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Kneser's theorem in σ-finite abelian groups

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Abstract. Let G be a σ-finite abelian group, i.e., $G = \bigcup_{n\geq 1} G_n$ where $(G_n)_{n\geq 1}$ is a nondecreasing sequence of finite subgroups. For any $A \subset G$, let $\underline{d}(A) := \liminf_{n\to\infty} \frac{|A\cap G_n|}{|G_n|}$ be its lower asymptotic density. We show that for any subsets A and B of G, whenever $\underline{d}(A+B) < \underline{d}(A) + \underline{d}(B)$, the sumset $A + B$ must be periodic, that is, a union of translates of a subgroup $H \leq G$ of finite index. This is exactly analogous to Kneser's theorem regarding the density of infinite sets of integers. Further, we show similar statements for the upper asymptotic density in the case where $A = \pm B$. An analagous statement had already been proven by Griesmer in the very general context of countable abelian groups, but the present paper provides a much simpler argument specifically tailored for the setting of σ-finite abelian groups. This argument relies on an appeal to another theorem of Kneser, namely the one regarding finite sumsets in an abelian group.

1 Introduction and statement of the results

Let A be a subset of an abelian group G, μ be a measure on G and $C > 0$ be a real number. Inverse results in additive number theory refer to those in which starting from the small doubling condition $\mu(A + A) < C\mu(A)$, it is possible to deduce structural information on A and its sumset $A + A$. For finite sets A, the measure of A can be its cardinality. More generally for infinite sets in a locally compact group, $\mu(A)$ can be chosen as the Haar measure of A. For infinite sets in a discrete semigroup, we may use various notions of density instead of measure.

One of the most popular inverse result is Kneser's theorem. In an abelian group with $\mu(\cdot) = |\cdot|$, the counting measure, and $C \leq 2$ it provides mainly a periodical structure for sumsets $A + B$ such that $|A + B| < |A| + |B| - 1$, yielding also a partial structure for A, B themselves.

In the particular semigroup $\mathbb{N}$ of positive integers with $\mu = \underline{d}$ being the lower asymptotic density, there exists again such a result due to Kneser when $\underline{d}(A + B) < \underline{d}(A) + \underline{d}(B)$ (see [8] or [3]); basically, $A + B$ is then a union of residue classes modulo some integer q. When μ is the upper asymptotic density or the upper Banach density, some structural information is still available (see [1, 7]).

Our goal is to investigate the validity of similar results in abelian σ-finite groups. A group G is said to be σ-finite if it is infinite and admits an *exhausting sequence*,

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that is a nondecreasing sequence $(G_n)_{n\geq 1}$ of finite subgroups such that $G = \bigcup_{n\geq 1} G_n$. As examples we have the polynomial ring $\mathbb{F}_{p^r}[t]$ for any prime p and integer $r \geq 1$. More generally, let $(C_n)_{n\geq 1}$ be a sequence of finite groups, $G_{\mathbb{N}} = \prod_{n\in\mathbb{N}} C_n$ and $G_n = \prod_{i\leq n} C_i \leq G_{\mathbb{N}}$. Then $G = \bigcup_{n\geq 1} G_n$ is σ-finite. Another class of σ-finite groups is given by the p-Prüfer groups $\mathbb{Z}(p^\infty) = \bigcup_{r\geq 1} \mathbb{U}_{p^r}$ where p is a prime number and $\mathbb{U}_q$ denotes the group of complex qth roots of unity. Further, if (d_n) is a sequence of integers satisfying $d_n \mid d_{n+1}$ and $G_n = \mathbb{U}_{d_n}$, then $G = \bigcup_{n\geq 1} G_n$ is σ-finite.

For any $A \subset G$, we define its lower and upper asymptotic densities as

$$\underline{\mathrm{d}}(A) := \liminf_{n\to\infty} \frac{|A \cap G_n|}{|G_n|} \quad \text{and} \quad \overline{\mathrm{d}}(A) := \limsup_{n\to\infty} \frac{|A \cap G_n|}{|G_n|},$$

respectively. We observe that this definition implicitly requires a particular exhausting sequence to be fixed. If both limits coincide, we denote by $\mathrm{d}(A)$ their common value. This type of groups and densities were already studied in the additive combinatorics literature; Hamidoune and Rödseth [4] proved that if $\langle A\rangle = G$ and $\alpha = \overline{\mathrm{d}}(A) > 0$, then $hA = G$ for some $h = O(\alpha^{-1})$. Hegyvári [5] showed that then $h(A-A) = \langle A - A\rangle$ for some $h = O(\log \alpha^{-1})$, where again $\alpha = \overline{\mathrm{d}}(A) > 0$; this was improved by Hegyvári and the second author [6].

For a subset X of an abelian group G, we denote by $\mathrm{Stab}_G(X)$ (or simply $\mathrm{Stab}(X)$ when no ambiguity is possible) the period or stabilizer of X, i.e., the subgroup $\{g \in G : \forall x \in X,\ g + x \in X\}$.

Theorem 1.1 *Let G be a σ-finite abelian group and $(G_n)_{n\geq 1}$ a fixed exhausting sequence. Let $A \subset G$ and $B \subset G$ satisfy one of the following hypotheses:*

(1) $\underline{\mathrm{d}}(A+B) < \underline{\mathrm{d}}(A) + \underline{\mathrm{d}}(B)$;
(2) $\overline{\mathrm{d}}(A+B) < \overline{\mathrm{d}}(A) + \overline{\mathrm{d}}(B)$ *and* $A = B$ *or* $A = -B$; *and*
(3) $\overline{\mathrm{d}}(A+B) < \overline{\mathrm{d}}(A) + \underline{\mathrm{d}}(B)$.

Then $H = \mathrm{Stab}(A+B)$ *has finite index* q *and admits a density equal to* q^{-1}. *Moreover, the numbers* a, b, c *of cosets of* H *met by* $A, B, A+B$ *respectively satisfy* $c = a + b - 1$.

We will refer to the statements obtained with hypotheses 1, 2, or 3 as Theorems 1, 2, or 3, respectively.

Let A, B, a, b, c be as in Theorem 1. Then $A + B$ admits a density which is rational, namely c/q. Further, let $\varepsilon = 1 - \frac{\mathrm{d}(A+B)}{\underline{\mathrm{d}}(A)+\underline{\mathrm{d}}(B)}$. Then we have

$$(1-\varepsilon)\frac{a+b}{q} \geq (1-\varepsilon)(\underline{\mathrm{d}}(A) + \underline{\mathrm{d}}(B)) = \mathrm{d}(A+B) = \frac{a+b-1}{q}.$$

Therefore, $a + b \leq \varepsilon^{-1}$. We also get

$$q \leq ((\underline{\mathrm{d}}(A) + \underline{\mathrm{d}}(B))\varepsilon)^{-1} = (\underline{\mathrm{d}}(A) + \underline{\mathrm{d}}(B) - \mathrm{d}(A+B))^{-1}.$$

We point out that a slightly weaker conclusion was obtained under a similar hypothesis with the upper Banach density in the general setting of countable abelian groups, by Griesmer [2]. To state his theorem, recall that a Følner sequence for an additive group G is a sequence $\Phi = (F_n)_{n\geq 1}$ of finite subsets in G such that

$$\lim_{n\to\infty} \frac{|(g+F_n)\Delta F_n|}{|F_n|} = 0, \quad \text{for all } g \in G,$$

where Δ is the symmetric difference operator. We let

$$\mathrm{d}^*(A) := \sup_{\Phi} \limsup_{n\to\infty} \frac{|A\cap F_n|}{|F_n|}$$

be the upper Banach density of A, where the supremum ranges over Følner sequences Φ. Under the hypothesis $\mathrm{d}^*(A+B) < \mathrm{d}^*(A) + \mathrm{d}^*(B)$, Griesmer obtained that $\mathrm{d}^*(A+B) = \mathrm{d}^*(A+B+H)$ for some subgroup H of finite index. The value of the present paper resides, besides the stronger conclusion that $A+B = A+B+H$, in the simplicity and brevity of our elementary argument, in contrast to Griesmer's involved, ergodic-theoretic method. We make full use of the σ-finiteness property in order to reduce the problem to passing to the limit in Kneser's theorem for finite sets.

Remark 1.2 In our theorem, the lower asymptotic density may not be replaced by the upper asymptotic nor upper Banach density without any extra hypothesis. Indeed, let G be a σ-finite abelian group. Let $(G_n)_{n\geq 0}$ be an exhausting sequence of finite subgroups such that $|G_{n+1}|/|G_n| \to \infty$. Let $A = \bigcup_{n\geq 1} G_{2n}\backslash G_{2n-1}$ and $B = \bigcup_{n\geq 0} G_{2n+1}\backslash G_{2n}$. Then $\overline{\mathrm{d}}(A) = \overline{\mathrm{d}}(B) = 1$, therefore $\overline{\mathrm{d}}(A+B) = 1$ and $\mathrm{d}^*(A) = \mathrm{d}^*(B) = \mathrm{d}^*(A+B) = 1$. Yet it is easy to see that $A+B = A\cup B = G\backslash G_0$ since $G_n\backslash G_{n-1} + G_m\backslash G_{m-1} = G_n\backslash G_{n-1}$ whenever $n > m$. Hence, the stabilizer of $A+B$ is G_0, which is a finite group, in particular it is not of finite index. We are thankful to the anonymous referee for this construction.

However, the fact that the extra hypothesis $B = \pm A$ suffices for the upper asymptotic density Kneser theorem is in contrast with the situation in the integers, cf [7]. Further, there exists a σ-finite group G and a set $A \subset G$ such that $\mathrm{d}^*(A+A) < 2\mathrm{d}^*(A)$ but $\mathrm{Stab}(A+A)$ is finite. Here is our construction. Let G be a σ-finite abelian group. Assume that G admits an exhausting sequence of finite subgroups $(G_n)_{n\geq 0}$ satisfying that none of the quotients G_{n+1}/G_n have exponent 2; this is the case of $G = \mathbb{F}_3[t]$ for instance. For any $n \geq 1$, let $x_n \in G_{n+1}\backslash G_n$ such that $2x_n \notin G_n$. Let

$$A = \bigcup_{n\geq 1} (x_n + G_n).$$

Then

$$A + A = \bigcup_{n\geq 1} (\{x_n, 2x_n\} + G_n).$$

Since the sequence $(x_n + G_n)_{n\in\mathbb{N}}$ is a Følner sequence, we see that $\mathrm{d}^*(A) = \mathrm{d}^*(A+A) = 1$ so $\mathrm{d}^*(A+A) < 2\mathrm{d}^*(A)$. Now, we prove that $\mathrm{Stab}(A+A) = G_0$, a finite subgroup.[1] Indeed, let $x \in G\backslash G_0$, thus $x \in G_{k+1}\backslash G_k$ for some $k \geq 0$. For any $g \in G_k$, we have $x + x_k + g \in G_{k+1}\backslash G_k$. Assume for a contradiction that $x + (\{x_k, 2x_k\} + G_k) \subset A + A$. In particular, $x + x_k + g \in A + A$ and $x + 2x_k + g \in$

[1] We may even choose it to be trivial.

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$A+A$ for some $g \in G_k$. Then $x + x_k + g = x_k + g'$ or $x + x_k + g = 2x_k + g'$, and $x + 2x_k + g = 2x_k + g''$ or $x + 2x_k + g = x_k + g''$ for some $g', g'' \in G_k$. Since $x \notin G_k$, it follows that $x + x_k + g = 2x_k + g'$ and $x + 2x_k + g = x_k + g''$. Therefore, $x = x_k + g' - g \in A$ and $x = -x_k + g'' - g \in -A$. But $A \cap (-A) = \varnothing$, whence the contradiction. We infer that $x \notin \mathrm{Stab}(A+A)$. Thus

$$\mathrm{Stab}(A+A) \leq G_0.$$

It is easy to check that $G_0 \leq \mathrm{Stab}(A+A)$ thus $\mathrm{Stab}(A+A) = G_0$ as announced. This remark shows that even in the context of σ-finite groups, a small doubling in Banach density of a set A does not imply that $A+A$ is periodic, thus Griesmer's weaker structural conclusion is optimal.

Remark 1.3 If G has no proper subgroup of finite index, which is the case of $\mathbb{Z}(p^\infty)$ for instance, the conclusion implies that $A+B=G$, so $\underline{\mathrm{d}}(A+B) < \underline{\mathrm{d}}(A) + \underline{\mathrm{d}}(B)$ is impossible except in the trivial case $\underline{\mathrm{d}}(A) + \underline{\mathrm{d}}(B) > 1$.

2 Proofs

We detail the proof of Theorem 1.1. Theorems 2 and 3 will be deduced straightforwardly by the same arguments.

Proof Let $\alpha := \underline{\mathrm{d}}(A)$ and $\beta := \underline{\mathrm{d}}(B)$. Let us assume that $\underline{\mathrm{d}}(A+B) < \alpha + \beta$. Therefore, $\alpha + \beta > 0$ and there exists $\varepsilon > 0$ such that

$$\underline{\mathrm{d}}(A+B) < (1-\varepsilon)(\alpha+\beta).$$

By definition of the lower limit, there exists an increasing sequence $(n_i)_{i\geq 1}$ of integers such that

$$|(A+B) \cap G_{n_i}| < (1-\varepsilon/2)|G_{n_i}|(\alpha+\beta).$$

Furthermore, denoting $A_n = A \cap G_n$ and $B_n = B \cap G_n$, for any i large enough, we have

$$|A_{n_i}| + |B_{n_i}| > (\alpha+\beta)|G_{n_i}|(1-\varepsilon/2)/(1-\varepsilon/3).$$

Combining these two bounds together with the fact that $A_{n_i} + B_{n_i} \subset (A+B) \cap G_{n_i}$, and reindexing the sequence (n_i) if necessary, we have

$$|A_{n_i} + B_{n_i}| < (1-\varepsilon/3)(|A_{n_i}| + |B_{n_i}|) \tag{2.1}$$

for all $i \geq 1$. In particular $|A_{n_i} + B_{n_i}| < |A_{n_i}| + |B_{n_i}| - 1$ for i large enough (by reindexing if necessary, for any $i \geq 1$), since $|A_{n_i}| + |B_{n_i}| \to \infty$ as a consequence of $\alpha + \beta > 0$.

We are now able to apply Kneser's theorem in the finite abelian group G_{n_i}. Let $H_i = \mathrm{Stab}_{G_{n_i}}(A_{n_i} + B_{n_i})$. We obtain that $H_i \neq \{0\}$ for each $i \geq 1$ and letting a_i, b_i, c_i be the number of cosets of H_i met by $A_{n_i}, B_{n_i}, A_{n_i} + B_{n_i}$ respectively, we get $c_i = a_i + b_i - 1$. Reformulating inequality (2.1) in terms of these quantities yields

$$\begin{aligned}(1-\varepsilon/3)(a_i+b_i)|H_i| &\geq (1-\varepsilon/3)(|A_{n_i}| + |B_{n_i}|)\\ &> |A_{n_i} + B_{n_i}| = (a_i + b_i - 1)|H_i|\end{aligned}$$

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from which we infer that $a_i + b_i < 3/\varepsilon$. Moreover,

$$(\alpha+\beta)|G_{n_i}|/2 < |A_{n_i}| + |B_{n_i}| \le (a_i + b_i)|H_i|$$

so $[G_{n_i} : H_i] = |G_{n_i}|/|H_i| < 2\frac{a_i+b_i}{\alpha+\beta} < 6/((\alpha+\beta)\varepsilon)$. By the pigeonhole principle, upon extracting again a suitable subsequence of (n_i), one may assume that $[G_{n_i} : H_i] = k$ for any $i \ge 1$ and some fixed $k < 6/((\alpha+\beta)\varepsilon)$.

Let us set

$$\mathcal{A}_i = \{K < G_{n_i} \,:\, [G_{n_i} : K] = k\}.$$

Thus, $H_i \in \mathcal{A}_i$ for any $n \ge 1$.

Lemma 2.1 *Let $i \ge 1$. If $L \in \mathcal{A}_{i+1}$, there exists $K \in \mathcal{A}_i$ such that $K \subset L$.*

Proof Let $L \in \mathcal{A}_{i+1}$. Let us set $K' = L \cap G_{n_i}$. Thus

$$G_{n_i}/K' \simeq (L + G_{n_i})/L \le G_{n_{i+1}}/L,$$

so $[G_{n_i} : K']$ divides k. Let us write

$$|G_n| = kg,\ [G_{n_i} : K'] = \frac{k}{h},\ |K'| = gh$$

for some positive integers g and h. Since K' is abelian and h divides $|K'|$, there exists a subgroup K of K' of index h. It satisfies

$$[G_{n_i} : K] = [G_{n_i} : K'] \times [K' : K] = k.$$

We draw from the lemma the following corollary by an easy induction.

Corollary *For all pairs of integers $j > i \ge 1$, there exists a sequence of subgroups $K_\ell \in \mathcal{A}_\ell$ for $\ell \in [i, j]$ such that $K_\ell \le K_{\ell+1}$ for all $\ell \in [i, j)$ and $K_j = H_j$.*

Borrowing terminology from graph theory, for $K \in \mathcal{A}_i$ and $L \in \mathcal{A}_j$ where $i \le j$, we call a *path* from K to L any non decreasing sequence of subgroups $K_\ell \in \mathcal{A}_\ell, \ell \in [i, j]$ for which $K_i = K$ and $K_j = L$. With this terminology, the conclusion of the corollary is that there exists a path from K_i to H_j.

We shall construct inductively a nondecreasing subsequence of subgroups $K_i \in \mathcal{A}_i$ for $i \ge 1$ such that for any i, the set of integers $j \ge i$ for which there exists a path from K_i to H_j is infinite.

To construct $K_1 \in \mathcal{A}_1$ with the desired property, let us observe that $\mathcal{A}_1$ is finite and invoke the corollary and the pigeonhole principle.

Suppose that $K_1, \dots, K_i$ are already constructed for some $i \ge 1$; one constructs K_{i+1} by observing again that $\mathcal{A}_{i+1}$ is finite and applying the pigeonhole principle. Indeed, there are infinitely many $j > i$ such that a path from K_i to H_j exists, but only finitely many $K \in \mathcal{A}_{i+1}$ through which these paths may pass, so there exists $K \in \mathcal{A}_{i+1}$ having the property that there exist infinitely many $j > i$ such that a path from K_i to H_j over $K \in \mathcal{A}_{i+1}$ exists. One then selects K_{i+1} to be such a subgroup K.

The sequence $(K_i)_{i\geq 1}$ being non decreasing, the union

$$K = \bigcup_{i\geq 1} K_i$$

is a subgroup of G. In fact, $K \leq \operatorname{Stab}(A+B) =: H$; indeed, let $g \in K$ and $x = a + b \in A + B$ where $(a, b) \in A \times B$. Let i satisfy $g \in K_i$ and $x \in A_{n_i} + B_{n_i}$. Since K_i is included in H_j for infinitely many $j \geq 1$, there exists in particular some $j \geq i$ for which $K_i \leq H_j$. Thus, $x + g \in A_{n_j} + B_{n_j}$ so $g \in H$.

Since $K_i \subset H \cap G_{n_i}$, the indices $[G_{n_i} : H \cap G_{n_i}]$ divide $k = [G_{n_i} : K_i]$. Now

$$G_{n_i}/(H \cap G_{n_i}) \simeq (G_{n_i} + H)/H,$$

and $((G_{n_i} + H)/H)_{i\in\mathbb{N}}$ is a non decreasing sequence of subgroups of bounded indices of G/H, whose union is G/H. It is therefore stationary, so $(G_{n_i} + H)/H = G/H$ for any large enough i. It follows that $[G : H]$ divides k, in particular $[G : H] \leq k < 6/((\alpha + \beta)\varepsilon)$.

Furthermore, if i is large enough, we have

$$\frac{|H \cap G_{n_i}|}{|G_{n_i}|} = |(G_{n_i} + H)/H|^{-1} = |G/H|^{-1},$$

whence we deduce that $\mathrm{d}(H)$ exists and equals $q^{-1} = [G : H]^{-1}$.

Let C, D and F be the projections of A, B, and $A + B$, respectively, in G/H. Then $F = C + D$. In view of the previous paragraph and the inequality $\underline{\mathrm{d}}(A + B) < \underline{\mathrm{d}}(A) + \underline{\mathrm{d}}(B)$, we obtain that $|C + D| < |C| + |D|$. Now $|C + D| \geq |C| + |D| - 1$, since otherwise by Kneser's theorem applied in the finite abelian group G/H, the set F would admit a non trivial period and $\operatorname{Stab}(A + B)$ would be strictly larger than H. Therefore $|C + D| = |C| + |D| - 1$, and we conclude.

We now prove the same result with the upper asymptotic density in place of the lower one in the case $B = \pm A$, that is, Theorem 2. Let $\alpha := \overline{\mathrm{d}}(A) = \overline{\mathrm{d}}(B) = \beta$ and suppose that $\overline{\mathrm{d}}(A + B) < 2\alpha(1 - \varepsilon)$ for some $\varepsilon > 0$. Then on the one hand we have for any large enough n

$$|A_n + B_n| \leq |(A + B) \cap G_n| < 2\alpha(1 - \varepsilon/2)|G_n|.$$

On the other hand, since $|A_n| = |B_n|$ for all n, there exist infinitely many n such that

$$|A_n| + |B_n| > \frac{1 - \varepsilon/2}{1 - \varepsilon/3}(\alpha + \beta)|G_n|. \tag{2.2}$$

Hence, equation (2.1) holds along some increasing sequence (n_i). From there, we are in position to conclude as in the proof of Theorem 1.1. The reader may observe that equation (2.2) fails for general A and B.

Finally, we prove Theorem 3. Let $\alpha := \overline{\mathrm{d}}(A)$ and $\beta = \underline{\mathrm{d}}(B)$ and suppose that $\overline{\mathrm{d}}(A + B) < (\alpha + \beta)(1 - \varepsilon)$ for some $\varepsilon > 0$. Then, on the one hand, we have for any large enough n

$$|A_n + B_n| \leq |(A + B) \cap G_n| < (\alpha + \beta)(1 - \varepsilon/2)|G_n|$$

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and

$$|B_n| > \frac{1-\varepsilon/2}{1-\varepsilon/3}\beta|G_n|.$$

On the other hand, there exist infinitely many n such that

$$|A_n| > \frac{1-\varepsilon/2}{1-\varepsilon/3}\alpha|G_n|.$$

Hence, again we recover equation (2.1) along some subsequence and can conclude.

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