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A $(5,5)$-Colouring of K_n with Few Colours

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For fixed integers p and q, let $f(n,p,q)$ denote the minimum number of colours needed to colour all of the edges of the complete graph K_n such that no clique of p vertices spans fewer than q distinct colours. Any edge-colouring with this property is known as a (p,q)-colouring. We construct an explicit $(5,5)$-colouring that shows that $f(n,5,5) \leqslant n^{1/3+o(1)}$ as $n \to \infty$. This improves upon the best known probabilistic upper bound of $O(n^{1/2})$ given by Erdős and Gyárfás, and comes close to matching the best known lower bound $\Omega(n^{1/3})$.

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1. Introduction

Let K_n denote the complete graph on n vertices. Fix positive integers p and q such that $1 \leqslant q \leqslant \binom{p}{2}$. A (p,q)-colouring of K_n is any edge-colouring such that every copy of K_p contains edges of at least q distinct colours. Let $f(n,p,q)$ denote the minimum number of colours needed to give a (p,q)-colouring of K_n.

This function generalizes the classical Ramsey problem. The diagonal Ramsey number $r_k(p)$ is the minimum number of vertices n for which any edge-colouring of K_n with at most k colours will contain a monochromatic copy of K_p. So $r_k(p) = n$ implies that $f(n-1,p,2) \leqslant k$ and $f(n,p,2) \geqslant k+1$. Similarly, $f(n,p,2) = k$ implies that $r_k(p) > n$ and $r_{k-1}(p) \leqslant n$. Therefore, determining $f(n,p,2)$ is equivalent to determining $r_k(p)$ which is well known to be very difficult in general.

Erdős and Shelah originally introduced the function $f(n,p,q)$ in 1975 [5, 6], but it was not studied systematically until 1997 when Erdős and Gyárfás [7] looked at the growth rate of $f(n,p,q)$ as $n \to \infty$ for fixed values of p and q. In particular, they determined the threshold values for q as a function of p for which $f(n,p,q)$ becomes linear in n, quadratic in n, and asymptotically equivalent to $\binom{n}{2}$. They also used the Lovász Local Lemma to give a general upper bound,

$$f(n,p,q) = O(n^{(p-2)/(1-q+\binom{p}{2})}).$$

Additionally, they found specific upper and lower bounds for $f(n,p,q)$ for various values of q when $p = 4,5$. Work on these small cases has continued. For instance, in 2000 Axenovich [1] gave an explicit $(5,9)$-colouring based on sets of integers containing no arithmetic progressions of length 3. This construction uses only $n^{1+o(1)}$ colours which comes close to matching the known linear lower bound on $f(n,5,9)$. Axenovich also worked on improving the coefficient for this linear lower bound, and further improvements have come more recently from E. Krop and I. Krop [8]. We will now discuss in more detail the work done on the cases where $q = p-1, p$ since these are closely connected to the main result of this paper.

1.1. Determining $f(n,p,p-1) = n^{o(1)}$

One of the main questions left open by Erdős and Gyárfás [7] was the determination of a threshold value of q in terms of p for which the function $f(n,p,q)$ becomes polynomial in n. They point out a simple induction argument which shows that

$$f(n,p,p) \geqslant n^{1/(p-2)} - 1,$$

but could not determine if $f(n,p,p-1) = n^{o(1)}$ even when $p = 4$, a problem they called 'the most annoying' of all the small cases.

In 1998, Mubayi [9] verified that this is indeed the case when $p = 4$ by giving an explicit $(4,3)$-colouring of K_n to show $f(n,4,3) \leqslant e^{O(\sqrt{\log n})}$. Eichhorn and Mubayi [4] later used a slight variation of this construction to show that $f(n,p,2\lceil \log p \rceil - 2) \leqslant e^{O(\sqrt{\log n})}$ for all $p \geqslant 5$ as well. In particular, this showed that $f(n,5,4)$ is also subpolynomial.

Recently, this problem was solved in general when Conlon, Fox, Lee and Sudakov [3] provided an explicit colouring which showed that

$$f(n,p,p-1) \leqslant 2^{16p(\log n)^{1-1/(p-2)} \log\log n}.$$

This construction is a generalization of the original $(4,3)$-colouring given by Mubayi [9], and we will use a simplified version of it as part of our $(5,5)$-colouring.

1.2. Determining $f(n,p,p)$

As previously stated, we know in general that $f(n,p,p) \geqslant \Omega(n^{1/(p-2)})$. However, the Local Lemma gives the best general upper bound,

$$f(n,p,p) \leqslant O(n^{2/(p-1)}).$$

Only for $p = 3,4$ do we know of a better upper bound.

A $(3,3)$-colouring is equivalent to a proper edge colouring, one in which no two incident edges can have the same colour. Therefore, it is well known that

$$f(n,3,3) = \begin{cases} n & n \text{ is odd}, \\ n-1 & n \text{ is even}. \end{cases}$$

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In 2004, Mubayi [10] provided an explicit $(4,4)$-colouring of K_n with only $n^{1/2}e^{O(\sqrt{\log n})}$ colours. This closed the gap for $p=4$ to

$$n^{1/2}-1 \leqslant f(n,4,4) \leqslant n^{1/2+o(1)}.$$

His construction was the product of two colourings. The first was his earlier $(4,3)$-colouring which used $n^{o(1)}$ colours. The second was an 'algebraic' colouring that assigned to each vertex a vector from a two-dimensional vector space over a finite field, and then coloured each edge with an element from the base field, giving $n^{1/2}$ colours. Some complicating factors needed to be addressed by splitting each of these colours a constant number of times so that ultimately the algebraic part of his colouring used only $O(n^{1/2})$ colours.

One such complication was the need to avoid what Mubayi called a 'striped K_4', four vertices with three distinct edge colours where each colour is a matching. Interestingly, this particular arrangement can actually be avoided with only $2^{O(\sqrt{\log n})}$ colours, as we will show in Section 2.

1.3. Summary of the $(5,5)$-colouring

Our $(5,5)$-colouring extends Mubayi's idea of combining a small $(p,p-1)$-colouring with an algebraic colouring to obtain the following result.

Theorem 1.1. *As $n\to\infty$,*

$$f(n,5,5) \leqslant n^{1/3}2^{O(\sqrt{\log n}\log\log n)}.$$

We begin in Section 2 by considering a particular instance of the general $(p,p-1)$-colouring of Conlon, Fox, Lee and Sudakov [3], which we will refer to as the CFLS colouring. We show that with few colours this construction avoids certain 'bad' configurations. We then modify it slightly so that it also avoids the striped K_4 configuration. By forbidding these specific configurations, we are able to show that there are only three possible edge-colourings of K_5 (up to isomorphism) with at most four colours that could still occur with this modified CFLS colouring.

In Section 3, we define the first part of an algebraic colouring which we call the Modified Inner Product (MIP) colouring. Under this construction, each vertex is associated with a vector in a three-dimensional space over a finite field. As in Mubayi's construction [10], each edge is coloured with a specific element in the base field. Some slight modifications are needed for special cases, but these will only split each colour a constant number of times, ultimately giving $O(n^{1/3})$ colours used in the MIP construction.

In Section 4, we will take the product of the modified CFLS colouring and the first part of the MIP to get a construction that uses $n^{1/3+o(1)}$ colours and eliminates the first two of the three remaining bad configurations. Finally, in Section 5 we define the rest of the MIP colouring to eliminate the third configuration.

2. The CFLS colouring

We will not define the CFLS [3] colouring in full generality since only a simple case is needed. We borrow part of the notation used in [3], but change it somewhat for clarity in this particular

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instance. Let $n = 2^{\beta^2}$ for some positive integer β. Associate each vertex of K_n with a unique binary string of length β^2. That is, we may assume that our vertex set is

$$V = \{0,1\}^{\beta^2}.$$

For any vertex $v \in V$, let $v^{(i)}$ denote the ith block of bits of length β in v so that

$$v = (v^{(1)}, \dots, v^{(\beta)})$$

where each $v^{(i)} \in \{0,1\}^{\beta}$.

Between two vertices $x, y \in V$, the CFLS colouring is defined by

$$\varphi_1(x,y) = ((i, \{x^{(i)}, y^{(i)}\}), i_1, \dots, i_\beta),$$

where i is the first index for which $x^{(i)} \neq y^{(i)}$, and for each $k = 1, \dots, \beta$, $i_k = 0$ if $x^{(k)} = y^{(k)}$ and otherwise is the first index at which a bit of $x^{(k)}$ differs from the corresponding bit in $y^{(k)}$.

For convenience, when discussing any edge colour α, we will let α_0 denote the first coordinate of the colour (of the form $(i, \{x^{(i)}, y^{(i)}\})$) and let α_k denote the index of the first bit difference of the kth block for $k = 1, \dots, \beta$. Furthermore, throughout this section, we will say that two vertices x and y *agree at* i if $x^{(i)} = y^{(i)}$ and that x and y *differ at* i if $x^{(i)} \neq y^{(i)}$.

2.1. Avoided configurations

We will show through the following series of lemmas that the CFLS colouring avoids certain specified arrangements of edge colours.

Lemma 2.1. *The CFLS colouring forbids monochromatic odd cycles.*

Proof. Suppose there exists a sequence of distinct vertices, $v_1, \dots, v_k$, for which k is odd and

$$\varphi_1(v_1,v_2) = \varphi_1(v_2,v_3) = \cdots = \varphi_1(v_{k-1},v_k) = \varphi_1(v_k,v_1) = \alpha.$$

Let $\alpha_0 = (i, \{x,y\})$. Without loss of generality we may assume that $v_1^{(i)} = x$ and $v_2^{(i)} = y$. It follows that

$$y = v_2^{(i)} = v_4^{(i)} = \cdots = v_{k-1}^{(i)} = v_1^{(i)} = x,$$

a contradiction. □

Lemma 2.2. *The CFLS colouring forbids four distinct vertices $a,b,c,d \in V$ for which*

$$\varphi_1(a,b) = \varphi_1(c,d) \quad \text{and} \quad \varphi_1(a,c) = \varphi_1(a,d).$$

See Figure 1(a).

Proof. Assume towards a contradiction that $\varphi_1(a,b) = \varphi_1(c,d) = \alpha$ and $\varphi_1(a,c) = \varphi_1(a,d) = \gamma$. Let $\alpha_0 = (i, \{x,y\})$. Without loss of generality, $a^{(i)} = c^{(i)} = x$ and $b^{(i)} = d^{(i)} = y$. Then $\gamma_i = 0$ since a and c agree at i, but $\gamma_i \neq 0$ as a and d differ at i, a contradiction. □

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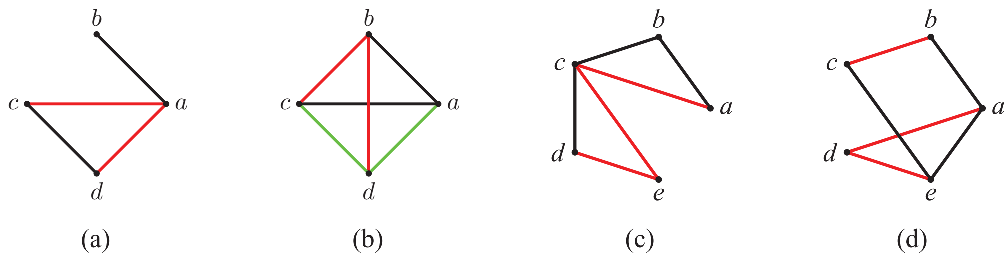

Figure 1. Four configurations avoided by the CFLS colouring.

Lemma 2.3. *The CFLS colouring forbids four distinct vertices $a,b,c,d \in V$ for which*

$$\varphi_1(a,b) = \varphi_1(a,c), \quad \varphi_1(b,d) = \varphi_1(b,c) \quad \textit{and} \quad \varphi_1(a,d) = \varphi_1(c,d).$$

See Figure 1(b).

Proof. Assume towards a contradiction that $\varphi_1(a,b) = \varphi_1(a,c) = \alpha$, $\varphi_1(b,d) = \varphi_1(b,c) = \gamma$ and $\varphi_1(a,d) = \varphi_1(c,d) = \pi$. Let $\alpha_0 = (i,\{x,y\})$, $\gamma_0 = (j,\{s,t\})$ and $\pi_0 = (k,\{w,v\})$. Without loss of generality we may assume that $a^{(i)} = x$ and $b^{(i)} = c^{(i)} = y$. Since b and c differ at j, then $i \neq j$. Without loss of generality we may assume that $b^{(j)} = s$ and $c^{(j)} = d^{(j)} = t$. So $\pi_j = 0$, and hence $a^{(j)} = t$ since $\varphi_1(a,d) = \pi$. Therefore $\alpha_j = 0$, which implies that $b^{(j)} = t$, a contradiction since $s \neq t$. □

Lemma 2.4. *The CFLS colouring forbids five distinct vertices $a,b,c,d,e \in V$ that contain two monochromatic paths of three edges each that share endpoints:*

$$\varphi_1(a,b) = \varphi_1(b,c) = \varphi_1(c,d) \quad \textit{and} \quad \varphi_1(a,c) = \varphi_1(c,e) = \varphi_1(e,d).$$

See Figure 1(c).

Proof. Assume towards a contradiction that

$$\varphi_1(a,b) = \varphi_1(b,c) = \varphi_1(c,d) = \alpha$$

and

$$\varphi_1(a,c) = \varphi_1(c,e) = \varphi_1(e,d) = \gamma.$$

Let $\alpha_0 = (i,\{x,y\})$ and $\gamma_0 = (j,\{s,t\})$. Without loss of generality we may assume that $a^{(i)} = c^{(i)} = x$ and $b^{(i)} = d^{(i)} = y$. Note that $\varphi_1(a,c) = \gamma$ implies $\gamma_i = 0$. Then $e^{(i)} = d^{(i)} = y$ and $e^{(i)} = c^{(i)} = x$. So $x = y$, a contradiction. □

Lemma 2.5. *The CFLS colouring forbids five distinct vertices $a,b,c,d,e \in V$ for which*

$$\varphi_1(a,b) = \varphi_1(a,e) = \varphi_1(e,c) \quad \textit{and} \quad \varphi_1(a,d) = \varphi_1(d,e) = \varphi_1(b,c).$$

See Figure 1(d).

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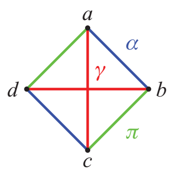

Figure 2. A striped K_4.

Proof. Assume towards a contradiction that $\varphi_1(a,b) = \varphi_1(a,e) = \varphi_1(e,c) = \alpha$ and $\varphi_1(a,d) = \varphi_1(d,e) = \varphi_1(b,c) = \gamma$. Let $\alpha_0 = (i,\{x,y\})$. We may assume without loss of generality that $b^{(i)} = e^{(i)} = x$ and $a^{(i)} = c^{(i)} = y$. We also know that $b^{(k)} = a^{(k)} = e^{(k)} = c^{(k)}$ for all $k < i$. Since $\varphi_1(b,c) = \gamma$, then $\gamma_0 = (i,\{x,y\})$. So either $d^{(i)} = x$ or $d^{(i)} = y$. Therefore, d must agree with either a or e at i, a contradiction. □

2.2. Modified CFLS

We will now add to the CFLS colouring to avoid the striped K_4, an edge-colouring of four distinct vertices a,b,c,d such that every pair of non-incident edges have the same colour (see Figure 2). The CFLS colouring alone will not avoid such arrangements, but the product of φ_1 with another small edge-colouring, φ_2, will.

We will define the colouring φ_2 on the same set of vertices as the CFLS colouring, $V = \{0,1\}^{\beta^2}$. However, we will also need to consider the vertices as an ordered set. Consider each vertex to be an integer represented in binary. Then order the vertices by the standard ordering of the integers. That is, $x < y$ if and only if the first bit at which x and y differ is zero in x and one in y. This ordering plays a large role in a recent construction by Mubayi [11] for a small case of the hypergraph version of the (p,q)-colouring problem. Note that each β-block is a binary representation of an integer from 0 to $2^\beta - 1$, so these blocks can be considered ordered in the same way. Moreover, note that if $x < y$ and if the first β-block at which x and y differ is i, then it must be the case that $x^{(i)} < y^{(i)}$.

Let $x,y \in V$ such that $x < y$. We define the second colouring as

$$\varphi_2(x,y) = (\delta_1(x,y),\dots,\delta_\beta(x,y))$$

where, for each i,

$$\delta_i(x,y) = \begin{cases} -1 & x^{(i)} > y^{(i)}, \\ +1 & x^{(i)} \leqslant y^{(i)}. \end{cases}$$

This construction uses 2^β colours. Therefore, the modified CFLS colouring, $\varphi = \varphi_1 \times \varphi_2$, uses

$$\beta^{\beta+1}2^{3\beta} = \sqrt{\log n}^{\sqrt{\log n}+1} 2^{3\sqrt{\log n}} = 2^{O(\sqrt{\log n}\log\log n)}$$

colours.

Lemma 2.6. *The modified CFLS colouring φ forbids four distinct vertices $a,b,c,d \in V$ with*

$$\varphi(a,b) = \varphi(c,d), \quad \varphi(a,c) = \varphi(b,d) \quad \textit{and} \quad \varphi(a,d) = \varphi(b,c).$$

See Figure 2.

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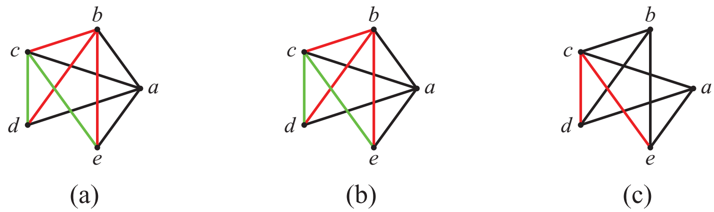

Figure 3. Three configurations not avoided by the modified CFLS colouring.

Proof. Assume towards a contradiction that a striped K_4 can occur. Then, $\varphi_1(a,b) = \varphi_1(c,d) = \alpha$, $\varphi_1(a,c) = \varphi_1(b,d) = \gamma$ and $\varphi_1(a,d) = \varphi_1(b,c) = \pi$. Let $\alpha_0 = (i,\{x,y\})$, $\gamma_0 = (j,\{s,t\})$ and $\pi_0 = (k,\{v,w\})$. Without loss of generality, assume that $i = \min\{i,j,k\}$. Since $\varphi_1(a,b) = \varphi_1(c,d)$, exactly one of $d^{(i)}$ and $c^{(i)}$ equals $a^{(i)}$. Say $d^{(i)} = a^{(i)}$ without loss of generality. Then, by the minimality of i, it must be the case that $j = i$ and that $i < k$.

Let $a^{(i)} = d^{(i)} = x$, $b^{(i)} = c^{(i)} = y$, $a^{(k)} = b^{(k)} = v$ and $c^{(k)} = d^{(k)} = w$. Without loss of generality we may assume that $x < y$. This implies that $a,d < b,c$ in the ordering of V as integers represented in binary. If $v < w$, then $\delta_k(a,c) = +1$ and $\delta_k(d,b) = -1$. Therefore, $\varphi_2(a,c) \neq \varphi_2(b,d)$, a contradiction. So, it must be the case that $w < v$. But then $\delta_k(a,c) = -1$ and $\delta_k(b,d) = +1$, which yields the same contradiction. □

Note that to eliminate the striped K_4 configuration we needed just

$$\beta 2^{3\beta} = \sqrt{\log n}\, 2^{3\sqrt{\log n}} = 2^{O(\sqrt{\log n})}$$

colours since only the first coordinate of the CFLS colouring was needed in the proof.

We can now systematically look at all edge-colourings of a K_5 up to isomorphism that use no more than four colours and do not contain any of these configurations to get a list of possible 'bad' colourings of a K_5 that could survive the modified CFLS colouring. A careful mathematician with a free day could work through these cases by hand. A simple computer program like the inelegant one detailed in the Appendix is easier to verify. However this process is executed, we end up with three possible bad colourings of K_5 (see Figure 3). Avoiding these will require both the CFLS colouring and the MIP colouring defined in Section 3.

Before we move on from discussing the modified CFLS colouring, we need to point out one nice fact that will be used in Section 4.

Lemma 2.7. *If $a < b < c$, then $\varphi(a,b) \neq \varphi(b,c)$.*

Proof. Suppose $\varphi_1(a,b) = \varphi_1(b,c) = \alpha$ and that $\alpha_0 = (i,\{x,y\})$ for $x < y$. Then $a^{(i)} = x$ and $b^{(i)} = y$. But then $c^{(i)} = x$. Therefore, $c < b$, a contradiction. □

3. The Modified Inner Product colouring

Let q be some odd prime power, and let $\mathbb{F}_q^*$ denote the non-zero elements of the finite field with q elements. The vertices of our graph will be the three-dimensional vectors over this set:

$$V = (\mathbb{F}_q^*)^3.$$

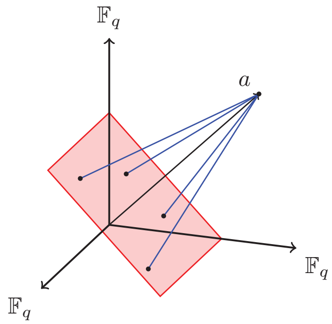

Figure 4. The geometric visualization can be misleading since we are working over a finite field, but monochromatic neighbourhoods are contained in affine planes.

All algebraic operations used in defining the MIP colouring are the standard ones from the underlying field, and $\cdot$ will denote the standard inner product of two vectors,

$$x \cdot y = x_1y_1 + x_2y_2 + x_3y_3,$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Additionally, let $<$ be any linear order on the elements of $\mathbb{F}_q$, and extend this to a linear order on the vectors so that

$$x < y \iff x_i < y_i,$$

where $i \in \{1,2,3\}$ is the first position at which $x_i \neq y_i$.

The MIP colouring will be broken up into two parts, $\chi = \chi_1 \times \chi_2$. The first part χ_1 uses at most $12n^{1/3}$ colours. The second part χ_2 uses only four colours and is used to split up colours from χ_1 in order to avoid one particularly difficult configuration. In this section, we will first define χ_1. Then, after a brief review of the necessary linear algebra concepts, we will prove some key properties of χ_1. The second part χ_2 will be defined in Section 5.

3.1. Motivation

The MIP colouring should be viewed as colouring each edge with the inner product of the two vectors with some adjustments for special cases. As motivation for this colouring, note that each of the three configurations with four colours that survive the modified CFLS colouring (see Figure 3) contains at least one pair of vertices in the intersection of monochromatic neighbourhoods of the other three vertices.

For instance, vertices d and e in Figure 3(a) are in the same monochromatic neighbourhood with respect to vertices a, b and c. Under the MIP colouring, the monochromatic neighbourhood of any vertex is contained in an affine plane of $\mathbb{F}_q^3$. So, if a, b and c are linearly independent, then these planes intersect in one point, not two. Therefore, the Figure 3(a) configuration could only happen under an inner product colouring if the span of a, b and c has dimension at most 2.

This same idea applies to the other two configurations, and the adjustments to colouring with the inner product are all dedicated to handling the cases for which the five vectors are not in general position. In these cases we frequently end up with three vectors that must all lie on the same affine line, and the offending configurations would be destroyed if the colouring could be modified to give a proper colouring on every affine line.

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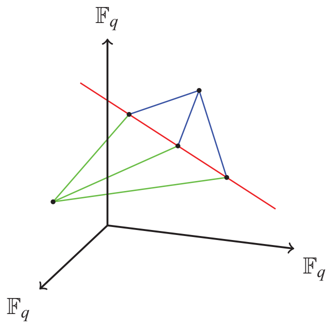

Figure 5. ‘Most’ of the time the intersection of two monochromatic neighbourhoods defines a subset of an affine line.

3.2. The colouring χ_1

In Lemma 3.6 we will show that χ_1 induces a proper edge colouring on every line, not just one-dimensional linear spaces but affine lines as well. This will be one of the key lemmas in showing that our construction avoids the remaining configurations. By itself, the inner product almost accomplishes this goal. However, a problem arises when one vector on a given line is orthogonal to the direction of the line. In this case, that particular vector has the same inner product with all other vectors on the line, so we must give these edges new colours. We accomplish this by replacing the inner product with another function.

The first part of χ_1 labels the type of edge-colouring we will have. For two distinct vectors, $x, y \in V$, let $T(x,y)$ be a function defined by

$$T(x,y) = \begin{cases} \mathrm{UP}_1 & x\cdot y = x\cdot x \text{ and } x_1 < y_1, \\ \mathrm{UP}_2 & x\cdot y = x\cdot x, x_1 = y_1, \text{ and } x < y, \\ \mathrm{DOWN}_1 & x\cdot y \neq x\cdot x, x\cdot y = y\cdot y, \text{ and } x_1 < y_1, \\ \mathrm{DOWN}_2 & x\cdot y \neq x\cdot x, x\cdot y = y\cdot y, x_1 = y_1, \text{ and } x < y \\ \mathrm{ZERO} & x\cdot y \notin \{x\cdot x, y\cdot y\} \text{ and } x\cdot y = 0, \\ \mathrm{DOT} & \text{otherwise.} \end{cases}$$

Here, the categories UP_i and DOWN_i let us know that at least one of the two vectors is orthogonal to the direction of the line between the two, and therefore this edge will need to receive something other than the inner product in the next part of the colour. The words UP and DOWN describe the edge from the perspective of the ‘special’ vertex. For instance, if x is orthogonal to the direction of the line it makes with y and $x < y$, then x looks up the edge to y. The need for different categories when $x_1 = y_1$ is a technical point. The category DOT stands for the inner product (or the ‘dot’ product), and ZERO is the special case where the inner product is zero. The need to split the colours with zero inner product is also a technical point.

Let $f_T(x,y) : \mathbb{F}_q^3 \to \mathbb{F}_q$ be a function defined by

$$f_T(x,y) = \begin{cases} x_1 + y_1 & T \in \{\mathrm{UP}_1, \mathrm{DOWN}_1, \mathrm{ZERO}\}, \\ x_2 + y_2 & T \in \{\mathrm{UP}_2, \mathrm{DOWN}_2\}, \\ x\cdot y & T = \mathrm{DOT}. \end{cases}$$

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One final technical point is to differentiate colours based on whether the two vectors are linearly dependent or independent. Let

$$\delta(x,y) = \begin{cases} 0 & \{x,y\} \text{ is linearly dependent,} \\ 1 & \{x,y\} \text{ is linearly independent.} \end{cases}$$

This is enough to define the colouring. For vertices $x < y$, let $T = T(x,y)$, and set

$$\chi_1(x,y) = (T, f_T(x,y), \delta(x,y)).$$

3.3. Algebraic definitions and facts

We assume that the reader has some familiarity with basic linear algebra notions such as dimension, linear independence, linear combination and span. The following definitions and facts are perhaps less familiar. All are reproduced from definitions and propositions in Chapter 2 of the great *Linear Algebra Methods in Combinatorics* book by László Babai and Péter Frankl [2].

Definition. Let $\mathbb{F}^n$ be a vector space, and let $S \subseteq \mathbb{F}^n$ be a set of vectors. The *rank* of S is the dimension of the linear space spanned by S.

Fact 3.1. *Let $\mathbb{F}$ be a field, and let A be a $k \times n$ matrix over $\mathbb{F}$. Then the rank of the set of column vectors as vectors in $\mathbb{F}^k$ is equal to the rank of the set of row vectors as vectors in $\mathbb{F}^n$. We know this value as the rank of the matrix A,* $\mathrm{rk}(A)$.

Definition. Let $\mathbb{F}^n$ be a vector space. An *affine combination* of vectors $v_1, \dots, v_k \in \mathbb{F}^n$ is a linear combination $\lambda_1 v_1 + \cdots + \lambda_k v_k$ for $\lambda_1, \dots, \lambda_k \in \mathbb{F}$ such that $\lambda_1 + \cdots + \lambda_k = 1$. An *affine subspace* is a subset of vectors that is closed under affine combinations.

Fact 3.2. *Any affine subspace U is either empty or the translation of some linear subspace V. That is, each vector $u \in U$ can be written in the form $u = v + t$ where v is some vector in V and t is a fixed translation vector.*

Definition. The *dimension* $\dim(U)$ of an affine subspace U is the dimension of the unique linear subspace of which U is a translate.

Definition. Let $\mathbb{F}^n$ be a vector space. Let $v_1, \dots, v_k \in \mathbb{F}^n$. We say that these vectors are *affine independent* if

$$\lambda_1 v_1 + \cdots + \lambda_k v_k = 0$$

implies that

$$\lambda_1 = \cdots = \lambda_k = 0$$

for any $\lambda_1, \dots, \lambda_k \in \mathbb{F}$ for which $\lambda_1 + \cdots + \lambda_k = 0$. Otherwise, these vectors are *affine dependent*. We say that a set of vectors S is a *basis* for an affine subspace if they are affine independent and every vector in the subspace is an affine combination of vectors in S.

Fact 3.3. *A basis of an affine subspace U contains exactly $\dim(U)+1$ elements.*

Fact 3.4. *Let $\mathbb{F}^n$ be some vector space. Let A be a $k\times n$ matrix over $\mathbb{F}$ and $b\in\mathbb{F}^n$. Then the solution set to $Ax=b$ is an affine subspace of dimension $n-\mathrm{rk}(A)$.*

Definition. A vector $x\in\mathbb{F}^n$ is *isotropic* if $x\cdot x=0$. A linear subspace $U\subseteq\mathbb{F}^n$ is *totally isotropic* if $x,y\in U$ implies that $x\cdot y=0$.

Fact 3.5. *For any non-zero vector $x\in\mathbb{F}^n$, the set of vectors $\{y:x\cdot y=0\}$ is a linear subspace of $\mathbb{F}^n$ with dimension $n-1$.*

3.4. Properties of χ_1

Lemma 3.6. *The colouring χ_1 induces a proper edge colouring on every one-dimensional affine subspace.*

Proof. Let $a,b,c\in\mathbb{F}_q^3$ be three distinct vectors in a one-dimensional affine subspace. Then there exists some $\lambda\in\mathbb{F}_q$ such that $c=\lambda a+(1-\lambda)b$. Suppose towards a contradiction that $\chi_1(a,b)=\chi_1(a,c)$, and let $T=T(a,b)=T(a,c)$. If $T\in\{\mathrm{ZERO},\mathrm{DOT}\}$, then

$$a\cdot b=a\cdot(\lambda a+(1-\lambda)b).$$

So $\lambda a\cdot(a-b)=0$. Since $c\neq b$, then $\lambda\neq 0$. Therefore, $a\cdot(a-b)=0$. But this contradicts the assumption that $T\in\{\mathrm{ZERO},\mathrm{DOT}\}$.

If $T\in\{\mathrm{UP}_1,\mathrm{DOWN}_1\}$, then $f_T(a,b)=f_T(a,c)$ gives

$$a_1+b_1=a_1+\lambda a_1+(1-\lambda)b_1.$$

So $b_1=a_1$, a contradiction since $T\in\{\mathrm{UP}_1,\mathrm{DOWN}_1\}$ implies that $a_1\neq b_1$. Similarly, if $T\in\{\mathrm{UP}_2,\mathrm{DOWN}_2\}$, then $a_2=b_2$ by the same argument and $a_1=b_1$ by definition. But then either $a_3(a_3-b_3)=0$ or $b_3(b_3-a_3)=0$. Both cases imply that $a_3=b_3$. So $a=b$, a contradiction. □

Definition. Given a vertex $a\in V$ and an edge-colour A, let

$$N_A(a)=\{x:\chi_1(a,x)=A\}$$

be the A-neighbourhood of a

Observation 3.7. *Given a vector $a\in V$ and a colour $A=(T,\alpha,i)$, the vectors in $N_A(a)$ all belong to the two-dimensional affine subspace defined by $\{x:f_T(a,x)=\alpha\}$. In particular, this plane can be defined as the solution space to either $a\cdot x=\alpha$ when $T=DOT$, $(1,0,0)\cdot x=\alpha-a_1$ when $T\in\{ZERO,UP_1,DOWN_1\}$, and $(0,1,0)\cdot x=\alpha-a_2$ when $T\in\{UP_2,DOWN_2\}$.*

In certain cases, we can actually say something a little stronger. First, note that if $T(a,x)\in\{\mathrm{UP}_1,\mathrm{DOWN}_1\}$, then we will have $a_1<x_1$ if and only if $a<x$. Therefore, if $f_T(a,x)=\alpha$, since $f_T(a,x)=a_1+x_1$, we will have $a_1<\alpha=a_1$ if and only if $a<x$.

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Lemma 3.8. *Given a vector $a \in V$, and a colour $A = (T, \alpha, i)$, the vectors of $N_A(a)$ all belong to a one-dimensional affine subspace if one of the following three cases holds for all $x \in N_A(a)$:*

(1) $T \in \{ZERO, UP_2, DOWN_2\}$,
(2) $T = UP_1$ *and* $a < x$,
(3) $T = DOWN_1$ *and* $a > x$.

Proof. In the first case, if $T = \text{ZERO}$, then every $x \in N_A(a)$ must satisfy the system of linear equations

$$a \cdot x = 0,$$
$$(1,0,0) \cdot x = \alpha - a_1.$$

Since a contains no zero components, then the rank of $\{a, (1,0,0)\}$ is two. Therefore, the solution space must be a one-dimensional affine subspace. If $T \in \{\text{UP}_2, \text{DOWN}_2\}$, then every $x \in N_A(a)$ must satisfy the system

$$(1,0,0) \cdot x = a_1,$$
$$(0,1,0) \cdot x = \alpha - a_2.$$

Since $(1,0,0)$ and $(0,1,0)$ are linearly independent, then, as before, the set of solutions is a one-dimensional affine subspace.

In each of the other two cases, we see that every $x \in N_A(a)$ must satisfy the system

$$a \cdot x = a \cdot a,$$
$$(1,0,0) \cdot x = \alpha - a_1.$$

As before, the solution space must be a one-dimensional affine subspace. □

Therefore, we immediately get the following corollary by Lemma 3.6.

Corollary 3.9. *Let $a, b, c, d \in V$ be four distinct vertices such that*

$$\chi_1(a,b) = \chi_1(a,c) = \chi_1(a,d) = (T, \alpha, i).$$

The set of vertices $\{b, c, d\}$ span three distinct edge colours under χ_1 if any of the following are true:

(1) $T \in \{ZERO, UP_2, DOWN_2\}$,
(2) $T = UP_1$ *and* $a < b, c, d$,
(3) $T = DOWN_1$ *and* $a > b, c, d$.

Lemma 3.10. *Let $a, b, c, d, e \in V$ be vectors such that $\{a, b\}$ is linearly independent, $\chi_1(a,c) = \chi_1(a,d) = \chi_1(a,e)$, and $\chi_1(b,c) = \chi_1(b,d) = \chi_1(b,e)$ (see Figure 6). Then the set $\{c, d, e\}$ spans three distinct edge colours.*

Proof. Let $\chi_1(a,c) = \chi_1(a,d) = \chi_1(a,e) = A$ and $\chi_1(b,c) = \chi_1(b,d) = \chi_1(b,e) = B$. The result is immediate if either pair (a, A) or (b, B) satisfies the conditions listed in Corollary 3.9. So

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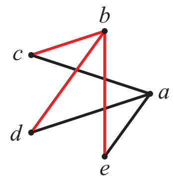

Figure 6.

assume not. If $A = (T_a, \alpha, i)$, then by Observation 3.7 we know that c, d and e must either satisfy $a \cdot x = \alpha$ or $(1,0,0) \cdot x = \alpha - a_1$. Similarly, if $B = (T_b, \beta, j)$, then c, d and e must either satisfy $b \cdot x = \beta$ or $(1,0,0) \cdot x = \beta - b_1$.

Since the sets $\{a,b\}$, $\{a,(1,0,0)\}$ and $\{(1,0,0),b\}$ are all linearly independent, then every case gives us the result immediately except when $T_a, T_b \in \{\mathrm{UP}_1, \mathrm{DOWN}_1\}$. Since we assume that none of the cases from Corollary 3.9 hold, then this can only happen when $x \cdot (x-a) = x \cdot (x-b) = 0$ for $x = c,d,e$. In this case, c, d and e all satisfy the two linear equations,

$$(a-b) \cdot x = 0,$$
$$(1,0,0) \cdot x = \alpha - a_1.$$

Hence, c, d and e are affine independent, and the result follows from Lemma 3.6 unless

$$a_2 - b_2 = a_3 - b_3 = 0.$$

But if this is true, then $c \cdot (c-a) = c \cdot (c-b)$ implies that $a = b$, a contradiction. □

4. Combining the colourings

Let $n = (q-1)^3$ where q is an odd prime power. To each $\alpha \in \mathbb{F}_q$ we associate the unique element $\alpha' \in \{0,1\}^{\lceil \log q \rceil}$ which represents in binary the rank of α under the linear order given to the elements of $\mathbb{F}_q$ in Section 3. Let β be the minimum positive integer for which

$$3\lceil \log q \rceil \leqslant \beta^2.$$

We associate each of the n vertices of K_n with a unique vector in $(\mathbb{F}_q^*)^3$ as in Section 3. To each vertex (x_1,x_2,x_3), we associate $(x_1',x_2',x_3',0) \in \{0,1\}^{\beta^2}$ as well, where for each i, x_i' is the binary representation of the rank of x_i, and 0 denotes a string of $\beta^2 - 3\lceil \log q \rceil$ zeros. Let

$$C = \varphi \times \chi_1.$$

Since

$$\beta = \Theta(\sqrt{3\log q}) = \Theta(\sqrt{\log n}),$$

it follows that the number of colours used in this combined colouring is at most

$$12q\beta 2^{3\beta} = n^{1/3} 2^{O(\sqrt{\log n}\log\log n)}$$

colours. This bound on the number of colours generalizes to all n by the standard density of primes argument [12].

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4.1. The first two configurations

Lemma 4.1. *Any distinct vertices $a,b,c,d,e \in V$ for which*

$$C(a,c) = C(a,d) = C(a,e), \quad C(b,c) = C(b,d) = C(b,e) \quad \textit{and} \quad C(a,c) \neq C(b,c)$$

(see Figure 6) span at least five distinct edge colours.

Proof. Lemma 2.1 implies that neither colour between $\{a,b\}$ and $\{c,d,e\}$ can be repeated on the edges spanned by $\{c,d,e\}$. Therefore, if $\{a,b\}$ is linearly independent it follows from Lemma 3.10 that $\{a,b,c,d,e\}$ span at least five colours.

Otherwise, $b = \lambda a$ for some $\lambda \in \mathbb{F}_q$. If $C(a,b)$ repeats one of the colours from the edges spanned by $\{c,d,e\}$, then this gives us the configuration forbidden by Lemma 2.2. If $C(a,b) = C(a,c)$ or $C(a,b) = C(b,c)$, then all five vectors belong to a one-dimensional linear subspace spanned by a which must be properly edge-coloured by Lemma 3.6. Therefore, the set of vertices $\{a,b,c,d,e\}$ spans at least five colours. □

This immediately shows that the first configuration will not appear under the combined colouring.

Corollary 4.2. *Let $a,b,c,d,e \in V$ be five distinct vertices. It cannot be the case that*

$$C(a,b) = C(a,c) = C(a,d) = C(a,e), \quad C(b,c) = C(b,d) = C(b,e) \quad \textit{and} \quad C(c,d) = C(c,e)$$

as in Figure 3(a).

The second configuration also will not appear under the combined colouring.

Lemma 4.3. *Let $a,b,c,d,e \in V$ be five distinct vertices. It cannot be the case that*

$$C(a,c) = C(a,d) = C(a,e) = C(b,c) = C(b,d) = C(b,e) \quad \textit{and} \quad C(c,d) = C(c,e)$$

as in Figure 3(b).

Proof. By Lemma 3.10, this can happen only if there exists some $\lambda \in \mathbb{F}_q$ such that $b = \lambda a$. In this case,

$$\chi_1(a,c) = \chi_1(a,d) = \chi_1(a,e) = \chi_1(\lambda a,c) = \chi_1(\lambda a,d) = \chi_1(\lambda a,e).$$

If this colour is in DOT, then $c \cdot a = c \cdot \lambda a$. So either $\lambda = 1$, a contradiction, or $c \cdot a = 0$, a contradiction that the colour is in DOT. If the colour is not in DOT, then it must be the case that $a_1 = \lambda a_1$. Since $a_1 \neq 0$, then this forces $\lambda = 1$, a contradiction. □

5. Splitting the colouring

Now we will split the colours of C to make a new colouring $C' = C \times \chi_2$, where χ_2 is the second part of the MIP colouring.

Let $U \subseteq \mathbb{F}_q^3$ be a two-dimensional linear subspace. Let G_U be an auxiliary graph where $V(G_U)$ is the set of non-isotropic vectors in U, and

$$xy \in E(G_U) \iff x \cdot y = 0.$$

We wish to show that G_U is bipartite. Note that $x \cdot y = 0$ implies that $\alpha x \cdot \beta y = 0$ for any $\alpha, \beta \in \mathbb{F}_q$. Suppose that $x \cdot z = 0$ for some $z \in V(G_U)$ such that $z \neq \beta y$ for any $\beta \in \mathbb{F}_q$. Then the intersection between U and the two-dimensional linear subspace orthogonal to x must also be a two-dimensional linear subspace. Therefore, x is contained in its own orthogonal linear subspace. So x is isotropic, a contradiction. Hence, G_U is comprised of disjoint complete bipartite graphs and so is itself bipartite.

For each two-dimensional linear subspace U, we label the vertices of G_U with A_U and B_U depending on their part in the bipartition, and then label all isotropic vectors in U with A_U as well.

For any two-dimensional linear subspace $U \subseteq \mathbb{F}_q^3$ and any $x \in U$ we define

$$S(x,U) = \begin{cases} A & x \in A_U, \\ B & x \in B_U. \end{cases}$$

For a given vector $a \in V$, and a given colour type T, define

$$a_T = \begin{cases} a & T = \text{DOT}, \\ (1,0,0) & T \in \{\text{UP}_1, \text{DOWN}_1, \text{ZERO}\}, \\ (0,1,0) & T \in \{\text{UP}_2, \text{DOWN}_2\}, \end{cases}$$

and let

$$U_{a,T} = \{x : a_T \cdot x = 0\}.$$

For convenience, let

$$a_b = \begin{cases} 0 & a_T \cdot a_T = 0, \\ (a_T \cdot b)(a_T \cdot a_T)^{-1} a_T & a_T \cdot a_T \neq 0, \end{cases}$$

for any vectors a and b where $T = T(a,b)$.

Now we can define the second part of the MIP colouring. For any two vectors, $a < b$ with $T = T(a,b)$, let

$$\chi_2(a,b) = (S(a - b_a, U_{b,T}), S(b - a_b, U_{a,T})).$$

5.1. The third configuration

Let $a,b,c,d,e \in V$ be five distinct vertices such that

$$C'(a,b) = C'(a,c) = C'(a,d) = C'(a,e) = \text{Black},$$

and let

$$C'(b,c) = C'(c,d) = C'(d,e) = C'(e,b) = \text{Red}$$

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as shown in Figure 3(c). By Lemma 2.7 we know that either $b,d < c,e$ or $c,e < b,d$. Similarly, we know that either $a < b,c,d,e$ or $b,c,d,e < a$. So without loss of generality, we can say that either $a < b,d < c,e$ or $b,d < c,e < a$. In either case,

$$S(b-a_b,U_{a,T}) = S(c-a_c,U_{a,T})$$

where $T = T(a,b) = T(a,c)$.

By Corollary 3.9 we know that one of the following three cases must be true:

(1) Black $\in$ DOT,

(2) Black $\in$ UP$_1$ such that $b,d < c,e < a$, or

(3) Black $\in$ DOWN$_1$ such that $a < b,d < c,e$.

This abuses our notation slightly, but the meaning is hopefully clear. For example, Black $\in$ DOT means that the first component of the χ_1 part of the colour Black is DOT.

We will show that none of these cases are possible through the following series of lemmas.

Lemma 5.1. *If Black $\in$ DOT and RED $\in$ DOT, then the configuration in Figure 3(c) is not possible under the colouring C'.*

Proof. Let the inner product part of colour Black be α and the inner product part of Red be β. Note that if either Black or Red encodes linear independence, then b, c, d and e would all belong to the same one-dimensional linear subspace, a contradiction of Lemma 3.6. Also, since $c-e$ satisfies the three linear equations, $a \cdot x = 0$, $b \cdot x = 0$ and $d \cdot x = 0$, then $\{a,b,d\}$ cannot be linearly independent since then $c = e$, a contradiction. So there exist non-zero $\lambda_1, \lambda_2 \in \mathbb{F}_q$ such that $d = \lambda_1 a + \lambda_2 b$.

Note that $a \cdot a \neq 0$ since otherwise

$$a \cdot d = a \cdot (\lambda_1 a + \lambda_2 b),$$
$$\alpha = \lambda_2 \alpha$$

implies that $\lambda_2 = 1$ since $\alpha \neq 0$. If $\lambda_2 = 1$, then we would reach a contradiction by taking the inner product of both sides of $d = \lambda_1 a + b$ with c to get that $\beta = \lambda_1 \alpha + \beta$, a contradiction since $d \neq b$.

So $a_b = a_c = \alpha (a \cdot a)^{-1} a$, then

$$d = \lambda_1' a_b + \lambda_2 b,$$

where $\lambda_1' = \lambda_1 \alpha^{-1} (a \cdot a)$. Taking the inner product of both sides of this with a gives that

$$\alpha = (\lambda_1' + \lambda_2)\alpha.$$

So it follows that a_b, b and d are affine dependent. By the same arguments we can conclude that a_b, c and e are also affine dependent.

Note that $(b-d) \cdot (c-e) = 0$. Therefore, $(b-a_b) \cdot (c-a_c) = 0$. Since

$$S(b-a_b,U_{a,\mathrm{DOT}}) = S(c-a_c,U_{a,\mathrm{DOT}}),$$

then $b-a_b$ and $c-a_c$ are contained in the same part of the bipartition of the auxiliary graph on $U_{a,\mathrm{DOT}}$. Therefore, either $b-a_b$ or $c-a_c$ must be isotropic since otherwise the fact that they are orthogonal would have made them adjacent in the auxiliary graph.

Assume without loss of generality that $b - a_b$ is isotropic. Since

$$(b - a_b) \cdot (b - a_b) = 0,$$

then $b \cdot b = \alpha^2 (a \cdot a)^{-1}$. Since $(b - a_b) \cdot (c - a_c) = 0$, then $\beta = \alpha^2 (a \cdot a)^{-1}$. Therefore, $b \cdot b = \beta$. Hence

$$b \cdot (b - c) = 0.$$

This contradicts our assumption that Red $\in$ DOT. □

Note in what follows that if Red $\notin$ DOT, then $b_1 = d_1$ and $c_1 = e_1$.

Lemma 5.2. *If Black $\in$ DOT and RED $\in$ ZERO, then the configuration in Figure 3(c) is not possible under the colouring C'.*

Proof. If Red $\in$ ZERO, then

$$b \cdot (c - e) = d \cdot (c - e) = 0.$$

Also, recall that

$$a \cdot (c - e) = 0.$$

If a, b, d are linearly independent, then $c = e$, a contradiction. So we must assume that a, b, d are linearly dependent.

If either b or d depends on a, then $\delta(a, x) = 0$ for $x = b, c, d, e$, which implies that all five vectors belong to a one-dimensional linear subspace spanned by a, contradicting Lemma 3.6. If $d = \lambda b$ for some $\lambda \in \mathbb{F}_q$, then $b_1 = d_1 = \lambda b_1$. So either $b_1 = 0$ or $\lambda = 1$, both contradictions. So we must assume that $d = \lambda_1 a + \lambda_2 b$ for non-zero $\lambda_1, \lambda_2 \in \mathbb{F}_q$. But then

$$\begin{aligned} d \cdot c &= \lambda_1 (a \cdot c) + \lambda_2 (b \cdot c), \\ 0 &= \lambda_1 (a \cdot c). \end{aligned}$$

Since $\lambda_1 \neq 0$, then $a \cdot c = 0$, which implies that Black $\notin$ DOT, a contradiction. □

Lemma 5.3. *If Black $\in$ DOT and RED $\notin \{DOT, ZERO\}$, then the configuration in Figure 3(c) is not possible under the colouring C'.*

Proof. If Red $\in \mathrm{UP}_2 \cup \mathrm{DOWN}_2$, then $b_1 = c_1 = d_1 = e_1$. So all four vectors b, c, d, e satisfy the linear equations $a \cdot x = \alpha$ and $(1, 0, 0) \cdot x = b_1$. Therefore, b, c, d and e all belong to a one-dimensional affine subspace, a contradiction of Lemma 3.6.

If Red $\in \mathrm{UP}_1$, then

$$b \cdot (b - c) = b \cdot (b - e) = d \cdot (d - c) = d \cdot (d - e) = 0$$

since we assume that $b, d < c, e$. Therefore,

$$b \cdot (c - e) = b \cdot c - b \cdot e = b \cdot b - b \cdot b = 0.$$

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Similarly, $d\cdot(c-e)=0$. Since $c_1=e_1$, then it follows that

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ b_1 & d_2 & d_3 \end{pmatrix}\begin{pmatrix} 0 \\ c_2-e_2 \\ c_3-e_3 \end{pmatrix}=0.$$

Therefore, if any two of (a_2,a_3), (b_2,b_3) and (d_2,d_3) are linearly independent as vectors in $\mathbb{F}_q^2$, then $c=e$, a contradiction. Hence, there must exist $\lambda_1,\lambda_2\in\mathbb{F}_q$ such that $(a_2,a_3)=\lambda_1(b_2,b_3)$ and $(d_2,d_3)=\lambda_2(b_2,b_3)$.

From the equations of the form $x\cdot(x-c)=0$ for $x=b,d$ we get

$$\begin{aligned} b_1(b_1-c_1)+b_2^2+b_3^2-b_2c_2-b_3c_3&=0,\\ b_1(b_1-c_1)+\lambda_2^2b_2^2+\lambda_2^2b_3^2-\lambda_2b_2c_2-\lambda_2b_3c_3&=0.\end{aligned}$$

So it follows that

$$c_3=b_3^{-1}(b_1(b_1-c_1)+b_2^2+b_3^2-b_2c_2),$$

which in turn gives that

$$(1-\lambda_2)b_1(b_1-c_1)=\lambda_2(1-\lambda_2)(b_2^2+b_3^2).$$

Since $\lambda_2\neq 1$, then

$$b_1(b_1-c_1)=\lambda_2(b_2^2+b_3^2).$$

Since $b_1\neq 0$ and $b_1\neq c_1$, then $b_2^2+b_3^2\neq 0$.

Now, since Black $\in$ DOT, we get

$$\begin{aligned} a\cdot b&=a\cdot d,\\ a_1b_1+\lambda_1b_2^2+\lambda_1b_3^2&=a_1b_1+\lambda_1\lambda_2b_2^2+\lambda_1\lambda_2b_3^2,\\ \lambda_1(1-\lambda_2)(b_2^2+b_3^2)&=0.\end{aligned}$$

But this is a contradiction, since none of these three terms are zero.

If Red $\in$ DOWN$_1$, then we swap b,d and e,c in the previous argument to obtain the same contradiction. □

Lemma 5.4. *If Black* $\notin$ *DOT, then the configuration in Figure 3(c) is not possible under the colouring* C'.

Proof. In this case, either $b,d<c,e<a$ and Black $\in$ UP$_1$, or $c,e>b,d>a$ and Black $\in$ DOWN$_1$. In both cases,

$$b\cdot(b-a)=c\cdot(c-a)=d\cdot(d-a)=e\cdot(e-a)=0.$$

Moreover, $b_1=c_1=d_1=e_1$, which implies that

$$\text{Red}\in \text{ZERO}\cup\text{UP}_2\cup\text{DOWN}_2\cup\text{DOT}.$$

If Red $\in$ ZERO, then

$$b\cdot(c-e)=d\cdot(c-e)=0.$$

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Therefore,

$$\begin{pmatrix} b_2 & b_3 \\ d_2 & d_3 \end{pmatrix} \begin{pmatrix} c_2 - e_2 \\ c_3 - e_3 \end{pmatrix} = 0.$$

So $(d_2, d_3) = \gamma(b_2, b_3)$ for some $\gamma \in \mathbb{F}_q$, and $b \cdot c = d \cdot c$ gives

$$b_1^2 + c_2 b_2 + c_3 b_3 = b_1^2 + \gamma c_2 b_2 + \gamma c_3 b_3.$$

Thus, $(1-\gamma)(c_2 b_2 + c_3 b_3) = 0$. Therefore, either $\gamma = 1$, a contradiction since $b \neq d$, or $c_2 b_2 + c_3 b_3 = 0$, also a contradiction since this implies that $b \cdot c = b_1^2 \neq 0$.

If Red $\in \mathrm{UP}_2 \cup \mathrm{DOWN}_2$, then we have $b_2 = d_2$, $c_2 = e_2$ and either $b \cdot (b-c) = b \cdot (b-e) = 0$ or $c \cdot (c-b) = c \cdot (c-d) = 0$. In the first case, $b \cdot (e-c) = 0$ so $b_3(e_3 - c_3) = 0$. So either $b_3 = 0$ or $e_3 = c_3$, both contradictions. Similarly, in the second case, $c \cdot (b-d) = 0$ means that $c_3(b_3 - d_3) = 0$, which gives the same contradictions.

Finally, if Red $\in$ DOT, then $b \cdot (c-e) = 0$ and $d \cdot (c-e) = 0$. So

$$\begin{pmatrix} b_2 & b_3 \\ d_2 & d_3 \end{pmatrix} \begin{pmatrix} c_2 - e_2 \\ c_3 - e_3 \end{pmatrix} = 0.$$

Therefore, either $c = e$, a contradiction, or $(d_2, d_3) = \lambda(b_2, b_3)$ for some non-zero $\lambda \in \mathbb{F}_q$.

If β is the inner product represented by Red, then we get that

$$\beta = b_1^2 + b_2 c_2 + b_3 c_3,$$
$$\beta = b_1^2 + \lambda b_2 c_2 + \lambda b_3 c_3.$$

So,

$$(1-\lambda)(b_2 c_2 + b_3 c_3) = 0.$$

Therefore, either $\lambda = 1$ or $b_2 c_2 + b_3 c_3 = 0$. If $\lambda = 1$, then $b = d$, a contradiction. So we must assume that $b_2 c_2 + b_3 c_3 = 0$. But then

$$(b - (b_1, 0, 0)) \cdot (c - (b_1, 0, 0)) = 0.$$

Since $a_b = a_c = (b_1, 0, 0)$ we know that

$$(b - a_b) \cdot (c - a_c) = 0.$$

Therefore, since

$$S(b - a_b, U_{a,T}) = S(c - a_c, U_{a,T}),$$

it must be that either $(0, b_2, b_3)$ or $(0, c_2, c_3)$ is isotropic.

First, assume that $(0, b_2, b_3)$ and $(0, c_2, c_3)$ are linearly independent. Then they must span the linear subspace $U_{a,T}$. Without loss of generality assume that $(0, b_2, b_3)$ is isotropic. Therefore, it is orthogonal to every vector in the subspace $U_{a,T}$. Since this space is defined to be orthogonal to $(1,0,0)$, then this means that $(0, b_2, b_3)$ is linearly dependent on $(1,0,0)$, a contradiction.

So we must assume that $(0, b_2, b_3)$ and $(0, c_2, c_3)$ are linearly dependent. Since at least one of them is isotropic, then they belong to a totally isotropic one-dimensional linear subspace. So $b_2^2 + b_3^2 = 0$ and $c = (b_1, \lambda b_2, \lambda b_3)$ for some $\lambda \in \mathbb{F}_q$. But then $b \cdot (b-c) = 0$, a contradiction of the assumption that Red $\in$ DOT. □

Since these lemmas show that the third and final configuration does not appear, the colouring C' is a $(5,5)$-colouring of K_n.

6. Conclusion

This construction provides additional evidence that a general strategy of combining a $(p,p-1)$-colouring with a $\mathbb{F}_q^{p-2} \to \mathbb{F}_q$ algebraic colouring might show that $f(n,p,p) \leqslant n^{1/(p-2)+o(1)}$. However, both Mubayi's proof for his $(4,4)$-colouring [10] and our proof for the $(5,5)$-colouring require case-checking. Already in this paper we found it far easier to appeal to an algorithm rather than present a logical elimination of all cases, but this problem will quickly become intractable as p increases. Some general principles will need to be identified before we can demonstrate that the analogous constructions work for all p.

Even if this type of construction were to demonstrate such a bound in general, a subpolynomial yet significant gap between the lower and upper bounds persists even for $p = 4$. It would be nice to find a way to avoid including the CFLS colouring and tighten the upper bound, or, perhaps more interestingly, show that the lower bound can be increased.

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Appendix: Algorithm for reducing cases

The following algorithm is not difficult to verify, so we present it here without proof. The specific implementation we rely on is a Python script that can be found (with comments) at http://homepages.math.uic.edu/~acamer4/EdgeColors.py.

Suppose we want to find every edge-colouring, up to isomorphism, of K_n that uses at most m colours and does not contain a copy of any $F \in \mathcal{F}$, a list of edge-coloured complete graphs on n or fewer vertices. The algorithm takes $\mathcal{F}$, n and m as input and returns a list $\mathcal{R}$ of edge-colourings of K_n satisfying these requirements.

For each $k = 3,\dots,n$, the algorithm creates a list L_k of acceptable edge-colourings of K_k by adding a new vertex to each K_{k-1} listed in L_{k-1} (where L_2 is the list of exactly one K_2 with its single edge given colour 1), and then colouring the $k-1$ new edges in all possible ways from the colour set $[m]$. For each graph in L_{k-1} and each way to colour the new edges, we test the resulting graph to see if it contains any of the forbidden edge-colourings. If it does, then we move on. If not, then we test it against the new list L_k to see if it is isomorphic to any of the colourings of K_k

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already on the list. If it is, then we move on. Otherwise, we add it to the list L_k. The algorithm terminates when it has tested all colourings of K_n.

Algorithm 1: List all edge-colourings with no forbidden subcolouring

```
Data: number of vertices n; maximum number of colours m; list of forbidden colourings F
initialize L_2 as list containing one K_2 with its edges coloured 1;
for k = 3,...,n do
    initialize empty list L_k;
    for H ∈ L_{k-1} do
        for each function f : [k-1] → [m] do
            let G be K_k with edge-colours same as H on the first k-1 vertices and colour
            f(i) on edge ki for i = 1,...,k-1;
            if G contains no element of F and is isomorphic to no element of L_k then
                add G to the list L_k
            end
        end
    end
end
return L_n
```

References

[1] Axenovich, M. (2000) A generalized Ramsey problem. *Discrete Math.* **222** 247–249.

[2] Babai, L. and Frankl, P. (1992) *Linear Algebra Methods in Combinatorics: With Applications to Geometry and Computer Science*, Department of Computer Science, University of Chicago.

[3] Conlon, D., Fox, J., Lee, C. and Sudakov, B. (2015) The Erdős–Gyárfás problem on generalized Ramsey numbers. *Proc. London Math. Soc.* **110** 1–18.

[4] Eichhorn, D. and Mubayi, D. (2000) Edge-coloring cliques with many colors on subcliques. *Combinatorica* **20** 441–444.

[5] Erdős, P. (1975) Problems and results on finite and infinite graphs. In *Recent Advances in Graph Theory (Proc. Second Czechoslovak Sympos., Prague, 1974)*, pp. 183–192.

[6] Erdős, P. (1981) Solved and unsolved problems in combinatorics and combinatorial number theory. *Europ. J. Combin.* **2** 1–11.

[7] Erdős, P. and Gyárfás, A. (1997) A variant of the classical Ramsey problem. *Combinatorica* **17** 459–467.

[8] Krop, E. and Krop, I. (2013) Almost-rainbow edge-colorings of some small subgraphs. *Discussiones Mathematicae Graph Theory* **33** 771–784.

[9] Mubayi, D. (1998) Edge-coloring cliques with three colors on all 4-cliques. *Combinatorica* **18** 293–296.

[10] Mubayi, D. (2004) An explicit construction for a Ramsey problem. *Combinatorica* **24** 313–324.

[11] Mubayi, D. (2016) Coloring triple systems with local conditions. *J. Graph Theory* **81** 307–311.

[12] Perelli, A., Pintz, J. and Salerno, S. (1984) Bombieri's theorem in short intervals. *Annali della Scuola Normale Superiore di Pisa–Classe di Scienze* **11** 529–539.