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RATIONAL EQUIVALENCE OF FIBRATIONS WITH FIBRE G/K

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1. Introduction. Let $\xi_\nu : F \rightarrow E_\nu \xrightarrow{\pi_\nu} B$ be two Serre fibrations with same base and fibre in which all the spaces have the homotopy type of simple CW complexes of finite type. We say they are *rationally homotopically equivalent* if there is a homotopy equivalence $(E_1)_{\mathbf{Q}} \xrightarrow{\simeq} (E_2)_{\mathbf{Q}}$ between the localizations at $\mathbf{Q}$ which covers the identity map of $B_{\mathbf{Q}}$.

Such an equivalence implies, of course, an isomorphism of cohomology algebras (over $\mathbf{Q}$) and of rational homotopy groups; on the other hand isomorphisms of these classical algebraic invariants are usually (by far) insufficient to establish the existence of a rational homotopy equivalence.

Nonetheless, as we shall show in this note, for certain fibrations rational homotopy equivalence is in fact implied by the existence of an isomorphism of cohomology algebras. While these fibrations are rare inside the class of all fibrations, they do include principal bundles with structure groups a connected Lie group G as well as many associated bundles with fibre G/K. (These, of course, are the fibrations which are basic to differential geometry.)

More precisely, call ξ_1 and ξ_2 *h-equivalent* if they are rationally homotopically equivalent, and *c-equivalent* if there is a commutative diagram

$$\text{(1.1)} \qquad \begin{array}{ccccc} & & H^*(E_1) & \xrightarrow{j_1^*} & H^*(F) \\ & \overset{\pi_1^*}{\nearrow} & \cong\big\downarrow f & & \cong\big\downarrow \bar{f} \\ H^*(B) & & & & \\ & \underset{\pi_2^*}{\searrow} & & & \\ & & H^*(E_2) & \xrightarrow[j_2^*]{} & H^*(F) \end{array}$$

in which f and $\bar{f}$ are isomorphisms of graded algebras. (Cohomology of spaces is singular, with rational coefficients.) If $\bar{f} = \text{id}$, ξ_1 and ξ_2 are *strictly c-equivalent*. Finally, if

$$\zeta^* : H^*(F) \rightarrow \text{Hom}_{\mathbf{Z}}(\pi_*(F); \mathbf{Q})$$

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is dual to the Hurewicz homomorphism we say ξ_1 and ξ_2 are *c-π-equivalent* if there is a commutative diagram

$$(1.2)\qquad \begin{array}{ccccc} & & H^*(E_1) & \xrightarrow{\zeta^* j_1^*} & \mathrm{Hom}\,(\pi_*(F);\mathbf{Q}) \\ & \overset{\pi_1^*}{\nearrow} & & & \\ H^*(B) & & \cong\Big\downarrow f & & \cong\Big\downarrow \eta \\ & \underset{\pi_2^*}{\searrow} & & & \\ & & H^*(E_2) & \xrightarrow[\zeta^* j_2^*]{} & \mathrm{Hom}\,(\pi_*(F);\mathbf{Q}) \end{array}$$

in which f (respectively, η) is an isomorphism of graded algebras (respectively graded spaces).

Evidently h-equivalence implies c-equivalence and c-π-equivalence, and it is easy to see that the converses usually fail. If, however, K is a closed connected subgroup of a connected Lie group G we have

THEOREM I. *Let $\xi_\nu : G/K \to E_\nu \to B$ be fibrations as described at the start of the introduction, and suppose that ξ_1 is associated with a principal G-bundle via the standard action of G on G/K. Then*

(i) *ξ_1 and ξ_2 are h-equivalent if and only if they are c-π-equivalent.*

(ii) *If j_1^* is surjective then ξ_1 and ξ_2 are h-equivalent if and only if they are strictly c-equivalent.*

COROLLARY. *Let $\xi : G/K \to E \to B$ be a Serre fibration of simple spaces which is c-equivalent to the trivial fibration $B \times G/K$. Then it is h-equivalent to $B \times G/K$.*

Proof. The isomorphism $f : H^*(E) \xrightarrow{\cong} H^*(B) \otimes H^*(G/K)$, inducing the automorphism $\bar{f}$ of $H^*(G/K)$ can be composed with id $\otimes \bar{f}^{-1}$ to show that ξ is strictly c-equivalent to the trivial fibration. Now apply Theorem I (ii) with ξ_1 the trivial fibration.

For many homogeneous spaces G/K, an automorphism of $H^*(G/K)$ automatically factors over ζ^* to yield an automorphism of Hom $(\pi_*(G/K);\mathbf{Q})$; see Section 4. (Indeed we know of no example where this fails although these presumably abound!) For such spaces as fibre c-equivalence implies c-π-equivalence and hence (when one fibration is associated with a principal bundle) h-equivalence.

By contrast, if $\alpha, \beta \in \pi_2(S^2 \vee S^2)$ are the obvious basis and $\phi = [\alpha, [\alpha, \beta]] \in \pi_4(S^2 \vee S^2)$, set

$$F = (S^2 \vee S^2 \cup_\phi e^5) \vee S^5.$$

Then

$$H^+(F) = H^+(S^2) \oplus H^+(S^2) \oplus H^+(S^5) \oplus H^+(S^5)$$

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and the automorphism which interchanges the two elements of degree 5 does not factor over ζ^*.

Theorem I is proved via minimal models. The proof applies verbatim to the larger class of *rational fibrations* ([**7**]) which are "two stage", and so we work in that setting. Rational fibrations and some necessary facts about models are recalled in Section 2 where also we define "two stage" and state the relevant generalization of Theorem I (Theorem II).

In Section 3 we derive the explicit form of the model of a fibration associated with a principal bundle via the action of G on G/K. (This is established in [**3**, Theorem IX, Section 12.30] for smooth bundles and real coefficients.) This, in particular, gives the rational model for G/K. (The real version is due to Cartan [**2**].) It also shows that such fibrations are two stage, so that Theorem I does follow from Theorem II. In Section 4 we show that for many homogeneous spaces c-equivalent fibrations with fibre G/K are automatically c-π-equivalent (so that Theorem I may be applied). Finally, Section 5 contains the proof of Theorem II.

2. Rational fibrations. Henceforth we adopt the terminology of [**7**; Sections 1–4] with $\mathbf{Q}$ as ground field. For more details see [**5**]. Thus for a topological space S, $H^*(S)$ denotes its rational singular cohomology algebra and $(A(S), d)$ the c.g.d.a. of rational polynomial differential forms on the singular simplices of S. If S is path connected and equipped with a basepoint $\pi_\psi{}^*(S)$ denotes its ψ-homotopy space: if m_S: $(\Lambda X, d) \xrightarrow{\simeq} (A(S), d)$ is a model (in the sense of Sullivan) we put

$$Q(\Lambda X) = \Lambda^+X/\Lambda^+X \cdot \Lambda^+X$$

and denote by $\zeta : \Lambda^+X \rightarrow Q(\Lambda X)$ the projection. A differential $Q(d)$ is induced in $Q(\Lambda X)$ and

$$\pi_\psi{}^*(S) = H(Q(\Lambda X), Q(d)).$$

(The decomposition $\Lambda^+X = X \oplus (\Lambda^+X \cdot \Lambda^+X)$ allows us to identify $X \cong Q(\Lambda X)$ but not $d|_X = Q(d)$). Identifying $H(\Lambda X)$ with $H^*(S)$ via $m_S{}^*$ we obtain $\zeta^* : H^+(S) \rightarrow \pi_\psi{}^*(S)$. When S is simple and $H^*(S)$ is a graded space of finite type then

$$\pi_\psi{}^*(S) = \mathrm{Hom}_{\mathbf{Z}}(\pi_*(S); \mathbf{Q})$$

and ζ^* is the dual of the Hurewicz homomorphism.

A *rational fibration* $\xi : F \xrightarrow{j} E \xrightarrow{\pi} B$ is (cf. [**7**, Definition 4.5]) a sequence of base-point preserving continuous maps between pointed, path connected topological spaces, such that a certain condition on the minimal models is satisfied. Rational fibrations include ([**5**, Theorem 20.3]) Serre

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fibrations of path connected spaces in which one of $H^*(B)$, $H^*(F)$ is a graded space of finite type, and $\pi_1(B)$ acts nilpotently in each $H^p(F)$. In particular fiber bundles associated with a G-principal bundle when G is a path connected group, and one of $H^*(B)$, $H^*(F)$ has finite type are rational fibrations.

With each rational fibration $\xi : F \xrightarrow{j} E \xrightarrow{\pi} B$ is associated ([**7**, Definition 4.8]) its Λ-minimal Λ model: a commutative diagram of c.g.d.a. morphisms

$$\begin{array}{ccccc} (\Lambda Y, d_B) & \xrightarrow{i} & (\Lambda Y \otimes \Lambda X, d_E) & \xrightarrow{\rho} & (\Lambda X, d_F) \\ m_B \downarrow \simeq & & \simeq \downarrow m_E & & \simeq \downarrow m_F \\ (A(B), d) & \xrightarrow[A(\pi)]{} & (A(E), d) & \xrightarrow[A(j)]{} & (A(F), d) \end{array} \tag{2.1}$$

in which the vertical arrows are models and m_B and m_F are minimal. Note in the upper row only the differential d in $\Lambda Y \otimes \Lambda X$ depends on ξ; the algebras and the other maps depend only on the fixed B and F.

2.2 *Definition*. A rational fibration is *two stage* if its Λ-minimal Λ-model (2.1) can be written

$$\Lambda Y \xrightarrow{i} \Lambda Y \otimes \Lambda X_0 \otimes \Lambda X_1 \xrightarrow{\rho} \Lambda X_0 \otimes \Lambda X_1$$

with $d_E(X_0) = 0$ and $d_E(X_1) \subset \Lambda Y \otimes \Lambda X_0$. A space F is *two stage* if its minimal model has the form $(\Lambda X_0 \otimes \Lambda X_1, d_F)$ with $d_F(X_0) = 0$ and $d_F(X_1) \subset \Lambda X_0$.

The fibre of a two stage fibration is a two stage space, as are H spaces, homogeneous spaces (Section 4) and pure spaces [**4**]. On the other hand the rational fibration

$$S_{\mathbf{Q}}^5 \times S_{\mathbf{Q}}^7 \times S_{\mathbf{Q}}^9 \to E \to S_{\mathbf{Q}}^3 \times S_{\mathbf{Q}}^3$$

with Λ-model

$$(\Lambda(b_3, b_3'), 0) \to (\Lambda(b_3, b_3', x_5, x_7, x_9), d_E) \to (\Lambda(x_5, x_7, x_9), 0)$$

and

$$d_E x_5 = b_3 b_3', \quad d_E x_7 = b_3 x_5, \quad d_E x_9 = b_3 x_4$$

is not two stage, even though its fibre is a two stage space.

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Now fix path connected pointed spaces F and B and consider the class of all rational fibrations ξ with fibre F and base B. We say that two such rational fibrations $\xi_\nu : F \to E_\nu \to B$, $\nu = 1, 2$, are *rationally homotopically equivalent* (*h-equivalent*) if their Λ-minimal Λ-models are connected by c.g.d.a. isomorphisms

$$\phi: (\Lambda Y \otimes \Lambda X, d_1) \xrightarrow{\cong} (\Lambda Y \otimes \Lambda X, d_2) \quad \text{and} \quad \bar{\phi}: (\Lambda X, d) \xrightarrow{\cong} (\Lambda X, d)$$

such that $\phi \circ i = i$ and $\bar{\phi} \circ \rho = \rho \circ \phi$. If we can choose ϕ so that $\bar{\phi} =$ id then we say ξ_1 and ξ_2 are *strictly h-equivalent.*

When ξ_1, ξ_2 are genuine fibrations in which F, E_ν and B have the homotopy type of simple CW complexes of finite type, then ξ_1 and ξ_2 are h-equivalent if and only if their localizations at $\mathbf{Q}$ have the same fibre homotopy type. Thus the definition above of h-equivalent extends the definition in the introduction for Serre fibrations of simple spaces.

The definitions of (strict) c-equivalence and of c-π-equivalence given in the introduction apply verbatim to rational fibrations, except that $\mathrm{Hom}_{\mathbf{Z}}(\pi_*(F); \mathbf{Q})$ has to be replaced by $\pi_\psi{}^*(F)$.

2.3 *Remark.* Suppose that ξ_1 and ξ_2 are c-equivalent and that the diagram (1.1) can be chosen so that $\bar{f} = \alpha^*$ for some automorphism α of the model $(\Lambda X, d_F)$ for F. Then ξ_1 and ξ_2 are c-π-equivalent. In particular strict c-equivalence implies c-π-equivalence.

If F is formal (cf. [**8**] or [**6**]) every automorphism of $H^*(F)$ is of the form $\bar{f} = \alpha^*$ and so in this case c-equivalence always implies c-π-equivalence.

In Section 3 we shall show that a fibration $G/K \to E \to B$ associated with a principal G-bundle (G/K as in Theorem I) is two stage. As well we recover the classical fact that dim $\pi_\psi{}^*(G/K) < \infty$. With these results Theorem I is a special case of

THEOREM II. *Let* $\xi_\nu : F \xrightarrow{j_\nu} E_\nu \xrightarrow{\pi_\nu} B$ *be rational fibrations with* ξ_1 *two-stage. Assume that* $\pi_\psi{}^*(F)$ *is a graded space of finite type. Then*

(i) ξ_1 *and* ξ_2 *are h-equivalent if and only if they are c-*π*-equivalent.*

(ii) *If* $j_1{}^*$ *is surjective, then* ξ_1 *and* ξ_2 *are strictly h-equivalent if and only if they are strictly c-equivalent.*

The exact same proof of the corollary in the introduction yields

COROLLARY. *Let* $\xi : F \to E \to B$ *be a rational fibration in which* F *is a two stage space and* $\pi_\psi{}^*(F)$ *is a graded space of finite type. If* ξ *is c-equivalent to the trivial fibration* $B \times F$ *then it is strictly h-equivalent to* $B \times F$.

2.4 *Remark.* Evidently (strict) h-equivalence implies (strict)-c-equivalence in any rational fibration. The reverse implication can easily fail. For instance the rational fibration $S_{\mathbf{Q}}{}^2 \vee S_{\mathbf{Q}}{}^2 \to E \to S_{\mathbf{Q}}{}^3$ of [**9**-VI.1, (6)] is strictly c-equivalent (and hence c-π-equivalent) to the trivial fibration.

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The minimal model of E, however, is not isomorphic with that of $(S_{\mathbf{Q}}^2 \vee S_{\mathbf{Q}}^2) \times S_{\mathbf{Q}}^3$.

3. Associated fibrations. Let $\lambda : K \to G$ be the inclusion of a closed connected subgroup of a connected Lie group G. Because they are connected the classifying spaces B_K, B_G are 1-connected, and it is a classical result of Borel [**1**, Theorem 19.1] that $H^*(B_K)$ and $H^*(B_G)$ are finitely generated polynomial algebras ΛQ_K and ΛQ_G. (Use Iawasawa's theorem to reduce to the compact case.) In particular, the minimal models are given by

$$(\Lambda Q_K, 0) \to (A(B_K), d) \quad \text{and} \quad (\Lambda Q_G, 0) \to (A(B_G), d).$$

Now the inclusion $\lambda : K \to G$ induces $B(\lambda) : B_K \to B_G$ and in the corresponding homotopy commutative diagram

$$(3.1) \qquad \begin{array}{ccc} A(B_G) & \xrightarrow{A(B(\lambda))} & A(B_K) \\ \uparrow & & \uparrow \\ \Lambda Q_G & \xrightarrow[\mu]{} & \Lambda Q_K \end{array}$$

we must have $\mu = B(\lambda)^*$. Define c.g.d.a. $(\Lambda Q_G \otimes \Lambda Q_K \otimes \Lambda P_G, D)$ as follows:

$$P_G^k = Q_G^{k+1};\ D(Q_G \oplus Q_K) = 0 \text{ and}$$
$$D(1 \otimes 1 \otimes x) = x \otimes 1 \otimes 1 - 1 \otimes B(\lambda)^* x \otimes 1,\ x \in P_G.$$

Define a commutative diagram of c.g.d.a. homomorphisms

$$(3.2) \qquad \begin{array}{ccc} & & \Lambda Q_K \\ & \nearrow^{B(\lambda)^*} & \uparrow \sigma \\ \Lambda Q_G & & \\ & \searrow & \\ & & \Lambda Q_G \otimes \Lambda Q_K \otimes \Lambda P_G \end{array}$$

with $\sigma(x) = 0$, $x \in P_G$; $\sigma(y) = y$, $y \in Q_K$; $\sigma(z) = B(\lambda)^* z$, $z \in Q_G$. A simple calculation shows that

$$\sigma^* : H(\Lambda Q_G \otimes \Lambda Q_K \otimes \Lambda P_G) \to \Lambda Q_K$$

is an isomorphism.

On the other hand if $G \to E_G \to B_G$ is the universal bundle for G we may take $B_K = E_G/K$ and $B(\lambda)$ the projection of the bundle

$$(3.3) \quad G/K \longrightarrow E_G/K \xrightarrow{B(\lambda)} B_G.$$

Combining (3.1) and (3.2) we can construct a commutative diagram

$$\begin{array}{ccccc} (A(B_G), d) & \xrightarrow{A(B(\lambda))} & (A(B_K), d) & \longrightarrow & (A(G/K), d) \\ \simeq \Big\uparrow \gamma & & \simeq \Big\uparrow \beta & & \Big\uparrow \alpha \\ (\Lambda Q_G, 0) & \xrightarrow[i]{} & \Lambda Q_G \otimes \Lambda Q_K \otimes \Lambda P_G, D) & \xrightarrow[\rho]{} & (\Lambda Q_K \otimes \Lambda P_G, \bar{D}) \end{array}$$

in which γ^*, β^* are isomorphisms and ρ is defined by $\rho(Q_G) = 0$, $\rho =$ identity in Q_K, P_G. Note that this determines $\bar{D}$.

Because B_G is 1-connected and $H^*(B_G)$ has finite type, [**5**, Theorem 20.3] shows that α^* is an isomorphism.

Now let $G \to P \to B$ be a principal G bundle. It pulls back from the universal bundle via a classifying map $\phi : B \to B_G$, and the associated bundle $\xi : G/K \to E \to B$ is then the pull-back of (3.3) via ϕ. Let

$$(\Lambda Y, d_B) \xrightarrow[m_B]{} (A(B), d)$$

be a minimal model and choose a homomorphism

$$(\Lambda Q_G, 0) \xrightarrow[\tau]{} (\Lambda Y, d_B)$$

so that $m_B^* \tau^* = \phi^*$. Then

$$\begin{array}{ccc} A(B_G) & \xrightarrow{A(\phi)} & A(B) \\ \Big\uparrow & & \Big\uparrow m_B \\ (\Lambda Q_G, 0) & \xrightarrow[\tau]{} & (\Lambda Y, d_B) \end{array}$$

is a homotopy commutative diagram.

Since B_G is 1-connected and $H^*(G/K)$ has finite type it follows that ξ is a rational fibration. Hence by [**5**, Section 20.5] a Λ-model (not necessarily Λ-minimal) for ξ is given by

$$\begin{array}{ccccc} (A(B), d) & \longrightarrow & (A(E), d) & \longrightarrow & (A(G/K), d) \\ m_B \Big\uparrow \simeq & & \Big\uparrow \simeq & & \Big\uparrow \simeq \\ (\Lambda Y, d_B) & \longrightarrow & (\Lambda Y \otimes \Lambda Q_K \otimes \Lambda P_G, D_\xi) & \longrightarrow & (\Lambda Q_K \otimes \Lambda P_G, \bar{D}), \end{array}$$

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where

$$D_\xi(y \otimes 1 \otimes 1) = d_B y \otimes 1 \otimes 1,\ D_\xi(1 \otimes z \otimes 1) = 0,$$
$$D_\xi(1 \otimes 1 \otimes x) = \tau x \otimes 1 \otimes 1 - 1 \otimes B(\lambda)^* x \otimes 1,$$
$$y \in Y, z \in Q_K, x \in P_G.$$

It follows (cf. [**3**, Proposition VII, Section 3.22]) that the Λ-minimal Λ-model has the form

$$(3.4) \quad (\Lambda Y, d_B) \xrightarrow{i} (\Lambda Y \otimes \Lambda Q \otimes \Lambda P, d) \xrightarrow{\rho} (\Lambda Q \otimes \Lambda P, \bar{d}),$$

where $d(1 \otimes Q \otimes 1) = 0$ and $d(1 \otimes 1 \otimes P) \subset \Lambda Y \otimes \Lambda Q \otimes 1$. In particular, ξ is two stage.

4. The model for G/K. Let K, G be as in Section 3. Specializing equation (3.4) to the case B = point ($Y = 0$) we obtain that the minimal model for G/K has the form $(\Lambda Q \otimes \Lambda P, \bar{d})$, where Q (respectively P) is evenly (respectively oddly) graded, $\bar{d}(Q) = 0$ and $\bar{d}(P) \subset \Lambda Q$. On the other hand, the preceding diagram gives us a (non-minimal) model of the form $(\Lambda Q_K \otimes \Lambda P_G, \bar{D})$ with $\bar{D}(Q_K) = 0$ and $\bar{D}x = -B(\lambda)^* x, x \in P_G$.

With these identifications many of the results in [**3**] go over from real to rational coefficients. We shall recall certain of these here. They will be applied to show that for certain classes of homogeneous spaces as fibre, cohomological equivalence of fibrations implies c-π-equivalence.

First, recall from [**3**] that

$$\chi_\pi(G/K) = \dim \pi_\psi{}^{\text{even}}(G/K) - \dim \pi_\psi{}^{\text{odd}}(G/K) = \dim Q - \dim P.$$

Since $\Lambda Q_K \otimes P_G$ is a model and since $\dim Q_K = rkK$, $\dim P_G = \dim Q_G = rkG$ we have

$$\chi_\pi(G/K) = rkK - rkG.$$

On the other hand, if we interpret ζ^* as a linear map $H^+(G/K) \to P \oplus Q$ we may write it as the sum of two linear maps

$$\zeta_0{}^* : H^{\text{odd}}(G/K) \to P \quad \text{and} \quad \zeta_e{}^* : (H^{\text{even}})^+(G/K) \to Q.$$

Denote their respective kernels and images by $N_0, \hat{P}$ and $N_e, \hat{Q}$. Using [**3**, Theorem II, Chapter 10 and diagram 11.1] we may identify $H^*(G) = \Lambda P_G$ and $\hat{P} = P_G \cap \operatorname{Im} p^*$, where $p : G \to G/K$.

Furthermore [**3**, Theorem V, Chapter 2] we may write

$$(\Lambda Q \otimes \Lambda P, \bar{d}) \cong (\Lambda \hat{P}, 0) \otimes (\Lambda Q \otimes \Lambda \tilde{P}, \bar{d});$$

here $P = \hat{P} \oplus \tilde{P}$. Because $H(\Lambda Q \otimes \Lambda \tilde{P})$ has finite dimension, $\dim \tilde{P} \geqq \dim Q$. If we set def $(G/K) = \dim \tilde{P} - \dim Q$ we have then

$$(4.1) \quad \operatorname{def}(G/K) = \dim \operatorname{Im} \zeta_0{}^* - \chi_\pi(G/K)$$
$$= \operatorname{rk} G - \operatorname{rk} K - \dim (P_G \cap \operatorname{Im} p^*).$$

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The first equation describes def (G/K) as a homotopy invariant; the second in terms of Lie group invariants. It follows from [**3**, Theorem XI, Chapter 3] that G/K is formal if and only if def $(G/K) = 0$.

4.2 PROPOSITION. *Let K be any closed connected subgroup of a connected Lie group G. Then any automorphism, f, of the graded algebra $H^*(G/K)$ embeds in a commutative diagram*

$$\begin{array}{ccc} H^{\text{odd}}(G/K) & \xrightarrow{\zeta_0^*} & P \\ f \Big\downarrow \cong & & \cong \Big\downarrow g \\ H^{\text{odd}}(G/K) & \xrightarrow[\zeta_0^*]{} & P \end{array}$$

(*g is a linear isomorphism of graded spaces.*)

Proof. We need only show that f preserves N_0. Let $\omega \in H^{\text{odd}}(G/K)$, and define a linear map $\delta : H^*(G/K) \to H^*(G/K)$ by $\delta(\beta) = \omega \cdot \beta$. Because ω has odd degree, $\delta^2 = 0$. We shall show that

(4.3) $\quad \omega \notin N_0 \Leftrightarrow H(H^*(G/K), \delta) = 0.$

Clearly this implies that $f(N_0) = N_0$.

If $\omega \notin N_0$ write $\zeta^*\omega = x$, $0 \neq x \in \hat{P}$. Write $\hat{P} = (x) \oplus P_1$; then

$$H^*(G/K) = H(\Lambda Q \otimes \Lambda \tilde{P}) \otimes \Lambda P_1 \otimes \Lambda(x) = A \otimes \Lambda(x).$$

Thus $\omega = a \otimes 1 + 1 \otimes x$, for some $a \in A$, and a simple calculation shows $H(H^*(G/K), \delta) = 0$.

Conversely, suppose $H(H^*(G/K), \delta) = 0$. If deg $\omega = 1$ it is obvious by inspection that $\omega \notin N_0$. Suppose deg $\omega > 1$. Let $u \in \Lambda Q \otimes \Lambda P$ be a cocycle representing ω. Define a c.g.d.a. $(\Lambda Q \otimes \Lambda P \otimes \Lambda v, \nabla)$ as follows: ∇ restricts to $\bar{d}$ in $\Lambda Q \otimes \Lambda P$, deg $v = (\deg \omega) - 1$, $\nabla v = u$. Set

$$F_p = \sum_{j=0}^{p} \Lambda Q \otimes \Lambda P \otimes \Lambda^j v;$$

this filtration defines a spectral sequence with E_1 term $(H^*(G/K) \otimes \Lambda v, \nabla_1)$, where ∇_1 is zero in $H^*(G/K)$ and $\nabla_1 v = \omega$. Thus the E_2 term is given by

$$E_2 = H^*(G/K)/\omega \cdot H^*(G/K) \oplus \sum_{j=1}^{\infty} H(H^*(G/K), \delta) \otimes \Lambda^j v$$
$$= H^*(G/K)/\omega \cdot H^*(G/K).$$

In particular, dim $E_2 < \infty$ and so

$$\dim H(\Lambda Q \otimes \Lambda P \otimes \Lambda v, \nabla) < \infty.$$

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Now consider [**4**, Proposition 1], applied to the c.g.d.a. $(\Lambda Q \otimes \Lambda P \otimes \Lambda v, \nabla_1)$. Conclusion (2) of that proposition is false for v but conclusion (1) holds. The only hypothesis which is possibly unsatisfied is minimality. Thus $(\Lambda Q \otimes \Lambda P \otimes \Lambda v, \nabla)$ cannot be minimal. Since $(\Lambda Q \otimes \Lambda P, \bar{d})$ is minimal we conclude that $0 \neq \zeta \nabla v = \zeta u = \zeta^* \omega$; i.e., $\omega \notin N_0$.

Consider now the following three classes of homogeneous spaces G/K:
(1) def $(G/K) = 0$ (i.e., G/K is formal)
(2) def $(G/K) = 1$
(3) K is a torus.
Note that class (1) contains the symmetric spaces [**3**, Section 11.5] as well as all the examples in [**3**, Chapter 11, Section 4]. In [**3**, Section 11.14] it is shown that $Q(n)/SU(n)$ is in class (2) for $n \geqq 5$.

4.4 Proposition. *Suppose G/K belongs to one of the above classes. Then every automorphism f of the graded algebra $H^*(G/K)$ embeds in a commutative diagram*

$$\begin{array}{ccc} H^+(G/K) & \xrightarrow{\zeta^*} & \pi_\psi^*(G/K) \\ f \big\downarrow \cong & & \cong \big\downarrow g \\ H^+(G/K) & \xrightarrow[\zeta^*]{} & \pi_\psi^*(G/K) \end{array}$$

in which g is an isomorphism of graded spaces.

In particular, for such spaces Theorem I remains valid when c-π-equivalence is replaced by c-equivalence in the statement of (i).

Proof. The proposition is obvious in case (1) (G/K formal) because in that case $f = \alpha^*$ for some automorphism, α, of the model. We now consider cases (2) and (3). It is sufficient to show that $f(N_0) = N_0$ and $f(N_e) = N_e$; the first is already established in Proposition 4.2.

In case (2) write $H^*(G/K) = H(\Lambda Q \otimes \Lambda \tilde{P}) \otimes \Lambda \hat{P}$ with dim $\tilde{P}$ − dim $Q = 1$. By [**3**, Theorem VII, Chapter 2] every cohomology class in $H(\Lambda Q \otimes \Lambda \tilde{P})$ can be represented by a cocycle of the form $\Phi \otimes 1 + \sum \Phi_i \otimes x_i$ with $\Phi, \Phi_i \in \Lambda Q$ and $x_i \in P$. Note that $[\Phi \otimes 1] \in H^{\text{even}}$ and $[\sum \Phi_i \otimes x_i] \in H^{\text{odd}}$. It follows easily that

$$N_e = H^+(G/K) \cdot H^+(G/K) \cap H^{\text{even}}(G/K)$$

and hence is preserved by f.

In case (3) observe that Q_K is concentrated in degree 2, and in fact that $Q_K \cong H^2(G/K)$. It follows that $N_e = \sum_{j>1} H^{2j}(G/K)$ and hence is preserved by f.

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5. Proof of theorem II. Let

$$(\Lambda Y, d_B) \xrightarrow{i_\nu} (\Lambda Y \otimes \Lambda X, d_\nu) \xrightarrow{\rho_\nu} (\Lambda X, d_F)$$

be the Λ-minimal Λ-models of ξ_ν, $\nu = 1, 2$.

(i) We need only prove that c-π-equivalence implies h-equivalence. Let

$$f : H(\Lambda Y \otimes \Lambda X, d_1) \overset{\cong}{\to} H(\Lambda Y \otimes \Lambda X, d_2) \quad \text{and} \quad \eta : X \overset{\cong}{\to} X$$

be isomorphisms such that $fi_1^* = i_2^*$ and $\eta\zeta^*\rho_1^* = \zeta^*\rho_2^*$. (We have written $X = \pi_\psi^*(F)$; cf. Section 2.)

Now set $Z_0 = \operatorname{Im}(\zeta^*\rho_1^*)$. It follows from the hypothesis that ξ_1 is two stage that $X = X_0 \oplus X_1$ as in Definition 2.2; clearly $X_0 \subset Z_0$. Thus we may assume that $X = Z_0 \oplus Z_1$ with $d_1(1 \otimes Z_0) = 0$ and $d_1(1 \otimes Z_1) \subset \Lambda Y \otimes \Lambda Z_0$.

Choose a linear map $\phi : 1 \otimes Z_0 \to (\Lambda Y \otimes \Lambda X) \cap \ker d_2$ such that $[\phi(1 \otimes z)] = f[1 \otimes z]$, $z \in Z_0$. Then extend ϕ to $\Lambda Y \otimes \Lambda Z_0$ by setting $\phi(y \otimes 1) = y \otimes 1$, $y \in Y$. Clearly

$$\phi : (\Lambda Y \otimes \Lambda Z_0, d_1) \to (\Lambda Y \otimes \Lambda X, d_2)$$

is a c.g.d.a. homomorphism, and the diagram

$$\begin{array}{ccc} H(\Lambda Y \otimes \Lambda Z_0) & \longrightarrow & H(\Lambda Y \otimes \Lambda X, d_1) \\ & \searrow^{\phi^*} & \cong \downarrow f \\ & & H(\Lambda Y \otimes \Lambda X, d_2) \end{array} \tag{5.1}$$

commutes.

Now for $z \in Z_1$, $d_1(1 \otimes z)$ is a cocycle in $\Lambda Y \otimes \Lambda Z_0$. The diagram (5.1) shows that $\phi^*[d_1(1 \otimes z)] = 0$. Thus if z_i is a homogeneous basis of Z_1 we can find $u_i \in \Lambda Y \otimes \Lambda X$ so that $d_2 u_i = \phi d_1(1 \otimes z_i)$. Extend ϕ to a c.g.d.a. homomorphism

$$\phi : (\Lambda Y \otimes \Lambda X, d_1) \to (\Lambda Y \otimes \Lambda X, d_2)$$

by setting $\phi z_i = u_i$.

It remains to show that ϕ is an isomorphism. Because ϕ is the identity in ΛY it induces a homomorphism of c.g.d.a.'s $\alpha : \Lambda X \to \Lambda X$ such that $\rho_2\phi = \alpha\rho_1$. Let $Q(\alpha) : X \to X$ be the linear map such that $\zeta\alpha = Q(\alpha)\zeta$. Since ΛY and ΛX are connected, it is clearly sufficient to prove $Q(\alpha)$ is an isomorphism. Since $X(\cong \pi_\psi^*(F))$ is a graded space of finite type we need only prove $Q(\alpha)$ injective.

We show first that if $Q(\alpha)$ becomes injective when restricted to some graded subspace $W \subset X$, then ϕ restricted to $\Lambda Y \otimes \Lambda W$ is also injective.

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Indeed write $X = U \oplus V$, where $U = Q(\alpha)W$ and define a homomorphism $l : \Lambda Y \otimes \Lambda X \to \Lambda Y \otimes \Lambda X$ by

$$l(y \otimes 1) = y \otimes 1, l(1 \otimes u) = \phi(1 \otimes Q(\alpha)^{-1}u),$$
$$l(1 \otimes v) = 1 \otimes v, y \in Y, u \in U, v \in V.$$

Then $\zeta\rho_2 l = \zeta\rho_2$ and it follows, as above, that l is an isomorphism. Since l coincides with ϕ in $\Lambda Y \otimes \Lambda W$ we conclude that ϕ is indeed injective in this subalgebra.

Next observe that for $z \in Z_0$

$$Q(\alpha)z = \zeta\rho_2\phi(1 \otimes z) = \zeta^*\rho_2^*[\phi(1 \otimes z)] = \zeta^*\rho_1^*f[1 \otimes z] = \eta z.$$

Thus $Q(\alpha)$ is injective in Z_0. Assume it is injective in X^q for $q < p$ and suppose for some $x \in X^p$ that $Q(\alpha)x = 0$. This implies that

$$\phi(1 \otimes x) \in \Lambda^+ Y \otimes \Lambda X + 1 \otimes \Lambda^+ X \cdot \Lambda^+ X.$$

Since $\Lambda Y \otimes \Lambda X$ is connected we obtain

$$\phi(1 \otimes x) \in \Lambda Y \otimes \Lambda(X^{<p}).$$

Now ϕ maps $\Lambda Y \otimes \Lambda(X^{<p})$ into $\Lambda Y \otimes \Lambda(X^{<p})$. Since by our hypothesis $Q(\alpha)$ is injective in $X^{<p}$ the argument given above (with $X^{<p}$ replacing X) shows that it is an automorphism of $\Lambda Y \otimes \Lambda(X^{<p})$. Thus for some $\Phi \in \Lambda Y \otimes \Lambda(X^{<p})$, $\phi(1 \otimes x + \Phi) = 0$, and so $\phi d_1(1 \otimes x + \phi) = 0$.

But

$$d_1(1 \otimes x + \Phi) \in d_1(1 \otimes x + \Lambda Y \otimes \Lambda(X^{<p}))$$
$$\subset \Lambda Y \otimes \Lambda(Z_0 + X^{<p}).$$

Moreover, since $Q(\alpha)$ is injective in Z_0 and in $X^{<p}$, it is injective in $Z_0 + X^{<p}$. As we observed above, this implies that ϕ is injective in $\Lambda Y \otimes \Lambda(Z_0 + X^{<p})$. In particular $d_1(1 \otimes x + \Phi) = 0$. Clearly

$$\zeta^*\rho_1^*[1 \otimes x + \Phi] = x.$$

We find, then, that

$$x \in \operatorname{Im} \zeta^*\rho_1^* \cap \ker Q(\alpha) = Z_0 \cap \ker Q(\alpha) = 0.$$

Thus $Q(\alpha)$ is injective in X^p.

This completes the proof of (i).

(ii) Here we assume ξ_1 and ξ_2 are strictly c-equivalent and prove they are strictly h-equivalent. Again write $X = X_0 \oplus X_1$ with $d_1(1 \otimes X_0) = 0$ and $d_1(1 \otimes X_1) \subset \Lambda Y \otimes \Lambda X_0$. By hypothesis we have an isomorphism

$$f: H(\Lambda Y \otimes \Lambda X, d_1) \xrightarrow{\cong} H(\Lambda Y \otimes \Lambda X, d_2)$$

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such that $f \circ i_1^* = i_2^*$ and $\rho_2^* \circ f = \rho_1^*$. Moreover ρ_1^* is surjective; hence so is ρ_2^*.

Choose a homogeneous basis z_i for X_0 and choose d_2-cocycles $w_i \in \Lambda Y \otimes \Lambda X$ so that $f[1 \otimes z_i] = [w_i]$. Then $\rho_2^*[w_i] = [z_i]$; i.e., $z_i - \rho_2 w_i = d_F \Psi_i$. Choose $\Omega_i \in \Lambda Y \otimes \Lambda X$ so that $\rho_2 \Omega_i = \Psi_i$.

Extend the identity in ΛY to a c.g.d.a. homomorphism

$$\phi : (\Lambda Y \otimes \Lambda X_0, d_1) \rightarrow (\Lambda Y \otimes \Lambda X, d_2)$$

by putting $\phi z_i = w_i + d_2 \Omega_i$. Let x_i be a homogeneous basis of X_1. Then $d_1(1 \otimes x_i) \in \Lambda Y \otimes \Lambda X_0$, and exactly as in (i) we can find $u_i \in \Lambda Y \otimes \Lambda X$ so that $d_2 u_i = \phi d_1(1 \otimes x_i)$.

Because of our construction we have

$$\rho_2 \phi = \rho_1 : \Lambda Y \otimes \Lambda X_0 \rightarrow X_0.$$

Thus applying ρ_2 to the last equation we find that

$$d_F(\rho_2 u_i - x_i) = 0.$$

Because ρ_2 and ρ_2^* are surjective there are d_2-cocycles $\Phi_i \in \Lambda Y \otimes \Lambda X$ such that $\rho_2 \phi_i = x_i - \rho_2 u_i$. Extend ϕ to a c.g.d.a. homomorphism

$$\phi : (\Lambda Y \otimes \Lambda X, d_1) \rightarrow (\Lambda Y \otimes \Lambda X, d_2)$$

by putting $\phi(1 \otimes x_i) = \Phi_i + u_i$. Then $\rho_2 \phi = \rho_1$ continues to hold. It follows that ϕ is an isomorphism.

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