AN EXTENSION OF THE CLASS OF ALTERNATIVE RINGS

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1. Introduction. Define R to be a ring of characteristic not 2 or 3 satisfying the identity

$$(x, y, z) = (y, z, x) \tag{1}$$

for all $x, y, z \in R$, where by characteristic not p is meant $x \to px$ is a one-to-one mapping of R upon R. The associator (a, b, c) of R is defined by $(a, b, c) = ab \cdot c - a \cdot bc$. If R also satisfies the identity

$$(x, x, x) = 0 \tag{2}$$

for all $x \in R$, then it is easy to see by linearizing (2) and applying (1) that R is alternative. Conversely, it is well known that (1) and (2) are satisfied by every alternative ring. That R need not be alternative is seen by (**4**, Ex. 1, p. 986); hence the class of rings satisfying (1) is a non-trivial extension of the class of alternative rings.

Some time ago, P. Jordan pointed out that $(x, x, x)^2 = 0$ is an identity in R (**7**, p. 395). This observation of Jordan's, which has apparently not been verified in the literature, yields the result that if $a^2 = 0$ implies $a = 0$ in R, then R is alternative. In this paper, we verify Jordan's observation in the process of establishing the following results on the structure of R. If R is either simple or primitive, then R is a Cayley-Dickson algebra or associative; and if R is semi-simple, then R is a subdirect sum of Cayley-Dickson algebras and primitive, associative rings.

We are using the standard terminology. R is simple provided R has no proper ideals and $R^2 \neq 0$; R is primitive in case R contains a regular, maximal right ideal which contains no non-zero ideals of R; the radical of R is the intersection of its regular, maximal right ideals; and R is semi-simple if the radical of R is zero.

Note that our results on the structure of R are identical with those for alternative rings (**2**, Th. 2, p. 729 and Th. 3, p. 730; **3**, Th. 3, p. 399). Also, that R is a Cayley-Dickson algebra or associative if R is simple, generalizes Kleinfeld's theorem that an assosymmetric ring of characteristic not 2 or 3 without ideals $I \neq 0$ such that $I^2 = 0$ is associative (**4**, p. 985). Finally, the

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result on simple rings is shared by many classes of associator dependent rings **(6**, Th. 6**)**. Associator dependent rings are defined to be rings satisfying (2) and

$$(3) \qquad \alpha_1(x, y, z) + \alpha_2(y, z, x) + \alpha_3(z, x, y) + \alpha_4(x, z, y) + \alpha_5(z, y, x) + \alpha_6(y, x, z) = 0$$

for fixed α_i in some field of scalars **(5)**. A natural extension of the study of associator dependent rings would be to drop the assumption of (2) and study rings satisfying a particular form of (3) which does not imply (2). Rings satisfying (1) are such a class of rings.

2. The f-function. In this section we shall study the properties in R of the f-function introduced in **(1**, p. 880**)**:

$$(4) \qquad f(w, x, y, z) = (wx, y, z) - x(w, y, z) - (x, y, z)w.$$

Then we shall use those properties to derive some useful identities.

Most of the computations which follow involve the use of equation (1). However, it will be easy to tell when (1) has been used; so its use will not be cited.

We need the Teichmüller identity, which holds in an arbitrary ring:

$$0 = h(w, x, y, z) = (wx, y, z) - (w, xy, z) + (w, x, yz) - w(x, y, z) - (w, x, y)z.$$

Now, expanding f and h and collecting terms yields

$$f(w, x, y, z) - h(w, x, y, z) - h(x, y, z, w) = -(zw, x, y) + w(z, x, y) + (w, x, y)z;$$

but

$$f(z, w, x, y) = (zw, x, y) - w(z, x, y) - (w, x, y)z;$$

hence

$$(5) \qquad f(w, x, y, z) = -f(z, w, x, y).$$

Define the commutator (a, b) by $(a, b) = ab - ba$. Then

$$f(w, x, y, z) - f(z, w, x, y) + f(y, z, w, x) - h(y, z, w, x) = (w, (x, y, z)) - (x, (y, z, w)) + (y, (z, w, x)) - (z, (w, x, y))$$

upon expanding f and h and collecting terms. However, by (5) we have

$$f(w, x, y, z) - f(z, w, x, y) + f(y, z, w, x) = 3f(w, x, y, z),$$

hence

$$(6) \qquad 3f(w, x, y, z) = (w, (x, y, z)) - (x, (y, z, w)) + (y, (z, w, x)) - (z, (w, x, y)).$$

Letting $w = x$ and $y = z$ in (6), we obtain $f(x, x, y, y) = 0$, which yields, upon linearizing,

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(7) $f(w, x, y, z) + f(x, w, y, z) + f(w, x, z, y) + f(x, w, z, y) = 0.$

Using (5) and (7), we can write

$$f(w, x, y, z) = f(x, w, z, y) + f(z, y, x, w) - f(w, x, z, y) - f(y, z, x, w) - 3f(x, w, z, y).$$

Expanding the first four terms using (4) and the last using (6), we obtain

(8) $$f(w, x, y, z) = ((x, w), z, y) + (x, w, (z, y)).$$

Using (4) and then (6) to expand $3f(x, y, x, z)$, we obtain, upon subtracting the resulting expressions,

$$A(x, y, z) = 3(xy, x, z) - x(y, x, z) - 2(y, x, z)x - 2y(x, x, z) - (x, x, z)y - (x, (x, y, z)) + (z, (y, x, x)) = 0.$$

Similarly, using (4) and then (6) to expand $3f(x, x, y, z)$, we obtain

$$B(x, y, z) = 3(x^2, y, z) - 3x(x, y, z) - 3(x, y, z)x - (y, (z, x, x)) + (z, (y, x, x)) = 0.$$

Also, by (7) we have

$$0 = 2f(y, z, x, x) - f(z, y, x, x) - 3f(y, z, x, x).$$

Expanding the first two terms using (4) and then the last term using (6), we obtain

(9) $$2(yz, x, x) - (zy, x, x) - z(y, x, x) - (z, x, x)y = 0.$$

Next, we wish to establish the important identity

(10) $$3((y, x, x), x, x) + ((x, x, x), x, y) + ((x, x, x), y, x) = 0.$$

Indeed, if we expand f by (4), we observe that

$$f(x, x, x^2, x) = f(x, x, x, x^2);$$

hence application of (5) yields

(11) $$f(x^2, x, x, x) = 0.$$

Note that in an arbitrary ring,

$$(x, x, x) = (x^2, x) = -(x, x^2).$$

Thus

$$((x, x, x), x, x) = -((x, x^2), x, x);$$

but by (8)

$$-((x, x^2), x, x) = -f(x^2, x, x, x),$$

hence by (11) we have

(12) $$((x, x, x), x, x) = 0.$$

Substituting $x + y$ for x in (12) and then $-x + y$ for x in (12) and adding yields

$$(13)\quad 3((x, x, y), x, x) + ((x, x, x), x, y) + ((x, x, x), y, x) + ((y, y, y), x, x) + 3((y, y, x), y, x) + 3((y, y, x), x, y) + 3((y, x, x), y, y) = 0,$$

since terms with one or three x's cancel, and terms with no x's or five x's go out by (12). Similarly, substituting $x + 2y$ for x in (12) and then $-x + 2y$ for x in (12) and adding yields

$$3((x, x, y), x, x) + ((x, x, x), x, y) + ((x, x, x), y, x) + 4((y, y, y), x, x) + 12((y, y, x), y, x) + 12((y, y, x), x, y) + 12((y, x, x), y, y) = 0.$$

Subtracting (13) from this last equation and cancelling by 3, we obtain

$$((y, y, y), x, x) + 3((y, y, x), y, x) + 3((y, y, x), x, y) + 3((y, x, x), y, y) = 0,$$

which yields (10) on comparing with (13).

Proceeding from (10) in somewhat the same way as we did from (12), substituting $x + z$ for x in (10) and then $x - z$ for x in (10), and adding yields

$$(14)\quad 0 = ((y, x, x), z, z) + ((y, x, z), z, x) + ((y, x, z), x, z) + ((y, z, x), z, x) + ((y, z, x), x, z) + ((y, z, z), x, x) + ((z, x, x), z, y) + ((z, z, x), x, y) + ((z, x, x), y, z) + ((z, z, x), y, x).$$

Letting $y = x$ in (14) yields

$$(15)\quad 0 = ((x, x, x), z, z) + 3((z, x, x), z, x) + 3((z, x, x), x, z) + 3((z, z, x), x, x).$$

Similarly, we substitute $x + y$ for x in (15) and then $-x + y$ for x in (15) and add to obtain

$$(16)\quad 0 = ((y, x, x), z, z) + ((z, x, x), z, y) + ((z, x, y), z, x) + ((z, y, x), z, x) + ((z, x, x), y, z) + ((z, x, y), x, z) + ((z, y, x), x, z) + ((z, z, x), x, y) + ((z, z, x), y, x) + ((z, z, y), x, x).$$

Next, we wish to substitute (x, x, x) for y in (16); however, before doing that we shall introduce some new notation to make the equations less cumbersome to work with.

3. The v-function.

For $x \in R$, define $v^n(a)$, $a \in R$, by

$$v(a) = (a, x, x), \qquad v^k(a) = v(v^{k-1}(a)).$$

It will cause no ambiguity in what follows to fail to take into account in the notation that $v^n(a)$ depends upon x as well as a.

Clearly,

$$v^m(v^n(a)) = v^n(v^m(a)) = v^{m+n}(a),$$

and

$$v^n(a+b) = v^n(a) + v^n(b).$$

Equations (9), (10), and (12) become, respectively,

$$C(y, z) = 2v(yz) - v(zy) - zv(y) - v(z)y = 0,$$
$$D(y) = 3v^2(y) + (v(x), x, y) + (v(x), y, x) = 0,$$

and

$$v^2(x) = 0. \tag{17}$$

In what follows, we shall derive some basic identities involving the v-function, and then we shall make the substitution $y = v(x)$ in (16) that we mentioned previously. This substitution will lead to the identity $v^4(y) = 0$, which in turn yields the identity $v^3(a)v^3(b) = 0$. In so doing, we shall also show that if $a^2 = 0$ implies $a = 0$ in R, then R is alternative.

Now, expanding $0 = B(y, x, x)$ yields

$$v(y^2) = yv(y) + v(y)y. \tag{18}$$

Also, expanding

$$0 = B(y, v(x), x) + B(y, x, v(x)) - 3D(y^2) + 3D(y)y + 3yD(y)$$

yields

$$0 = -9v^2(y^2) + 9v^2(y)y + 9yv^2(y);$$

hence, by (18) we have

$$v^2(y^2) = v(yv(y) + v(y)y) = v^2(y)y + yv^2(y). \tag{19}$$

Finally, computing

$$0 = C(v(y), y) + C(y, v(y)),$$

we obtain

$$0 = v(v(y)y + yv(y)) - yv^2(y) - v^2(y)y - 2(v(y))^2;$$

hence (19) yields

$$(v(y))^2 = 0. \tag{20}$$

Equation (20) has far-reaching consequences. In the first place, Jordan's result that $(v(x))^2 = 0$, which we mentioned in §1, is a special case of (20), and we have by the comments in §1, the following theorem.

THEOREM 1. *If $a^2 = 0$ implies $a = 0$ in R, then R is alternative.*

Linearizing (20) yields

$$E(a, b) = v(a)v(b) + v(b)v(a) = 0.$$

Since

$$(v(a), v(a), v(b)) = (v(a), v(b), v(a)) = (v(b), v(a), v(a)),$$

using (20) we obtain

$$\begin{aligned}3(v(a), v(a), v(b)) &= -v(a)\cdot v(a)v(b) + v(a)v(b)\cdot v(a) \\ &\quad -v(a)\cdot v(b)v(a) + v(b)v(a)\cdot v(a) \\ &= -v(a)E(a, b) + E(a, b)v(a) = 0.\end{aligned}$$

Hence

$$(v(a), v(a), v(b)) = 0. \tag{21}$$

Linearizing (21) we obtain

$$(v(a), v(b), v(c)) + (v(b), v(a), v(c)) = 0. \tag{22}$$

Finally, by (22),

$$0 = v(b)E(a, c) - E(a, b)v(c) + (v(a), v(b), v(c)) + (v(b), v(a), v(c)),$$

which yields, upon expansion,

$$v(a)\cdot v(b)v(c) = v(b)\cdot v(c)v(a). \tag{23}$$

Now, computing $0 = 2C(a, b) + C(b, a)$, we obtain

$$F(a, b) = 3v(ab) - av(b) - 2v(b)a - v(a)b - 2bv(a) = 0.$$

Next, we expand

$$\begin{aligned}0 = 3v(F(a, b)) + F(a, v(b)) + 2F(v(b), a) + F(v(a), b) \\ + 2F(b, v(a)) + 8E(a, b)\end{aligned}$$

to obtain

$$G(a, b) = 9v^2(ab) - 5av^2(b) - 4v^2(b)a - 5v^2(a)b - 4bv^2(a) - 2v(a)v(b) = 0.$$

Linearizing (19), we have

$$H(a, b) = v^2(ab + ba) - av^2(b) - v^2(a)b - bv^2(a) - v^2(b)a = 0.$$

Then, we compute $0 = G(a, b) - 4H(a, b)$ to obtain

$$I(a, b) = 5v^2(ab) - 4v^2(ba) - av^2(b) - v^2(a)b - 2v(a)v(b) = 0.$$

Finally, expanding

$$\begin{aligned}0 = 3v(G(a, b)) + 5F(a, v^2(b)) + 4F(v^2(b), a) + 5F(v^2(a), b) \\ + 4F(b, v^2(a)) + 2F(v(a), v(b)) + 15E(a, v(b)) + 15E(v(a), b)\end{aligned}$$

yields

$$\begin{aligned}J(a, b) = 27v^3(ab) - 13av^3(b) - 14v^3(b)a - 13v^3(a)b - 14bv^3(a) \\ - 3v^2(b)v(a) - 3v(b)v^2(a) = 0.\end{aligned}$$

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Letting $y = v(x)$ in (16), subtracting

$$0 = (D(z), x, z) + (D(z), z, x) + D((z, z, x)),$$

and applying (17), we obtain

$$\begin{aligned}(24)\quad 0 = (v(z), z, v(x)) + (v(z), v(x), z) + v((z, z, v(x))) - 3(v^2(z), x, z)\\ - 3(v^2(z), z, x) - 3v^2((z, z, x)).\end{aligned}$$

Substituting $x + y$ for z in (24), subtracting $0 = D(v(y)) + v(D(y))$, and applying (17) yields

$$(25)\qquad 9v^3(y) = (v(x), v(x), y).$$

We obtain

$$(26)\qquad v^4(y) = 0$$

by letting $y = v(y)$ in (25) and applying (21). Next, we expand

$$\begin{aligned}0 = 3v(J(a, b)) + 13F(a, v^3(b)) + 14F(v^3(b), a) + 13F(v^3(a), b)\\ + 14F(b, v^3(a)) + 3F(v^2(b), v(a)) + 3F(v(b), v^2(a))\\ + 43E(a, v^2(b)) + 43E(v^2(a), b) + 6E(v(a), v(b))\end{aligned}$$

to obtain, upon applying (26),

$$(27)\qquad 2v(a)v^3(b) + 2v^3(a)v(b) + 3v^2(a)v^2(b) = 0.$$

Finally, letting $a = v(a)$ and $b = v(b)$ in (27) and applying (26), we obtain

$$(28)\qquad v^3(a)v^3(b) = 0.$$

4. An ideal. Define

$$U(x) = \{u \in R | u(r, x, x) = (r, x, x)u = (u, x, x) = 0 \text{ for all } r \in R\}.$$

In terms of the v-function, this would be written

$$uv(r) = v(r)u = v(u) = 0.$$

LEMMA 1. *$U(x)$ is an ideal of R for all $x \in R$.*

Proof. Clearly $U(x)$ is a subgroup of the additive group of R. Now,

$$(29)\qquad v(ur) = v(ru) = 0$$

for $u \in U(x)$ and $r \in R$ since

$$3v(ur) = F(u, r) = 0 = F(r, u) = 3v(ru).$$

Let $u \in U(x)$ and $r, s \in R$. Then computing

$$0 = uC(r, s) + C(r, s)u - (u, s, v(r)) - (u, v(s), r) + (s, v(r), u) + (v(s), r, u)$$

yields

$$us \cdot v(r) = -v(s) \cdot ru. \tag{30}$$

Next, we obtain

$$us \cdot v(r) = -2v(us \cdot r) \tag{31}$$

upon expanding

$$0 = C(r, us) - C(ru, s) + 2v((r, u, s)) - 3v((u, s, r)) + v((s, r, u))$$

and applying (29) and (30). Furthermore, expanding $0 = 2F(us, r)$ and applying (29) and (31), we have

$$5us \cdot v(r) = -4v(r) \cdot us. \tag{32}$$

We establish that

$$us \cdot v(r) = -4ru \cdot v(s) \tag{33}$$

by expanding

$$0 = 4C(r, s)u + 4(s, v(r), u) + 4(v(s), r, u) - 4(v(r), u, s) - 4(r, u, v(s))$$

and applying (30) and (32). Finally, expanding

$$0 = 4F(ru, s) - 4F(r, us) - 12v((r, u, s)) + 12v((u, s, r))$$

and applying equations (29) through (33), we obtain $6us \cdot v(r) = 0$. This latter, along with (29), (30), (32), and (33), completes the proof.

5. Property P. R will be said to have property P providing there is a subgroup N of the additive group of R such that $v^2(N) = 0$ for all $x \in R$ and such that $N + I = R$ for every ideal $I \neq 0$ of R. Property P is of interest because clearly R has property P if R is simple (take $N = 0$), and we shall see that R has property P if R is primitive.

LEMMA 2. *If R has property P, then $v(R) \subset U(x)$ for all $x \in R$.*

Proof. Letting $a = x$ and $b = v^2(a)$ in (21), we have

$$(v(x), v(x), v^3(a)) = 0,$$

and by (25) and (28),

$$v^3(a)(v(x), v(x), r) = 0 = (v(x), v(x), r)v^3(a);$$

hence $v^3(a) \in U(v(x))$. If $U(v(x)) = 0$, then $v^3(a) = 0$; if $U(v(x)) \neq 0$, then $N + U(v(x)) = R$. For $n \in N$, $u \in U(v(x))$, we have, using (25),

$$9v^3(n + u) = 9v^3(n) + (v(x), v(x), u) = 0.$$

Hence, in either case we have

$$v^3(a) = 0. \tag{34}$$

Applying (34) to $J(a, b) = 0$ yields

$$v^2(b)v(a) + v(b)v^2(a) = 0, \tag{35}$$

and we obtain

$$v^2(b)v^2(a) = 0 \tag{36}$$

if we let $a = v(a)$ in (35) and apply (34).

The next sequence of identities will show that

$$v^2(b) \in U(v^2(a)).$$

Expanding

$$0 = B(v^2(a), v^2(a), r)$$

and applying (20) and (21) with $a = b = v(a)$ yields

$$2v^2(a)(v^2(a), v^2(a), r) + (v^2(a), v^2(a), r)v^2(a) = 0. \tag{37}$$

Similarly, expanding

$$0 = B(v^2(a), r, v^2(a))$$

yields

$$-2v^2(a)(v^2(a), v^2(a), r) - 4(v^2(a), v^2(a), r)v^2(a) = 0. \tag{38}$$

By adding (37) to (38) we obtain

$$v^2(a)(v^2(a), v^2(a), r) = 0 = (v^2(a), v^2(a), r)v^2(a) \tag{39}$$

upon referring to (38). Substituting $a + b$ for a in (39) and then $-a + b$ for a in (39) and adding yields

$$\begin{aligned} v^2(a)[(v^2(a), v^2(b), r) + (v^2(b), v^2(a), r)] &= -v^2(b)(v^2(a), v^2(a), r), \\ [(v^2(a), v^2(b), r) + (v^2(b), v^2(a), r)]v^2(a) &= -(v^2(a), v^2(a), r)v^2(b) \end{aligned} \tag{40}$$

since terms with one or three a's cancel, and terms with no a's go out by (39). Now, applying (20) to the expansion of

$$0 = B(v^2(a), v^2(b), r) + B(v^2(a), r, v^2(b))$$

yields

$$\begin{aligned} 0 = -3v^2(a)[(v^2(a), v^2(b), r) + (v^2(b), v^2(a), r)] \\ - 3[(v^2(a), v^2(b), r) + (v^2(b), v^2(a), r)]v^2(a), \end{aligned}$$

to which we apply (40) to obtain

$$v^2(b)(v^2(a), v^2(a), r) + (v^2(a), v^2(a), r)v^2(b) = 0. \tag{41}$$

Furthermore, computing

$$0 = A(v^2(a), r, v^2(b)) + B(v^2(a), r, v^2(b)) - 3h(v^2(a), v^2(b), v^2(a), r)$$

and applying (20), (21), (36), and (40), we have

$$0 = 3v^2(b)(v^2(a), v^2(a), r).$$

Hence, referring to (41), we see that

$$v^2(b)(v^2(a), v^2(a), r) = 0 = (v^2(a), v^2(a), r)v^2(b). \tag{42}$$

Equations (21) and (42) show that $v^2(b) \in U(v^2(a))$. If $U(v^2(a)) = 0$, then $v^2(b) = 0$; if $U(v^2(a)) \neq 0$, then $N + U(v^2(a)) = R$. For $n \in N$, $u \in U(v^2(a))$, we have, using (25) with $x = v(a)$,

$$\begin{aligned}(n + u, v^2(a), v^2(a)) &= (n, v^2(a), v^2(a)) \\ &= 9(((n, v(a), v(a)), v(a), v(a)), v(a), v(a)) = 0\end{aligned}$$

since $((n, v(a), v(a)), v(a), v(a)) = 0$ by the definition of N. Hence, in either case we have

$$(v^2(a), v^2(a), r) = 0. \tag{43}$$

Evaluating the associator in (43) and applying (20), we obtain

$$v^2(a) \cdot v^2(a)r = 0, \tag{44}$$

and linearizing (43) yields

$$0 = (v^2(a), v^2(b), r) + (v^2(b), v^2(a), r). \tag{45}$$

Since by (45) we have

$$0 = I(a, a)v^2(r) - v^2(a)I(a, r) + (v^2(a), a, v^2(r)) + (a, v^2(a), v^2(r)),$$

we obtain, upon expanding and applying (20), (36), and (44),

$$0 = 2v^2(a) \cdot v(a)v(r);$$

but then, using (23) with $a = r$, $b = v(a)$, and $c = a$ yields

$$0 = v(r) \cdot v^2(a)v(a). \tag{46}$$

It is easy to see that (1) holds in the anti-isomorphic copy R' of R and that R' has property P if R does; hence (46) holds in R'. But then we have in R the identity

$$0 = v(a)v^2(a) \cdot v(r),$$

from which we subtract

$$0 = E(a, v(a))v(r)$$

to obtain

$$0 = v^2(a)v(a) \cdot v(r). \tag{47}$$

Computing $0 = F(v^2(a), v(a))$ and applying (20) and (34) yields

$$v(v^2(a)v(a)) = 0. \tag{48}$$

Equations (46), (47), and (48) imply that $v^2(a)v(a) \in U(x)$. If $U(x) = 0$, then $v^2(a)v(a) = 0$; if $U(x) \neq 0$, then $N + U(x) = R$. For $n \in N$, $u \in U(x)$, we have $v^2(n + u) = 0$. Hence, in either case we have

$$v^2(a)v(a) = 0. \tag{49}$$

Linearizing (49) yields $v^2(a)v(b) + v^2(b)v(a) = 0$, from which we subtract (35) to obtain

$$0 = v^2(a)v(b) - v(b)v^2(a) + E(b, v(a)) = 2v^2(a)v(b);$$

hence

$$v^2(a)v(b) = v(b)v^2(a) = 0. \tag{50}$$

Equations (34) and (50) imply that $v^2(a) \in U(x)$. If $U(x) = 0$, then $v^2(a) = 0$; if $U(x) \neq 0$, then $N + U(x) = R$. For $n \in N$, $u \in U(x)$, we have as before $v^2(n + u) = 0$. Hence, in either case we have $v^2(a) = 0$. But then computing $0 = G(a, b)$ yields $v(a)v(b) = 0 = v(b)v(a)$. Therefore, $v(a) \in U(x)$.

6. Structure theorems.

THEOREM 2. *A simple ring R of characteristic not 2 or 3 satisfying (1) is alternative and hence a Cayley-Dickson algebra or associative.*

Proof. Clearly R has property P if R is simple. Hence, by Lemma 2, $v(a) \in U(x)$, which implies that $v(a) = 0$ if $U(x) = 0$ or if $U(x) = R$. Therefore, R is alternative and we can apply (**3**, Th. 3, p. 399).

THEOREM 3. *A primitive ring R of characteristic not 2 or 3 satisfying (1) is alternative and hence a Cayley-Dickson algebra or associative.*

Proof. First we shall show that R has property P if R is primitive. Let M be a maximal right ideal of R which contains no ideals $I \neq 0$ of R. The letter m will denote an element of M; other letters will denote arbitrary elements of R. By $r \equiv s$ we shall mean that $r - s \in M$. Clearly

$$(m, x, y) \equiv 0 \tag{51}$$

since M is a right ideal. In particular, it follows from (51) that $v(m) \equiv 0$. Also, since M is a right ideal, (51) yields

$$v(my) \equiv v(m)y \equiv 0. \tag{52}$$

Expanding $0 = F(y, m) - C(m, y)$ and applying (52), we have

$$v(ym) \equiv 0. \tag{53}$$

Furthermore, expanding $0 = F(y, m)$ and applying both (52) and (53) yields

$$yv(m) + v(y)m \equiv 0. \tag{54}$$

Next, we expand $0 = E(m, y)$ and apply (52) to obtain

$$v(y)v(m) \equiv 0. \tag{55}$$

Letting $m = v(m)$ in (54), we have by (55)

$$yv^2(m) \equiv 0. \tag{56}$$

Now define

$$S = \{s \in M | Rs \equiv 0\}.$$

S is an ideal of R. Indeed, for $s \in S$, $a, b \in R$, (51) yields $a \cdot sb \equiv as \cdot b \equiv 0$ and $a \cdot bs \equiv ab \cdot s \equiv 0$. Hence, $S = 0$ since $S \subset M$. But then $v^2(m) = 0$ since $v^2(m) \in S$ by (56). Hence R has property P since $M + I = R$ for all ideals $I \neq 0$ of R. Therefore, $v(a) \in U(x)$ by Lemma 2. If $U(x) \subset M$, then $U(x) = 0$ and $v(a) = 0$; if $U(x) \not\subset M$, then $M + U(x) = R$. For $m \in M$, $u \in U(x)$, $v(u + m) = v(m)$. Hence in either case, (51) yields

$$v(a) \equiv 0. \tag{57}$$

Computing $0 = C(a, b)$ and applying (57) yields $bv(a) \equiv 0$. Thus $v(a) \in S$ which implies $v(a) = 0$. Hence R is alternative and we can apply (**2**, Th. 2, p. 729) to complete the proof.

The proof of (**2**, Th. 3, p. 730) can be seen to hold for every class of rings in which the primitive rings are Cayley-Dickson algebras or associative. Therefore, that theorem holds in R:

THEOREM 4. *The radical Q of a ring R of characteristic not 2 or 3 satisfying (1) is an ideal having the property that R/Q is a subdirect sum of Cayley-Dickson algebras and primitive, associative rings.*

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