homotopy data $F^{r,s}$ is closed if for some $\mathbf{R} > 0$,

$$F^{r,s} = \begin{cases} F^{0,s} \in \mathbf{F}_{\text{reg}}^s(F^0, F^1) & \text{for all } r \leq -\mathbf{R}, \\ F^{1,s} \in \mathbf{F}_{\text{reg}}^s(F^0, F^1) & \text{for all } r \geq \mathbf{R}. \end{cases}$$

PROPOSITION 2.18. *Suppose that $F^{r,s} = (H^{r,s}, J^{r,s})$ is regular and closed and that $\mathcal{A}_{H^0}(X^0) - \mathcal{A}_{H^1}(X^1) + \text{cost}(H^{r,s}) < T_{\min}(\lambda_0)$. If X^0 and X^1 are both simple and $\text{cost}(H^{r,s}) < |\mathcal{A}_H(X^0)|$, then $\mathcal{M}_{r,s}^1(X^0, X^1; F^{r,s})$ admits a compactification $\overline{\mathcal{M}}_{r,s}^1(X^0, X^1; F^{r,s})$ which is a one-dimensional manifold with boundary equal to*

$$\begin{aligned} & \mathcal{M}_s^0(X^0, X^1; F^{0,s}) \cup \mathcal{M}_s^0(X^0, X^1; F^{1,s}) \\ & \cup \bigcup_{Y^0 \in \mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H^0)} \mathcal{M}^0(X^0, Y^0; F^0) \times \mathcal{M}_{r,s}^0(Y^0, X^1; F^{r,s}) \\ & \cup \bigcup_{Y^1 \in \mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H^1)} \mathcal{M}_{r,s}^0(X^0, Y^1; F^{r,s}) \times \mathcal{M}^0(Y^1, X^1; F^1). \end{aligned}$$

Version 2: A half-open homotopy of homotopies. We now consider a more explicit homotopy of homotopies with an open end. Let H^0, H^1 and G be nondegenerate and admissible Hamiltonians. Consider two admissible homotopies, H_0^s from H^0 to G , and H_1^s from G to H^1 . Now consider a homotopy of homotopies of the form

$$H_{0\#1}^{r,s} = \begin{cases} H_0^{s+\xi(r)} & \text{for } s \leq 0, \\ H_1^{s-\xi(r)} & \text{for } s > 0, \end{cases}$$

where $\xi(r)$ is a smooth, positive and nondecreasing function which equals r for $r \gg \xi(0)$ and which equals $\xi(0)$ for $r \leq \xi(0)/2$. It is easy to check that this is an admissible homotopy of homotopies from H^0 to H^1 if we choose $\xi(0)$ to be sufficiently large.

Fixing regular Floer data sets $F^0 = (H^0, J^0)$, $F^1 = (H^1, J^1)$ and $F^G = (G, J^G)$, we extend these to Floer continuation data sets

$$F_0^s = (H_0^s, J_0^s) \in \mathbf{F}^s(F^0, F^G)$$

and

$$F_1^s = (H_1^s, J_1^s) \in \mathbf{F}^s(F^G, F^1),$$

which we use to form the Floer homotopy data set

$$F_{0\#1}^{r,s} = \begin{cases} F_0^{s+\xi(r)} & \text{for } s \leq 0, \\ F_1^{s-\xi(r)} & \text{for } s > 0 \end{cases}$$

in $\mathbf{F}^{r,s}(F^0, F^1)$. Perturbing these, if necessary, we may assume that the three previous data sets are all regular.

PROPOSITION 2.19. *Suppose that $\mathcal{A}_{H^0}(X^0) - \mathcal{A}_{H^1}(X^1) + \text{cost}(H^{r,s}) < T_{\min}(\lambda_0)$. If X^0 and X^1 are both simple and $\text{cost}(H_{0\#1}^{r,s}) < -\mathcal{A}_H(X^0)$, then the manifold*

$$\mathcal{M}_{r,s}^1(X^0, X^1; F_{0\#1}^{r,s})$$

admits a compactification

$$\overline{\mathcal{M}}_{r,s}^1(X^0, X^1; F_{0\#1}^{r,s})$$

which is a one-dimensional manifold whose boundary is

$$\begin{aligned} & \mathcal{M}_s^0(X^0, X^1; F_{0\#1}^{0,s}) \\ & \cup \bigcup_{Z \in \mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(G)} \mathcal{M}_s^0(X^0, Z; F_0^s) \times \mathcal{M}_s^0(Z, X^1; F_1^s) \\ & \cup \bigcup_{Y^0 \in \mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H^0)} \mathcal{M}^0(X^0, Y^0; F^0) \times \mathcal{M}_{r,s}^0(Y^0, X^1; F_{0\#1}^{r,s}) \\ & \cup \bigcup_{Y^1 \in \mathcal{P}_{\mathbb{R}/\mathbb{Z}}^-(H^1)} \mathcal{M}_{r,s}^0(X^0, Y^1; F_{0\#1}^{r,s}) \times \mathcal{M}^0(Y^1, X^1; F^1). \end{aligned}$$

2.7 Floer theory for finely tuned Hamiltonians

Throughout this section we consider a fixed nondegenerate rigid constellation $\mathcal{C}_{\lambda_0, \alpha}(T)$.

2.7.1 *Homology.* To every Hamiltonian H which is finely tuned to $\mathcal{C}_{\lambda_0, \alpha}(T)$ and every regular Floer data set $F = (H, J) \in \mathbf{F}_{\text{reg}}$, we associate a version of Floer homology. Let $\mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^-(H)$ be the set of \mathbb{R}/\mathbb{Z} -families of closed 1-periodic orbits of H which have negative action and which represent the class α . Since H is tuned, each family X in $\mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^-(H)$ corresponds to a unique \mathbb{R}/\mathbb{Z} -family of closed Reeb orbits Γ_X in $\mathcal{C}_{\lambda_0, \alpha}(T)$ and vice versa. Since each family Γ_X in $\mathcal{C}_{\lambda_0, \alpha}(T)$ is simple, so are the families X in $\mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^-(H)$.

Define the chain group by

$$\text{CF}(H; \alpha) = \text{Span}_{\mathbb{Z}/2\mathbb{Z}}\{X : X \in \mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^-(H)\}$$

and the corresponding boundary map $\partial_F : \text{CF}(H; \alpha) \rightarrow \text{CF}(H; \alpha)$ to be the linear operator defined on generators by

$$\partial_F(X) = \sum_{Y \in \mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^-(H)} \#\mathcal{M}^0(X, Y; F)Y.$$

Here, and in what follows, for any finite set \mathcal{M} the notation $\#\mathcal{M}$ will denote the number of elements modulo 2. By the definition of *finely tuned*, we have $\Delta(H) < T_{\min}(\lambda_0)$ (see Lemma 2.7). It then follows from Proposition 2.15 that ∂_F is well defined and satisfies $\partial_F \circ \partial_F = 0$. Standard arguments imply that the resulting homology is independent of the choice of regular $J \in \mathcal{J}(H)$ and so we denote this homology by $\text{HF}(H; \alpha)$.

2.7.2 *Continuation maps.* Let H^0 and H^1 both be finely tuned to $\mathcal{C}_{\lambda_0, \alpha}(T)$. We now construct tools which allow us to compare $\text{HF}(H^0; \alpha)$ and $\text{HF}(H^1; \alpha)$. Let

$$\Delta_s(H^0, H^1) = \min\left\{\frac{T_{\min}(\lambda_0, \alpha)}{2}, T_{\min}(\lambda_0) - \Delta(H^0, H^1)\right\}. \tag{34}$$

By Lemma 2.7, $\Delta_s(H^0, H^1) > 0$. Consider an admissible homotopy H^s from H^0 to H^1 such that

$$\text{cost}(H^s) < \Delta_s(H^0, H^1). \tag{35}$$

Perturbing H^s if necessary, we choose regular Floer continuation data $F^s = (H^s, J^s)$ between regular Floer data $F^0 = (H^0, J^0)$ and $F^1 = (H^1, J^1)$ and define the linear map

$$\theta_{F^s} : \text{CF}(H^0; \alpha) \rightarrow \text{CF}(H^1; \alpha)$$

on generators by

$$\theta_{F^s}(X^i) = \sum_{Y^j \in \mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^-(H^1)} \#\mathcal{M}_s^0(X^i, Y^j; F^s)Y^j.$$

Proposition 2.16 implies that θ_{F^s} is a well-defined chain map. The usual arguments again imply that the resulting map in homology is independent of the choice of J^s and so we denote this map by

$$\Theta_{H^s} : \text{HF}(H^0; \alpha) \rightarrow \text{HF}(H^1; \alpha).$$

A convex combination of two homotopies that satisfy (35) also satisfies the same bound. Hence, the usual homotopy of homotopies argument in Floer theory can be used to show that the map Θ_{H^s} does not depend on the choice of homotopy H^s with cost less than $\Delta_s(H^0, H^1)$.

LEMMA 2.20. *If H^s and \tilde{H}^s are two admissible homotopies from H^0 to H^1 with cost less than $\Delta_s(H^0, H^1)$, then the maps Θ_{H^s} and $\Theta_{\tilde{H}^s}$ are equal.*

In the present setting, the usual composition rule for continuation maps has the following form.

LEMMA 2.21. *Suppose that H^0, H^1 and H^2 are Hamiltonians that are finely tuned to $\mathcal{C}_{\lambda_0, \alpha}(T)$ and that H_{10}^s is an admissible homotopy from H^0 to H^1 and H_{21}^s is an admissible homotopy from H^1 to H^2 . If*

$$\text{cost}(H_{10}^s) < \Delta_s(H^0, H^1), \tag{36}$$

$$\text{cost}(H_{21}^s) < \Delta_s(H^1, H^2) \tag{37}$$

and

$$\text{cost}(H_{10}^s) + \text{cost}(H_{21}^s) < \Delta_s(H^0, H^2), \tag{38}$$

then there is an admissible homotopy H_{20}^s from H^0 to H^2 with cost at most $\text{cost}(H_{10}^s) + \text{cost}(H_{21}^s)$ such that

$$\Theta_{H_{20}^s} = \Theta_{H_{21}^s} \circ \Theta_{H_{10}^s}.$$

With these tools in place we can now begin the process of identifying the Floer homology groups associated to different finely tuned Hamiltonians. As above, the arguments we use are standard, but are complicated by the need to manage the cost of homotopies at each step.

Given a function $G : \mathbb{R} \times M \rightarrow \mathbb{R}$, set

$$\|G\| = \max_{\mathbb{R} \times M} G - \min_{\mathbb{R} \times M} G. \tag{39}$$

COROLLARY 2.22. *Suppose that the Hamiltonians H^0 and H^1 are finely tuned to $\mathcal{C}_{\lambda_0, \alpha}(T)$. If*

$$\max_{\mathbb{R} \times M} (H^1 - H^0) < \Delta_s(H^0, H^1), \tag{40}$$

$$\max_{\mathbb{R} \times M} (H^0 - H^1) < \Delta_s(H^1, H^0) \tag{41}$$

and

$$\|H^1 - H^0\| < \Delta_s(H^0, H^0), \tag{42}$$

then $\text{HF}(H^0; \alpha)$ and $\text{HF}(H^1; \alpha)$ are isomorphic.

Proof. Fix a smooth nondecreasing step function $\mathbf{step} : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\mathbf{step}(\tau) = \begin{cases} 0 & \text{for } \tau \leq -1, \\ 1 & \text{for } \tau \geq 0. \end{cases}$$

Let

$$H^s = (1 - \mathbf{step}(s))H^0 + \mathbf{step}(s)H^1.$$

We then have

$$\text{cost}(H^s) = \max_{\mathbb{R} \times M} (H^1 - H^0)$$

and

$$\text{cost}(H^{-s}) = \max_{\mathbb{R} \times M} (H^0 - H^1).$$

By Lemma 2.21, there is an admissible homotopy \tilde{H}^s from H^0 to H^0 with

$$\text{cost}(\tilde{H}^s) \leq \max_{\mathbb{R} \times M}(H^1 - H^0) - \min_{\mathbb{R} \times M}(H^1 - H^0) = \|H^1 - H^0\| < \Delta_s(H^0, H^0) \tag{43}$$

such that

$$\Theta_{\tilde{H}^s} = \Theta_{H^{-s}} \circ \Theta_{H^s} : \text{HF}(H^0; \alpha) \rightarrow \text{HF}(H^0; \alpha).$$

By Lemma 2.20, the map $\Theta_{\tilde{H}^s}$ is the same as that corresponding to the constant homotopy from H^0 to itself and so $\Theta_{\tilde{H}^s}$ is an isomorphism. Thus, Θ_{H^s} is injective. Applying the same argument to the composition $\Theta_{H^s} \circ \Theta_{H^{-s}}$, we see that Θ_{H^s} is also surjective. \square

COROLLARY 2.23. *If H^0 and H^1 are finely tuned to $\mathcal{C}_{\lambda_0, \alpha}(T)$ and H^1 is sufficiently C^1 -close to H^0 , then $\text{HF}(H^0; \alpha)$ is isomorphic to $\text{HF}(H^1; \alpha)$.*

Proof. This follows easily from Corollary 2.22 since the hypotheses of the theorem are met for all H^1 sufficiently C^1 -close to H^0 . For example, if H^k is a sequence of finely tuned Hamiltonians converging to H^0 in the C^1 -topology, then $\max_{\mathbb{R} \times M}(H^k - H^0) \rightarrow 0$ whereas $\Delta_s(H^0, H^k) \rightarrow \Delta_s(H^0, H^0) > 0$. \square

COROLLARY 2.24. *Suppose that the Hamiltonians H^0 and H^1 are finely tuned to $\mathcal{C}_{\lambda_0, \alpha}(T)$ and that H^s is an admissible homotopy from H^0 to H^1 such that for each s the Hamiltonian H^s is also finely tuned to $\mathcal{C}_{\lambda_0, \alpha}(T)$. Then the following statements hold.*

- (i) *The groups $\text{HF}(H^s; \alpha)$ are isomorphic to one another for all s .*
- (ii) *If, in addition, $\text{cost}(H^s) = 0$, then the map*

$$\Theta_{H^s} : \text{HF}(H^0; \alpha) \rightarrow \text{HF}(H^1; \alpha)$$

is an isomorphism.

Proof. Reparameterizing if necessary, we may assume that $H^s = H^0$ for all $s \leq 0$ and $H^s = H^1$ for all $s \geq 1$. It follows from Corollary 2.22, and continuity, that for each $s' \in [0, 1]$ there is a $\delta_{s'} > 0$ such that $\text{HF}(H^s; \alpha)$ is isomorphic to $\text{HF}(H^{s'}; \alpha)$ for all $s \in (s' - \delta_{s'}, s' + \delta_{s'})$. Covering $[0, 1]$ by finitely many such intervals, it follows that for all $s \in [0, 1]$ the groups $\text{HF}(H^s; \alpha)$ are isomorphic to one another.

To prove the second assertion of the corollary, it suffices (by Lemma 2.20) to find an admissible homotopy \tilde{H}^s from H^0 to H^1 with $\text{cost}(\tilde{H}^s) < \Delta_s(H^0, H^1)$ such that $\Theta_{\tilde{H}^s} : \text{HF}(H^0; \alpha) \rightarrow \text{HF}(H^1; \alpha)$ is an isomorphism. We will use H^s and Lemma 2.21 to construct this \tilde{H}^s .

Arguing as above, and invoking the proof of Corollary 2.22, we can find numbers $s_0 = 1 < s_1 < \dots < s_N = 1$ such that for $k = 0, \dots, N - 1$, each linear homotopy

$$G_k^s = H^{s_k}(1 - \text{step}(s)) + H^{s_{k+1}}\text{step}(s)$$

induces an isomorphism $\Theta_{G_k^s} : \text{HF}(H^{s_k}; \alpha) \rightarrow \text{HF}(H^{s_{k+1}}; \alpha)$. From H^s , we can also construct a homotopy

$$H_k^s = H^{(s_k + (s_{k+1} - s_k)\text{step}(s))}$$

from H^{s_k} to $H^{s_{k+1}}$ which is admissible and cost free. It follows from Lemma 2.20 that $\Theta_{H_k^s} = \Theta_{G_k^s}$, so that each $\Theta_{H_k^s}$ is an isomorphism. On the other hand, since each H_k^s is cost free and each

$\Delta_s(H^{s_k}, H^{s_{k+1}})$ is positive, we can invoke Lemma 2.21 $N+1$ times to obtain a cost-free admissible homotopy \tilde{H}^s from H^0 to H^1 such that

$$\Theta_{\tilde{H}^s} = \Theta_{H_{N-1}^s} \circ \cdots \circ \Theta_{H_0^s}.$$

This completes the proof. □

Now we get to our main invariance result.

PROPOSITION 2.25. *The rank of $\text{HF}(H; \alpha)$ is the same for every Hamiltonian H that is finely tuned to the rigid constellation $\mathcal{C}_{\lambda_0, \alpha}(T)$.*

Proof. We first observe that for all sufficiently small $\epsilon > 0$, every Hamiltonian in the space

$$\mathcal{H}_\epsilon(T) = \mathcal{H}_{\epsilon, T+\epsilon, 3(T+\epsilon)/\epsilon, 0}$$

is finely tuned to $\mathcal{C}_{\lambda_0, \alpha}(T)$. Moreover, there is an $\epsilon_0 > 0$ such that for any $\epsilon, \epsilon' < \epsilon_0$ and any Hamiltonians $H_\epsilon \in \mathcal{H}_\epsilon(T)$ and $H_{\epsilon'} \in \mathcal{H}_{\epsilon'}(T)$, we have

$$\Delta_s(H_\epsilon, H_{\epsilon'}) > (T_{\min}(\lambda_0) + T_{\min}(\lambda_0, \alpha) - T)/2 > 0.$$

It then follows from Corollary 2.22 that for all $\epsilon < \epsilon_0$ and every Hamiltonian H_ϵ in $\mathcal{H}_\epsilon(T)$, the rank of $\text{HF}(H_\epsilon; \alpha)$ is the same.

By Corollary 2.24, it now suffices to show that given any finely tuned Hamiltonian H there is an admissible homotopy H^s which consists of finely tuned Hamiltonians and connects H to some $H_\epsilon \in \mathcal{H}_\epsilon(T)$ with $\epsilon < \epsilon_0$. The starting point of H^s , H , belongs to $\mathfrak{H}_{a,b,c,\kappa}$ for some h in $\mathfrak{h}_{a,b,c}$. For the end point H_ϵ , we choose $\epsilon < \epsilon_0$ small enough so that the following inequalities hold:

$$\epsilon < a, \quad T + \epsilon < b \quad \text{and} \quad 3(T + \epsilon)/\epsilon > c.$$

We now view h as belonging to a smooth family of functions $h(A, B, C)$ such that $h(A, B, C)$ belongs to $\mathfrak{h}_{A,B,C}$. The segments of the path H^s will then be defined by varying the parameters A, B, C and κ one at a time.

We begin with κ . The distinction between a tuned and a finely tuned Hamiltonian involves only the relationship between a and κ and, if a works for κ , then it also works for all smaller values of κ . So, the first segment of the path H^s will be

$$s \in [0, 1] \mapsto h(a, b, c)(e^{\tau-(1-s)\kappa}).$$

The next segment increases C from c to $3(T + \epsilon)/\epsilon$. Again it follows easily from the definitions that the intermediate Hamiltonians remain finely tuned. Continuing in this way, we decrease B from b to $T + \epsilon$ and finally decrease A from a to ϵ . By joining these four segments in order and reparameterizing to smooth the transitions between them, we obtain the desired homotopy H^s . □

3. The proof of Theorem 1.12

In the set-up of Theorem 1.12, we are given a nondegenerate rigid constellation $\mathcal{C}_{\lambda_0, \alpha}(T)$ and a pinched contact form $\lambda = f\lambda_0$ such that the function f is positive. By rescaling, we may assume that $\min(f) = 1$ and so the pinching condition becomes

$$\max(f) < \min \left\{ \frac{T^+}{T}, \frac{T_{\min}(\lambda_0) + T_{\min}(\lambda_0, \alpha)}{T} \right\}.$$

We also have the condition that every closed Reeb orbit of λ in the class α and with period in $[T_{\min}(\lambda_0, \alpha), T \max(f)]$ is nondegenerate.

Define \widehat{T} by

$$\widehat{T} = \min\{T^+, T_{\min}(\lambda_0) + T_{\min}(\lambda_0, \alpha)\}.$$

Every b in the open interval $(T \max(f), \widehat{T})$ is not the period of a closed Reeb orbit of λ_0 . Choose such a b which also lies in the complement of $\mathcal{T}(\lambda, \alpha)$. For a profile h in $\mathfrak{h}_{a,b,c}$, we set

$$G(\tau, p) = h(e^\tau / f(p)).$$

While G is admissible, it is clearly not radial. However, for the diffeomorphism $\Psi_\lambda : \mathbb{R} \times M \rightarrow \mathbb{R} \times M$ defined by

$$(\tau, p) \mapsto (\tau + f(p), p)$$

we have

$$\Psi_\lambda^*(e^\tau \lambda_0) = e^\tau \lambda \tag{44}$$

and

$$\Psi_\lambda^* G(\tau, p) = h(e^\tau). \tag{45}$$

Thus, $\Psi_\lambda^* G$ is radial. By our choice of b above, we may therefore assume that for all sufficiently small a and sufficiently large c , the Hamiltonian $\Psi_\lambda^* G$ is dividing (see Lemma 2.2). More precisely, by (44), the Hamiltonian $\Psi_\lambda^* G$ is dividing for λ and not for λ_0 (see Remark 2.5). As a consequence, there is a bijection between $\mathcal{P}_\alpha^-(\Psi_\lambda^* G)$ and $\mathcal{R}_\alpha^b(\lambda)$ and so between $\mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^-(\Psi_\lambda^* G)$ and $\mathcal{R}_{\alpha, \mathbb{R}/\mathbb{Z}}^b(\lambda)$, as well.

By (44) and the fact that Ψ_λ is isotopic to the identity (and so preserves $[\mathbb{R}/\mathbb{Z}, \mathbb{R} \times M]$), we also know that Ψ_λ maps $\mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^-(\Psi_\lambda^* G)$ bijectively onto $\mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^-(G)$ and preserves actions. Thus, every family in $\mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^-(G)$ also corresponds to a unique family in $\mathcal{R}_{\alpha, \mathbb{R}/\mathbb{Z}}^b(\lambda)$.

Milepost 1. To prove the first assertion of Theorem 1.12, it suffices to find at least $\text{rank}(\mathcal{C}_{\lambda_0, \alpha}(T))$ elements of $\mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^-(G)$ which correspond to distinct closed Reeb orbits of λ with periods in $[T_{\min}(\lambda_0, \alpha), T \max(f)]$.

We now refine this task. For the constant a from the definition of the profile function h , consider the following subset of $\mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^-(G)$:

$$\mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^a(G) = \{Y \in \mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^-(G) : \mathcal{A}_G(Y) \in (-(1+a)T \max(f), -T_{\min}(\lambda_0, \alpha) + a^2)\}.$$

LEMMA 3.1. *If $a > 0$ is sufficiently small, then every family Y in $\mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^a(G)$ is nondegenerate and the corresponding family Γ_Y in $\mathcal{R}_{\alpha, \mathbb{R}/\mathbb{Z}}^b(\lambda)$ has period in the interval*

$$[T_{\min}(\lambda_0, \alpha), T \max(f)].$$

Proof. Consider a family $Y \in \mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^a(G)$ and the corresponding family Γ_Y in $\mathcal{R}_{\alpha, \mathbb{R}/\mathbb{Z}}^b(\lambda)$. As described above, the preimage $\Psi_\lambda^{-1}(Y)$ is an \mathbb{R}/\mathbb{Z} -family in $\mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^-(\Psi_\lambda^* G)$ with the same action. So,

$$\mathcal{A}_{\Psi_\lambda^* G}(\Psi_\lambda^{-1}(Y)) = \mathcal{A}_G(Y) \in (-(1+a)T \max(f), -T_{\min}(\lambda_0, \alpha) + a^2).$$

Since the Hamiltonian $\Psi_\lambda^* G$ is dividing, we also have

$$\mathcal{A}_G(Y) = \mathcal{A}_{\Psi_\lambda^* G}(\Psi_\lambda^{-1}(Y)) = -e^{\tau_Y} T_{\Gamma_Y} + h(e^{\tau_Y}),$$

where T_{Γ_Y} is the common period of the family Γ_Y and e^{τ_Y} is in $(1, 1 + a)$. Thus,

$$T_{\Gamma_Y} \in \left(\frac{T_{\min}(\lambda_0, \alpha) - a^2}{1 + a}, (1 + a)T \max(f) + a^2 \right). \tag{46}$$

So, if the assertion of the lemma does not hold, then for every small $a > 0$ there is a Y in $\mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^a(G)$ such that (46) holds but T_{Γ_Y} lies outside the closed subinterval

$$[T_{\min}(\lambda_0, \alpha), T \max(f)] \subset \left(\frac{T_{\min}(\lambda_0, \alpha) - a^2}{1 + a}, (1 + a)T \max(f) + a^2 \right).$$

Thus, there is sequence of closed Reeb orbits of λ in class α whose periods are monotonically converging to one of the end points $T_{\min}(\lambda_0, \alpha)$ or $T \max(f)$ as $a \rightarrow 0$. By Arzela–Ascoli, a subsequence of these orbits must converge to a closed Reeb orbit of λ in class α with period equal to either $T_{\min}(\lambda_0, \alpha)$ or $T \max(f)$. This contradicts our assumption that every closed Reeb orbit of λ in class α and with period in $[T_{\min}(\lambda_0, \alpha), T \max(f)]$ is nondegenerate (and hence isolated). □

Henceforth we will assume that $a > 0$ is sufficiently small in the sense of Lemma 3.1.

Milepost 2. To prove the first assertion of Theorem 1.12, it suffices to find at least $\text{rank}(\mathcal{C}_{\lambda_0, \alpha}(T))$ distinct elements of $\mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^a(G)$.

To achieve this, we now use the Floer-theoretic machinery developed in the previous section. We will argue as in [Ker08] by adapting a technique introduced by Chekanov in [Che98].

Starting with the profile h used to define G , we set

$$H^0 = h(e^\tau)$$

and

$$H^1 = h(e^\tau / \max(f)).$$

By our choice of $b \in (T \max(f), \widehat{T})$, both H^0 and H^1 are finely tuned to $\mathcal{C}_{\lambda_0, \alpha}(T)$ for all sufficiently small a and sufficiently large c . Since h is nondecreasing and $\min(f) = 1$, we also have

$$h(e^\tau) \geq h(e^\tau / f(p)) \geq h(e^\tau / \max(f))$$

for all $(\tau, p) \in \mathbb{R} \times M$ and thus

$$H^0 \geq G \geq H^1. \tag{47}$$

Using, again, the simple function **step**, we define two admissible homotopies:

$$H_0^s = (1 - \mathbf{step}(s))H^0 + \mathbf{step}(s)G$$

from H^0 to G and

$$H_1^s = (1 - \mathbf{step}(s))G + \mathbf{step}(s)H^1$$

from G to H^1 . Inequality (47) implies that

$$\partial_s(H_0^s), \partial_s(H_1^s) \leq 0 \tag{48}$$

and so

$$\text{cost}(H_0^s) = \text{cost}(H_1^s) = 0.$$

Consider the $\mathbb{Z}/2$ -vector space spanned by the elements of $\mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^a(G)$,

$$V^a(G; \alpha) = \text{Span}_{\mathbb{Z}/2}\{Y \in \mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^a(G)\}.$$

For regular Floer continuation data sets $F_0^s = (H_0^s, J_0^s)$ and $F_1^s = (H_1^s, J_1^s)$, we define two linear maps. The first,

$$\chi_{F_0^s} : \text{CF}(H^0; \alpha) \rightarrow V^a(G; \alpha),$$

is defined on generators by

$$\chi_{F_0^s}(X^0) = \sum_{Y \in \mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^a(G)} \#\mathcal{M}_s^0(X^0, Y; F_0^s)Y$$

and the second map

$$\chi_{F_1^s} : V^a(G; \alpha) \rightarrow \text{CF}(H^1; \alpha)$$

is defined on generators by

$$\chi_{F_1^s}(Y) = \sum_{X^1 \in \mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^-(H^1)} \#\mathcal{M}_s^0(Y, X^1; F_1^s)X^1.$$

Since the elements of $\mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^-(H^0)$ and $\mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^-(H^1)$ are all simple, it follows from Proposition 2.16 that the maps $\chi_{F_0^s}$ and $\chi_{F_1^s}$ are well defined.

Note that no claim is being made that $\chi_{F_0^s}$ and $\chi_{F_1^s}$ are chain maps. Indeed, the relevant version of Floer homology can only be defined for G by imposing prohibitively restrictive assumptions on it and hence on λ . This reflects one of the important observations of Chekanov in [Che98]. We now prove the following result.

PROPOSITION 3.2. *The composition $\chi_{F_1^s} \circ \chi_{F_0^s} : \text{CF}(H^0; \alpha) \rightarrow \text{CF}(H^1; \alpha)$ is a chain map which induces an isomorphism in homology.*

Proof. To prove this, we return to the setting of Proposition 2.19. For the homotopies H_0^s and H_1^s above, consider the half-open homotopy of homotopies

$$H_{0\#1}^{r,s} = \begin{cases} H_0^{s+\xi(r)} & \text{for } s \leq 0, \\ H_1^{s-\xi(r)} & \text{for } s > 0, \end{cases}$$

where $\xi(r)$ is a smooth, positive and nondecreasing function which equals r for $r \gg 2$ and which equals 2 for $r \leq 0$. It follows from (47) and the choices above that

$$\text{cost}(H_{0\#1}^{r,s}) = 0. \tag{49}$$

Fixing regular Floer data sets $F^0 = (H^0, J^0)$, $F^1 = (H^1, J^1)$ and $F^G = (G, J^G)$, we extend these to Floer continuation data sets

$$\begin{aligned} F_0^s &= (H_0^s, J_0^s) \in \mathbf{F}^s(F^0, F^G), \\ F_1^s &= (H_1^s, J_1^s) \in \mathbf{F}^s(F^G, F^1), \end{aligned}$$

which we use to form the Floer homotopy data set

$$F_{0\#1}^{r,s} = \begin{cases} F_0^{s+\xi(r)} & \text{for } s \leq 0, \\ F_1^{s-\xi(r)} & \text{for } s > 0 \end{cases}$$

in $\mathbf{F}^{r,s}(F^0, F^1)$. Perturbing again, if necessary, we assume that these data sets are all regular.

Given $X^0 \in \mathcal{P}^-_{\alpha, \mathbb{R}/\mathbb{Z}}(H^0)$ and $X^1 \in \mathcal{P}^-_{\alpha, \mathbb{R}/\mathbb{Z}}(H^1)$, Proposition 2.19 implies that the boundary of the compactification $\overline{\mathcal{M}}^1_{r,s}(X^0, X^1; F_{0\neq 1}^{r,s})$ can be identified with the elements of the following four sets:

- (I) $\mathcal{M}_s^0(X^0, X^1; F_{0\neq 1}^{0,s});$
- (II) $\bigcup_{Z \in \mathcal{P}^-_{\alpha, \mathbb{R}/\mathbb{Z}}(G)} \mathcal{M}_s^0(X^0, Z; F_0^s) \times \mathcal{M}_s^0(Z, X^1; F_1^s);$
- (III) $\bigcup_{Y^0 \in \mathcal{P}^-_{\alpha, \mathbb{R}/\mathbb{Z}}(H)} \mathcal{M}^0(X^0, Y^0; F^0) \times \mathcal{M}_{r,s}^0(Y^0, X^1; F_{0\neq 1}^{r,s});$
- (IV) $\bigcup_{Y^1 \in \mathcal{P}^-_{\alpha, \mathbb{R}/\mathbb{Z}}(H^1)} \mathcal{M}_{r,s}^0(X^0, Y^1; F_{0\neq 1}^{r,s}) \times \mathcal{M}^0(Y^1, X^1; F^1).$

By definition, the number of elements in set (I), modulo 2, is the coefficient of X^1 in the image of X^0 under the map

$$\theta_{F_{0\neq 1}^{0,s}} : \text{CF}(H^0; \alpha) \rightarrow \text{CF}(H^1; \alpha),$$

which is well defined by (49). If we can show that every Z which contributes a term to the set (II) must belong to the subset $\mathcal{P}^a_{\alpha, \mathbb{R}/\mathbb{Z}}(G)$ of $\mathcal{P}^-_{\mathbb{R}/\mathbb{Z}}(G; \alpha)$, then the number of elements in set (II), modulo 2, will be the coefficient of X^1 in the image of X^0 under the map

$$\chi_{F_1^s} \circ \chi_{F_0^s} : \text{CF}(H^0; \alpha) \rightarrow \text{CF}(H^1; \alpha).$$

With this, the fact that $\theta_{F_{0\neq 1}^{0,s}}$ and $\chi_{F_1^s} \circ \chi_{F_0^s}$ are chain homotopic will follow from the usual arguments.

Suppose then that $Z \in \mathcal{P}^-_{\mathbb{R}/\mathbb{Z}}(G; \alpha)$ contributes a nontrivial term to the set (II). In this case both $\mathcal{M}_s^0(X^0, Z; F_0^s)$ and $\mathcal{M}_s^0(Z, X^1; F_1^s)$ must be nonempty. By (29) and (48), we then have

$$\mathcal{A}_{H^0}(X^0) > \mathcal{A}_G(Z) > \mathcal{A}_{H^1}(X^1)$$

and so, by (20), the following holds:

$$-(1+a)T \max(f) < \mathcal{A}_G(Z) < -T_{\min}(\lambda_0, \alpha) + a^2.$$

Thus, Z belongs to $\mathcal{P}^a_{\alpha, \mathbb{R}/\mathbb{Z}}(G)$, as desired.

Since $\chi_{F_1^s} \circ \chi_{F_0^s}$ is chain homotopic to $\theta_{F_{0\neq 1}^{0,s}}$, it only remains to show that $\Theta_{H_{0\neq 1}^{0,s}}$ (the map that $\theta_{F_{0\neq 1}^{0,s}}$ induces in homology) is an isomorphism. Consider the admissible homotopy

$$H^s(\tau, p) = h\left(\frac{e^\tau}{1 - \text{step}(s) + \text{step}(s) \max(f)}\right)$$

from H^0 to H^1 . For all small enough a and large enough c , each function H^s is finely tuned to the rigid constellation $\mathcal{C}_{\lambda_0, \alpha}(T)$. Moreover, $\partial_s(H^s) \leq 0$ and so Corollary 2.24 implies that the map

$$\Theta_{H^s} : \text{HF}(H^0; \alpha) \rightarrow \text{HF}(H^1; \alpha)$$

is an isomorphism. Lemma 2.20 then implies that

$$\Theta_{H_{0\neq 1}^{0,s}} = \Theta_{H^s},$$

which concludes the proof. □

At this point we can complete the task described in Milepost 2 and thus the proof of the first assertion of Theorem 1.12.

LEMMA 3.3. *There are at least $\text{rank}(\mathcal{C}_{\lambda_0, \alpha}(T))$ distinct elements of $\mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^a(G)$.*

Proof. Let V_0 be a subspace of $\text{CF}(H; \alpha)$ that represents the homology $\text{HF}(F^0; \alpha)$. By Proposition 3.2, the restriction of $\chi_{F_R^s} \circ \chi_{F_L^s}$ to V_0 has no kernel. Thus, the restriction of $\chi_{F_L^s}$ to V_0 also has no kernel. We therefore have

$$\dim(V^a(G; \alpha)) \geq \dim(\chi_{F_L^s}(V_0)) = \dim(V_0) = \text{rank}(\text{HF}(F^0; \alpha)).$$

With this we are done. □

Finally, we prove the second assertion of Theorem 1.12.

LEMMA 3.4. *If the class α is either primitive or of infinite order, then the closed Reeb orbits of λ corresponding to the distinct elements of $\mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^a(G)$ are geometrically distinct. Otherwise, they are geometrically distinct if there are no closed Reeb orbits of λ with period at most*

$$\frac{1}{|\alpha|}(T \max(f) - T_{\min}(\lambda_0, \alpha))$$

that represent a class β in $[\mathbb{S}^1, M]$ such that $\beta^k = \alpha$ for some integer $k > 1$.

Proof. Let Y and Y' be distinct elements of $\mathcal{P}_{\alpha, \mathbb{R}/\mathbb{Z}}^a(G)$ and let Γ_Y and $\Gamma_{Y'}$ be the corresponding (distinct) elements of $\mathcal{R}_{\alpha, \mathbb{R}/\mathbb{Z}}^b(\lambda)$. Assume that Γ_Y and $\Gamma_{Y'}$ are not geometrically distinct. Then there must be an \mathbb{R}/\mathbb{Z} -family Γ_R of closed Reeb orbits of λ and integers $k \geq 1$ and $l \geq 1$ such that

$$\Gamma_Y = (\Gamma_R)^k \quad \text{and} \quad \Gamma_{Y'} = (\Gamma_R)^{k+l}. \tag{50}$$

Let β be the class in $[\mathbb{R}/\mathbb{Z}, M]$ represented by Γ_R . By (50), we then have

$$\beta^k = \beta^{k+l} = \alpha. \tag{51}$$

This implies that β^l is equal to the trivial element $e \in [\mathbb{R}/\mathbb{Z}, M]$ and so

$$\alpha^l = (\beta^k)^l = (\beta^l)^k = e. \tag{52}$$

In this case the class α can be neither primitive, by (51), or of infinite order, by (52). This implies the first statement of the lemma.

Suppose then that Γ_Y and $\Gamma_{Y'}$ are not geometrically distinct and that α is not primitive and of finite order. By Lemma 3.1, we have

$$T_{\Gamma_Y}, T_{\Gamma_{Y'}} \in [T_{\min}(\lambda_0, \alpha), T \max(f)]. \tag{53}$$

Together with (50), this implies that

$$kT_{\Gamma_R}, (k+l)T_{\Gamma_R} \in [T_{\min}(\lambda_0, \alpha), T \max(f)]$$

and so

$$lT_{\Gamma_R} \leq T \max(f) - T_{\min}(\lambda_0, \alpha). \tag{54}$$

Equation (52) implies that $l \geq |\alpha|$ and so we conclude from (54) that

$$T_{\Gamma_R} \leq \frac{1}{|\alpha|}(T \max(f) - T_{\min}(\lambda_0, \alpha)).$$

This implies the second statement of the lemma and concludes the proof. □

4. The proof of Theorem 1.6

To begin, we recall the setting and the statement of the theorem. Let (Q, ω) be a symplectic manifold of dimension $2n$ such that the class $-\omega/2\pi \in H^2(Q; \mathbb{R})$ is the image of an integral class $\mathbf{e} \in H^2(Q; \mathbb{Z})$. Let

$$p_Q : M \rightarrow Q$$

be an \mathbb{S}^1 -bundle over Q with first Chern class equal to \mathbf{e} and let λ_Q be the corresponding Boothby–Wang contact form on M . Denote by $\alpha_{\mathbf{f}} \in [\mathbb{S}^1, M]$ the free homotopy class corresponding to the fibres of the bundle p_Q .

THEOREM 4.1. *Let $\lambda = f\lambda_Q$ for some positive function f . If*

$$\frac{\max(f)}{\min(f)} < 2,$$

then there are at least $n + 1$ distinct closed Reeb orbits of λ which represent the class $\alpha_{\mathbf{f}}$ and have period in the interval

$$[2\pi \min(f), 2\pi \max(f)].$$

These orbits are geometrically distinct from one another if the class $\alpha_{\mathbf{f}}$ is either primitive or of infinite order. Otherwise, they are geometrically distinct if there are no closed Reeb orbits of λ which have period less than or equal to

$$\frac{2\pi}{|\alpha_{\mathbf{f}}|}(\max(f) - \min(f))$$

and which represent a class β such that $\beta^k = \alpha_{\mathbf{f}}$ for some integer $k > 1$.

The assertions concerning the conditions under which the detected orbits are geometrically distinct follow as in Lemma 3.4 and so their proof is left to the reader. It remains for us to detect $n + 1$ closed Reeb orbits of λ in class $\alpha_{\mathbf{f}}$ which are distinct and whose periods lie in the interval $[2\pi \min(f), 2\pi \max(f)]$.

We may assume that λ has finitely many, say N , distinct \mathbb{R}/\mathbb{Z} -families of such closed orbits. We denote them by

$$\Xi_1, \dots, \Xi_N.$$

We may also assume, by a simple rescaling, that $\min(f) = 1$. It remains to prove that $N \geq n + 1$.

Step 1. First we derive a Morse-theoretic implication of the fact that ω_Q is a symplectic form (Lemma 4.4 below). Let $F : Q \rightarrow \mathbb{R}$ be a Morse function, let q be a critical point of F with Morse index $\text{index}(q)$ and let g be a Riemannian metric on Q . We denote the stable submanifold of q , for the negative gradient flow of F with respect to g , by $W^s(q, (F, g))$, or just $W^s(q)$ assuming that the gradient data is clear from the context. For a generic choice of the metric g , each $W^s(q)$ is an embedded submanifold diffeomorphic to $\mathbb{R}^{2n - \text{index}(q)}$ and admits a compactification as a manifold with corners whose boundary faces comprise stable submanifolds of critical points of F with Morse index greater than that of q .

The following result is implied by standard transversality arguments.

LEMMA 4.2. *For a generic collection of Morse functions F_1, \dots, F_n and metrics g_1, \dots, g_n , the stable and unstable submanifolds of the critical points of the F_j all intersect transversally as do all of their repeated intersections. Moreover, for any closed Reeb orbit of $\xi_i(t)$ of λ belonging to one of the families Ξ_i , and any critical point q_j of any F_j , we have*

$$W^s(q_j) \pitchfork p_Q(\xi_i(t)). \tag{55}$$

From now on we fix Morse functions F_1, \dots, F_n and metrics g_1, \dots, g_n as in Lemma 4.2. We note, for later purposes, the following immediate consequence of condition (55) and the description of the closure of stable submanifolds above.

COROLLARY 4.3. *If q is a critical point of one of the F_j and $\text{index}(q) \geq 2$, then for all $i = 1, \dots, N$ we have*

$$\overline{W^s(q)} \cap p_Q(\Xi_i) = \emptyset. \tag{56}$$

Since ω_Q is a symplectic form, its n -fold wedge product

$$\omega_Q \wedge \dots \wedge \omega_Q$$

is a volume form on Q . When expressed in the Morse-theoretic version of the cup product from [BC94], for example, the existence of this nontrivial wedge product has the following implication.

LEMMA 4.4. *Let F_1, \dots, F_n and g_1, \dots, g_n be a collection of Morse functions and metrics as in Lemma 4.2. There are critical points q_j of the F_j such that $\text{index}(q_j) = 2$ and*

$$W^s(q_1) \cap \dots \cap W^s(q_n)$$

is a compact manifold of dimension zero with an odd number of elements.

We will denote the set $W^s(q_1) \cap \dots \cap W^s(q_n)$ by \mathcal{M}_0 and its elements by

$$\{\widehat{q}_1, \dots, \widehat{q}_{2K+1}\}.$$

Step 2. We now define a useful lift of the set $\mathcal{M}_0 \subset Q$ to M . Since each stable manifold $W^s(q_i)$ is contractible, the restriction of the bundle $p_Q : M \rightarrow Q$ to $W^s(q_i)$ is trivial. Fix such a trivialization for each $W^s(q_i)$. Then, given any point $q \in W^s(q_i)$ and any point $m \in p_Q^{-1}(q)$, there is a unique lift of $W^s(q_i)$ to M which intersects $p_Q^{-1}(q)$ at m . We denote this lift by

$$[W^s(q_i)]_m$$

and note that

$$[W^s(q_i)]_m \cap [W^s(q_i)]_{m'} = \emptyset \iff m \neq m' \in p_Q^{-1}(q).$$

Choose an $m_1 \in p_Q^{-1}(\widehat{q}_1)$ and consider the set

$$[W^s(q_1)]_{m_1} \cap \dots \cap [W^s(q_n)]_{m_1}.$$

Since it projects to \mathcal{M}_0 , we have

$$[W^s(q_1)]_{m_1} \cap \dots \cap [W^s(q_n)]_{m_1} = \bigcup_{j \in [1, 2K+1]} \left(\bigcap_{i \in [1, n]} [W^s(q_i)]_{m_1} \cap p_Q^{-1}(\widehat{q}_j) \right).$$

In particular, $[W^s(q_1)]_{m_1} \cap \dots \cap [W^s(q_n)]_{m_1}$ is a finite set of points each of which is a point on a fibre $p_Q^{-1}(\widehat{q}_j)$ at which all the $[W^s(q_i)]_{m_1}$ meet. By construction, m_1 is one of these points. Relabelling the \widehat{q}_j , if necessary, we may assume that

$$[W^s(q_1)]_{m_1} \cap \dots \cap [W^s(q_n)]_{m_1} = \{m_1, \dots, m_{k_1-1}\}$$

where $2 \leq k_1 \leq 2K + 2$ and

$$m_j \in p_Q^{-1}(\hat{q}_j).$$

To proceed we now choose a point $m_{k_1} \in p_Q^{-1}(\hat{q}_{k_1})$ and consider the lifts $[W^s(q_i)]_{m_{k_1}}$. For a generic such point, we may assume that

$$[W^s(q_i)]_{m_1} \cap [W^s(q_l)]_{m_{k_1}} \cap p_Q^{-1}(\hat{q}_j) = \emptyset$$

for all i, l and j . The intersection

$$[W^s(q_1)]_{m_{k_1}} \cap \dots \cap [W^s(q_n)]_{m_{k_1}}$$

is again a finite set consisting of points on the fibres $p_Q^{-1}(\hat{q}_j)$ at which all the $[W^s(q_i)]_{m_{k_1}}$ meet. Since

$$\bigcap_{1 \in [1, n]} [W^s(q_i)]_{m_1} \cap p_Q^{-1}(\hat{q}_{k_1}) = \emptyset,$$

it follows that none of the points in $[W^s(q_1)]_{m_{k_1}} \cup \dots \cup [W^s(q_n)]_{m_{k_1}}$ lie in the fibres $p_Q^{-1}(\hat{q}_j)$ for $j = 1, \dots, k_1 - 1$. Thus, relabelling again if needed, we may assume that

$$[W^s(q_1)]_{m_{k_1}} \cap \dots \cap [W^s(q_n)]_{m_{k_1}} = \{m_{k_1}, \dots, m_{k_2-1}\}$$

where $k_1 + 1 \leq k_2 \leq 2K + 2$ and again

$$m_j \in p_Q^{-1}(\hat{q}_j).$$

Continuing in this way, we obtain a set of points

$$\{m_1, \dots, m_{k_1}, \dots, m_{k_2}, \dots, m_{k_L}, \dots, m_{2K+1}\}$$

such that

$$m_j \in p_Q^{-1}(\hat{q}_j) \tag{57}$$

for all $j = 1, \dots, 2K + 1$. Setting $k_0 = 1$ and $k_{L+1} = 2K + 1$, we have

$$[W^s(q_1)]_{m_{k_j}} \cup \dots \cup [W^s(q_n)]_{m_{k_j}} = \{m_{k_j}, \dots, m_{k_{j+1}}\}$$

for $j = 0, \dots, k_L$. We may also assume that for $d \neq d'$,

$$[W^s(q_i)]_{m_{k_d}} \cap [W^s(q_l)]_{m_{k_{d'}}} \cap p_Q^{-1}(\hat{q}_j) = \emptyset \tag{58}$$

for all i, l and j .

We set

$$[\mathcal{M}_0] = \{m_1, \dots, m_{2K+1}\}$$

and note, for future reference, that

$$[\mathcal{M}_0] = \bigcup_{j=0}^L ([W^s(q_1)]_{m_{k_j}} \cap [W^s(q_2)]_{m_{k_j}} \cap \dots \cap [W^s(q_n)]_{m_{k_j}}). \tag{59}$$

Step 3. Here we identify the set $[\mathcal{M}_0]$ with a space of solutions to Floer's equation. Recall that the Reeb flow of λ_Q generates the natural \mathbb{S}^1 -action on the bundle M with (minimal) period 2π . In particular, every $q \in Q$ can be identified with the \mathbb{R}/\mathbb{Z} -family of closed Reeb orbits of λ_Q of period 2π whose image is $p_Q^{-1}(q)$. We will denote this family by Γ_q and will denote an element of Γ_q by $\gamma_q(t)$.

To proceed, we now utilize some of the machinery developed in the proof of Theorem 1.12. The collection $\mathcal{C}_{\lambda_Q, \alpha_f}(2\pi)$ is a rigid constellation with

$$\min\left\{\frac{T^+}{T}, \frac{T_{\min}(\lambda_Q) + T_{\min}(\lambda_Q, \alpha_f)}{T}\right\} = 2.$$

Choose a constant b which lies in the open interval $(2\pi \max(f), 4\pi)$ and which is not the period of a closed Reeb orbit of λ . Since $\max(f) < 2$, we can choose a sufficiently small and c sufficiently large so that for any profile h in $\mathfrak{h}_{a,b,c}$, the functions $H^0(\tau, p) = h(e^\tau)$ and $H^1(\tau, p) = h(e^\tau/\max(f))$ are finely tuned to $\mathcal{C}_{\lambda_Q, \alpha_f}(2\pi)$ and the function $\Psi_\lambda^*G(\tau, p) = h(e^\tau)$ is dividing with respect to λ .

The nonconstant 1-periodic orbits of H^0 with negative action are of the form

$$x(t) = (\tau_0, \gamma(2\pi t)), \tag{60}$$

where τ_0 is the unique solution of $h'(e^\tau) = 2\pi$ in the interval $(0, \ln(1+a))$ and $\gamma(t)$ belongs to one of the families Γ_q for $q \in Q$. We denote the collection of all 1-periodic orbits of the form (60) by $X(H^0)$.

Choosing a J^0 in $\mathcal{J}(H^0)$, we define \mathcal{M}_1 to be the set of smooth maps $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R} \times M$ such that

$$\begin{aligned} \partial_s u + J^0(u)(\partial_t u - V_{H^0}(u)) &= 0, \\ \lim_{s \rightarrow \pm\infty} u(s, t) &= x^\pm(t) \in X(H^0) \end{aligned} \tag{61}$$

and

$$p_M(u(0, 0)) \in [\mathcal{M}_0], \tag{62}$$

where $p_M : \mathbb{R} \times M \rightarrow M$ is the obvious projection.

The set of Floer trajectories \mathcal{M}_1 is in bijection with $[\mathcal{M}_0]$. To see this, let u belong to \mathcal{M}_1 and suppose that $p_M(u(0, 0)) = m_j \in [\mathcal{M}_0]$. Since the action \mathcal{A}_{H^0} is constant on $X(H^0)$, the energy identity

$$\int_{\mathbb{R} \times \mathbb{R}/\mathbb{Z}} d(e^\tau \lambda_0)(\partial_s u, J^0(u)\partial_s u) ds dt = \mathcal{A}_{H^0}(x^-) - \mathcal{A}_{H^0}(x^+)$$

together with the limiting conditions (61) imply that $\partial_s u(s, t) = 0$ for all (s, t) . Thus, $u(s, t) = (\tau_0, \gamma(2\pi t))$, where $\gamma(t)$ belongs to one of the families Γ_q for $q \in Q$. It then follows from condition (62) and property (57) that

$$u(s, t) = (\tau_0, \gamma_{\hat{q}_j}(2\pi t)),$$

where $\gamma_{\hat{q}_j}$ is the unique element of the family $\Gamma_{\hat{q}_j}$ satisfying $\gamma_{\hat{q}_j}(0) = m_j$. Conversely, every such map belongs to \mathcal{M}_1 and so it is in bijection with \mathcal{M}_0 . In particular, \mathcal{M}_1 has an odd number of elements.

Step 4. We now begin to deform the space \mathcal{M}_1 as a set of Floer trajectories. The first deformation involves moving the orbits that define the right asymptotic limits of the curves of \mathcal{M}_1 . Here we use the family of Hamiltonians

$$H^\varrho(\tau, p) = h\left(\frac{e^\tau}{1 + \varrho(\max(f) - 1)}\right),$$

which decreases from H^0 to H^1 as ϱ goes from zero to one.

LEMMA 4.5. For all $\rho \in [0, 1]$, the nonconstant 1-periodic orbits of H^ρ with negative action are of the form

$$t \mapsto (\tau_\rho, \gamma(2\pi t)), \tag{63}$$

where τ_ρ is the unique solution of

$$h' \left(\frac{e^\tau}{1 + \rho(\max(f) - 1)} \right) = 2\pi(1 + \rho(\max(f) - 1))$$

in the interval $(1 + \rho(\max(f) - 1), 1 + \rho(\max(f) - 1) + \ln(1 + a))$ and $\gamma(t)$ belongs to one of the families Γ_q for $q \in Q$.

Let $X(H^\rho)$ be the collection of all 1-periodic orbits of the form (63). For each ρ , let $H^{\rho,s}$ be the homotopy from H^0 to H^ρ of the form

$$H^{\rho,s} = [(1 - \text{step}(s))H^0 + \text{step}(s)\frac{1}{2}(H^0 + H^\rho)](1 - \text{step}(s)) + \text{step}(s)H^\rho.$$

Each function appearing in this family is finely tuned to $\mathcal{C}_{\lambda_Q, \alpha_f}(2\pi)$. So, for each $\rho \in [0, 1]$, we have

$$\Delta(H^0, H^\rho) < T_{\min}(\lambda_Q) = 2\pi. \tag{64}$$

As is easily checked, $\partial_s(H^{\rho,s}) \leq s$ and hence for each $\rho \in [0, 1]$ we also have

$$\text{cost}(H^{\rho,s}) = 0 \tag{65}$$

for the corresponding homotopy.

Choose a smooth two-parameter family $J^{\rho,s}$ of almost complex structures such that $(H^{0,s}, J^{0,s})$ and $(H^{1,s}, J^{1,s})$ are regular. Let $\mathcal{M}_{1+\rho}$ be the set of smooth maps $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R} \times M$ such that

$$\begin{aligned} \partial_s u + J^{\rho,s}(u)(\partial_t u - V_{H^{\rho,s}}(u)) &= 0, \\ \lim_{s \rightarrow -\infty} u(s, t) &\in X(H^0), \\ \lim_{s \rightarrow +\infty} u(s, t) &\in X(H^\rho) \end{aligned}$$

and

$$p_M(u(0, 0)) \in [\mathcal{M}_0].$$

By (64) and (65), each $\mathcal{M}_{1+\rho}$ is compact. So too is the collection

$$\mathcal{M}_{[1,2]} = \{(\rho, u) \mid \rho \in [0, 1], u \in \mathcal{M}_{1+\rho}\}.$$

In a standard way, $\mathcal{M}_{[1,2]}$ can also be described as the intersection of the zero section of an appropriate Banach space bundle with another Fredholm section. As described by Albers and Hein in [AH16, p. 21], one can then use (compact) abstract perturbations in this setting to perturb $\mathcal{M}_{[1,2]}$, away from the values $\rho = 0, 1$, to obtain a compact cobordism between the (zero-dimensional) spaces \mathcal{M}_1 and \mathcal{M}_2 .

By definition, the space \mathcal{M}_2 consists of maps $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R} \times M$ satisfying

$$\begin{aligned} \partial_s u + J^{1,s}(u)(\partial_t u - V_{H^{1,s}}(u)) &= 0, \\ \lim_{s \rightarrow -\infty} u(s, t) &\in X(H^0), \\ \lim_{s \rightarrow +\infty} u(s, t) &\in X(H^1) \end{aligned}$$

and

$$p_M(u(0, 0)) \in [\mathcal{M}_0].$$

It follows from the discussion above that \mathcal{M}_2 also has an odd number of elements.

Step 5. Now we deform \mathcal{M}_2 by deforming the monotone homotopy which defines it,

$$H^{1,s} = [(1 - \text{step}(s))H^0 + \text{step}(s)\frac{1}{2}(H^0 + H^1)](1 - \text{step}(s)) + \text{step}(s)H^1,$$

to another monotone homotopy from H^0 to H^1 that lingers on the Hamiltonian G .

Let

$$G^r = (1 - \text{step}(r - 1))\frac{1}{2}(H^0 + H^1) + \text{step}(r - 1)G$$

and set

$$\mathbf{STEP}(r, s) = \text{step}(s - (n + 1)r + \text{step}(r - 1)).$$

With these pieces, define

$$G^{r,s} = [(1 - \text{step}(s))H^0 + \text{step}(s)G^r](1 - \mathbf{STEP}(r, s)) + \mathbf{STEP}(r, s)H^1.$$

This is an admissible homotopy of homotopies from H^0 to H^1 and the following addition properties of $G^{r,s}$ are easily verified:

- ($G^{r,s}1$) $G^{0,s} = H^{1,s}$;
- ($G^{r,s}2$) $\partial_s(G^{r,s}) \leq 0$;
- ($G^{r,s}3$) $G^{r,s} = G$ whenever $r \geq 1$ and $s \in [0, (n + 1)r]$.

Choose a smooth family of almost complex structures $\bar{J}^{r,s}$ on $\mathbb{R} \times M$ such that for all $r \leq 0$ we have $\bar{J}^{r,s} = J^{1,s}$, where $J^{1,s}$ is the path of almost complex structures used in the definition of \mathcal{M}_2 and, for $r \in \mathbb{N}$, the continuation data set $(G^{r,s}, \bar{J}^{r,s})$ is regular. For each $r \geq 0$, define \mathcal{M}_3^r to be the space of maps $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R} \times M$ such that

$$\begin{aligned} \partial_s u + \bar{J}^{r,s}(u)(\partial_t u - V_{G^{r,s}}(u)) &= 0, \\ \lim_{s \rightarrow -\infty} u(s, t) &\in X(H^0), \\ \lim_{s \rightarrow +\infty} u(s, t) &\in X(H^1) \end{aligned}$$

and

$$p_M(u(jr, 0)) \in [W^s(q_j)]_{m_{k_0}} \sqcup \cdots \sqcup [W^s(q_j)]_{m_{k_L}} \quad \text{for all } j = 1, \dots, n. \tag{66}$$

LEMMA 4.6. *The space \mathcal{M}_3^0 is identical to \mathcal{M}_2 .*

Proof. Since $G^{0,s} = H^{1,s}$ and $\bar{J}^{0,s} = J^{1,s}$, it suffices to show that when $r = 0$ condition (66) is equivalent to $p_M(u(0, 0)) \in [\mathcal{M}_0]$. Condition (66) can be rewritten as

$$p_M(u(0, 0)) \in \bigcap_{j=1}^n ([W^s(q_j)]_{m_{k_0}} \sqcup \cdots \sqcup [W^s(q_j)]_{m_{k_L}}).$$

Distributing the intersections for the set appearing on the right, it becomes the union of sets of the form

$$[W^s(q_1)]_{m_{k_{d_0}}} \cap \cdots \cap [W^s(q_n)]_{m_{k_{d_L}}}. \tag{67}$$

By definition, the projection of any set of this form to Q is \mathcal{M}_0 . Thus, each such set is contained in $p_Q^{-1}(\mathcal{M}_0)$. Condition (58) then implies that the set (67) is empty unless

$$d_0 = d_1 = \dots = d_L.$$

So, for $r = 0$, condition (66) becomes

$$p_M(u(0, 0)) \in \bigcup_{j=0}^L ([W^s(q_1)]_{m_{k_j}} \cap [W^s(q_2)]_{m_{k_j}} \cap \dots \cap [W^s(q_n)]_{m_{k_j}})$$

and the set on the right equals $[\mathcal{M}_0]$ by (59). □

LEMMA 4.7. *For every $\ell \in \mathbb{N}$, the space \mathcal{M}_3^ℓ is a compact zero-dimensional manifold which is cobordant to \mathcal{M}_2 and hence is nonempty.*

Proof. Property $(G^{r,s}2)$ implies that $\text{cost}(G^{\ell,s}2) = 0$. Compactness then follows from this and the fact that the common end points of the homotopies, H^0 and H^1 , are finely tuned. For a fixed $\ell \in \mathbb{N}$, the desired cobordism is again obtained using abstract perturbations of the Fredholm section of the appropriate Banach space bundle which cuts out the compact set

$$\mathcal{M}_3^{[0,\ell]} = \{(r, u) \mid r \in [0, \ell], u \in \mathcal{M}_3^r\}. \quad \square$$

End game. By Lemma 4.7, we can consider a sequence of maps u_ℓ in \mathcal{M}_3^ℓ for $\ell \in \mathbb{N}$. The L^2 -energy of each u_ℓ is bounded by $\Delta(H^0, H^1) < T_{\min}(\lambda_Q) = 2\pi$. For $j = 1, \dots, n$, consider the sequence of maps

$$v_\ell^j(s, t) = u_\ell(s + j\ell, t), \quad \ell \in \mathbb{N}.$$

By the uniform energy bound above, each of these n sequences converges in $C_{\text{loc}}^\infty(\mathbb{R} \times \mathbb{R}/\mathbb{Z}, \mathbb{R} \times M)$ after passing to subsequences. Denote the n th limit obtained in this process by v^j . It follows from condition $(G^{r,s}3)$ that v_j satisfies the equation

$$\partial_s v^j + J(v^j)(\partial_t v^j - V_G(v^j)) = 0 \tag{68}$$

and again has L^2 -energy less than 2π . Recall that we have assumed that $\lambda = f\lambda_Q$ has finitely many families of closed Reeb orbits in class $\alpha_f, \Xi_1, \dots, \Xi_N$. Recall also, from §3, that the Hamiltonian Ψ_λ^*G is dividing for λ , where $\Psi_\lambda : \mathbb{R} \times M \rightarrow \mathbb{R} \times M$ is the diffeomorphism defined by

$$(\tau, p) \mapsto (\tau + f(p), p).$$

In particular, we have the following.

LEMMA 4.8. *The nonconstant 1-periodic orbits of G in class α_f with negative action are of the form*

$$x(t) = (\tau^i - f(\xi_i(T_i t)), \xi_i(T_i t)) \tag{69}$$

for $i = 1, \dots, N$, where τ^i is the unique solution of $h'(e^{\tau^i}) = T_i$ in the interval $(0, \ln(1 + a))$ and $\xi_i(t)$ belongs to the family Ξ_i whose common period is T_i .

Hence, for each $j = 1, \dots, n$, the limits

$$\lim_{s \rightarrow \pm\infty} v^j(s, t) = x_{\pm}^j(t)$$

exist and each $x_{\pm}^j(t)$ is a 1-periodic orbit of G of the form (69). If all the limits v^j depend nontrivially on s , then

$$\mathcal{A}_G(x_-^1) < \mathcal{A}_G(x_+^1) < \mathcal{A}_G(x_+^2) < \dots < \mathcal{A}_G(x_+^n)$$

and the orbits

$$x_-^1, x_+^1, x_+^2, \dots, x_+^n$$

are all distinct. By Lemma 4.8, the corresponding closed Reeb orbits

$$\xi_-^1, \xi_+^1, \xi_+^2, \dots, \xi_+^n$$

of λ are also distinct and so we will be done.

Assume then that one of the limits v_j does not depend on s . By Lemma 4.8, we then have

$$v^j(s, t) = v^j(0, t) = x^j(t) = (\tau^k - f(\xi_k(t + \theta)), \xi_k(t + \theta)) \tag{70}$$

for some $1 \leq k \leq N$ and $\theta \in [0, T_k)$. On the other hand, we have

$$v^j(0, 0) = \lim_{\ell \rightarrow \infty} u_{\ell}(j\ell, 0) \tag{71}$$

and

$$p_M(u_{\ell}(j\ell, 0)) \in [W^s(q_j)]_{m_1} \sqcup [W^s(q_j)]_{m_2} \sqcup \dots \sqcup [W^s(q_j)]_{m_L}.$$

Together, these conditions imply that

$$\xi_k(\theta) = \lim_{\ell \rightarrow \infty} p_M(u_{\ell}(j\ell, 0)) \in \overline{([W^s(q_j)]_{m_1} \sqcup [W^s(q_j)]_{m_2} \sqcup \dots \sqcup [W^s(q_j)]_{m_L})}$$

and so

$$p_Q(\xi_k(\theta)) \in \overline{W^s(q_j)}.$$

Since the Morse index of each q_j is two, this contradicts Corollary 4.3. Thus, the limits v^j above all depend nontrivially on s and the proof of Theorem 1.6 is complete.

5. The proof of Theorem 1.19

Recall the statement of Theorem 1.19.

THEOREM 5.1. *Let (M, λ_0) be a contact manifold. For any free homotopy class $\alpha \in [\mathbb{S}^1, M]$ and any positive constants $c_1, c_2 > 0$, there is a contact form $\lambda = f\lambda_0$ on M such that $\min(f) = 1$, $\max(f) < 1 + c_1$ and λ has a closed Reeb orbit in class α of period less than c_2 .*

We will first give the proof for the case when M is three dimensional. Here all the essential ideas are present and unobscured. We then give the proof for higher dimensional contact manifolds. In each case there are two steps. The first step involves the construction of a new and elementary Wilson semi-plug for Reeb flows which inserts fast closed Reeb orbits. In the second step, the contact form resulting from the insertion of the new plug is deformed into the desired contact form (in the correct conformal class).

Dimension three. Suppose that (M, λ_0) is three dimensional. Recall that a Legendrian knot in M is an embedded closed curve which is everywhere tangent to $\xi = \ker \lambda_0$. Recall also that there is a Legendrian knot arbitrarily C^0 -close to any closed loop in M . This allows us to start by fixing a Legendrian knot L in M which represents the class α .

We now consider a flow box for the Reeb flow of λ_0 around L using the following normal neighborhood theorem from [Wei71, Wei91].²

THEOREM 5.2. *For every small enough $\epsilon > 0$, there is a neighborhood P_ϵ of L in M of the form*

$$\{(t, x, \theta) \in [-2\epsilon, 2\epsilon] \times [-2\epsilon, 2\epsilon] \times \mathbb{R}/\mathbb{Z}\}$$

in which

$$\lambda_0 = dt + x d\theta.$$

The Reeb vector field of λ_0 in P_ϵ is just ∂_t and so P_ϵ is our flow box. Following [Gin95] and [Cie97], we now consider a deformation of λ_0 within P_ϵ of the form

$$\lambda_{\delta, \epsilon} = (1 - \delta \mathcal{A}) dt + \mathcal{B} d\theta.$$

Here \mathcal{A} and \mathcal{B} are smooth functions of t and x described below, and δ is a suitably small positive constant to be chosen later. Depending on the context, \mathcal{A} and \mathcal{B} will be considered as functions on either P_ϵ or the square $Q_\epsilon = [-2\epsilon, 2\epsilon] \times [-2\epsilon, 2\epsilon]$.

We choose $\mathcal{A}(t, x)$ so that it has the following simple properties:

- (A1) \mathcal{A} is supported in Q_ϵ ;
- (A2) $-1 < \mathcal{A} \leq 0$;
- (A3) $\mathcal{A}_x(0, \epsilon) = 1$ on the rectangle $[-\epsilon, \epsilon] \times [\epsilon/2, 3\epsilon/2]$.

The function $\mathcal{B}(t, x)$ is constructed as a perturbation of the function $(t, x) \mapsto x$ of the form

$$\mathcal{B}(t, x) = (1 - \mathcal{J}(t))x + \mathcal{J}(t)\mathcal{X}(x), \tag{72}$$

where $\mathcal{J} : [-2\epsilon, 2\epsilon] \rightarrow [0, 1]$ is a smooth function such that:

- (J1) the support of \mathcal{J} is $[-\epsilon, \epsilon]$;
- (J2) \mathcal{J} is an even function;
- (J3) $\mathcal{J}^{-1}(1) = \{0\}$

and $\mathcal{X} : [-2\epsilon, 2\epsilon] \rightarrow [-2\epsilon, 2\epsilon]$ is chosen so that:

- (X1) $\mathcal{X}(x) \geq x$ with equality only outside $(\epsilon/2, 3\epsilon/2)$;
- (X2) $\mathcal{X}'(x) \geq 0$ with equality only at ϵ ;
- (X3) $\mathcal{X}(\epsilon) = \epsilon + \epsilon^2$;
- (X4) $\mathcal{X}(x) - x < 2\epsilon^2$.

For these choices, the function \mathcal{B} inherits the following properties:

- (B1) $(0, \epsilon)$ is the only critical point of \mathcal{B} and the only point where \mathcal{B}_x is not positive;
- (B2) $0 \leq \mathcal{B}(t, x) - x < 2\epsilon^2$ for all $(t, x) \in Q_\epsilon$.

² A self-contained and detailed account of the proof is also presented in [Cie97], where the relevant result appears as Lemma A5.

By extending $\lambda_{\delta,\epsilon}$ as λ_0 outside of P_ϵ , we may view it as a one-form on M .

LEMMA 5.3. *For all sufficiently small $\delta > 0$, the form $\lambda_{\delta,\epsilon}$ is contact.*

Proof. It suffices to check this in P_ϵ , where we have

$$\lambda_{\delta,\epsilon} \wedge d(\lambda_{\delta,\epsilon}) = (\mathcal{B}_x(1 - \delta\mathcal{A}) + \delta\mathcal{A}_x\mathcal{B}) dt \wedge dx \wedge d\theta.$$

By property (A2),

$$\mathcal{B}_x(1 - \delta\mathcal{A}) + \delta\mathcal{A}_x\mathcal{B} \geq \mathcal{B}_x + \delta\mathcal{A}_x\mathcal{B}$$

and so it suffices to show that for all sufficiently small $\delta > 0$, the function $\mathcal{B}_x + \delta\mathcal{A}_x\mathcal{B}$ is strictly positive on Q_ϵ . In the subrectangle $[-\epsilon, \epsilon] \times [\epsilon/2, 3\epsilon/2] \subset Q_\epsilon$, this follows easily from (A3) and (B2), which imply that

$$\mathcal{B}_x + \delta\mathcal{A}_x\mathcal{B} = \mathcal{B}_x + \delta\mathcal{B} > \delta\frac{\epsilon}{2}.$$

Outside of this subrectangle, we have

$$\mathcal{B}_x + \delta\mathcal{A}_x\mathcal{B} = 1 + \delta x\mathcal{A}_x.$$

Choosing $\delta < |\min(x\mathcal{A}_x)|^{-1}$, we are done. □

The crucial feature of the new contact form $\lambda_{\delta,\epsilon}$ is the following.

LEMMA 5.4. *There is exactly one \mathbb{R}/\mathbb{Z} -family of simple periodic Reeb orbits of $\lambda_{\delta,\epsilon}$, $\Gamma_{\delta,\epsilon}$, which is contained in P_ϵ . The orbits in the family $\Gamma_{\delta,\epsilon}$ represent the class α and their (common) period is $2\pi(\epsilon + \epsilon^2)$.*

Proof. The kernel of $d\lambda_{\delta,\epsilon}$ in P_ϵ is spanned by the vector field

$$K = \mathcal{B}_x\partial_t - \mathcal{B}_t\partial_x + \delta\mathcal{A}_x\partial_\theta.$$

Since $(0, \epsilon)$ is the only critical point of \mathcal{B} , it follows easily from the formula for K that the embedded circle $S_\epsilon = \{0\} \times \{\epsilon\} \times \mathbb{R}/\mathbb{Z}$ corresponds to a unique \mathbb{R}/\mathbb{Z} -family, $\Gamma_{\delta,\epsilon}$, of simple periodic Reeb orbits of $\lambda_{\delta,\epsilon}$. Since the ∂_t -component of K is positive away from S_ϵ , there are no other closed Reeb orbits in P_ϵ .

Since S_ϵ is C^∞ -close to $\{0\} \times \{0\} \times \mathbb{R}/\mathbb{Z}$ in P_ϵ , it is also C^∞ -close to our original Legendrian knot L . Hence, orbits in $\Gamma_{\delta,\epsilon}$ represent α . Finally, the common periods of these orbits is

$$\left| \int_{S_\epsilon} \lambda_{\delta,\epsilon} \right| = 2\pi\mathcal{B}(0, \epsilon) = 2\pi(\epsilon + \epsilon^2). \tag{73}$$

□

Remark 5.5. The deformation of λ_0 to $\lambda_{\delta,\epsilon}$ within the flow box P_ϵ corresponds to the insertion of a Wilson semi-plug. Here the *semi* refers to the fact that our plug fails to have the *matching end point property*. This means that a trajectory which enters P_ϵ at $(-2\epsilon, x, \theta)$, then follows our new Reeb flow and exits P_ϵ at $t = 2\epsilon$ and does not need to do so at the point $(2\epsilon, x, \theta)$ (as the trajectories of the original Reeb vector field R_{λ_0} did). Thus, it is possible, indeed inevitable in some cases, that we have created new closed Reeb orbits which pass through P_ϵ but are not contained therein. This cannot be remedied. As shown by Rechtman in her thesis [Rec09], Sullivan’s characterization of geodesible vector fields from [Sul78] implies that no Wilson plug is geodesible and hence no Wilson plug is Reeb. In dimension three, a similar conclusion was also reached by Cieliebak in [Cie97]. There it is observed that if one could construct a Wilson plug for Reeb flows (centered about an arbitrary Legendrian knot) then one could immediately construct a counterexample to the following result.

THEOREM 5.6 (Hofer *et al.* [HWZ96]). *Every Reeb vector field on \mathbb{S}^3 has a periodic orbit which is unknotted and has self-linking number -1 .*

Having inserted the desired family of closed Reeb orbits with our semi-plug, we must now reckon with the fact that the form $\lambda_{\delta,\epsilon}$ need not be of the form $f\lambda_0$ for some positive function f .

LEMMA 5.7. *For all sufficiently small $\delta > 0$, there exists a diffeomorphism Ψ of M such that Ψ is isotopic to the identity and $\Psi^*\lambda_{\delta,\epsilon} = f\lambda_0$ for some positive function f such that $\min(f) = 1$ and*

$$\max(f) < e^{2\delta+4\epsilon}.$$

Proof. The desired diffeomorphism will be constructed as a composition of two maps. Each of these will be obtained using Gray’s stability theorem. We begin by refining the standard phrasing of this result to include some relevant quantitative information, which is easily extracted from the standard proof.

THEOREM 5.8 (Quantitative Gray stability). *Let $(\lambda_s)_{s \in [0,1]}$ be a smooth family of contact forms on M . Denote their Reeb vector fields by R_s and set*

$$r_s = i_{R_s} \left(\frac{d}{ds} \lambda_s \right).$$

There exists a one-parameter family of diffeomorphisms $(\psi_s)_{s \in [0,1]}$ of M which starts at the identity and satisfies

$$\psi_s^*(\lambda_s) = \exp \left(\int_0^s r_\sigma \circ \psi_\sigma d\sigma \right) \lambda_0.$$

Hence,

$$\psi_1^*(\lambda_1) = f_1 \lambda_0,$$

where

$$\min(f_1) \geq \exp \left(\int_0^1 \min_M r_\sigma d\sigma \right)$$

and

$$\max(f_1) \leq \exp \left(\int_0^1 \max_M r_\sigma d\sigma \right).$$

To obtain the first of our maps, we apply this version of Gray’s theorem to the family of one-forms

$$\bar{\lambda}_s = (1 - s\delta\mathcal{A}) dt + x d\theta.$$

Since

$$\bar{\lambda}_s \wedge d\bar{\lambda}_s = (1 + \delta s(x\mathcal{A}_x - \mathcal{A})) dt \wedge dx \wedge d\theta,$$

this is a family of contact forms for all sufficiently small $\delta > 0$. Their Reeb vector fields are given by

$$\bar{R}_s = \frac{1}{1 + \delta s(x\mathcal{A}_x - \mathcal{A})} (\partial_t + s\delta\mathcal{A}_x \partial_\theta).$$

We then have

$$\bar{r}_s = i_{\bar{R}_s} \left(\frac{d}{ds} \bar{\lambda}_s \right) = \frac{-\delta\mathcal{A}}{1 + \delta s(x\mathcal{A}_x - \mathcal{A})}.$$

Property (A2) implies that the numerator in the last expression is in $[0, \delta)$. It is also clear that for all sufficiently small $\delta > 0$, the denominator is greater than $1/2$. Thus, for all such small δ , we have

$$0 \leq \bar{r}_s < 2\delta$$

and Gray's theorem (as stated above) yields a diffeomorphism $\bar{\psi}_1$ such that

$$\bar{\psi}_1^*(\bar{\lambda}_1) = \bar{f}_1 \lambda_0$$

and

$$1 \leq \bar{f}_1 < e^{2\delta}. \tag{74}$$

To obtain our second map, we now consider the family of one-forms

$$\hat{\lambda}_s = \bar{\lambda}_1 + s(\mathcal{B} - x) d\theta = (1 - \delta\mathcal{A}) dt + \mathcal{B}^s d\theta,$$

where $\mathcal{B}^s = x + s(\mathcal{B} - x)$. These forms are contact whenever the functions $\mathcal{B}_x^s(1 - \delta\mathcal{A}) + \delta\mathcal{A}_x\mathcal{B}^s$ are strictly positive. Arguing as in Lemma 5.3, one can easily verify that this holds for all sufficiently small δ . Assuming this then, the Reeb vector field of $\hat{\lambda}_s$ is

$$\hat{R}_s = \frac{1}{\mathcal{B}_x^s(1 - \delta\mathcal{A}) + \delta\mathcal{A}_x\mathcal{B}^s} (\mathcal{B}_x^s \partial_t - \mathcal{B}_x^s \partial_x + \delta\mathcal{A}_x \partial_\theta)$$

and the relevant family of functions is

$$\hat{r}_s = i_{\hat{R}_s} \left(\frac{d}{ds} \hat{\lambda}_s \right) = \frac{(\mathcal{B} - x) \delta \mathcal{A}_x}{\mathcal{B}_x^s(1 - \delta\mathcal{A}) + \delta\mathcal{A}_x\mathcal{B}^s}.$$

Outside the subrectangle $[-\epsilon, \epsilon] \times [\epsilon/2, 3\epsilon/2] \subset Q_\epsilon$, where $B = x$, we have $\hat{r}_s = 0$. Inside this subrectangle, we have $\mathcal{A}_x = 1$ and so

$$\hat{r}_s = \frac{(\mathcal{B} - x) \delta}{\mathcal{B}_x^s(1 - \delta\mathcal{A}) + \delta\mathcal{B}^s}, \tag{75}$$

which is nonnegative by property (B2). Thus, each function \hat{r}_s is nonnegative on all of Q_ϵ .

To obtain an upper bound for the \hat{r}_s , it suffices to do so inside $[-\epsilon, \epsilon] \times [\epsilon/2, 3\epsilon/2]$. It follows from (75) and properties (A2) and (B2) that here we have

$$\frac{(\mathcal{B} - x) \delta}{\mathcal{B}_x^s(1 - \delta\mathcal{A}) + \delta\mathcal{B}^s} < \frac{2\epsilon^2 \delta}{\mathcal{B}_x^s + \delta\mathcal{B}^s}.$$

We also have $\mathcal{B}_x^s \geq 0$ and $\mathcal{B}^s \geq \epsilon/2$ in this subrectangle and so

$$0 \leq \hat{r}_s < 4\epsilon.$$

Thus, for all sufficiently small $\delta > 0$, there is a diffeomorphism $\hat{\psi}_1$ such that

$$\hat{\psi}_1^*(\hat{\lambda}_1) = \hat{f}_1 \bar{\lambda}_1$$

and

$$1 \leq \hat{f}_1 < e^{4\epsilon}. \tag{76}$$

To conclude, we set $\Psi = \widehat{\psi} \circ \bar{\psi}$. Since both factors are isotopic to the identity, so is Ψ . We also have

$$\begin{aligned} \Psi^* \lambda_{\delta, \epsilon} &= (\widehat{\psi} \circ \bar{\psi})^* (\widehat{\lambda}_1) \\ &= \bar{\psi}^* (\widehat{f}_1 \bar{\lambda}_1) \\ &= (\widehat{f}_1 \circ \bar{\psi}) \bar{f}_1 \lambda_0. \end{aligned}$$

By (74) and (76), the function $(\widehat{f}_1 \circ \bar{\psi}) \bar{f}_1$ satisfies

$$1 \leq (\widehat{f}_1 \circ \bar{\psi}) \bar{f}_1 \leq e^{2\delta + 4\epsilon}. \quad \square$$

At this point we can finish the proof of Theorem 1.19 in the three-dimensional case. Choose δ and ϵ so that

$$2\delta + 4\epsilon < c_1$$

and

$$2\pi(\epsilon + \epsilon^2) < c_2.$$

Assuming also that δ is small enough for Lemma 5.7 to hold, we set

$$\lambda = \Psi^* \lambda_{\delta, \epsilon}.$$

Given an orbit $\gamma_{\delta, \epsilon}$ in the family $\Gamma_{\delta, \epsilon}$ from Lemma 5.4 the closed curve $\Psi^{-1}(\gamma_{\delta, \epsilon}(t))$ is then a closed Reeb orbit of λ with (the same) period, $2\pi(\epsilon + \epsilon^2)$. Since Ψ is isotopic to the identity, the periodic orbit $\Psi^{-1}(\gamma_{\delta, \epsilon}(t))$ also represents the class α and, with this, we are done.

Remark 5.9. Rather than looking for specific types of periodic orbits that are forced to exist by the Reeb condition, as in Theorem 5.6, one might instead ask (in the spirit of [EG00]): what collections of periodic orbits can be generated by Reeb vector fields on a fixed three-manifold? In this direction the construction above yields the following result.

THEOREM 5.10. *For any co-oriented contact three-manifold (M, ξ) and any link type \mathcal{L} in M , there is a contact form $\lambda \in \Lambda(\xi)$ whose Reeb vector field has a collection of closed orbits which represents \mathcal{L} .*

Higher dimensions. We now show that the proof above for dimension three extends easily to higher dimensions. In practice, this amounts to showing that the extra variables can be cut off with no meaningful effects.

Consider then a contact manifold (M, λ_0) of dimension $2n - 1 > 3$ and let L be a simple closed curve in M which is everywhere tangent to ξ and represents the class α . Theorem 5.2 generalizes in the obvious way and yields, for small enough $\epsilon > 0$, a normal neighborhood of L in M of the form

$$P_\epsilon = \{(t, x, \theta, z) \in [-2\epsilon, 2\epsilon] \times [-2\epsilon, 2\epsilon] \times \mathbb{R}/\mathbb{Z} \times B^{2n-4}(\epsilon)\}$$

in which

$$\lambda_0 = dt + x d\theta + \kappa_0.$$

Here $B^{2n-4}(\epsilon)$ is the open unit ball of radius ϵ in \mathbb{R}^{2n-4} , we have

$$z = ((q_1, p_1), \dots, (q_{n-2}, p_{n-2}))$$

and κ_0 is the standard Liouville form

$$\kappa_0 = \frac{1}{2} \sum_{i=1}^{n-2} q_i dp_i - p_i dq_i.$$

In what follows we will only need to consider functions whose z -dependence is radial, i.e. which depend only on

$$\rho = \frac{|z|^2}{2}.$$

Our tool to cut off the extra variables will then be a smooth function $\mathbf{cut} : [0, \epsilon^2/2] \rightarrow [0, 1]$ with the following properties:

- (c1) $\mathbf{cut} = 0$ near $\epsilon^2/2$;
- (c2) $\mathbf{cut}(\rho) = 1 - \rho$ near 0;
- (c3) $-(4/\epsilon^2) < \mathbf{cut}'(\rho) \leq 0$.

Given the functions $\mathcal{A}(t, x)$ and $\mathcal{B}(t, x)$ from the previous section, we set

$$\widehat{\mathcal{A}}(t, x, \rho) = \mathbf{cut}(\rho)\mathcal{A}(t, x)$$

and

$$\widehat{\mathcal{B}}(t, x, \rho) = (1 - \mathbf{cut}(\rho))x + \mathbf{cut}(\rho)\mathcal{B}(t, x).$$

Now $(0, \epsilon, 0)$ is the only critical point of $\widehat{\mathcal{B}}$ and the only point at which $\widehat{\mathcal{B}}_x$ fails to be positive.

As before, for $\delta > 0$, we consider deformations of λ_0 of the form

$$\lambda_{\delta, \epsilon} = (1 - \delta\widehat{\mathcal{A}}) dt + \widehat{\mathcal{B}} d\theta + \kappa_0.$$

LEMMA 5.11. *For all sufficiently small $\delta > 0$, the form $\lambda_{\delta, \epsilon}$ is contact.*

Proof. Simple computations yield

$$d\lambda_{\delta, \epsilon} = \delta\widehat{\mathcal{A}}_x dt \wedge dx + \delta\widehat{\mathcal{A}}_\rho dt \wedge d\rho + \widehat{\mathcal{B}}_t dt \wedge d\theta + \widehat{\mathcal{B}}_x dx \wedge d\theta + \widehat{\mathcal{B}}_\rho d\rho \wedge d\theta + d\kappa_0$$

and

$$\begin{aligned} (d\lambda_{\delta, \epsilon})^{n-1} &= (n-1)(\delta\widehat{\mathcal{A}}_x dt \wedge dx + \widehat{\mathcal{B}}_t dt \wedge d\theta + \widehat{\mathcal{B}}_x dx \wedge d\theta) \wedge (d\kappa_0)^{n-2} \\ &\quad + (n-1)(n-2)\delta(\widehat{\mathcal{A}}_x \widehat{\mathcal{B}}_\rho - \widehat{\mathcal{A}}_\rho \widehat{\mathcal{B}}_x) dt \wedge dx \wedge d\rho \wedge d\theta \wedge \kappa_0 \wedge (d\kappa_0)^{n-3}. \end{aligned}$$

Using the identity

$$d\rho \wedge \kappa_0 \wedge (d\kappa_0)^{n-3} = \rho(d\kappa_0)^{n-2},$$

we then arrive at the following expression for $\lambda_{\delta, \epsilon} \wedge (d\lambda_{\delta, \epsilon})^{n-1}$:

$$(n-1)[((1 - \delta\widehat{\mathcal{A}})\widehat{\mathcal{B}}_x + \delta\widehat{\mathcal{A}}_x \widehat{\mathcal{B}}) + (n-2)\rho\delta(\widehat{\mathcal{A}}_x \widehat{\mathcal{B}}_\rho - \widehat{\mathcal{A}}_\rho \widehat{\mathcal{B}}_x)] dt \wedge dx \wedge d\theta \wedge (d\kappa_0)^{n-2}.$$

The function $((1 - \delta\widehat{\mathcal{A}})\widehat{\mathcal{B}}_x + \delta\widehat{\mathcal{A}}_x \widehat{\mathcal{B}}) + (n-2)\rho\delta(\widehat{\mathcal{A}}_x \widehat{\mathcal{B}}_\rho - \widehat{\mathcal{A}}_\rho \widehat{\mathcal{B}}_x)$ can be rewritten in the form

$$\widehat{\mathcal{B}}_x + \delta\mathcal{E},$$

where $\mathcal{E}(0, \epsilon, 0) = \widehat{\mathcal{B}}(0, \epsilon, 0) = \epsilon + \epsilon^2 > 0$. Since $\widehat{\mathcal{B}}_x \geq 0$ with equality only at the point $(0, \epsilon, 0)$, it follows from continuity that for all sufficiently small $\delta > 0$ the form $\lambda_{\delta, \epsilon} \wedge (d\lambda_{\delta, \epsilon})^{n-1}$ is nonvanishing and hence $\lambda_{\delta, \epsilon}$ is a contact form. \square

LEMMA 5.12. *There is exactly one \mathbb{R}/\mathbb{Z} -family of simple periodic Reeb orbits of $\lambda_{\delta,\epsilon}$, $\Gamma_{\delta,\epsilon}$ which is contained in P_ϵ . The orbits in the family $\Gamma_{\delta,\epsilon}$ represent the class α and their (common) period is $2\pi(\epsilon + \epsilon^2)$.*

Proof. For a fixed t and x , let $V^{t,x}$ be the Hamiltonian vector field on $(B^{2n-4}(\epsilon), d\kappa_0)$ defined by the function $z \mapsto \delta\widehat{\mathcal{A}}(t, x, |z|^2/2)$. That is, $V^{t,x}(z)$ is defined by the equation

$$d\kappa_0(z)(V^{t,x}(z), \cdot) = \mathbf{cut}'(|z|^2/2)\delta\mathcal{A}(t, x) d\rho(\cdot).$$

Let $V(t, x, z, \theta)$ be the vector field whose projection to $(B^{2n-4}(\epsilon), d\kappa_0)$ is $V^{t,x}$ and whose other components are trivial. Define the vector field U by replacing $\delta\widehat{\mathcal{A}}$ above by $\widehat{\mathcal{B}}$. The kernel of $d\lambda_{\delta,\epsilon}$ is then spanned by the vector field

$$K = \widehat{\mathcal{B}}_x \partial_t - \widehat{\mathcal{B}}_t \partial_x + \delta\widehat{\mathcal{A}}_x \partial_\theta + \delta\widehat{\mathcal{A}}_x U - \widehat{\mathcal{B}}_x V.$$

To see this, note first that $d\rho(V) = d\rho(U) = 0$ since the corresponding Hamiltonian vector fields are defined by functions of ρ . A simple computation then yields

$$\begin{aligned} i_K d\lambda_{\delta,\epsilon} &= \delta\widehat{\mathcal{A}}_\rho \widehat{\mathcal{B}}_x d\rho - \delta\widehat{\mathcal{B}}_\rho \widehat{\mathcal{A}}_x d\rho + i_{\delta\widehat{\mathcal{A}}_x U} d\kappa_0 - i_{\widehat{\mathcal{B}}_x V} d\kappa_0 \\ &= \delta\widehat{\mathcal{A}}_\rho \widehat{\mathcal{B}}_x d\rho - \delta\widehat{\mathcal{B}}_\rho \widehat{\mathcal{A}}_x d\rho + \delta\widehat{\mathcal{A}}_x \widehat{\mathcal{B}}_\rho d\rho - \delta\widehat{\mathcal{B}}_x \widehat{\mathcal{A}}_\rho d\rho \\ &= 0. \end{aligned}$$

The t -component of K vanishes only when $(t, x, z) = (0, \epsilon, 0)$. At this point both $\widehat{\mathcal{B}}_x$ and $\widehat{\mathcal{B}}_t$ vanish as do V and U since $d\rho = 0$ when $z = 0$. Thus, $\lambda_{\delta,\epsilon}$ has exactly one \mathbb{R}/\mathbb{Z} -family of simple closed Reeb orbits in P_ϵ . These orbits all have image

$$S_\epsilon = \{0\} \times \{\epsilon\} \times \mathbb{R}/\mathbb{Z} \times \{0\}$$

and period

$$\left| \int_{S_\epsilon} \lambda_{\delta,\epsilon} \right| = 2\pi(\epsilon + \epsilon^2). \quad \square$$

LEMMA 5.13. *For all sufficiently small $\delta > 0$, there exists a diffeomorphism Ψ of M such that Ψ is isotopic to the identity and $\Psi^* \lambda_{\delta,\epsilon} = f \lambda_0$ for some positive function f such that $\min(f) = 1$ and*

$$\max(f) < e^{2\delta+4\epsilon}.$$

Proof. As in the proof of Lemma 5.7, we first apply the quantitative version of Gray’s theorem to the family of one-forms

$$\bar{\lambda}_s = (1 - s\delta\widehat{\mathcal{A}}) dt + x d\theta + \kappa_0.$$

These are contact for all sufficiently small $\delta > 0$ and their Reeb vector fields are given by

$$\bar{R}_s = \frac{1}{(n-1)[1 - s\delta\widehat{\mathcal{A}} + s\delta\widehat{\mathcal{A}}_x x - s(n-2)\rho\delta\widehat{\mathcal{A}}_\rho]} (\partial_t + s\delta\widehat{\mathcal{A}}_x \partial_\theta + s\delta\widehat{\mathcal{A}}_x Y - X).$$

The relevant functions are then

$$\bar{r}_s = i_{\bar{R}_s} \left(\frac{d}{ds} \bar{\lambda}_s \right) = \frac{-\delta\widehat{\mathcal{A}}}{(n-1)[1 - s\delta\widehat{\mathcal{A}} + s\delta\widehat{\mathcal{A}}_x x - s(n-2)\rho\delta\widehat{\mathcal{A}}_\rho]}.$$

Property (A2) implies that the numerator in the last expression is in $[0, \delta)$. It is also clear that for all sufficiently small $\delta > 0$, the denominator is greater than $1/2$. Thus, for all such small δ , we have

$$0 \leq \bar{r}_s < 2\delta$$

and Gray's theorem (as stated above) yields a diffeomorphism $\bar{\psi}_1$ such that

$$\bar{\psi}_1^*(\bar{\lambda}_1) = \bar{f}_1 \lambda_0$$

and

$$1 \leq \bar{f}_1 < e^{2\delta}. \tag{77}$$

Next we consider the family of one-forms

$$\widehat{\lambda}_s = \bar{\lambda}_1 + s(\widehat{\mathcal{B}} - x) d\theta = (1 - \delta\widehat{\mathcal{A}}) dt + \widehat{\mathcal{B}}^s d\theta.$$

They are also contact for all sufficiently small δ and the Reeb vector field \widehat{R}_s of $\widehat{\lambda}_s$ is equal to

$$\widehat{\mathcal{B}}_x^s \partial_t - \widehat{\mathcal{B}}_t^s \partial_x + \delta\widehat{\mathcal{A}}_x \partial_\theta + \delta\widehat{\mathcal{A}}_x Y - \widehat{\mathcal{B}}_x^s X$$

multiplied by the function

$$((n - 1)[(1 - \delta\widehat{\mathcal{A}})\widehat{\mathcal{B}}_x^s + \delta\widehat{\mathcal{A}}_x \widehat{\mathcal{B}}^s] + (n - 2)\rho\delta(\widehat{\mathcal{A}}_x \widehat{\mathcal{B}}_\rho^s - \widehat{\mathcal{A}}_\rho \widehat{\mathcal{B}}_x^s))^{-1}.$$

This makes the relevant family of functions

$$\widehat{r}_s = \frac{(\widehat{\mathcal{B}} - x)\delta\widehat{\mathcal{A}}_x}{(n - 1)[(1 - \delta\widehat{\mathcal{A}})\widehat{\mathcal{B}}_x^s + \delta\widehat{\mathcal{A}}_x \widehat{\mathcal{B}}^s] + (n - 2)\rho\delta(\widehat{\mathcal{A}}_x \widehat{\mathcal{B}}_\rho^s - \widehat{\mathcal{A}}_\rho \widehat{\mathcal{B}}_x^s)}.$$

Outside $[-\epsilon, \epsilon] \times [\epsilon/2, 3\epsilon/2] \times B^{2n-4}(\epsilon)$, where $\widehat{\mathcal{B}} = x$, we have $\widehat{r}_s = 0$. Inside this region, we have $\widehat{\mathcal{A}}_x = \mathbf{cut}(\rho)$ and so

$$\widehat{r}_s = \frac{(\widehat{\mathcal{B}} - x)\delta\mathbf{cut}(\rho)}{(n - 1)[(1 - \delta\widehat{\mathcal{A}})\widehat{\mathcal{B}}_x^s + \delta\mathbf{cut}(\rho)\widehat{\mathcal{B}}^s] + (n - 2)\rho\delta(\mathbf{cut}(\rho)\widehat{\mathcal{B}}_\rho^s - \mathbf{cut}'(\rho)\widehat{\mathcal{A}}\widehat{\mathcal{B}}_x^s)}.$$

Property (B2) implies that this expression is nonnegative and so each function \widehat{r}_s is nonnegative on all of $Q_\epsilon \times B^{2n-4}(\epsilon)$.

To obtain the desired upper bound for the \widehat{r}_s , it suffices to do so inside $[-\epsilon, \epsilon] \times [\epsilon/2, 3\epsilon/2] \times B^{2n-4}(\epsilon)$. Here we have

$$\begin{aligned} \widehat{r}_s &= \frac{(\widehat{\mathcal{B}} - x)\delta\mathbf{cut}(\rho)}{(n - 1)[(1 - \delta\widehat{\mathcal{A}})\widehat{\mathcal{B}}_x^s + \delta\mathbf{cut}(\rho)\widehat{\mathcal{B}}^s] + (n - 2)\rho\delta(\mathbf{cut}(\rho)\widehat{\mathcal{B}}_\rho^s - \mathbf{cut}'(\rho)\widehat{\mathcal{A}}\widehat{\mathcal{B}}_x^s)} \\ &= \frac{(\widehat{\mathcal{B}} - x)\delta\mathbf{cut}(\rho)}{(n - 1)[(1 - \delta\widehat{\mathcal{A}}(1 + (n - 2)\rho\mathbf{cut}(\rho)\mathbf{cut}'(\rho)))\widehat{\mathcal{B}}_x^s + \delta\mathbf{cut}(\rho)(\widehat{\mathcal{B}}^s + (n - 2)\rho\widehat{\mathcal{B}}_\rho^s)]}. \end{aligned}$$

For all sufficiently small δ , we may assume that the coefficient of $\widehat{\mathcal{B}}_x^s$ in the denominator is greater than $1/2$. Using this and the formula defining $\widehat{\mathcal{B}}^s$, we get

$$\widehat{r}_s \leq \frac{(\widehat{\mathcal{B}} - x)\delta\mathbf{cut}(\rho)}{(n - 1)[\frac{1}{2}\widehat{\mathcal{B}}_x^s + \delta\mathbf{cut}(\rho)(x + [\mathbf{cut}(\rho) + (n - 2)\rho\mathbf{cut}'(\rho)]s(B - x))]}.$$

By condition (B2), the function $B - x$ (and thus $\widehat{B} - x$) takes values in $[0, 2\epsilon^2]$. For sufficiently small $\epsilon > 0$, we may therefore assume that

$$|[\mathbf{cut}(\rho) + (n - 2)\rho\mathbf{cut}'(\rho)](B - x)| < \frac{\epsilon}{4}.$$

(Here we have used the fact that (c3) implies that $0 \geq \rho\mathbf{cut}'(\rho) > -2$.) It then follows that on the subset of interest, $[-\epsilon, \epsilon] \times [\epsilon/2, 3\epsilon/2] \times B^{2n-4}(\epsilon)$, where $x \geq \epsilon/2$, we have

$$\widehat{r}_s < \frac{2\epsilon^2\delta\mathbf{cut}(\rho)}{(n - 1)\left[\frac{1}{2}\widehat{B}_x^s + \delta\mathbf{cut}(\rho)\frac{\epsilon}{4}\right]}.$$

Using the fact that $\widehat{B}_x^s \geq 0$ and $n \geq 3$, we arrive at the upper bound

$$\widehat{r}_s < 4\epsilon.$$

Thus, for all sufficiently small $\delta > 0$ and $\epsilon > 0$, it follows from Gray's theorem that there is a diffeomorphism $\widehat{\psi}_1$ such that

$$\widehat{\psi}_1^*(\widehat{\lambda}_1) = \widehat{f}_1\bar{\lambda}_1$$

and

$$1 \leq \widehat{f}_1 < e^{4\epsilon}.$$

The desired diffeomorphism is then $\Psi = \widehat{\psi} \circ \bar{\psi}$. □

The rest of the proof of Theorem 1.19 now follows exactly as it did for dimension three.

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RIGID CONSTELLATIONS OF CLOSED REEB ORBITS

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