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TRIVIAL ZEROS OF P-ADIC L-FUNCTIONS AT NEAR-CENTRAL POINTS

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Abstract Using the ℓ-invariant constructed in our previous paper we prove a Mazur–Tate–Teitelbaum-style formula for derivatives of p-adic L-functions of modular forms at trivial zeros. The novelty of this result is to cover the near-central point case. In the central point case our formula coincides with the Mazur–Tate–Teitelbaum conjecture proved by Greenberg and Stevens and by Kato, Kurihara and Tsuji at the end of the 1990s.

Keywords: modular form; p-adic representation; (φ, Γ)-module; p-adic L-function

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0. Introduction

0.1. Trivial zeros of modular forms

In this paper we prove a Mazur–Tate–Teitelbaum-style formula for the values of derivatives of p-adic L-functions of modular forms at near-central points. Together with the results of Kato, Kurihara and Tsuji and of Greenberg and Stevens on the Mazur–Tate–Teitelbaum conjecture this gives a complete proof of the trivial zero conjecture formulated in [7] for elliptic modular forms. Namely, let $f = \sum_{n=1}^{\infty} a_n q^n$ be a normalized newform on $\Gamma_0(N)$ of weight $k \geqslant 2$ and character ε and let $L(f, s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be the complex L-function associated with f. It is well known that $L(f, s)$ converges for $\mathrm{Re}(s) > \frac{k+1}{2}$ and decomposes into an Euler product

$$L(f, s) = \prod_{l} E_l(f, l^{-s})^{-1}$$

where l runs over all primes and $E_l(f, X) = 1 - a_l X + \varepsilon(l) l^{k-1} X^2$. Moreover $L(f, s)$ has an analytic continuation on the whole complex plane and satisfies the functional equation

$$(2\pi)^{-s}\, \Gamma(s)\, L(f, s) = i^k c N^{k/2-s} (2\pi)^{s-k}\, \Gamma(k-s)\, L(f^*, k-s)$$

where $f^* = \sum_{n=1}^{\infty} \bar{a}_n q^n$ is the dual cusp form and c is some constant (see for example [56, Theorems 4.3.12 and 4.6.15]). More generally, with any Dirichlet character η we can associate the L-function

$$L(f,\eta,s)=\sum_{n=1}^{\infty}\frac{\eta(n)\,a_n}{n^s}.$$

The theory of modular symbols implies that there exist non-zero complex numbers Ω_f^+ and Ω_f^- such that for any Dirichlet character η one has

$$\widetilde{L}(f,\eta,j)=\frac{\Gamma(j)}{(2\pi i)^{j-1}\,\Omega_f^{\pm}}\,L(f,\eta,j)\in\overline{\mathbb{Q}},\quad 1\leqslant j\leqslant k-1, \tag{1}$$

where $\pm=(-1)^{j-1}\eta(-1)$. Fix a prime number $p>2$ such that the Euler factor $E_p(f,X)$ is not equal to 1. Let α be a root of the polynomial $X^2-a_pX+\varepsilon(p)p^{k-1}$ in $\overline{\mathbb{Q}}_p$. Denote by v_p the p-adic valuation on $\overline{\mathbb{Q}}_p$ normalized such that $v_p(p)=1$. Assume that α is not critical, i.e. that $v_p(\alpha)<k-1$. Let $\omega:(\mathbb{Z}/p\mathbb{Z})^*\to\mathbb{Q}_p^*$ denote the Teichmüller character. Manin [53], Vishik [76] and independently Amice and Vélu [1] constructed analytic p-adic L-functions $L_{p,\alpha}(f,\omega^m,s)$ which interpolate algebraic parts of special values of $L(f,s)$.[1] Namely, the interpolation property reads

$$L_{p,\alpha}(f,\omega^m,j)=\mathcal{E}_\alpha(f,\omega^m,j)\,\widetilde{L}(f,\omega^{j-m},j),\quad 1\leqslant j\leqslant k-1$$

where $\mathcal{E}_\alpha(f,\omega^m,j)$ is an explicit Euler-like factor. One says that $L_{p,\alpha}(f,\omega^m,s)$ has a trivial zero at $s=j$ if $\mathcal{E}_\alpha(f,\omega^m,j)=0$ and $\widetilde{L}(f,\omega^{j-m},j)\neq 0$. This phenomenon was first studied by Mazur *et al.* in [55] where the following cases were distinguished:

- *The semistable case*: $p\parallel N$, k *is even and* $\alpha=a_p=p^{k/2-1}$. The p-adic L-function $L_{p,\alpha}(f,\omega^{k/2},s)$ has a trivial zero at the central point $s=k/2$;
- *The crystalline case*: $p\nmid N$, k *is odd and either* $\alpha=p^{\frac{k-1}{2}}$ *or* $\alpha=\varepsilon(p)\,p^{\frac{k-1}{2}}$. The p-adic L-function $L_{p,\alpha}(f,\omega^{\frac{k+1}{2}},s)$ (respectively $L_{p,\alpha}(f,\omega^{\frac{k-1}{2}},s)$) has a trivial zero at the near-central point $s=\frac{k+1}{2}$ (respectively $s=\frac{k-1}{2}$).
- *The potentially crystalline case*: $p|N$, k *is odd and* $\alpha=a_p=p^{\frac{k-1}{2}}$. The p-adic L-function $L_{p,\alpha}(f,\omega^{\frac{k+1}{2}},s)$ has a trivial zero at the near-central point $s=\frac{k+1}{2}$.

Let

$$\rho_f:\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to\mathrm{GL}(W_f)$$

be the p-adic Galois representation associated with f by Deligne [26]. We remark that in the semistable (resp. crystalline, potentially crystalline) case the restriction of ρ_f to the decomposition group at p is semistable non-crystalline (resp. crystalline, potentially crystalline) in the sense of Fontaine [33].

0.2. The semistable case

Assume that k is even, $p\parallel N$ and $a_p=p^{k/2-1}$. Then the associated filtered (φ,N)-module $\mathbf{D}_{\mathrm{st}}(W_f)$ has a basis e_α,e_β such that $e_\alpha=Ne_\beta$, $\varphi(e_\alpha)=a_pe_\alpha$ and $\varphi(e_\beta)=p\,a_pe_\beta$.

[1] This construction was recently generalized to the critical case by Pollack and Stevens [69] and by Bellaiche [3].

The jumps of the canonical decreasing filtration of $\mathbf{D}_{\mathrm{st}}(W_f)$ are 0 and $k-1$ and the $\mathscr{L}$-invariant of Fontaine and Mazur is defined to be the unique element $\mathscr{L}_{\mathrm{FM}}(f)\in\overline{\mathbb{Q}}_p$ such that $\mathrm{Fil}^{k-1}\mathbf{D}_{\mathrm{st}}(V_f)$ is generated by $e_\beta+\mathscr{L}_{\mathrm{FM}}(f)e_\alpha$. In [55] Mazur *et al.* conjectured that

$$L'_{p,\alpha}(f,\omega^{k/2},k/2)=\mathscr{L}_{\mathrm{FM}}(f)\,\widetilde{L}(f,k/2). \tag{2}$$

We remark that $L(f,k/2)$ can vanish. The conjecture (2) was proved in [37] in the weight 2 case and in [74] in general using the theory of p-adic families of modular forms. Another proof, based on the theory of Euler systems was found by Kato, Kurihara and Tsuji (unpublished, but see [46, 67, 21]). Note that in [74], Stevens uses another definition of the $\mathscr{L}$-invariant proposed by Coleman [17]. We refer the reader to [18] and to the survey article [22] for further information and references.

0.3. The general case

Our main aim in this paper is to prove an analogue of the formula (2) in the crystalline and potentially crystalline cases. In fact, we will treat all three cases simultaneously. Let f be a newform of weight k. Fix an odd prime p and assume that the p-adic L-function $L_{p,\alpha}(f,\omega^{k_0},s)$ has a trivial zero at $s=k_0$. Assume further that $k_0\geqslant k/2$. Note that the last assumption holds automatically in the semistable and potentially crystalline cases and in the crystalline case it is not restrictive because we can use the functional equation (see remark (3) below). Let $\mathbf{D}_{\mathrm{cris}}(W_f)$ denote the crystalline module associated with W_f. By Saito's theorem [73] the vanishing of the Euler-like factor $\mathcal{E}_\alpha(f,\omega^{k_0},k_0)$ implies that $\mathbf{D}_{\mathrm{cris}}\left(W_f(k_0)\right)^{\varphi=p^{-1}}$ is a one-dimensional vector space which we denote by D_α (see Lemma 4.2.2 below). The main construction of [7] associates with D_α an element $\ell\left(W_f(k_0),D_\alpha\right)\in\overline{\mathbb{Q}}_p$ which can be viewed as a direct generalization of Greenberg's ℓ-invariant [35] to the non-ordinary case.[2] To simplify notation we set $\ell_\alpha(f)=\ell\left(W_f(k_0),D_\alpha\right)$. The main result of this paper can be stated as follows.

Theorem. *Let f be a newform on $\Gamma_0(N)$ of character ε and weight k and let p be an odd prime. Assume that the p-adic L-function $L_{p,\alpha}\left(f,\omega^{k_0},s\right)$ has a trivial zero at $s=k_0\geqslant k/2$. Then*

$$L'_{p,\alpha}\left(f,\omega^{k_0},k_0\right)=\ell_\alpha(f)\left(1-\frac{\varepsilon(p)}{p}\right)\widetilde{L}\left(f,k_0\right).$$

Remarks. (1) In the semistable case $\ell_\alpha(f)=\mathscr{L}_{\mathrm{FM}}(f)$ (see [7, Proposition 2.3.7]), $\varepsilon(p)=0$ and we recover the Mazur–Tate–Teitelbaum conjecture. Our proof in this case can be seen as an interpretation of Kato, Kurihara and Tsuji's approach in terms of (φ,Γ)-modules. For the crystalline case some version of our formula was proved by Orton [61] (unpublished). She does not use (φ,Γ)-modules and works

[2] Strictly speaking, in [7] we define the ℓ-invariant for p-adic representations which are semistable at p. In our setting this covers the semistable and crystalline cases, but the definition can be extended to include the potentially crystalline case too (see § 2.1 below).

with an *ad hoc* definition of the ℓ-invariant in terms of the Bloch–Kato exponential map in the spirit of [66].

(2) In the crystalline and potentially crystalline cases $\widetilde{L}(f,k_0)$ does not vanish, by the theorem of Jacquet and Shalika [44].

(3) In the crystalline case, trivial zeros at the symmetric point $s=\frac{k-1}{2}$ can be easily studied using the functional equation for p-adic L-functions [55, §17]. If $\alpha=p^{(k-1)/2}$ and therefore $L_{p,\alpha}(f,\omega^{\frac{k+1}{2}},s)$ has a trivial zero at $s=\frac{k+1}{2}$, then $\alpha^*=\varepsilon^{-1}(p)\,\alpha$ is a root of the Hecke polynomial associated with the dual form $f^*=\sum_{n=1}^{\infty}\bar{a}_nq^n$ and $L_{p,\alpha^*}(f^*,\omega^{\frac{k-1}{2}},s)$ has a trivial zero at $s=\frac{k-1}{2}$. Using the compatibility of the trivial zero conjecture with the functional equation [7, §2.3.5] (or just repeating the proof of the main theorem with obvious modifications), we obtain a trivial zero formula for $L'_{p,\alpha^*}\left(f^*,\omega^{\frac{k-1}{2}},\frac{k-1}{2}\right)$.

(4) The $\mathscr{L}$-invariant of Fontaine and Mazur which appears in semistable case (2) is local, i.e. it depends only on the restriction of the p-adic representation ρ_f to the decomposition group at p. However, in the crystalline and potentially crystalline cases our ℓ-invariant is global and contains information about the localization map $H^1(\mathbb{Q},W_f(\frac{k+1}{2}))\to H^1(\mathbb{Q}_p,W_f(\frac{k+1}{2}))$. We remark that in the semistable case the p-adic L-function has a zero at the central point and in the crystalline and potentially crystalline cases it has a zero at a near-central point.

(5) Let η be a Dirichlet character of conductor M with $(p,M)=1$. The study of trivial zeros of $L_{p,\alpha}\left(f,\eta\omega^{\frac{k+1}{2}},s\right)$ reduces to our situation on considering the newform $f\otimes\eta$ associated with $f_\eta=\sum_{n=1}^{\infty}\eta(n)\,a_nq^n$ (see § 4.1.3).

Our theorem follows from a formula for the derivative of the Perrin-Riou exponential map [63] in terms of the ℓ-invariant which we prove in Propositions 1.3.7 and 2.2.2 below, applied to the Euler system constructed by Kato [46].

0.4. Trivial zeros of Dirichlet L-functions

Let η be a primitive Dirichlet character modulo N and let $p\nmid N$ be a fixed prime. The p-adic L-function of Kubota and Leopoldt, $L_p(\eta\omega,s)$, satisfies the interpolation property

$$L_p(\eta\omega,-j)=(1-(\eta\omega^{-j})(p)\,p^j)\,L(\eta\omega^{-j},-j),\quad j\geqslant 0.$$

Assume that η is odd and $\eta(p)=1$. Then $L(\eta,0)\neq 0$ but the Euler-like factor $1-(\eta\omega^{-j})\,(p)p^j$ vanishes at $j=0$ and $L_p(\eta\omega,s)$ has a trivial zero at $s=0$. Fix a finite extension $L/\mathbb{Q}_p$ containing the values of η. Let χ denote the cyclotomic character and let $\mathrm{ord}_p:\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)\to L$ be the character defined by $\mathrm{ord}_p(\mathrm{Fr}_p)=-1$ where Fr_p is the geometric Frobenius. Then $H^1(\mathbb{Q}_p,L)=\mathrm{Hom}(\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p),L)$ is the two-dimensional L-vector space generated by $\log\chi$ and ord_p. Since $p\nmid N$ and $\eta(p)=1$ the restriction of $L(\eta)$ to the decomposition group at p is a trivial representation. The localization map

$$\kappa_\eta:H^1(\mathbb{Q},L(\eta))\to H^1(\mathbb{Q}_p,L)$$

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is injective and identifies $H^1(\mathbb{Q}, L(\eta))$ with a one-dimensional subspace of $H^1(\mathbb{Q}_p, L)$. It can be shown that $\operatorname{Im}(\kappa_\eta)$ is generated by an element of the form

$$\log\chi + \mathscr{L}(\eta)\operatorname{ord}_p \tag{3}$$

where $\mathscr{L}(\eta) \in L$ is necessarily unique. Applying Proposition 2.2.2 to the Euler system of cyclotomic units we obtain a new proof of the trivial zero conjecture for Dirichlet L-functions

$$L_p'(\eta\omega, 0) = -\mathscr{L}(\eta)\, L(\eta, 0). \tag{4}$$

This formula was first proved in [38] as the combination of the result of Ferrero and Greenberg [28] giving an explicit formula for $L_p'(\eta\omega, 0)$ in terms of the p-adic Γ-function and the Gross–Koblitz formula [39]. We also remark that Dasgupta *et al.* [25] recently generalized (4) to totally real number fields F assuming Leopoldt's conjecture and some additional condition on the vanishing of p-adic L-functions.

0.5. The plan of the paper

The main content of this article is as follows. In § 1 we review the necessary preliminaries. Sections 1.1 and 1.2 are devoted to the theory of (φ, Γ)-modules which plays a key role in our definition of the ℓ-invariant. In particular, in § 1.2.4 we define the Bloch–Kato exponential map for potentially semistable (φ, Γ)-modules. An alternative but equivalent definition can be found in [57, 71]. In § 1.3 we review the construction and main properties of Perrin-Riou's large exponential map and give an explicit formula for its derivative. The ℓ-invariant is introduced and studied in § 2. In § 2.1 we review and slightly generalize the construction of $\ell(V, D)$ from [7]. In § 2.2 we prove an explicit formula for the derivative of the large logarithmic map in terms of $\ell(V, D)$ and the dual exponential map. The rest of the paper is devoted to applications of this result. In § 3 we consider Dirichlet L-functions and give a new proof of (4). Trivial zeros of modular forms are studied in § 4.

1. Preliminaries

1.1. (φ, Γ)-modules

1.1.1. Definition of (φ, Γ)-modules (see [31, 15, 23]). Let $\overline{\mathbb{Q}}_p$ be a fixed algebraic closure of $\mathbb{Q}_p$. We denote by C the p-adic completion of $\overline{\mathbb{Q}}_p$ and by $v_p : C \to \mathbb{R} \cup \{\infty\}$ the p-adic valuation normalized so that $v_p(p) = 1$ and set $|x|_p = \left(\frac{1}{p}\right)^{v_p(x)}$. Write $B(r, 1)$ for the p-adic annulus $B(r, 1) = \{x \in C \mid r \leqslant |x|_p < 1\}$. Fix a system of primitive roots of unity $\varepsilon = (\zeta_{p^n})_{n\geqslant 0}$ such that $\zeta_{p^n}^p = \zeta_{p^{n-1}}$ for all n. If K is a finite extension of $\mathbb{Q}_p$ we set $G_K = \operatorname{Gal}(\overline{\mathbb{Q}}_p/K)$, $K_n = K(\zeta_{p^n})$ and $K_\infty = \bigcup_{n=0}^{\infty} K_n$. Put $\Gamma_K = \operatorname{Gal}(K_\infty/K)$ and denote by $\chi : \Gamma_K \to \mathbb{Z}_p^*$ the cyclotomic character. We write O_K for the ring of integers of K, K_0 for the maximal unramified subextension of K and σ for the absolute Frobenius of $K_0/\mathbb{Q}_p$.

For any $r > 0$ let $\mathbf{B}_K^{\dagger,r}$ denote the ring of overconvergent elements of Fontaine's ring $\mathbf{B}_K$ (see [15, 9]). Note that $\mathbf{B}_K^{\dagger,r_1} \subset \mathbf{B}_K^{\dagger,r_2}$ if $r_1 \leqslant r_2$. The ring $\mathbf{B}_K$ is equipped with a continuous action of Γ_K and a Frobenius operator φ which commute with each other.

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We remark that the $\mathbf{B}_K^{\dagger,r}$ are stable under the action of Γ_K and that $\varphi(\mathbf{B}_K^{\dagger,r}) \subset \mathbf{B}_K^{\dagger,pr}$ for all r. The following description of $\mathbf{B}_K^{\dagger,r}$ is sufficient for the goals of this paper. Let F denote the maximal unramified subextension of K_∞/K_0 and let $e = [K_\infty : K_0(\zeta_{p^\infty})]$. For any $0 \leqslant s < 1$ define

$$\mathscr{R}^{(s)}(K) = \left\{ f(X_K) = \sum_{k\in\mathbb{Z}} a_k X_K^k \mid a_k \in F \text{ and } f \text{ is holomorphic on } B(s,1) \right\},$$

$$\mathscr{E}^{(s)}(K) = \left\{ f(X_K) = \sum_{k\in\mathbb{Z}} a_k X_K^k \mid a_k \in F \text{ and } f \text{ is holomorphic and bounded on } B(s,1) \right\}.$$

Then there exists $r(K) > 0$ such that for all $r > r(K)$ the ring $\mathbf{B}_K^{\dagger,r}$ is isomorphic to $\mathscr{E}^{(p^{-1/er})}(K)$. The union $\mathbf{B}_K^{\dagger} = \cup_{r>0}\mathbf{B}_K^{\dagger,r}$ is a field which is stable under the actions of Γ_K and φ and is isomorphic to $\mathscr{E}^{\dagger}(K) = \cup_{0\leqslant s<1}\mathscr{E}^{(s)}(K)$. In general, the action of Γ_K and φ on $\mathscr{E}^{\dagger}(K)$ is quite complicated but the ring $\mathbf{B}_K^{\dagger,0}$ contains an element X such that

$$\varphi(X) = (1+X)^p - 1, \quad \tau(X) = (1+X)^{\chi(\tau)} - 1, \quad \tau \in \Gamma_K.$$

If $K = K_0$, i.e. K is absolutely unramified, then $F = K_0$ and one can take $X_{K_0} = X$. The action of Γ_{K_0} and φ on $\mathscr{E}^{(s)}(K_0)$ is given by

$$\varphi f(X) = f^{\sigma}(\varphi(X)), \quad \tau f(X) = f(\tau(X)), \quad \tau \in \Gamma_{K_0}.$$

We come back to the general case. The field $\mathscr{E}^{\dagger}(K)$ is endowed with the valuation

$$w\left(\sum_{k\in\mathbb{Z}} a_k X^k\right) = \min\{v_p(a_k) | k \in \mathbb{Z}\}$$

and we denote by $\mathcal{O}_{\mathscr{E}^{\dagger}(K)}$ its ring of integers. The operator φ has a left inverse given by

$$\psi(f) = \frac{1}{p}\varphi^{-1}\left(\mathrm{Tr}_{\mathscr{E}^{\dagger}(K)/\varphi(\mathscr{E}^{\dagger}(K))}(f)\right).$$

If $K = K_0$ we can also write

$$\psi(f(X)) = \frac{1}{p}\varphi^{-1}\left(\sum_{\zeta^p=1} f(\zeta(1+X)-1)\right).$$

Set $\mathscr{R}(K) = \cup_{0\leqslant s<1}\mathscr{R}^{(s)}(K)$. The actions of Γ_K, φ and ψ can be extended to $\mathscr{R}(K)$ by continuity. If $K \subset K'$ then the natural inclusions $\mathbf{B}_K^{\dagger,r} \subset \mathbf{B}_{K'}^{\dagger,r}$ induce embeddings $\mathscr{E}^{\dagger}(K) \subset \mathscr{E}^{\dagger}(K')$ and $\mathscr{R}(K) \subset \mathscr{R}(K')$. Let $t = \log(1+X) = \sum_{k=1}^{\infty}(-1)^{k+1}X^k/k \in \mathscr{R}(\mathbb{Q}_p)$. Note that $\varphi(t) = pt$ and $\tau(t) = \chi(\tau)t$ for all $\tau \in \Gamma_K$.

In this paper we deal with p-adic representations with coefficients in a finite extension L of $\mathbb{Q}_p$. By this reason it is convenient to set $\mathscr{E}_L^{\dagger}(K) = \mathscr{E}^{\dagger}(K)\otimes_{\mathbb{Q}_p}L$, $\mathscr{R}_L^{\dagger}(K) = \mathscr{R}^{\dagger}(K)\otimes_{\mathbb{Q}_p}L$ and $\mathcal{O}_{\mathscr{E}_L^{\dagger}(K)} = \mathcal{O}_{\mathscr{E}^{\dagger}(K)}\otimes_{\mathbb{Z}_p}O_L$.

Definition. (i) A (φ,Γ_K)-module over $\mathscr{E}_L^{\dagger}(K)$ (resp. $\mathscr{R}_L(K)$) is a free $\mathscr{E}_L^{\dagger}(K)$-module (resp. $\mathscr{R}_L(K)$-module) $\mathbf{D}$ of finite rank d equipped with semilinear actions of Γ_K

and φ which commute with each other and such that the induced linear map $\mathscr{E}_L^{\dagger}(K)\otimes_{\varphi}\mathbf{D}\to\mathbf{D}$ (resp. $\mathscr{R}_L(K)\otimes_{\varphi}\mathbf{D}\to\mathbf{D}$) is an isomorphism.

(ii) A (φ,Γ_K)-module $\mathbf{D}$ over $\mathscr{E}_L^{\dagger}(K)$ is said to be étale if there exists a basis of $\mathbf{D}$ such that the matrix of φ in this basis is in $\mathrm{GL}_d(\mathcal{O}_{\mathscr{E}_L^{\dagger}(K)})$.

If $\mathbf{D}$ is a (φ,Γ_K)-module over $A=\mathscr{E}_L^{\dagger}(K)$ or $\mathscr{R}_L(K)$ we write $\mathbf{D}^*$ for the dual module $\mathrm{Hom}_A(\mathbf{D},A)$ and $\mathbf{D}(\chi)$ for the module obtained from $\mathbf{D}$ by twisting the action of Γ_K by the cyclotomic character.

Let $\mathbf{Rep}_L(G_K)$ be the category of p-adic representations of G_K with coefficients in L, i.e. the category of finite dimensional L-vector spaces equipped with a continuous linear action of G_K.

Theorem 1.1.2 ([31, 15]). *There exists a natural functor $V\to\mathbf{D}^{\dagger}(V)$ which induces an equivalence between $\mathbf{Rep}_L(G_K)$ and the category of étale (φ,Γ_K)-modules over $\mathscr{E}_L^{\dagger}(K)$.*

From Kedlaya's theory it follows [23, Proposition 1.4 and Corollary 1.5] that the functor $\mathbf{D}\to\mathscr{R}_L(K)\otimes_{\mathscr{E}_L^{\dagger}(K)}\mathbf{D}$ establishes an equivalence between the category of étale (φ,Γ_K)-modules over $\mathscr{E}_L^{\dagger}(K)$ and the category of (φ,Γ_K)-modules over $\mathscr{R}_L(K)$ of slope 0 in the sense of [48]. Together with Theorem 1.1.2 this implies that the functor $V\to\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ defined by $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)=\mathscr{R}_L(K)\otimes_{\mathscr{E}_L^{\dagger}(K)}\mathbf{D}^{\dagger}(V)$ induces an equivalence between the category of p-adic representations and the category of (φ,Γ_K)-modules over $\mathscr{R}_L(K)$ of slope 0.

1.1.3. Crystalline and semistable (φ,Γ_K)-modules (see [33, 11, 12]). Recall that a filtered (φ,N)-module over K with coefficients in L is a free $K_0\otimes_{\mathbb{Q}_p}L$-vector space M equipped with the following structures:

- a σ-semilinear isomorphism $\varphi: M\to M$ (σ acts trivially on L);
- a $K_0\otimes_{\mathbb{Q}_p}L$-linear nilpotent operator N such that $N\varphi=p\varphi N$;
- an exhaustive decreasing filtration $(\mathrm{Fil}^iM_K)_{i\in\mathbb{Z}}$ on $M_K=K\otimes_{K_0}M$ by $(K\otimes_{\mathbb{Q}_p}L)$-submodules.

If K'/K is a finite Galois extension with Galois group $G_{K'/K}$, then a filtered $(\varphi,N,G_{K'/K})$-module is a filtered (φ,N)-module M over K' equipped with a semilinear action of $G_{K'/K}$ such that the filtration $\mathrm{Fil}^iM_{K'}$ is $G_{K'/K}$-stable. We say that M is a filtered (φ,N,G_K)-module if it is a filtered $(\varphi,N,G_{K'/K})$-module for some K'/K. It is well known (see for example [33]) that filtered (φ,N,G_K)-modules form a tensor category $\mathbf{MF}_K^{\varphi,N,G_K}$ which is additive, and has kernels and cokernels but is not abelian. The unit object $\mathbf{1}$ of $\mathbf{MF}_K^{\varphi,N,G_K}$ is the module $K_0\otimes_{\mathbb{Q}_p}L$ with the natural action of φ and the filtration given by

$$\mathrm{Fil}^i\mathbf{1}_K=\begin{cases}K\otimes_{\mathbb{Q}_p}L, & \text{if } i\leqslant 0,\\ 0, & \text{if } i>0.\end{cases}$$

A filtered (φ,N)-module can be viewed as a filtered (φ,N,G_K)-module with the trivial action of G_K and we denote by $\mathbf{MF}_K^{\varphi,N}$ the resulting subcategory. A filtered Dieudonné

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module is an object M of $\mathbf{MF}_K^{\varphi,N}$ such that $N=0$ on M. Filtered Dieudonné modules form a full subcategory $\mathbf{MF}_K^{\varphi}$ of $\mathbf{MF}_K^{\varphi,N}$.

If M is a filtered (φ,N,G_K)-module of rank 1 and m is a basis vector of M, then $\varphi(m)=\alpha\, m$ for some $\alpha\in L$. Set $t_N(M)=v_p(\alpha)$ and denote by $t_H(M)$ the unique jump in the filtration of M. If M has rank $d\geqslant 1$, set $t_N(M)=t_N(\wedge^d M)$ and $t_H(M)=t_H(\wedge^d M)$. A filtered (φ,N,G_K)-module M is said to be weakly admissible if $t_H(M)=t_N(M)$ and $t_H(M')\leqslant t_N(M')$ for any (φ,N,G_K)-submodule M' of M. Weakly admissible modules form a subcategory of $\mathbf{MF}_K^{\varphi,N,G_K}$ which we denote by $\mathbf{MF}_{K,f}^{\varphi,N,G_K}$.

Let $\mathscr{R}_{L,\log}(K)=\mathscr{R}_L(K)[\log X]$ where $\log X$ is transcendental over $\mathscr{R}_L(K)$ and

$$\tau(\log X)=\log X+\log\left(\tau(X)/X\right),\quad \tau\in\Gamma_K,\quad \varphi(\log X)=p\log X+\log\left(\varphi(X)/X^p\right).$$

Define a monodromy operator $N:\mathscr{R}_{L,\log}(K)\to\mathscr{R}_{L,\log}(K)$ by $N=-\left(1-\frac{1}{p}\right)^{-1}\frac{d}{d\log X}$. For any (φ,Γ_K)-module $\mathbf{D}$ over $\mathscr{R}_L(K)$ we set

$$\mathscr{D}_{\mathrm{cris}}(\mathbf{D})=\left(\mathbf{D}\otimes_{\mathscr{R}_L(K)}\mathscr{R}_L(K)[1/t]\right)^{\Gamma_K},\quad \mathscr{D}_{\mathrm{st}}(\mathbf{D})=\left(\mathbf{D}\otimes_{\mathscr{R}_L(K)}\mathscr{R}_{L,\log}(K)[1/t]\right)^{\Gamma_K}.$$

Then $\mathscr{D}_{\mathrm{cris}}(\mathbf{D})$ (resp. $\mathscr{D}_{\mathrm{st}}(\mathbf{D})$) is a $K\otimes_{\mathbb{Q}_p}L$-module of finite rank equipped with a natural action of φ (resp. with natural actions of φ and N). There exists a compatible system of embeddings $\varphi^{-m}:\mathscr{R}_L(K)^{(r)}\to(L\otimes K_\infty)[[t]]$ which allows one to define exhaustive decreasing filtrations on $\mathscr{D}_{\mathrm{cris}}(\mathbf{D})_K$ and $\mathscr{D}_{\mathrm{st}}(\mathbf{D})_K$ (see [12, proof of Theorem III.2.3]). Moreover $\mathscr{D}_{\mathrm{cris}}(\mathbf{D})=\mathscr{D}_{\mathrm{st}}(\mathbf{D})^{N=0}$ and

$$\mathrm{rg}_{L\otimes K_0}(\mathscr{D}_{\mathrm{cris}}(\mathbf{D}))\leqslant\mathrm{rg}_{L\otimes K_0}(\mathscr{D}_{\mathrm{st}}(\mathbf{D}))\leqslant\mathrm{rg}_{\mathscr{R}_L(K)}(\mathbf{D}).$$

We say that $\mathbf{D}$ is crystalline (resp. semistable) if $\mathrm{rg}_{L\otimes K_0}(\mathscr{D}_{\mathrm{cris}}(\mathbf{D}))=\mathrm{rg}_{\mathscr{R}_L(K)}(\mathbf{D})$ (resp. $\mathrm{rg}_{L\otimes K_0}(\mathscr{D}_{\mathrm{st}}(\mathbf{D}))=\mathrm{rg}_{\mathscr{R}_L(K)}(\mathbf{D})$).

If K'/K is a finite extension, then $\mathscr{R}_L(K)\subset\mathscr{R}_L(K')$ and we set $\mathbf{D}_{K'}=\mathscr{R}_L(K')\otimes_{\mathscr{R}_L(K)}\mathbf{D}$. The direct limit

$$\mathscr{D}_{\mathrm{pst}}(\mathbf{D})=\varinjlim_{K'/K}\mathscr{D}_{\mathrm{st}/K'}(\mathbf{D}_{K'})$$

is a $K_0^{ur}\otimes_{\mathbb{Q}_p}L$-module of finite rank equipped with a discrete action of G_K and we say that $\mathbf{D}$ is potentially semistable if $\mathrm{rg}_{L\otimes K_0}(\mathscr{D}_{\mathrm{pst}}(\mathbf{D}))=\mathrm{rg}_{\mathscr{R}_L(K)}(\mathbf{D})$. Denote by $\mathbf{M}_{\mathrm{pst},K}^{\varphi,\Gamma}$, $\mathbf{M}_{\mathrm{st},K}^{\varphi,\Gamma}$ and $\mathbf{M}_{\mathrm{cris},K}^{\varphi,\Gamma}$ the categories of potentially semistable, semistable and crystalline (φ,Γ_K)-modules respectively.

Proposition 1.1.4. (i) *The functors* $\mathscr{D}_{\mathrm{cris}}:\mathbf{M}_{\mathrm{cris},K}^{\varphi,\Gamma}\to\mathbf{MF}_K^{\varphi}$, $\mathscr{D}_{\mathrm{st}}:\mathbf{M}_{\mathrm{st},K}^{\varphi,\Gamma}\to\mathbf{MF}_K^{\varphi,N}$ *and* $\mathscr{D}_{\mathrm{pst}}:\mathbf{M}_{\mathrm{pst},K}^{\varphi,\Gamma}\to\mathbf{MF}_K^{\varphi,N,G_K}$ *are equivalences of categories.*

(ii) *If V is a p-adic representation of G_K then* $\mathscr{D}_{\mathrm{cris}}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))$ *(resp.* $\mathscr{D}_{\mathrm{st}}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))$, *resp.* $\mathscr{D}_{\mathrm{pst}}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))$*) is canonically and functorially isomorphic to Fontaine's module* $\mathbf{D}_{\mathrm{cris}}(V)$ *(resp.* $\mathbf{D}_{\mathrm{st}}(V)$, *resp.* $\mathbf{D}_{\mathrm{pst}}(V)$*).*

Proof. The first statement is the main result of [12]. The second statement follows from [9, Theorem 0.2]. □

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1.2. Cohomology of (φ, Γ)-modules

1.2.1. Fontaine–Herr complexes (see [40, 41, 52]). Let A be either $\mathscr{E}_L^{\dagger}(K)$ or $\mathscr{R}_L(K)$. We fix a generator $\gamma_K \in \Gamma_K$. If $\mathbf{D}$ is a (φ, Γ_K)-module over A we shall write $H^*(\mathbf{D})$ for the cohomology of the complex

$$C_{\varphi,\gamma_K}(\mathbf{D}) : 0 \to \mathbf{D} \xrightarrow{f} \mathbf{D} \oplus \mathbf{D} \xrightarrow{g} \mathbf{D} \to 0$$

where $f(x) = ((\varphi - 1)\,x, (\gamma_K - 1)\,x)$ and $g(y, z) = (\gamma_K - 1)\,y - (\varphi - 1)\,z$. A short exact sequence of (φ, Γ_K)-modules

$$0 \to \mathbf{D}' \to \mathbf{D} \to \mathbf{D}'' \to 0$$

gives rise to an exact cohomology sequence:

$$0 \to H^0(\mathbf{D}') \to H^0(\mathbf{D}) \to H^0(\mathbf{D}'') \to H^1(\mathbf{D}') \to \cdots \to H^2(\mathbf{D}'') \to 0.$$

The cohomology of (φ, Γ_K)-modules over $\mathscr{R}_L(K)$ satisfies the following fundamental properties (see [52, Theorem 0.2]):

- *Euler characteristic formula.* $H^*(\mathbf{D})$ are finite dimensional L-vector spaces and

$$\sum_{i=0}^{2} (-1)^i \dim_L H^i(\mathbf{D}) = -[K : \mathbb{Q}_p]\,\mathrm{rg}_{\mathscr{R}_L(K)}(\mathbf{D}).$$

- *Poincaré duality.* For each $i = 0, 1, 2$ there exist functorial pairings

$$H^i(\mathbf{D}) \times H^{2-i}(\mathbf{D}^*(\chi)) \xrightarrow{\cup} H^2(\mathscr{R}_L(K)(\chi)) \simeq L$$

  which are compatible with the connecting homomorphisms in the usual sense.

Proposition 1.2.2. *Let V be a p-adic representation of G_K. Then:*

(i) *The continuous Galois cohomology $H^*(K, V)$ is canonically (up to the choice of γ_K) and functorially isomorphic to $H^*(\mathbf{D}^{\dagger}(V))$.*

(ii) *The natural map $\mathbf{D}^{\dagger}(V) \to \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ induces a quasi-isomorphism of complexes $C_{\varphi,\gamma_K}(\mathbf{D}^{\dagger}(V)) \to C_{\varphi,\gamma_K}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))$.*

Proof. See [40] and [52, Theorem 1.1]. □

1.2.3. Iwasawa cohomology (see [16]). If V is a p-adic representation of G_K and T is an O_L-lattice of V stable under G_K we define

$$H^i_{\mathrm{Iw}}(K, T) = \varprojlim_{\mathrm{cor}_{K_n/K_{n-1}}} H(K_n, T)$$

and $H^i_{\mathrm{Iw}}(K, V) = H^i_{\mathrm{Iw}}(K, T) \otimes_{O_L} L$. Since $\mathbf{D}^{\dagger}(V)$ is étale, each $x \in \mathbf{D}^{\dagger}(V)$ can be written in the form $x = \sum_{i=1}^d a_i \varphi(e_i)$ where $\{e_i\}_{i=1}^d$ is a basis of $\mathbf{D}^{\dagger}(V)$ and $a_i \in \mathscr{E}_L^{\dagger}(K)$. Therefore the formula

$$\psi\left(\sum_{i=1}^{d} a_i\,\varphi(e_i)\right) = \sum_{i=1}^{d} \psi(a_i)\,e_i$$

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defines an operator $\psi : \mathbf{D}^{\dagger}(V) \to \mathbf{D}^{\dagger}(V)$ which is a left inverse for φ. The Iwasawa cohomology $H^*_{\mathrm{Iw}}(K, V)$ is canonically (up to the choice of γ_K) and functorially isomorphic to the cohomology of the complex

$$C^{\dagger}_{\mathrm{Iw},\psi}(V) : \mathbf{D}^{\dagger}(V) \xrightarrow{\psi-1} \mathbf{D}^{\dagger}(V).$$

The projection map $\mathrm{pr}_{V,n} : H^1_{\mathrm{Iw}}(K, V) \to H^1(K_n, V)$ has the following explicit description. Set $\gamma_{K,n} = \gamma_K^{[K_n:K]}$. Let $x \in \mathbf{D}^{\dagger}(V)^{\psi=1}$. Then $(\varphi - 1)\,x \in \mathbf{D}^{\dagger}(V)^{\psi=0}$ and by [15, Lemma 1.5.1] there exists $y \in \mathbf{D}^{\dagger}(V)$ such that $(\gamma_{K,n} - 1)\,y = (\varphi - 1)\,x$. Then $\mathrm{pr}_{V,n}$ sends x to $\mathrm{cl}(y, x)$. This interpretation of the Iwasawa cohomology was found by Fontaine (unpublished but see [16]).

1.2.4. The exponential map (see [13, 59, 7]). Let $\mathbf{D}$ be a (φ, Γ_K)-module. With any cocycle $\alpha = (a, b) \in Z^1(C_{\varphi,\gamma}(\mathbf{D}))$ one can associate the extension

$$0 \to \mathbf{D} \to \mathbf{D}_{\alpha} \to \mathscr{R}_L(K) \to 0$$

defined by

$$\mathbf{D}_{\alpha} = \mathbf{D} \oplus \mathscr{R}_L(K)\,e, \quad (\varphi - 1)\,e = a, \quad (\gamma_K - 1)\,e = b.$$

As usual, this gives rise to a canonical isomorphism $H^1(\mathbf{D}) \simeq \mathrm{Ext}^1_{(\varphi,\Gamma_K)}(\mathscr{R}_L(K), \mathbf{D})$. We say that the class $\mathrm{cl}(\alpha)$ of α in $H^1(\mathbf{D})$ is crystalline if $\mathrm{rg}_{L\otimes K_0}\mathscr{D}_{\mathrm{cris}}(\mathbf{D}_{\alpha}) = \mathrm{rg}_{L\otimes K_0}\mathscr{D}_{\mathrm{cris}}(\mathbf{D}) + 1$ and define

$$H^1_f(\mathbf{D}) = \{\mathrm{cl}(\alpha) \in H^1(\mathbf{D}) | \mathrm{cl}(\alpha) \text{ is crystalline}\}$$

(see [7, §1.4]). Now assume that $\mathbf{D}$ is potentially semistable and define the tangent space of $\mathbf{D}$ as

$$t_{\mathbf{D}}(K) = \mathscr{D}_{\mathrm{dR}}(\mathbf{D})/\mathrm{Fil}^0\mathscr{D}_{\mathrm{dR}}(\mathbf{D}).$$

Consider the complex

$$C^{\bullet}_{\mathrm{cris}}(\mathbf{D}) : \mathscr{D}_{\mathrm{cris}}(\mathbf{D}) \xrightarrow{f} t_{\mathbf{D}}(K) \oplus \mathscr{D}_{\mathrm{cris}}(\mathbf{D})$$

where the modules are placed in degrees 0 and 1 and $f(d) = (d(\mathrm{mod}\ \mathrm{Fil}^0\mathscr{D}_{\mathrm{dR}}(\mathbf{D})), (1 - \varphi)\,(d))$ (see [59, 34]). From Proposition 1.1.4 there follows the existence of a canonical isomorphism

$$H^1(C^{\bullet}_{\mathrm{cris}}(\mathbf{D})) \to H^1_f(\mathbf{D})$$

(see [7, Proposition 1.4.4] for the proof). We define the exponential map

$$\exp_{\mathbf{D},K} : t_{\mathbf{D}}(K) \oplus \mathscr{D}_{\mathrm{cris}}(\mathbf{D}) \to H^1(\mathbf{D})$$

as the composition of this isomorphism with the natural projection $t_{\mathbf{D}}(K) \oplus \mathscr{D}_{\mathrm{cris}}(\mathbf{D}) \to H^1(C^{\bullet}_{\mathrm{cris}}(\mathbf{D}))$ and the embedding $H^1_f(\mathbf{D}) \hookrightarrow H^1(\mathbf{D})$.

If V is a potentially semistable representation and $\mathbf{D} = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ then the isomorphism $H^1(\mathbf{D}) \simeq H^1(K, V)$ identifies $H^1_f(\mathbf{D})$ with $H^1_f(K, V)$ of Bloch and Kato [7, Proposition 1.4.2]. Let

$$t_V(K) = \mathbf{D}_{\mathrm{dR}}(V)/\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V)$$

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denote the tangent space of V. By [59, Proposition 1.21] the following diagram commutes and identifies our exponential map with the exponential map $\exp_{V,K}$ of Bloch and Kato [13, §4]:

$$\begin{array}{ccc} t_{\mathbf{D}}(K) & \xrightarrow{\exp_{\mathbf{D},K}} & H^1(\mathbf{D}) \\ \downarrow & & \downarrow \\ t_V(K) & \xrightarrow{\exp_{V,K_n}} & H^1(K,V). \end{array}$$

Let

$$[\ ,\ ] : \mathscr{D}_{\mathrm{dR}}(\mathbf{D}) \times \mathscr{D}_{\mathrm{dR}}(\mathbf{D}^*(\chi)) \to L \otimes_{\mathbb{Q}_p} K$$

be the canonical duality. The dual exponential map

$$\exp^*_{\mathbf{D}^*(\chi),K} : H^1(\mathbf{D}^*(\chi)) \to \mathrm{Fil}^0 \mathscr{D}_{\mathrm{dR}}(\mathbf{D}^*(\chi))$$

is defined to be the unique linear map such that

$$\exp_{\mathbf{D},K}(x) \cup y = \mathrm{Tr}_{K/\mathbb{Q}_p}[x, \exp^*_{\mathbf{D}^*(\chi),K}(y)]$$

for all $x \in \mathscr{D}_{\mathrm{dR}}(\mathbf{D})$, $y \in \mathscr{D}_{\mathrm{dR}}(\mathbf{D}^*(\chi))$.

Remark 1.2.5. An alternative definition of the exponential map for (φ, Γ)-modules, based on Berger's theory of B-pairs can be found in Nakamura's paper [57] and Riedel's PhD thesis [71]. It is not difficult to see that the two definitions are equivalent.

1.2.6. (φ, Γ)-modules of rank 1 (see [23, 7]). In this paper we deal with potentially semistable representations of $G_{\mathbb{Q}_p}$. To simplify notation we set $K_n = \mathbb{Q}_p(\zeta_{p^n})$, $\mathscr{E}_L^\dagger = \mathscr{E}_L^\dagger(\mathbb{Q}_p)$, $\mathscr{R}_L = \mathscr{R}_L(\mathbb{Q}_p)$, $\Gamma = \Gamma_{\mathbb{Q}_p}$ and we fix a topological generator γ of Γ. With each continuous character $\delta : \mathbb{Q}_p^* \to L^*$ one can associate the (φ, Γ)-module of rank 1, $\mathscr{R}_L(\delta) = \mathscr{R}_L e_\delta$, defined by $\gamma(e_\delta) = \delta(\chi(\gamma))e_\delta$ and $\varphi(e_\delta) = \delta(p)e_\delta$. Colmez proved that any (φ, Γ)-module of rank 1 over $\mathscr{R}_L$ is isomorphic to one and only one of the $\mathscr{R}_L(\delta)$ [23, Proposition 3.1]. It is easy to see that $\mathscr{R}_L(\delta)$ is crystalline if and only if there exists $m \in \mathbb{Z}$ such that $\delta(u) = u^m$ for all $u \in \mathbb{Z}_p^*$ [7, Lemma 1.5.2]. In this case $\mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\delta))$ is the one-dimensional vector space generated by $t^{-m}e_\delta$ with Hodge–Tate weight equal to $-m$ and φ acts on $\mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\delta))$ as multiplication by $p^{-m}\delta(p)$. The computation of the cohomology of crystalline (φ, Γ)-modules of rank 1 reduces to the following four cases. We refer the reader to [23, §§2.3–2.5] and to [7, Proposition 1.5.3 and Theorem 1.5.7] for proofs and more details.

- $\delta(u) = u^{-m}$ $(u \in \mathbb{Z}_p^*)$ *for some* $m \geqslant 0$ *but* $\delta(x) \neq x^{-m}$. In this case $H^1(\mathscr{R}_L(\delta))$ is a one-dimensional L-vector space, $H^1_f(\mathscr{R}_L(\delta)) = 0$ and $H^i(\mathscr{R}_L(\delta)) = 0$ for $i = 0, 2$.
- $\delta(x) = x^{-m}$ *for some* $m \geqslant 0$. In this case $H^0(\mathscr{R}_L(\delta)) = \mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\delta))$ and $H^2(\mathscr{R}_L(\delta)) = 0$. The map

$$i_\delta : \mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\delta)) \oplus \mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\delta)) \to H^1(\mathscr{R}_L(\delta)),$$

$$i_\delta(x, y) = \mathrm{cl}(-x, \log \chi(\gamma) y)$$

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is an isomorphism. We let $i_{\delta,f}$ and $i_{\delta,c}$ denote its restrictions to the first and second direct summand respectively. Then $\mathrm{Im}(i_{\delta,f}) = H^1_f(\mathscr{R}_L(\delta))$ and we have a canonical decomposition

$$H^1(\mathscr{R}_L(\delta)) \simeq H^1_f(\mathscr{R}_L(\delta)) \oplus H^1_c(\mathscr{R}_L(\delta)) \tag{5}$$

where $H^1_c(\mathscr{R}_L(\delta)) = \mathrm{Im}(i_{\delta,c})$. Set

$$\mathbf{x}_m = i_{\delta,f}(t^m e_\delta) = -\mathrm{cl}(t^m, 0)\, e_\delta,$$

$$\mathbf{y}_m = i_{\delta,c}(t^m e_\delta) = \log\chi(\gamma)\,\mathrm{cl}(0, t^m)\, e_\delta.$$

- $\delta(u) = u^m$ $(u \in \mathbb{Z}_p^*)$ *for some* $m \geqslant 1$ *but* $\delta(x) \neq |x|x^m$. Then $H^1(\mathscr{R}_L(\delta))$ is a one-dimensional L-vector space, $H^1_f(\mathscr{R}_L(\delta)) = H^1(\mathscr{R}_L(\delta))$ and $H^i(\mathscr{R}_L(\delta)) = 0$ for $i = 0, 2$.
- $\delta(x) = |x|x^m$ *for some* $m \geqslant 1$. Then $H^0(\mathscr{R}_L(\delta)) = 0$ and $H^2(\mathscr{R}_L(\delta))$ is a one-dimensional L-vector space. Moreover $\chi\delta^{-1}(x) = x^{1-m}$ and there exists a unique isomorphism

$$i_\delta : \mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\delta)) \oplus \mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\delta)) \to H^1(\mathscr{R}_L(\delta))$$

such that

$$i_\delta(\alpha, \beta) \cup i_{\chi\delta^{-1}}(x, y) = [\beta, x] - [\alpha, y]$$

where $[\ ,\ ] : \mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\delta)) \times \mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\chi\delta^{-1})) \to L$ is the canonical pairing. Denote as $i_{\delta,f}$ and $i_{\delta,c}$ the restrictions of i_δ to the first and second direct summand respectively. Then $\mathrm{Im}(i_{\delta,f}) = H^1_f(\mathscr{R}_L(\delta))$ and again we have a canonical decomposition

$$H^1(\mathscr{R}_L(\delta)) \simeq H^1_f(\mathscr{R}_L(\delta)) \oplus H^1_c(\mathscr{R}_L(\delta)) \tag{6}$$

where $H^1_c(\mathscr{R}_L(\delta)) = \mathrm{Im}(i_{\delta,c})$.

More explicitly, let $\boldsymbol{\alpha}_m = -\left(1 - \frac{1}{p}\right)\mathrm{cl}(\alpha_m)$ and $\boldsymbol{\beta}_m = \left(1 - \frac{1}{p}\right)\log\chi(\gamma)\,\mathrm{cl}(\beta_m)$ where

$$\alpha_m = \frac{(-1)^{m-1}}{(m-1)!}\,\partial^{m-1}\left(\frac{1}{X} + \frac{1}{2}, a\right) e_\delta, \quad (1-\varphi)\, a = (1 - \chi(\gamma)\gamma)\left(\frac{1}{X} + \frac{1}{2}\right),$$

$$\beta_m = \frac{(-1)^{m-1}}{(m-1)!}\,\partial^{m-1}\left(b, \frac{1}{X}\right) e_\delta, \quad (1-\varphi)\left(\frac{1}{X}\right) = (1 - \chi(\gamma)\,\gamma)\, b$$

and $\partial = (1+X)\frac{d}{dX}$. Then $H^1_f(\mathscr{R}_L(\delta))$ and $H^1_c(\mathscr{R}_L(\delta))$ are generated by $\boldsymbol{\alpha}_m$ and $\boldsymbol{\beta}_m$ respectively and one has

$$\boldsymbol{\alpha}_m \cup \mathbf{x}_{m-1} = \boldsymbol{\beta}_m \cup \mathbf{y}_{m-1} = 0, \quad \boldsymbol{\alpha}_m \cup \mathbf{y}_{m-1} = -1, \quad \boldsymbol{\beta}_m \cup \mathbf{x}_{m-1} = 1. \tag{7}$$

Proposition 1.2.7. *Let* $\delta(x) = |x|x^m$ *where* $m \geqslant 1$. *Then* $d_m = t^{-m} e_\delta$ *is a basis of* $\mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\delta))$ *and the exponential map sends* $(d_m, 0)$ *to* $\boldsymbol{\alpha}_m$.

Proof. See [7, Theorem 1.5.7]. □

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1.3. The large exponential map

1.3.1. The large exponential map (see [63, 19, 5, 10]). In this section we review the construction and basic properties of Perrin-Riou's large exponential map [63]. We work with p-adic representations of $G_{\mathbb{Q}_p}$ and keep the notation of § 1.2.6. Let p be an odd prime number. We let $\Lambda = O_L[[\Gamma]]$ denote the Iwasawa algebra of Γ over O_L and set $\mathscr{R}_L^+ = \mathscr{R}_L \cap L[[X]]$. We remark that $\mathscr{R}_L^+$ is the ring of power series with coefficients in L which converge on the open unit disk. Fix a topological generator γ of Γ and define a compartible system of generators of Γ_n setting $\gamma_1 = \gamma^{p-1}$ and $\gamma_{n+1} = \gamma_n^p$ for $n \geqslant 1$. Let $\Delta = \mathrm{Gal}(K_1/\mathbb{Q}_p)$. Define

$$\mathscr{H} = \{f(\gamma_1 - 1) \mid f \in \mathscr{R}_L^+\}, \quad \mathscr{H}(\Gamma) = \mathbb{Z}_p[\Delta] \otimes_{\mathbb{Z}_p} \mathscr{H}.$$

Thus $\mathscr{H}(\Gamma) = \oplus_{i=0}^{p-2} \mathscr{H}\delta_i$ where $\delta_i = \frac{1}{|\Delta|}\sum_{g\in\Delta} \omega^{-i}(g)g$. We equip $\mathscr{H}(\Gamma)$ with twist operators $\mathrm{Tw}_m : \mathscr{H}(\Gamma) \to \mathscr{H}(\Gamma)$ defined by $\mathrm{Tw}_m(f(\gamma_1 - 1)\,\delta_i) = f(\chi(\gamma_1)^m\gamma_1 - 1)\,\delta_{i-m}$. The ring $\mathscr{H}(\Gamma)$ acts on $\mathscr{R}_L^+$ and $(\mathscr{R}_L^+)^{\psi=0}$ is the free $\mathscr{H}(\Gamma)$-module generated by $(1+X)$ [63, Proposition 1.2.7].

Let V be a potentially semistable representation of $G_{\mathbb{Q}_p}$. Set $\mathcal{D}(V) = (\mathscr{R}_L^+)^{\psi=0} \otimes_L \mathbf{D}_{\mathrm{cris}}(V)$ and define a map

$$\Xi_{V,n}^{\varepsilon} : \mathcal{D}(V) \to H^1(K_n, C^{\bullet}_{\mathrm{cris}}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))) = \mathrm{coker}(\mathbf{D}_{\mathrm{cris}}(V) \xrightarrow{f} t_V(K_n) \oplus \mathbf{D}_{\mathrm{cris}}(V))$$

by

$$\Xi_{V,n}^{\varepsilon}(\alpha) = \begin{cases} p^{-n}\left(\sum_{k=1}^{n} (\sigma\otimes\varphi)^{-k}\alpha(\zeta_{p^k} - 1), -\alpha(0)\right) & \text{if } n \geqslant 1, \\ -\left(0, (1-p^{-1}\varphi^{-1})\,\alpha(0)\right) & \text{if } n = 0. \end{cases} \tag{8}$$

In particular, if $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1} = 0$ the operator $1-\varphi$ is invertible on $\mathbf{D}_{\mathrm{cris}}(V)$ and

$$\Xi_{V,0}^{\varepsilon}(\alpha) = \left(\frac{1-p^{-1}\varphi^{-1}}{1-\varphi}\,\alpha(0), 0\right).$$

For any $m \in \mathbb{Z}$ let $\mathrm{Tw}_{V,m}^{\varepsilon} : H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V) \to H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V(m))$ denote the twist map $\mathrm{Tw}_{V,m}^{\varepsilon}(x) = x \otimes \varepsilon^{\otimes m}$.

Theorem 1.3.2 (**Perrin-Riou**). *Let V be a potentially semistable representation of $G_{\mathbb{Q}_p}$ such that $H^0(K_\infty, V) = 0$. Then for any integers h and m such that $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{dR}}(V) = \mathbf{D}_{\mathrm{dR}}(V)$ and $m+h \geqslant 1$ there exists a unique $\mathscr{H}(\Gamma)$-homomorphism*

$$\mathrm{Exp}_{V(m),h}^{\varepsilon} : \mathcal{D}(V(m)) \to \mathscr{H}(\Gamma) \otimes_{\Lambda_{\mathbb{Q}_p}} H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V(m))$$

satisfying the following properties:

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(i) *For any $n \geqslant 0$ the diagram*

$$\begin{array}{ccc} \mathcal{D}(V(m)) & \xrightarrow{\mathrm{Exp}^{\varepsilon}_{V(m),h}} & \mathscr{H}(\Gamma) \otimes_{\Lambda_{\mathbb{Q}_p}} H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V(m)) \\ \downarrow{\scriptstyle \Xi^{\varepsilon}_{V(m),n}} & & \downarrow{\scriptstyle \mathrm{pr}_{V(m),n}} \\ H^1(K_n, C^{\bullet}_{\mathrm{cris}}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V(m))) & \xrightarrow{(h-1)!\,\exp_{V(m),K_n}} & H^1(K_n, V(m)) \end{array}$$

commutes.

(ii) *Let $e_1 = \varepsilon^{-1} \otimes t$ denote the canonical generator of $\mathbf{D}_{\mathrm{cris}}(\mathbb{Q}_p(-1))$. Then*

$$\mathrm{Exp}^{\varepsilon}_{V(m+1),h+1} = -\mathrm{Tw}^{\varepsilon}_{V(m),1} \circ \mathrm{Exp}^{\varepsilon}_{V(m),h} \circ (\partial \otimes e_1).$$

(iii) *One has*

$$\mathrm{Exp}^{\varepsilon}_{V(m),h+1} = \ell_h \mathrm{Exp}^{\varepsilon}_{V(m),h}$$

where $\ell_m = m - \frac{\log(\gamma_1)}{\log \chi(\gamma_1)}$.

Proof. We sketch the construction of the large exponential map $\mathrm{Exp}^{\varepsilon}_{V(m),h}$ found by Berger [10]. This gives the most natural proof of Perrin-Riou's theorem. Berger assumes that V is crystalline, but his arguments work in our case without modifications. See also [70] for some detail and an interpretation of this proof entirely in terms of (φ, Γ)-modules.

The action of $\mathscr{H}(\Gamma)$ on $\mathbf{D}^{\dagger}(V)^{\psi=1}$ induces an isomorphism $\mathscr{H}(\Gamma) \otimes_{\Lambda_{\mathbb{Q}_p}} \mathbf{D}^{\dagger}(V)^{\psi=1} \to \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)^{\psi=1}$ (see [70, §6.4]). Composing this map with the canonical isomorphism $H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V) \simeq \mathbf{D}^{\dagger}(V)^{\psi=1}$ we obtain an isomorphism $\mathscr{H}(\Gamma) \otimes_{\Lambda_{\mathbb{Q}_p}} H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V) \to \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)^{\psi=1}$. One can check that ℓ_m acts on $\mathscr{R}_L$ as $m - t\partial$ and an easy induction shows that $\prod_{k=0}^{h-1} \ell_k = (-1)^h t^h \partial^h$. Let $h \geqslant 1$ be such that $\mathrm{Fil}^{-h} \mathbf{D}_{\mathrm{dR}}(V) = \mathbf{D}_{\mathrm{dR}}(V)$. To simplify the formulation, assume that $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1} = 0$. For any $\alpha \in \mathcal{D}(V)$ the equation

$$(\varphi - 1)\, F = \alpha - \sum_{m=1}^{h} \frac{\partial^m \alpha(0)}{m!}\, t^m$$

has a solution in $\mathscr{R}_L^+ \otimes \mathbf{D}_{\mathrm{cris}}(V)$ and we define

$$\Omega^{\varepsilon}_{V,h}(\alpha) = \frac{\log \chi(\gamma_1)}{p}\, \ell_{h-1} \ell_{h-2} \cdots \ell_0(F(X)).$$

It is not difficult to see that $\Omega^{\varepsilon}_{V,h}(\alpha) \in \mathbf{D}^{+}_{\mathrm{rig}}(V)^{\psi=1}$ and an explicit computation shows that $\Omega^{\varepsilon}_{V,h}(\alpha)$ satisfies properties (i)–(iii) [10, Theorem II.13].

This construction of $\mathrm{Exp}^{\varepsilon}_{V,h}(\alpha)$ will be used in the proof of Proposition 1.3.7 below. □

Remark 1.3.3. (1) Theorem 1.3.2 was first proved in [63] for crystalline representations. Other proofs can be found in [19, 47, 5]. The fact that the statement is still valid if we take the 'crystalline' part of an arbitrary potentially semistable representation was first pointed out in [47] (see also [46]). We also remark that

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in [63], $\mathrm{Exp}^{\varepsilon}_{V,h}(\alpha)$ was defined only for α such that $\partial^m\alpha(0) \in (1-p^m\varphi)\,\mathbf{D}_{\mathrm{cris}}(V)$ for all $m \in \mathbb{Z}$. This assumption can be omitted (see [66] or [5, §5.1]).

(2) It is very important to define the large exponential map to the 'non-crystalline' part of V and develop a natural generalization of Perrin-Riou's theory (see [63, 65, 8]) in this context. In [68], Perrin-Riou constructed the large exponential map for absolutely semistable representations. Recently Nakamura [57] and, independently and in a slightly different form, Riedel [71] extended Berger's construction to all de Rham (φ, Γ)-modules. The results of Pottharst concerning the Iwasawa cohomology of (φ, Γ)-modules [70] play an important role in these papers. See also [58, 49]. The relationship between these constructions and the theory of p-adic L-functions is an interesting open question. We refer the reader to the Introduction to [68] for some remarks on this subject.

1.3.4. The logarithmic maps. The Iwasawa algebra Λ is equipped with an involution $\iota : \Lambda \to \Lambda$ defined by $\iota(\tau) = \tau^{-1}$, $\tau \in \Gamma$. If M is a Λ-module we set $M^\iota = \Lambda \otimes_\iota M$ and denote by $m \mapsto m^\iota$ the canonical bijection of M onto M^ι. Thus $\lambda m^\iota = (\iota(\lambda)\, m)^\iota$ for all $\lambda \in \Lambda$, $m \in M$. Let T be an O_L-lattice of V stable under the action of $G_{\mathbb{Q}_p}$. The cohomological pairings

$$(\ ,\ )_{T,n} : H^1(K_n, T) \times H^1(K_n, T^*(1)) \to O_L$$

give rise to a Λ-bilinear pairing

$$\langle \ ,\ \rangle_T : H^1_{\mathrm{Iw}}(\mathbb{Q}_p, T) \times H^1_{\mathrm{Iw}}(\mathbb{Q}_p, T^*(1))^\iota \to \Lambda$$

defined by

$$\left\langle x, y^\iota \right\rangle_T \equiv \sum_{\tau \in \Gamma/\Gamma_n} (\tau^{-1}x_n, y_n)_{T,n}\tau \quad \mod (\gamma_n - 1), \quad n \geqslant 1$$

(see [63, §3.6.1]). By linearity we extend this pairing to

$$\langle \ ,\ \rangle_V : \mathscr{H}(\Gamma) \otimes_\Lambda H^1_{\mathrm{Iw}}(\mathbb{Q}_p, T) \times \mathscr{H}(\Gamma) \otimes_\Lambda H^1_{\mathrm{Iw}}(\mathbb{Q}_p, T^*(1))^\iota \to \mathscr{H}(\Gamma).$$

For any $\eta \in \mathbf{D}_{\mathrm{cris}}(V^*(1))$ the element $\widetilde{\eta} = \eta \otimes (1+X)$ lies in $\mathcal{D}(V^*(1))$ and we define a map

$$\mathfrak{L}^{\varepsilon}_{V,1-h,\eta} : H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V) \to \mathscr{H}(\Gamma)$$

by

$$\mathfrak{L}^{\varepsilon}_{V,1-h,\eta}(x) = \left\langle x, \mathrm{Exp}^{\varepsilon^{-1}}_{V^*(1),h}(\tilde{\eta})^\iota \right\rangle_V.$$

Lemma 1.3.5. *For any $j \in \mathbb{Z}$ one has*

$$\mathfrak{L}^{\varepsilon}_{V(-1),-h,\eta\otimes e_1}(\mathrm{Tw}^{\varepsilon}_{V,-1}(x)) = \mathrm{Tw}_1(\mathfrak{L}^{\varepsilon}_{V,1-h,\eta}(x)).$$

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Proof. A short computation shows that $\langle \mathrm{Tw}^{\varepsilon}_{V,j}(x), \mathrm{Tw}^{\varepsilon}_{V^*(1),-j}(y)\rangle_{V(j)} = \mathrm{Tw}_{-j}\langle x, y\rangle_V$. Taking into account that $\mathrm{Tw}^{\varepsilon^{-1}}_{V^*(1),1} = -\mathrm{Tw}^{\varepsilon}_{V^*(1),1}$ we have

$$\begin{aligned}
\mathfrak{L}^{\varepsilon}_{V(-1),-h,\eta\otimes e_1}(\mathrm{Tw}^{\varepsilon}_{V,-1}(x)) &= \left\langle \mathrm{Tw}^{\varepsilon}_{V,-1}(x), \mathrm{Exp}^{\varepsilon^{-1}}_{V^*(2),h+1}(\widetilde{\eta\otimes e_1})^{\iota}\right\rangle_{V(-1)} \\
&= \left\langle \mathrm{Tw}^{\varepsilon}_{V,-1}(x), -\mathrm{Tw}^{\varepsilon^{-1}}_{V^*(1),1}\left(\mathrm{Exp}^{\varepsilon^{-1}}_{V^*(1),h}(\widetilde{\eta})\right)^{\iota}\right\rangle_{V(-1)} \\
&= \left\langle \mathrm{Tw}^{\varepsilon}_{V,-1}(x), \mathrm{Tw}^{\varepsilon}_{V^*(1),1}\left(\mathrm{Exp}^{\varepsilon^{-1}}_{V^*(1),h}(\widetilde{\eta})\right)^{\iota}\right\rangle_{V(-1)} \\
&= \mathrm{Tw}_1\left\langle x, \mathrm{Exp}^{\varepsilon^{-1}}_{V^*(1),h}(\widetilde{\eta})^{\iota}\right\rangle_V = \mathrm{Tw}_1(\mathfrak{L}^{\varepsilon}_{V,1-h,\eta}(x))
\end{aligned}$$

and the lemma is proved. □

1.3.6. The derivative of the large exponential map. In this section we give an explicit formula for the derivative of the large exponential map. This result will be used in § 2.2 to relate the derivative of the p-adic L-function associated with global cohomology classes to the ℓ-invariant.

Let D be a one-dimensional subspace of $\mathbf{D}_{\mathrm{cris}}(V)$ on which φ acts as multiplication by p^{-1}. Set $D_{\mathbb{Q}_p^{\mathrm{ur}}} = D\otimes_{\mathbb{Q}_p}\mathbb{Q}_p^{\mathrm{ur}}$ where $\mathbb{Q}_p^{\mathrm{ur}}$ denotes the maximal unramified extension of $\mathbb{Q}_p$. Using the weak admissibility of $\mathbf{D}_{\mathrm{pst}}(V)$ it is easy to see that D is not contained in $\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V)$ and therefore

$$\mathbf{D}_{\mathrm{pst}}(V) = \mathrm{Fil}^0\mathbf{D}_{\mathrm{pst}}(V)\oplus D_{\mathbb{Q}_p^{\mathrm{ur}}} \tag{9}$$

as $\mathbb{Q}_p^{\mathrm{ur}}\otimes_{\mathbb{Q}_p}L$-modules. Let $m\geqslant 1$ denote the unique Hodge–Tate weight of D. By Berger's theory [12] (see also [4, Section 2.4.2]), the intersection $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\cap\left(D\otimes_L\mathscr{R}_L[1/t]\right)$ is a saturated (φ,Γ)-submodule of $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ of rank 1 which is isomorphic to $\mathscr{R}_L(\delta)$ with $\delta(x) = |x|x^m$.

Recall (see the proof of Theorem 1.3.2) that $H^1(\mathbb{Q}_p,\mathscr{H}(\Gamma)\otimes_{\mathbb{Q}_p}V) = \mathscr{H}(\Gamma)\otimes_{\Lambda_{\mathbb{Q}_p}} H^1_{\mathrm{Iw}}(\mathbb{Q}_p,V)$ injects into $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$. Set

$$H^1_{\delta}(\mathbb{Q}_p,\mathscr{H}(\Gamma)\otimes_{\mathbb{Q}_p}V) = \mathscr{R}_L(\delta)\cap H^1(\mathbb{Q}_p,\mathscr{H}(\Gamma)\otimes_{\mathbb{Q}_p}V).$$

The projection map induces a commutative diagram

$$\begin{array}{ccc}
H^1_{\delta}(\mathbb{Q}_p,\mathscr{H}(\Gamma)\otimes_{\mathbb{Q}_p}V) & \longrightarrow & H^1(\mathbb{Q}_p,\mathscr{H}(\Gamma)\otimes_{\mathbb{Q}_p}V) \\
\downarrow & & \downarrow {\scriptstyle \mathrm{pr}_0} \\
H^1(\mathscr{R}_L(\delta)) & \longrightarrow & H^1(\mathbb{Q}_p,V).
\end{array}$$

Note that the bottom map is not injective in general. Fix a generator $\gamma\in\Gamma$ and an integer $h\geqslant 1$ such that $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{dR}}(V) = \mathbf{D}_{\mathrm{dR}}(V)$.

Proposition 1.3.7. *Let V be a potentially semistable representation of $G_{\mathbb{Q}_p}$ such that $H^0(K_\infty,V)=0$. Assume that D is a one-dimensional subspace of $\mathbf{D}_{\mathrm{cris}}(V)$ on which φ acts as multiplication by p^{-1}. For any $a\in D$ let $x\in\mathcal{D}(V)$ be such that $x(0)=a$. Then:*

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(i) *There exists a unique* $F \in H^1_\delta(\mathbb{Q}_p, \mathscr{H}(\Gamma) \otimes V)$ *such that*

$$(\gamma - 1)\, F = \mathrm{Exp}^{\varepsilon}_{V,h}(x).$$

(ii) *The composition map*

$$\delta_D : D \to H^1_\delta(\mathbb{Q}_p, \mathscr{H}(\Gamma) \otimes V) \to H^1(\mathscr{R}_L(\delta))$$
$$\delta_D(a) = \mathrm{pr}_0(F)$$

is well defined and is explicitly given by the following formula:

$$\delta_D(a) = \Gamma(h) \left(1 - \frac{1}{p}\right)^{-1} (\log \chi(\gamma))^{-1}\, i_c(a).$$

Proof. (1) Consider the diagram

$$\begin{array}{ccc} \mathcal{D}(V) & \xrightarrow{\ \mathrm{Exp}^{\varepsilon}_{V,h}\ } & H^1(\mathbb{Q}_p, \mathscr{H}(\Gamma) \otimes V) \\ \big\downarrow{\scriptstyle \Xi^{\varepsilon}_{V,0}} & & \big\downarrow{\scriptstyle \mathrm{pr}_0} \\ t_V(\mathbb{Q}_p) \otimes \mathbf{D}_{\mathrm{cris}}(V) & \xrightarrow{\ (h-1)!\, \exp_V\ } & H^1(\mathbb{Q}_p, V) \end{array}$$

where $\Xi^{\varepsilon}_{V,0}$ is given by (8). If $x \in D \otimes \mathscr{R}_L^{\psi=0}$ then $\Xi^{\varepsilon}_{V,0}(x) = -(0, (1-p^{-1}\varphi^{-1})\, x(0)) = 0$ and $\mathrm{pr}_0\left(\mathrm{Exp}^{\varepsilon}_{V,h}(x)\right) = 0$. On the other hand, as $H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V)$ is $\Lambda_{\mathbb{Q}_p}$-free, the map

$$(\mathscr{H}(\Gamma) \otimes_{\Lambda_{\mathbb{Q}_p}} H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V))_\Gamma \to H^1(\mathbb{Q}_p, V)$$

is injective and therefore there exists a unique $F \in \mathscr{H}(\Gamma) \otimes_{\Lambda_{\mathbb{Q}_p}} H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V)$ such that $\mathrm{Exp}^{\varepsilon}_{V,h}(x) = (\gamma - 1)\, F$. Let $y \in D \otimes \mathscr{R}_L^{\psi=0}$ be another element such that $y(0) = a$ and let $\mathrm{Exp}^{\varepsilon}_{V,h}(y) = (\gamma - 1)\, G$. Since $\mathscr{R}_L^{\psi=0} = \mathscr{H}(\Gamma)\,(1 + X)$ we have $y = x + (\gamma - 1)g$ for some $g \in D \otimes \mathscr{R}_L^{\psi=0}$. As $\mathrm{Exp}^{\varepsilon}_{V,h}(g) = 0$, we obtain immediately that $\mathrm{pr}_0(G) = \mathrm{pr}_0(F)$ and we have proved that the map δ_D is well defined.

(2) Take $a \in D$ and set

$$x = a \otimes \ell\left(\frac{(1 + X)^{\chi(\gamma)} - 1}{X}\right),$$

where $\ell(u) = \frac{1}{p} \log\left(\frac{u^p}{\varphi(u)}\right)$. An easy computation shows that

$$\sum_{\zeta^p = 1} \ell\left(\frac{\zeta^{\chi(\gamma)}(1 + X)^{\chi(\gamma)} - 1}{\zeta(1 + X) - 1}\right) = 0.$$

Thus $x \in D \otimes O_L[[X]]^{\psi=0}$. Write x in the form $f = (1 - \varphi)\,(\gamma - 1)\,(a \otimes \log(X))$. Then

$$\Omega^{\varepsilon}_{V,h}(x) = (-1)^{h-1} \frac{\log \chi(\gamma_1)}{p}\, t^h \partial^h((\gamma - 1)\,(a \log(X)) = \left(1 - \frac{1}{p}\right) \log \chi(\gamma)\,(\gamma - 1)\, F$$

where

$$F = (-1)^{h-1} t^h \partial^h (a \log(X)) = (-1)^{h-1} a t^h \partial^{h-1}\left(\frac{1 + X}{X}\right).$$

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This implies immediately that $F \in H^1_\delta(\mathbb{Q}_p, \mathscr{H}(\Gamma) \otimes V)$. On the other hand, as $D = \mathcal{D}_{\mathrm{cris}}(\mathscr{R}_L(\delta))$, without loss of generality we may assume that $a = t^{-m} e_\delta$ where $\delta(x) = |x| x^m$. Then

$$F = (-1)^{h-1} t^{h-m} \partial^h \log(X)\, e_\delta.$$

One has $\mathrm{pr}_0(F) = \mathrm{cl}(G, F)$ where $(1-\gamma)\, G = (1-\varphi)\, F$ (see § 1.2.3) and by [15, Lemma 1.5.1] there exists a unique $b \in \mathscr{E}_L^{\dagger, \psi=0}$ such that $(1-\gamma)\, b = \ell(X)$. One has

$$(1-\gamma)\ \left(t^{h-m} \partial^h b e_\delta\right) = (1-\varphi)\ \left(t^{h-m} \partial^h \log(X) e_\delta\right) = (-1)^{h-1} (1-\varphi)\, F.$$

Thus $G = (-1)^{h-1} t^{h-m} \partial^h(b) e_\delta$ and $\mathrm{res}\left(G\, t^{m-1} dt\right) = (-1)^{h-1} \mathrm{res}\left(t^{h-1} \partial^h(b) dt\right)\ e_\delta = 0$. Next from the congruence $F \equiv (h-1)!\, t^{-m} e_\delta \pmod{\mathbb{Q}_p[[X]]\, e_\delta}$ it follows that $\mathrm{res}(F t^{m-1} dt) = (h-1)!\, e_\delta$. Therefore by [7, Corollary 1.5.5] we have

$$\left(1 - \frac{1}{p}\right)\ (\log \chi(\gamma))\, \mathrm{cl}(G, F) = (h-1)!\, \mathrm{cl}(\boldsymbol{\beta}_m) = (h-1)!\, i_c(a). \tag{10}$$

On the other hand

$$x(0) = a \otimes \ell \left(\frac{(1+X)^{\chi(\gamma)} - 1}{X} \right)\Bigg|_{X=0} = a \left(1 - \frac{1}{p}\right)\ \log \chi(\gamma). \tag{11}$$

The formulas (10) and (11) imply that

$$\delta_D(a) = (h-1)! \left(1 - \frac{1}{p}\right)^{-1} (\log \chi(\gamma))^{-1}\, i_c(a).$$

and the proposition is proved. □

1.4. *p*-adic distributions (see [24, Chapitre II], [63, §§1.1, 1.2])

Let $\mathcal{D}(\mathbb{Z}_p^*, L)$ be the space of distributions on $\mathbb{Z}_p^*$ with values in a finite extension L of $\mathbb{Q}_p$. With each $\mu \in \mathcal{D}(\mathbb{Z}_p^*, L)$ one can associate its Amice transform $\mathscr{A}_\mu(X) \in L[[X]]$ by

$$\mathscr{A}_\mu(X) = \int_{\mathbb{Z}_p^*} (1+X)^x \mu(x) = \sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p^*} \binom{x}{n} \mu(x) \right) X^n.$$

The map $\mu \mapsto \mathscr{A}_\mu(X)$ establishes an isomorphism between $\mathcal{D}(\mathbb{Z}_p^*, L)$ and $(\mathscr{R}_L^+)^{\psi=0}$. We will denote by $\mathbf{M}(\mu)$ the unique element of $\mathscr{H}(\Gamma)$ such that

$$\mathbf{M}(\mu)\, (1+X) = \mathcal{A}_\mu(X).$$

For each $m \in \mathbb{Z}$ the character $\chi^m : \Gamma \to \mathbb{Z}_p^*$ can be extended to a unique continuous L-linear map $\chi^m : \mathscr{H}(\Gamma) \to L^*$. If $h = \sum_{i=0}^{p-2} \delta_i h_i(\gamma_1 - 1)$, then $\chi^m(h) = h_i(\chi^m(\gamma_1) - 1)$ with $i \equiv m \pmod{(p-1)}$. An easy computation shows that

$$\int_{\mathbb{Z}_p^*} x^m \mu(x) = \partial^m \mathscr{A}_\mu(0) = \chi^m(\mathbf{M}(\mu)).$$

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If $x \in \mathbb{Z}_p^*$ we set $\langle x \rangle = \omega^{-1}(x)\, x$ where ω denotes the Teichmüller character. With any $\mu \in \mathcal{D}(\mathbb{Z}_p^*, L)$ we associate p-adic functions

$$L_p(\mu, \omega^i, s) = \int_{\mathbb{Z}_p^*} \omega^i(x) \langle x \rangle^s \mu(x), \quad 0 \leqslant i \leqslant p-2.$$

Write $\mathbf{M}(\mu) = \sum_{i=0}^{p-2} \delta_i h_i(\gamma_1 - 1)$. Then

$$L_p(\mu, \omega^i, s) = h_i\left(\chi(\gamma_1)^s - 1\right). \tag{12}$$

To prove this formula it is enough to compare the values of both sides at the integers $s \equiv i \pmod{(p-1)}$.

We say that μ is of order $r > 0$ if its Amice transform $\mathscr{A}_\mu(X) = \sum_{n=1}^{\infty} a_n X^n$ is of order r, i.e. if the sequence $|a_n|_p / n^r$ is bounded above. A distribution of order r is completely determined by the values of the integrals

$$\int_{\mathbb{Z}_p^*} \zeta_{p^n}^x x^i \mu(x), \quad n \in \mathbb{N}, \quad 0 \leqslant i \leqslant [r],$$

where $[r]$ is the largest integer no greater then r.

Set $\hat{\mathbb{Z}}^{(p)} = \mathbb{Z}_p^* \times \prod_{l \neq p} \mathbb{Z}_l$. A locally analytic function on $\prod_{l \neq p} \mathbb{Z}_l$ is locally constant and we say that a distribution μ on $\hat{\mathbb{Z}}^{(p)}$ is of order r if for any locally constant function $g(y)$ on $\prod_{l \neq p} \mathbb{Z}_l$ the linear map $f \mapsto \int_{\hat{\mathbb{Z}}^{(p)}} f(x) g(y) \mu(x, y)$ is a distribution of order r on $\mathbb{Z}_p^*$.

2. The ℓ-invariant

2.1. The ℓ-invariant

2.1.1. Definition of the ℓ-invariant. In this section we review and slightly generalize the definition of the ℓ-invariant proposed in our previous article [7] in order to cover the case of potentially crystalline reduction of modular forms. Let S be a finite set of primes and $\mathbb{Q}^{(S)}/\mathbb{Q}$ be the maximal Galois extension of $\mathbb{Q}$ unramified outside $S \cup \{\infty\}$. Fix a finite extension $L/\mathbb{Q}_p$. Let V be an L-adic representation of $G_S = \mathrm{Gal}(\mathbb{Q}^{(S)}/\mathbb{Q})$, i.e. a finite dimensional L-vector space equipped with a continuous linear action of G_S. We write $H_S^*(\mathbb{Q}, V)$ for the continuous cohomology of G_S with coefficients in V. We will always assume that the restriction of V to the decomposition group at p is potentially semistable. For all primes $l \neq p$ (resp. for $l = p$), Greenberg [35] (resp. Bloch and Kato [13]) defined a subgroup $H_f^1(\mathbb{Q}_l, V)$ of $H^1(\mathbb{Q}_l, V)$ by

$$H_f^1(\mathbb{Q}_l, V) = \begin{cases} \ker(H^1(\mathbb{Q}_l, V) \to H^1(\mathbb{Q}_l^{\mathrm{ur}}, V)) & \text{if } l \neq p, \\ \ker(H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, V \otimes \mathbf{B}_{\mathrm{cris}})) & \text{if } l = p \end{cases}$$

where $\mathbf{B}_{\mathrm{cris}}$ is the ring of crystalline periods [32]. The Selmer group of V is defined as

$$H_f^1(\mathbb{Q}, V) = \ker\left(H_S^1(\mathbb{Q}, V) \to \bigoplus_{l \in S} \frac{H^1(\mathbb{Q}_l, V)}{H_f^1(\mathbb{Q}_l, V)}\right).$$

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We also define

$$H^1_{f,\{p\}}(\mathbb{Q},V)=\ker\left(H^1_S(\mathbb{Q},V)\to\bigoplus_{l\in S-\{p\}}\frac{H^1(\mathbb{Q}_l,V)}{H^1_f(\mathbb{Q}_l,V)}\right).$$

Note that these definitions do not depend on the choice of S. From now until the end of this section we assume that V satisfies the following conditions:

(1) $H^1_f(\mathbb{Q},V)=H^1_f(\mathbb{Q},V^*(1))=0$.

(2) The action of φ on $\mathbf{D}_{\mathrm{st}}(V)$ is semisimple at 1.

(3) $\dim_L t_V(\mathbb{Q}_p)=1$.

We remark that the last condition can be relaxed but it simplifies the formulation of Proposition 2.2.2 below and holds for the situations considered in §§ 3 and 4.

The condition (1) together with the Poitou–Tate exact sequence

$$\cdots\to H^1_f(\mathbb{Q},V)\to H^1_S(\mathbb{Q},V)\to\bigoplus_{l\in S}\frac{H^1(\mathbb{Q}_l,V)}{H^1_f(\mathbb{Q}_l,V)}\to H^1_f(\mathbb{Q},V^*_f(1))^*\to\cdots$$

(see [34, Proposition 2.2.1]) gives an isomorphism

$$H^1_S(\mathbb{Q},V)\simeq\bigoplus_{l\in S}\frac{H^1(\mathbb{Q}_l,V)}{H^1_f(\mathbb{Q}_l,V)}.$$

In particular, we have

$$H^1_{f,\{p\}}(\mathbb{Q},V)\simeq\frac{H^1(\mathbb{Q}_p,V)}{H^1_f(\mathbb{Q}_p,V)}. \tag{13}$$

Let D be a one-dimensional subspace of $\mathbf{D}_{\mathrm{cris}}(V)$ on which φ acts as multiplication by p^{-1}. Recall (see §1.3.5) that $D=\mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\delta))$ where $\delta(x)=|x|x^m$ and $m\geqslant 1$ denotes the unique Hodge–Tate weight of D. Thus we have an exact sequence of (φ,Γ)-modules

$$0\to\mathscr{R}_L(\delta)\to\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\to\mathbf{D}\to 0 \tag{14}$$

where $\mathbf{D}=\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)/\mathscr{R}_L(\delta)$. Passing to duals and taking the long exact cohomology sequence we obtain an exact sequence

$$H^1(\mathbb{Q}_p,V^*(1))\to H^1(\mathscr{R}_L(\chi\delta^{-1}))\to H^2(\mathbf{D}^*(\chi)). \tag{15}$$

Proposition 2.1.2. *Let V be an L-adic representation of G_S which satisfies the conditions (1)–(3) above. Assume that D is a one-dimensional subspace of $\mathbf{D}_{\mathrm{cris}}(V)$ such that $D^{\varphi=p^{-1}}=D$ and that one of the following conditions holds:*

(a) *D is not contained in the image of the monodromy operator $N:\mathbf{D}_{\mathrm{pst}}(V)\to\mathbf{D}_{\mathrm{pst}}(V)$ and $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1}=0$.*

(b) *D is contained in the image of N and $N^{-1}(D)\cap\mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}$ is a one-dimensional L-vector space.*

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Then the composition

$$\varkappa : H^1_{f,\{p\}}(\mathbb{Q}, V^*(1)) \to H^1(\mathscr{R}_L(\chi\delta^{-1}))$$

of the localization map $H^1_{f,\{p\}}(\mathbb{Q}, V^*(1)) \to H^1(\mathbb{Q}_p, V^*(1))$ *with* $H^1(\mathbb{Q}_p, V^*(1)) \to H^1(\mathscr{R}_L(\chi\delta^{-1}))$ *is injective. Moreover,* $\mathrm{Im}(\varkappa)$ *is a one-dimensional* L*-vector space such that*

$$\mathrm{Im}(\varkappa) \cap H^1_f(\mathscr{R}_L(\chi\delta^{-1})) = \{0\}.$$

Proof. We consider the cases (a) and (b) separately.

First assume that D satisfies (a). Applying the functor $\mathscr{D}_{\mathrm{pst}}$ to (14) we obtain an exact sequence

$$0 \to D_{\mathbb{Q}_p^{\mathrm{ur}}} \to \mathbf{D}_{\mathrm{pst}}(V) \xrightarrow{f} \mathscr{D}_{\mathrm{pst}}(\mathbf{D}) \to 0. \tag{16}$$

From (9) it follows that $\mathrm{Fil}^0\mathscr{D}_{\mathrm{pst}}(\mathbf{D}) = \mathscr{D}_{\mathrm{pst}}(\mathbf{D})$ and by [7, Proposition 1.4.4]

$$H^0(\mathbf{D}) \simeq \mathscr{D}_{\mathrm{pst}}(\mathbf{D})^{\varphi=1,N=0,G_{\mathbb{Q}_p}} = \mathscr{D}_{\mathrm{cris}}(\mathbf{D})^{\varphi=1}.$$

Applying the snake lemma to (16) we obtain an isomorphism of $G_{\mathbb{Q}_p}$-modules $\mathbf{D}_{\mathrm{pst}}(V)^{\varphi=1} \simeq \mathscr{D}_{\mathrm{pst}}(\mathbf{D})^{\varphi=1}$. Thus $\mathbf{D}_{\mathrm{st}}(V)^{\varphi=1} \simeq \mathscr{D}_{\mathrm{st}}(\mathbf{D})^{\varphi=1}$. Let $x \in \mathscr{D}_{\mathrm{cris}}(\mathbf{D})^{\varphi=1}$. There exists a unique $y \in \mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}$ such that $f(y) = x$. Since $f(N(y)) = N(x) = 0$, one has $N(y) \in D$ and by (a) $N(y) = 0$, i.e. $y \in \mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1}$. Since $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1} = 0$ by assumption (a), we have proved that $H^0(\mathbf{D}) = 0$.

Now $H^2(\mathbf{D}^*(\chi)) = 0$ by Poincaré duality and from the exact sequence (15) we obtain that the map $H^1(\mathbb{Q}_p, V^*(1)) \to H^1(\mathscr{R}_L(\chi\delta^{-1}))$ is surjective. Since $\chi\delta^{-1}(x) = x^{1-m}$, the cohomology $H^1(\mathscr{R}_L(\chi\delta^{-1}))$ decomposes into the direct sum of one-dimensional subspaces

$$H^1(\mathscr{R}_L(\chi\delta^{-1})) \simeq H^1_f(\mathscr{R}_L(\chi\delta^{-1})) \oplus H^1_c(\mathscr{R}_L(\chi\delta^{-1})).$$

The image of $H^1_f(\mathbb{Q}_p, V^*(1))$ in $H^1(\mathscr{R}_L(\chi\delta^{-1}))$ is contained in $H^1_f(\mathscr{R}_L(\chi\delta^{-1}))$ and we have a surjective map

$$\frac{H^1(\mathbb{Q}_p, V^*(1))}{H^1_f(\mathbb{Q}_p, V^*(1))} \to \frac{H^1(\mathscr{R}_L(\chi\delta^{-1}))}{H^1_f(\mathscr{R}_L(\chi\delta^{-1}))}. \tag{17}$$

From $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1} = 0$ it follows that $H^0(\mathbb{Q}_p, V) = 0$ and

$$\dim_L\left(H^1_f(\mathbb{Q}_p, V)\right) = \dim_L(t_V(L)) + \dim_L(H^0(\mathbb{Q}_p, V)) = 1.$$

Therefore $H^1(\mathbb{Q}_p, V^*(1))/H^1_f(\mathbb{Q}_p, V^*(1))$ is one-dimensional and the map (17) is an isomorphism. Combining (17) with the isomorphism (13) for the cohomology with coefficients in $V^*(1)$ instead of V we obtain an isomorphism

$$H^1_{f,\{p\}}(\mathbb{Q}, V^*(1)) \simeq \frac{H^1(\mathscr{R}_L(\chi\delta^{-1}))}{H^1_f(\mathscr{R}_L(\chi\delta^{-1}))}.$$

This proves the proposition in the case (a).

Now assume that D satisfies (b). We follow the approach of [7, §§2.1 and 2.2] with some modifications (see especially the proofs of Proposition 2.1.7 and Lemma 2.1.8 op. cit.). The proposition will be proved in several steps.

(b1) Consider the filtration on $\mathbf{D}_{\mathrm{pst}}(V)$ given by

$$D_i = \begin{cases} 0 & \text{if } i=-1 \\ D_{\mathbb{Q}_p^{\mathrm{ur}}} & \text{if } i=0 \\ (D+N^{-1}(D)\cap \mathbf{D}_{\mathrm{st}}(V)^{\varphi=1})_{\mathbb{Q}_p^{\mathrm{ur}}} & \text{if } i=1 \\ \mathbf{D}_{\mathrm{pst}}(V) & \text{if } i=2. \end{cases}$$

By [12] this filtration induces a unique filtration on $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$

$$\{0\} = F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \subset F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \subset F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \subset F_2\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$$

such that $\mathscr{D}_{\mathrm{pst}}(F_i\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) = D_i$. Note that $F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ is a semistable (φ,Γ)-submodule of $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$. To simplify notation set $M_0 = F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$, $M = F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ and $M_1 = \mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$. We remark that $M_0 \simeq \mathscr{R}_L(\delta)$ and since $\mathrm{Fil}^0(D_1/D_0) = D_1/D_0$ and $(D_0/D_1)^{\varphi=1} = D_0/D_1$, Proposition 1.5.9 of [7] implies that $M_1 \simeq \mathscr{R}_L(x^{-k})$ for some $k \geqslant 0$. By the assumption (b) the monodromy operator N acts non-trivially on D_1 and therefore we have a non-crystalline extension

$$0 \to \mathscr{R}_L(\delta) \to M \to \mathscr{R}_L(x^{-k}) \to 0$$

which is a particular case of the exact sequence from [7, Proposition 2.1.7]. Passing to duals and taking the long cohomology sequence we obtain a diagram

$$\begin{array}{ccccccccc} H^1(\mathscr{R}_L(|x|x^{k+1})) & \xrightarrow{f_1} & H^1(M^*(\chi)) & \xrightarrow{g_1} & H^1(\mathscr{R}_L(\chi\delta^{-1})) & \xrightarrow{\delta_1} & H^2(\mathscr{R}_L(|x|x^{k+1})) & \longrightarrow & 0. \\ & & \uparrow{\scriptstyle \eta} & \nearrow{\scriptstyle \varkappa} & & & & & \\ & & H^1_{f,\{p\}}(\mathbb{Q}, V^*(1)) & & & & & & \end{array} \qquad (18)$$

(b2) The quotient $\widetilde{\mathbf{D}} = \mathrm{gr}_2\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ is a potentially semistable (φ,Γ)-module with Hodge–Tate weights $\geqslant 0$. Thus $H^0(\widetilde{\mathbf{D}}) \simeq (\mathbf{D}_{\mathrm{pst}}(V)/D_1)^{G_{\mathbb{Q}_p},\varphi=1,N=0}$. Let $\bar{x} = x + D_1 \in (\mathbf{D}_{\mathrm{pst}}(V)/D_1)^{G_{\mathbb{Q}_p},\varphi=1,N=0}$. For each $g \in G_{\mathbb{Q}_p}$ we can write $g(x) = x + d$ for some $d \in D_1$. Since the inertia subgroup $I_p \subset G_{\mathbb{Q}_p}$ acts on $\mathbf{D}_{\mathrm{pst}}(V)$ through a finite quotient and since the restriction of this action to D_1 is trivial, we obtain that $x \in \mathbf{D}_{\mathrm{pst}}(V)^{I_p} = \mathbf{D}_{\mathrm{st}}(V)\otimes_{\mathbb{Q}_p}\mathbb{Q}_p^{ur}$. Thus

$$\begin{aligned} \bar{x} &\in \left(\left(\frac{\mathbf{D}_{\mathrm{st}}(V)}{D+N^{-1}(D)\cap \mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}}\right)\otimes \mathbb{Q}_p^{ur}\right)^{G_{\mathbb{Q}_p},\varphi=1,N=0} \\ &= \left(\frac{\mathbf{D}_{\mathrm{st}}(V)}{D+N^{-1}(D)\cap \mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}}\right)^{\varphi=1,N=0}. \end{aligned}$$

Since φ is semisimple at 1, we can assume that $x \in \mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}$. Then $N(x) \in D$ and therefore $x \in N^{-1}(D) \cap \mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}$. This shows that $\bar{x} = 0$ and we have proved that $H^0(\widetilde{\mathbf{D}})=0$. By Poincaré duality we obtain immediately that $H^2(\widetilde{\mathbf{D}}^*(\chi))=0$. Now the Euler characteristic formula together with [7, Corollary 1.4.5] give

$$\dim_L H^1(\widetilde{\mathbf{D}}^*(\chi))=\mathrm{rg}(\widetilde{\mathbf{D}}^*(\chi))+\dim_L H^0(\widetilde{\mathbf{D}}^*(\chi))=\dim_L H^1_f(\widetilde{\mathbf{D}}^*(\chi))$$

and therefore

$$H^1_f(\widetilde{\mathbf{D}}^*(\chi))=H^1(\widetilde{\mathbf{D}}^*(\chi)). \tag{19}$$

(b3) Consider the exact sequence

$$0\to\widetilde{\mathbf{D}}^*(\chi)\to\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\to M^*(\chi)\to 0.$$

Since $H^0(M^*(\chi))=0$, this sequence together with the isomorphism (19) gives an exact sequence

$$0\to H^1_f(\widetilde{\mathbf{D}}^*(\chi))\to H^1(\mathbb{Q}_p,V^*(1))\to H^1(M^*(\chi))\to 0. \tag{20}$$

On the other hand, by [7, Corollary 1.4.6] the sequence

$$0\to H^1_f(\widetilde{\mathbf{D}}^*(\chi))\to H^1_f(\mathbb{Q}_p,V^*(1))\to H^1_f(M^*(\chi))\to 0 \tag{21}$$

is also exact.

(b4) We come back to the diagram (18). The sequences (20) and (21) together with the isomorphism (13) show that the map η is injective. By [7, Lemma 2.1.8] one has $\ker(g_1)=H^1_f(M^*(\chi))$ and $H^1(\mathscr{R}_L(\chi\delta^{-1}))=H^1_f(\mathscr{R}_L(\chi\delta^{-1}))\oplus\mathrm{Im}(g_1)$. Since

$$H^1(\mathbb{Q}_p,V^*(1))/H^1_f(\mathbb{Q}_p,V^*(1))\simeq H^1(M^*(\chi))/H^1_f(M^*(\chi))$$

we obtain that $\ker(\varkappa)=\mathrm{Im}(\eta)\cap\ker(g_1)=0$ and that $\mathrm{Im}(\varkappa)\cap H^1_f(\mathscr{R}_L(\chi\delta^{-1}))=0$. The proposition is proved. □

Definition. The ℓ-invariant associated with V and D is the unique element $\ell(V,D)\in L$ such that

$$\mathrm{Im}(\varkappa)=L(\mathbf{y}_{m-1}+\ell(V,D)\,\mathbf{x}_{m-1}).$$

Here $\{\mathbf{x}_{m-1},\mathbf{y}_{m-1}\}$ is the canonical basis of $H^1(\mathscr{R}_L(\chi\delta^{-1}))$ constructed in § 1.2.6.

Remarks 2.1.3. (1) If V is semistable at p this definition agrees with the definition of $\ell(V,D)$ proposed in [7, §§2.2.2 and 2.3.3].

(2) In the both cases (a) and (b) our assumptions imply that $H^0(\mathbb{Q}_p,V)=0$.

(3) One can express $\ell(V,D)$ directly in terms of V and D. Assume that D satisfies the condition (a) of Proposition 2.1.2. Since $H^0(\mathbf{D})=0$ the sequence (14) shows that $H^1(\mathscr{R}_L(\delta))$ injects into $H^1(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))\simeq H^1(\mathbb{Q}_p,V)$. Moreover, from $\dim_L H^1_f(\mathbb{Q}_p,V)=1$ and the fact that $\dim_L H^1_f(\mathscr{R}_L(\delta))=1$ it follows that $H^1_f(\mathbb{Q}_p,V)\simeq H^1_f(\mathscr{R}_L(\delta))$. Let $H^1_D(V)$ denote the inverse image of $H^1(\mathscr{R}_L(\delta))/H^1_f(\mathscr{R}_L(\delta))$ under the isomorphism (13). Then

$$H^1_D(V)\simeq\frac{H^1(\mathscr{R}_L(\delta))}{H^1_f(\mathscr{R}_L(\delta))}$$

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and the localization map $H^1_S(V) \to H^1(\mathbb{Q}_p, V)$ induces an injection $H^1_D(V) \to H^1(\mathscr{R}_L(\delta))$. Using the decomposition (6) we define $\mathscr{L}(V, D)$ as the unique element of L such that

$$\mathrm{Im}(H^1_D(V) \to H^1(\mathscr{R}_L(\delta))) = L(\boldsymbol{\beta}_m + \mathscr{L}(V, D)\boldsymbol{\alpha}_m)$$

where $\{\boldsymbol{\alpha}_m, \boldsymbol{\beta}_m\}$ denotes the canonical basis of $H^1(\mathscr{R}_L(\delta))$. Then

$$\ell(V, D) = -\mathscr{L}(V, D) \tag{22}$$

(see [7, Proposition 2.2.7]). Note that in the cited work, V is assumed to be semistable, but in the potentially semistable case the proof is exactly the same.

A similar duality formula can be proved in the case (b) too, but it will not be used in this paper. We refer the reader to [7, §2.2.3] for more detail.

(4) The diagram (18) shows that in the case (b) the image of $H^1_{f,\{p\}}(\mathbb{Q}, V^*(1))$ in $H^1(\mathscr{R}_L(\chi\delta^{-1}))$ coincides with $\mathrm{Im}(g_1)$ and therefore that $\ell(V, D)$ depends only on the local properties of V at p. On the other hand, in the case (a) the ℓ-invariant is global and contains information about the localization map $H^1_{f,\{p\}}(\mathbb{Q}, V^*(1)) \to H^1(\mathscr{R}_L(\chi\delta^{-1}))$.

2.2. Relation to the large logarithmic map

2.2.1. Global Iwasawa cohomology. We keep the notation and conventions of § 2.1. Consider the global Iwasawa cohomology

$$H^1_{\mathrm{Iw},S}(\mathbb{Q}, T^*(1)) = \varprojlim_{\mathrm{cor}} H^1_S(\mathbb{Q}(\zeta_{p^n}), T^*(1))$$

and

$$H^1_{\mathrm{Iw},S}(\mathbb{Q}, V^*(1)) = H^1_{\mathrm{Iw},S}(\mathbb{Q}, T^*(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

By [62, §2.1.7] for $l \neq p$ one has $H^1_{\mathrm{Iw}}(\mathbb{Q}_l, V^*(1)) \simeq H^0(\mathbb{Q}_l(\zeta_{p^\infty}), V^*(1))$ and therefore $H^1_{\mathrm{Iw}}(\mathbb{Q}_l, V^*(1))_\Gamma$ is contained in $H^1_f(\mathbb{Q}_l, V^*(1))$. Thus $H^1_{\mathrm{Iw},S}(\mathbb{Q}, V^*(1))_\Gamma$ injects into $H^1_{f,\{p\}}(\mathbb{Q}, V^*(1))$ and we have a commutative diagram

$$\begin{array}{ccc}
H^1_{\mathrm{Iw},S}(\mathbb{Q}, V^*(1)) & \xrightarrow{\mathrm{loc}_p} & H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V^*(1)) \\
\downarrow{\scriptstyle \mathrm{pr}_{V,0}} & & \downarrow{\scriptstyle \mathrm{pr}_{V,0}} \\
H^1_{f,\{p\}}(\mathbb{Q}, V^*(1)) & \xrightarrow{\mathrm{loc}_p} & H^1(\mathbb{Q}_p, V^*(1))
\end{array}$$

where loc_p and $\mathrm{pr}_{V,0}$ denote localization maps and projections respectively. From (13) (applied to $V^*(1)$) it follows that the bottom localization map is injective. One expects that the upper localization map is also injective. This is a consequence of the weak Leopoldt conjecture which predicts the vanishing of $H^2_S(\mathbb{Q}(\zeta_{p^\infty}), V/T)$ (see [65, Appendice B]). Let δ_0 denote the idempotent of $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ associated with the trivial character. It is not difficult to see (see [66, Proposition B5]) that if $H^1_f(V^*(1)) = 0$ and $H^0(\mathbb{Q}_p, V^*(1)) = 0$ then the

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δ_0-component $H^2_S(\mathbb{Q}(\zeta_{p^\infty}), V/T)^{(\delta_0)}$ of $H^2_S(\mathbb{Q}(\zeta_{p^\infty}), V/T)$ vanishes and therefore the map $H^1_{\mathrm{Iw},S}(\mathbb{Q}, V^*(1))^{(\delta_0)} \to H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V^*(1))^{(\delta_0)}$ is injective. This result will not be used in the remainder of the paper.

Fix a generator $\gamma \in \Gamma$ and an integer $h \geqslant 1$ such that $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{dR}}(V) = \mathbf{D}_{\mathrm{dR}}(V)$. Recall (see § 1.3.4) that with each $d \in \mathbf{D}_{\mathrm{cris}}(V)$ one can associate the large logarithmic map

$$\mathfrak{L}^{\varepsilon}_{V^*(1),1-h,d} : H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V^*(1)) \to \mathscr{H}(\Gamma).$$

The main results of this paper will be directly deduced from the following statement.

Proposition 2.2.2. *Let V be an L-adic representation which satisfies the conditions (1)–(3). Assume that D is a one-dimensional subspace of $\mathbf{D}_{\mathrm{cris}}(V)$ on which φ acts as multiplication by p^{-1} and which satisfies one of the conditions (a), (b) of Proposition 2.1.2. Fix a non-zero element $d \in D$. Let $\mathbf{z} \in H^1_{\mathrm{Iw},S}(\mathbb{Q}, V^*(1))$ be such that $\mathbf{z}_0 = \mathrm{pr}_0(\mathbf{z}) \in H^1_S(\mathbb{Q}, V^*(1))$ is non-zero. Denote by $\mu_{\mathbf{z}} \in \mathcal{D}(\mathbb{Z}_p^*, L)$ the distribution defined by*

$$\mathbf{M}(\mu_{\mathbf{z}}) = \mathfrak{L}^{\varepsilon}_{V^*(1),1-h,d}(\mathrm{loc}_p(\mathbf{z}))$$

and consider the p-adic function

$$L_p(\mu_{\mathbf{z}}, s) = \int_{\mathbb{Z}_p^*} \langle x \rangle^s \mu_{\mathbf{z}}(x).$$

Then $L_p(\mu_{\mathbf{z}}, 0) = 0$ and

$$L_p'(\mu_{\mathbf{z}}, 0) = \ell(V, D)\ \Gamma(h) \left(1 - \frac{1}{p}\right)^{-1} \left[d, \exp^*_{V^*(1)}(\mathrm{loc}_p(\mathbf{z}_0))\right]_V$$

where $[\ ,\ ]_V : \mathbf{D}_{\mathrm{cris}}(V) \times \mathbf{D}_{\mathrm{cris}}(V^*(1)) \to L$ is the canonical duality.

Proof. We fixed a basis d of the one-dimensional L-vector space $D = \mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\delta))$. Let d^* be the basis of $\mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\chi\delta^{-1}))$ which is dual to d. Denote by $\tilde{\mathbf{z}}_0$ the image of $\mathrm{loc}_p(\mathbf{z}_0)$ under the projection map $H^1(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))) \to H^1(\mathscr{R}_L(\chi\delta^{-1}))$. Write $\tilde{\mathbf{z}}_0 = a\, i_f(d^*) + b\, i_c(d^*)$. Then $\ell(V, D) = a/b$. By Proposition 1.2.7 and (7) we have

$$\begin{aligned}\left[d, \exp^*_{V^*(1)}(\mathrm{loc}_p(\mathbf{z}_0))\right]_V &= -\exp_V(d) \cup \mathrm{loc}_p(\mathbf{z}_0) = -\exp_{\mathscr{R}_L(\delta)}(d) \cup \tilde{\mathbf{z}}_0 \\ &= -b\left(i_f(d) \cup i_c(d^*)\right) = -b\ (\boldsymbol{\alpha}_m \cup \mathbf{y}_{m-1}) = b. \end{aligned} \tag{23}$$

Let $\mathbf{M}(\mu_{\mathbf{z}}) = \sum_{i=0}^{p-2} \delta_i h_i(\gamma_1 - 1)$. Then $L_p(\mu_{\mathbf{z}}, s) = h_0(\chi(\gamma_1)^s - 1)$ by (8). From Proposition 1.3.7 it follows that there exists $F \in H^1_\delta(\mathbb{Q}_p, \mathscr{H}(\Gamma) \otimes V)$ such that $\mathrm{Exp}^{\varepsilon^{-1}}_{V,h}(d \otimes (1+X)) = (\gamma - 1)\, F$ and

$$\begin{aligned}\mathbf{M}(\mu_{\mathbf{z}}) = \mathfrak{L}^{\varepsilon}_{V^*(1),1-h,d}(\mathrm{loc}_p(\mathbf{z})) &= \left\langle \mathrm{loc}_p(\mathbf{z}), \mathrm{Exp}^{\varepsilon^{-1}}_{V,h}(d \otimes (1+X)^{\iota} \right\rangle_V \\ &= (\gamma^{-1} - 1)\left\langle \mathrm{loc}_p(\mathbf{z}), F^{\iota} \right\rangle_V.\end{aligned}$$

Put $\left\langle \mathrm{loc}_p(\mathbf{z}), F^{\iota} \right\rangle_{V_f} = \sum_{i=0}^{p-2} \delta_i H_i(\gamma_1 - 1)$. Then $L_p(\mu_{\mathbf{z}}, s) = (\chi(\gamma)^{-s} - 1)\, H_0(\chi(\gamma_1)^s - 1)$. Since $\chi(\gamma_1) = \chi(\gamma)^{p-1}$ the last formula implies that $L_p(\mu_{\mathbf{z}}, s)$ has a zero at $s = 0$ and

$$L_p'(\mu_{\mathbf{z}}, 0) = -(\log \chi(\gamma))\, H_0(0). \tag{24}$$

On the other hand, by Proposition 1.3.7,

$$\begin{aligned} H_0(0) &= \mathrm{loc}_p(\mathbf{z}_0) \cup (\mathrm{pr}_0 F) = \tilde{\mathbf{z}}_0 \cup \delta_D(d) \\ &= \Gamma(h)\left(1-\frac{1}{p}\right)^{-1} (\log \chi(\gamma))^{-1} \left(\tilde{\mathbf{z}}_0 \cup i_c(d)\right) \\ &= -\Gamma(h)\left(1-\frac{1}{p}\right)^{-1} (\log \chi(\gamma))^{-1} a. \end{aligned} \tag{25}$$

From (23)–(25) we obtain that

$$L'_p(\mu_{\mathbf{z}}, 0) = \Gamma(h)\left(1-\frac{1}{p}\right)^{-1} a = \ell(V, D)\ \Gamma(h)\left(1-\frac{1}{p}\right)^{-1} \left[d, \exp^*_{V^*(1)}(\mathrm{loc}_p(\mathbf{z}_0))\right]_V$$

and the proposition is proved. □

3. Trivial zeros of Dirichlet L-functions

3.1. Dirichlet L-functions

Let $\eta : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ be a Dirichlet character of conductor N. We fix a primitive N th root of unity ζ_N and set $\tau(\eta) = \sum_{a \bmod N} \eta(a)\, \zeta_N^a$. The Dirichlet L-function

$$L(\eta, s) = \sum_{n=1}^{\infty} \frac{\eta(n)}{n^s}, \qquad \mathrm{Re}(s) > 1$$

has a meromorphic continuation on the whole complex plane and satisfies the functional equation

$$\left(\frac{N}{\pi}\right)^{s/2} \Gamma\left(\frac{s+\delta_\eta}{2}\right) L(\eta, s) = W_\eta \left(\frac{N}{\pi}\right)^{(1-s)/2} \Gamma\left(\frac{1-s+\delta_\eta}{2}\right) L(\bar{\eta}, 1-s)$$

where $W_\eta = i^{-\delta_\eta} N^{-1/2} \tau(\eta)$ and $\delta_\eta = \frac{1-\eta(-1)}{2}$. From now until the end of this section we assume that η is not trivial. For any $j \geqslant 0$ the special value $L(\eta, -j)$ is the algebraic integer given by

$$L(\eta, -j) = \frac{d^j F_\eta(0)}{dt^j} \tag{26}$$

where

$$F_\eta(t) = \frac{1}{\tau(\eta^{-1})} \sum_{a \bmod N} \frac{\eta^{-1}(a)}{1 - \zeta_N^a e^t}$$

(see for example [64, proof of Proposition 3.1.4]). In particular,

$$L(\eta, 0) = \frac{1}{\tau(\eta^{-1})} \sum_{a \bmod N} \frac{\eta^{-1}(a)}{1 - \zeta_N^a}. \tag{27}$$

Moreover $L(\eta, -j) = 0$ if and only if $j \equiv \delta_\eta \pmod 2$.

Let p be a prime number such that $(p,N)=1$. We fix a finite extension L of $\mathbb{Q}_p$ containing the values of all Dirichlet characters η of conductor N. The power series

$$\mathscr{A}_{\mu_\eta}(X)=-\frac{1}{\tau(\eta^{-1})}\sum_{a \bmod N}\left(\frac{\eta^{-1}(a)}{(1+X)\,\zeta_N^a-1}-\frac{\eta^{-1}(a)}{(1+X)^p\zeta_N^{pa}-1}\right)$$

lies in $O_L[[X]]^{\psi=0}$ and therefore can be viewed as the Amice transform of a unique measure μ_η on $\mathbb{Z}_p^*$. The p-adic L-functions associated with η are defined to be

$$L_p(\eta\,\omega^m,s)=\int_{\mathbb{Z}_p^*}\omega^{m-1}(x)\langle x\rangle^{-s}\mu_\eta(x),\quad 0\leqslant m\leqslant p-2.$$

From (12) and (26) it follows that these functions satisfy the following interpolation property (the Iwasawa theorem):

$$L_p(\eta\,\omega^m,1-j)=(1-(\eta\,\omega^{m-j})(p)p^{1-j})\,L(\eta\,\omega^{m-j},1-j)\quad j\geqslant 1.$$

Note that the Euler factor $1-(\eta\,\omega^{m-j})(p)p^{1-j}$ vanishes if $m=j=1$ and $\eta(p)=1$ and that $L(\eta,0)$ does not vanish if and only if η is odd, i.e. $\eta(-1)=-1$.

3.2. p-adic representations associated with Dirichlet characters

We continue to assume that $(p,N)=1$. Set $F=\mathbb{Q}(\zeta_N)$, $G=\mathrm{Gal}(F/\mathbb{Q})$ and let $\rho: G\simeq(\mathbb{Z}/N\mathbb{Z})^*$ denote the canonical isomorphism normalized by $g(\zeta_N)=\zeta_N^{\rho(g)^{-1}}$. Fix a finite extension $L/\mathbb{Q}_p$ containing the values of all Dirichlet characters modulo N. If η is such a character, we identify η with the character $\psi\circ\rho$ of G and denote by $L(\psi)$ the associated one-dimensional Galois representation. Let S denote the set of primes dividing N.

Assume that η is a non-trivial character of conductor N. We need the following well known results concerning the Galois cohomology of $L(\eta)$:

(i) $H^*(\mathbb{Q}_l,L(\eta))=H^*(\mathbb{Q}_l,L(\chi\eta^{-1}))=0$ for $l\in S$.

(ii) $H^1_f(\mathbb{Q},L(\eta))=0$ and $H^1_f(\mathbb{Q},L(\chi\eta^{-1}))\simeq(O_F^*\otimes_{\mathbb{Z}}L)^{(\eta)}$. In particular, $H^1_f(\mathbb{Q},L(\chi\eta^{-1}))=0$ if η is odd.

(iii) The restriction of $L(\eta)$ to the decomposition group at p is crystalline. More precisely, φ acts on $\mathbf{D}_{\mathrm{cris}}(L(\eta))$ as multiplication by $\eta(p)$ and the unique Hodge–Tate weight of $L(\eta)$ is 0.

Note that $H^0(\mathbb{Q}_l,L(\eta))=0$ if $l|N$ because in this case the inertia group acts non-trivially on $L(\eta)$. Together with Poincaré duality and the Euler characteristic formula this gives (i). To prove (ii) it is enough to remark that $H^1_f(F,\mathbb{Q}_p(1))\simeq O_F^*\hat{\otimes}\mathbb{Q}_p$ (see for example [45, §5]). Finally (iii) follows immediately from the definition of $\mathbf{D}_{\mathrm{cris}}$.

Assume now that η is odd and $\eta(p)=1$. Then φ acts on $\mathbf{D}_{\mathrm{cris}}(L(\chi\eta^{-1}))$ as multiplication by p^{-1} and $D=\mathbf{D}_{\mathrm{cris}}(L(\chi\eta^{-1}))$ satisfies the conditions (1)–(4) from § 2.1.1. The isomorphism (13) takes the form

$$H^1_S(\mathbb{Q},L(\chi\eta^{-1}))\simeq\frac{H^1(\mathbb{Q}_p,L(\chi))}{H^1_f(\mathbb{Q}_p,L(\chi))}.$$

3.3. Trivial zeros

3.3.1. Cyclotomic units. Set $F_n = F(\zeta_{p^n})$. The collection $\mathbf{z}_{\mathrm{cycl}} = (1 - \zeta_N^{p^{-n}} \zeta_{p^n})_{n\geqslant 1}$ form a norm compartible system of units which can be viewed as an element of $H^1_{\mathrm{Iw},S}(F, L(\chi))$ using Kummer maps $F_n^* \to H^1_S(F_n, L(\chi))$. Twisting by ε^{-1} we obtain an element $\mathbf{z}_{\mathrm{cycl}}(-1) \in H^1_{\mathrm{Iw},S}(F, L)$. Shapiro's lemma gives an isomorphism of G-modules $H^1_{\mathrm{Iw},S}(F, L) \simeq H^1_{\mathrm{Iw},S}(\mathbb{Q}, L[G]^\iota)$. Let $e_\eta = \frac{1}{|G|} \sum_{g\in G} \eta^{-1}(g)\, g$. Since $e_\eta L[G]^\iota = Le_{\eta^{-1}}$ is isomorphic to $L(\eta^{-1})$ we have

$$e_\eta H^1_{\mathrm{Iw},S}(F, L) \simeq H^1_{\mathrm{Iw},S}(\mathbb{Q}, L(\eta^{-1})).$$

Moreover $\mathbf{D}_{\mathrm{cris}}(L[G]) \simeq (L[G] \otimes F)^G \simeq L \otimes F$. The isomorphism $\mathbb{Q}[G] \simeq F$ defined by $\lambda \mapsto \lambda(\zeta_N)$ induces an isomorphism $\mathbf{D}_{\mathrm{cris}}(L[G]) \simeq L[G]$ and therefore we can consider e_η as a basis of $\mathbf{D}_{\mathrm{cris}}(L(\eta^{-1}))$. Let $\mathbf{z}^{(\eta)}_{\mathrm{cycl}}(-1)$ denote the image of $\mathbf{z}_{\mathrm{cycl}}(-1)$ in $H^1_{\mathrm{Iw}}(\mathbb{Q}, L(\eta^{-1}))$. We need the following properties of these elements:

(1) *Relation to the complex L-function.* Let $\mathbf{z}^{(\eta^{-1})}_{\mathrm{cycl}}(-1)_0$ denote the projection of $\mathbf{z}^{(\eta^{-1})}_{\mathrm{cycl}}(-1)$ on $H^1(\mathbb{Q}, L(\eta))$. Then

$$\exp^*_{L(\eta)}\left(\mathrm{loc}_p\left(\mathbf{z}^{(\eta^{-1})}_{\mathrm{cycl}}(-1)_0\right)\right) = -\left(1 - \frac{\eta^{-1}(p)}{p}\right) L(\eta, 0)\, e_{\eta^{-1}}.$$

(2) *Relation to the p-adic L-function.* Let $e^*_{\eta^{-1}} \in \mathbf{D}_{\mathrm{cris}}(L(\chi\eta^{-1}))$ be the dual basis of the basis $e_{\eta^{-1}}$ of $\mathbf{D}_{\mathrm{cris}}(L(\eta))$ and let $\mathfrak{L}^{(\varepsilon)}_{L(\eta),0} : H^1_{\mathrm{Iw}}(\mathbb{Q}_p, L(\eta)) \to \mathscr{H}(\Gamma)$ denote the associated logarithmic map. Then

$$\mathfrak{L}^{\varepsilon}_{L(\eta),0}\left(\mathrm{loc}_p\left(\mathbf{z}^{(\eta^{-1})}_{\mathrm{cycl}}(-1)\right)\right) = -\mathbf{M}(\mu_\eta).$$

We remark that (1) follows from the explicit reciprocity law of Iwasawa [43] together with (27). See also [45, Theorem 5.12] and [42, Corollary 3.2.7] where a more general statement is proved using the explicit reciprocity law for $\mathbb{Q}_p(r)$. The statement (2) is Coleman's construction of p-adic L-functions reformulated in terms of the large logarithmic map [64, Proposition 3.1.4].

Theorem 3.3.2. *Let η be an odd character of conductor N. Assume that p is a prime odd number such that $p \nmid N$ and $\eta(p) = 1$. Then*

$$L'_p(\eta\,\omega, 0) = -\mathscr{L}(\eta)\, L(\eta, 0)$$

where $\mathscr{L}(\eta)$ is the invariant defined by (3).

Proof. We apply Proposition 2.2.2 to $V = L(\chi\eta^{-1})$, $D = \mathbf{D}_{\mathrm{cris}}(L(\chi\eta^{-1}))$ and $\mathbf{z} = \mathbf{z}^{(\eta^{-1})}_{\mathrm{cycl}}(-1)$. It is easy to see that $\mathscr{L}(\eta)$ coincides with $\ell(L(\chi\eta^{-1}), D)$. Taking into account (1), (2) above we obtain that

$$\begin{aligned} L'_p(\eta\,\omega, 0) &= L'_p(\mu_z, 0) = \ell(L(\chi\eta^{-1}), D)\left(1 - \frac{1}{p}\right)^{-1} \left[e^*_{\eta^{-1}}, \exp^*_{L(\eta)}\left(\mathrm{loc}_p(\mathbf{z}_0)\right)\right] \\ &= -\mathscr{L}(\eta)\, L(\eta, 0) \end{aligned}$$

and the theorem is proved. □

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4. Trivial zeros of modular forms

4.1. p-adic L-functions

4.1.1. Construction of p-adic L-functions (see [1, 53, 76, 55]). Let $f=\sum_{n=1}^{\infty} a_n q^n$ be a normalized newform on $\Gamma_0(N)$ of weight k and character ε. The complex L-function $L(f,s)=\sum_{n=1}^{\infty} a_n n^{-s}$ decomposes into an Euler product

$$L(f,s)=\prod_p E_p(f,p^{-s})^{-1}$$

with $E_p(f,X)=1-a_pX+\varepsilon(p)p^{k-1}X^2$. Let $p>2$ be a prime such that the Euler factor $E_p(f,X)$ is not equal to 1 and let $\alpha\in\overline{\mathbb{Q}}_p$ be a root of the polynomial $X^2-a_pX+\varepsilon(p)p^{k-1}$. Assume that α is not critical, i.e. that $v_p(\alpha)<k-1$. Manin and Vishik [53, 76] and, independently, Amice and Vélu [1] proved that there exists a unique distribution $\mu_{f,\alpha}$ on $\hat{\mathbb{Z}}^{(p)}$ of order $v_p(\alpha)$ such that for any Dirichlet character η of conductor M prime to p and any Dirichlet character ξ of conductor p^m,

$$\int_{\hat{\mathbb{Z}}^{(p)}} \eta(x)\xi(x)x^{j-1}\mu_{f,\alpha}(x)=\begin{cases}\left(1-\dfrac{\bar\eta(p)p^{j-1}}{\alpha}\right)\left(1-\dfrac{\beta\eta(p)}{p^j}\right)\widetilde{L}(f,\eta,j)\\ \quad\text{if } 1\leqslant j\leqslant k-1 \text{ and } m=0,\\ \dfrac{p^{mj}\,\bar\eta(p^m)}{\alpha^m\tau(\bar\xi)}\,\widetilde{L}(f,\eta\xi^{-1},j)\\ \quad\text{if } 1\leqslant j\leqslant k-1 \text{ and } m\geqslant 1\end{cases}$$

where $\tau(\bar\xi)=\sum_{a=1}^{p^m-1}\bar\xi(a)\,\zeta_{p^m}^a$ and $\widetilde{L}(f,\eta,j)$ is the algebraic part of $L(f,\eta,j)$ (see (1)). For us it will be more convenient to work with the distribution $\lambda_{f,\alpha}=x^{-1}\mu_{f,\alpha}$. The p-adic L-functions associated with $\eta:(\mathbb{Z}/M\mathbb{Z})^*\to\overline{\mathbb{Q}}_p^*$ are defined by[3]

$$L_{p,\alpha}(f,\eta\omega^m,s)=\int_{\hat{\mathbb{Z}}^{(p)}}\eta\omega^m(x)\langle x\rangle^s\lambda_{f,\alpha}(x)\quad 0\leqslant m\leqslant p-2. \tag{28}$$

It is easy to see that $L_{p,\alpha}(f,\eta\omega^m,s)$ is a p-adic analytic function which satisfies the following interpolation property:

$$L_{p,\alpha}(f,\eta\omega^m,j)=\mathcal{E}_\alpha(f,\eta\omega^m,j)\,\widetilde{L}(f,\eta\omega^{j-m},j),\quad 1\leqslant j\leqslant k-1 \tag{29}$$

where

$$\mathcal{E}_\alpha(f,\eta\omega^m,j)=\begin{cases}\left(1-\dfrac{\bar\eta(p)p^{j-1}}{\alpha}\right)\left(1-\dfrac{\eta(p)\varepsilon(p)p^{k-j-1}}{\alpha}\right) & \text{if } j\equiv m \pmod{p-1}\\ \dfrac{\bar\eta(p)p^j}{\alpha\,\tau(\omega^{j-m})} & \text{if } j\not\equiv m \pmod{p-1}.\end{cases} \tag{30}$$

4.1.2. p-adic representations associated with modular forms. For each prime p, Deligne [26] constructed a p-adic representation

$$\rho_f:\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to\mathrm{GL}(W_f)$$

[3] Our $L_{p,\alpha}(f,\eta\omega^m,s)$ coincides with $L_p(f,\alpha,\bar\eta\omega^{m-1},s-1)$ of [55].

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with coefficients in a finite extension L of $\mathbb{Q}_p$. This representation has the following properties:

(i) $\det \rho_f$ is isomorphic to $\varepsilon\chi^{k-1}$ where χ is the cyclotomic character.

(ii) If $l \nmid Np$ then the restriction of ρ_f to the decomposition group at l is unramified and

$$\det(1 - \mathrm{Fr}_l X | W_f) = 1 - a_l X + \varepsilon(l)\, l^{k-1} X^2$$

(the Deligne–Langlands–Carayol theorem [14, 50]).

(iii) The restriction of ρ_f to the decomposition group at p is potentially semistable with Hodge–Tate weights $(0, k-1)$ [29]. It is crystalline if $p \nmid N$ and semistable non-crystalline if $p \parallel N$ and $(p, \mathrm{cond}(\varepsilon)) = 1$. If $p|N$ and $\mathrm{ord}_p(N) = \mathrm{ord}_p(\mathrm{cond}(\varepsilon))$ the restriction of ρ_f to the decomposition group at p is potentially crystalline and $\mathbf{D}_{\mathrm{cris}}(W_f) = \mathbf{D}_{\mathrm{pcris}}(W_f)^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is one-dimensional. In all three cases,

$$\det(1 - \varphi X | \mathbf{D}_{\mathrm{cris}}(W_f)) = 1 - a_p X + \varepsilon(p)\, p^{k-1} X^2$$

(Saito's theorem [73]; see also [30, 75]).

4.1.3. Trivial zeros (see [55]).

Definition. We say that $L_{p,\alpha}(f, \eta\omega^m, s)$ has a trivial zero at $s = j$ if

$$L(f, \eta\omega^{j-m}, j) \neq 0 \quad \text{and} \quad \mathcal{E}_\alpha(f, \eta\omega^m, j) = 0. \tag{31}$$

From (30) it is not difficult to deduce that this occurs in the following three cases [55, §15]:

- *The semistable case*: $p \parallel N$, k *is even and* $(p, \mathrm{cond}(\varepsilon)) = 1$. Thus $\varepsilon(p) = 0$, $E_p(f, X) = 1 - a_p X$ and a_p is the unique non-critical root of $X^2 - a_p X$. The restriction of W_f to the decomposition group at p is semistable non-crystalline and the eigenvalues of φ acting on $\mathbf{D}_{\mathrm{st}}(W_f)$ are $\alpha = a_p$ and $\beta = p\alpha$. The module $\mathbf{D}_{\mathrm{st}}(W_f)$ has a basis $\{e_\alpha, e_\beta\}$ such that $\varphi(e_\alpha) = a_p e_\alpha$, $\varphi(e_\beta) = \beta e_\beta$ and $N(e_\beta) = e_\alpha$. Moreover $\mathbf{D}_{\mathrm{cris}}(W_f) = L e_\alpha$. Let $\tilde{\varepsilon}$ be the primitive character associated with ε. Then $\tilde{\varepsilon}(p) \neq 0$ and $a_p^2 = \tilde{\varepsilon}(p) p^{k-2}$ [51, Theorem 3]. Write $a_p = \xi p^{k/2-1}$ where ξ is a root of unity. Then $\mathcal{E}_\alpha(f, \eta\omega^m, j) = 0$ if and only if $j = k/2$, $m \equiv k/2 \bmod (p-1)$ and $\bar{\eta}(p) = \xi$. Therefore the p-adic L-function $L_{p,\alpha}(f, \eta\omega^{k/2}, s)$ has a trivial zero at the central point $s = k/2$ if and only if $\bar{\eta}(p) = \xi$.
- *The crystalline case*: $p \nmid N$. The restriction of W_f to the decomposition group at p is crystalline and by Deligne [27] one has $|\alpha| = p^{(k-1)/2}$. Write $\alpha = \xi p^{\frac{k-1}{2}}$ with $|\xi| = 1$. Then $\mathcal{E}_\alpha(f, \eta\omega^m, j)$ vanishes if and only if $m \equiv j \pmod{p-1}$, k is odd, and either $j = \frac{k+1}{2}$ and $\bar{\eta}(p) = \xi$ or $j = \frac{k-1}{2}$ and $\eta(p)\varepsilon(p) = \xi$. The p-adic L-function $L_{p,\alpha}(f, \eta\omega^{\frac{k+1}{2}}, s)$ has a trivial zero at the near-central point $s = \frac{k+1}{2}$ if and only if $\alpha = \bar{\eta}(p)\, p^{\frac{k-1}{2}}$ and $L_{p,\alpha}(f, \eta\omega^{\frac{k-1}{2}}, s)$ has a trivial zero at $s = \frac{k-1}{2}$ if and only if $\alpha = \eta(p)\, \varepsilon(p)\, p^{\frac{k-1}{2}}$.
- *The potentially crystalline case*: $p|N$ *and* $\mathrm{ord}_p(N) = \mathrm{ord}_p(\mathrm{cond}(\varepsilon))$. One has $E_p(f, X) = 1 - a_p X$ and $\alpha = a_p$ is the unique non-critical root of $X^2 - a_p X$. Moreover $\tilde{\varepsilon}(p) = 0$ and

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it can be shown that $|a_p| = p^{\frac{k-1}{2}}$ [60, 51]. The restriction of W_f to the decomposition group at p is potentially crystalline and $\mathbf{D}_{\mathrm{cris}}(W_f)$ is a one-dimensional vector space on which φ acts as multiplication by a_p. The factor $\mathcal{E}_\alpha(f, \eta\omega^m, j)$ vanishes if and only if k is odd, $j = m = \frac{k+1}{2}$ and $a_p = \bar{\eta}(p)p^{\frac{k-1}{2}}$. The p-adic L-function $L_{\alpha,p}(f, \eta\omega^{\frac{k+1}{2}}, s)$ has a trivial zero at the near-central point $s = \frac{k+1}{2}$ if and only if $a_p = \eta(p)p^{\frac{k-1}{2}}$.

If η is a Dirichlet character of conductor M, the twisted modular form $f_\eta = \sum_{n=1}^{\infty} \eta(n)\, a_n q^n$ is not necessarily primitive, but there exists a unique normalized newform $f \otimes \eta$ such that

$$L(f, \eta, s) = L(f \otimes \eta, s) \prod_{l|M} E_l(f \otimes \eta, l^{-s})$$

(see for example [2]). Write $L(f \otimes \eta, s) = \sum_{n=1}^{\infty} \frac{a_{\eta,n}}{n^s}$. If $p \nmid M$, the Euler factors at p of $L_M(f \otimes \eta, s)$ and $L(f, \eta, s)$ coincide and $\alpha_\eta = \alpha\eta(p)$ is a root of $X^2 - a_{\eta,p}X + \varepsilon(p)\eta^2(p)p^{k-1}$. It is easy to see that $\mathcal{E}_{\alpha_\eta}(f_\eta, \omega^m, j) = \mathcal{E}_\alpha(f, \eta\omega^m, j)$ and from the interpolation formula (29) it follows immediately that the behavior of $L_{p,\alpha}(f, \eta\omega^m, s)$ and $L_{p,\alpha_\eta}(f \otimes \eta, \omega^m, s)$ is essentially the same. Therefore the general case reduces to the case of the trivial character η.

4.2. Selmer groups and ℓ-invariants of modular forms

4.2.1. The Selmer group. From now until the end of this section, we assume that $L_{p,\alpha}(f, \omega^{k_0}, s)$ has a trivial zero at k_0. Thus $k_0 = k/2$ in the semistable case and $k_0 = \frac{k\pm1}{2}$ in the crystalline or potentially crystalline case. Set $V_f = W_f\,(k_0)$. Let f^* denote the complex conjugation of f, i.e. $f^* = \sum_{n=1}^{\infty} \bar{a}_n q^n$. The canonical pairing $W_f \times W_{f^*} \to L(1-k)$ induces an isomorphism $W_{f^*}\,(k - k_0) \simeq V_f^*(1)$. We need the following basic results concerning the Galois cohomology of V_f:

(i) $H^0(\mathbb{Q}_p, V_f) = H^0(\mathbb{Q}_p, V_f^*(1)) = 0$ and $\dim_L H^1(\mathbb{Q}_p, V_f) = \dim_L H^1(\mathbb{Q}_p, V_f^*(1)) = 2$.

(ii) $H^1_f(\mathbb{Q}_p, V_f)$ and $H^1_f(\mathbb{Q}_p, V_f^*(1))$ are one-dimensional L-vector spaces.

(iii) $H^1_f(\mathbb{Q}, V_f) = H^1_f(\mathbb{Q}, V_f^*(1)) = 0$.

We remark that using the fact that the Hodge–Tate weights of W_f are 0 and $k-1$ and the eigenvalues of φ on $\mathbf{D}_{\mathrm{cris}}(W_f)$ have absolute value $p^{(k-1)/2}$ (respectively $p^{k/2}$) in the crystalline and potentially crystalline case (respectively in the semistable case), one deduces that $H^0(\mathbb{Q}_p, W_f(m)) = 0$ for all $1 \leqslant m \leqslant k-1$ (see [46, Proposition 14.12 and §13.3]). Applying Poincaré duality and the Euler characteristic formula we obtain (i). Next (ii) follows from (i) together with the formula

$$\dim_L H^1_f(\mathbb{Q}_p, V_f) = \dim_L t_{V_f}(L) + \dim_L H^0(\mathbb{Q}_p, V_f).$$

Finally (iii) is a deep result of Kato [46, Theorem 14.2]. Note that in the semistable case we assume (see (31)) that $L(f, k/2) \neq 0$ and therefore the assumptions of Kato's theorem hold.

From (i)–(iii) above it follows that V_f satisfies the conditions (1)–(3) of § 2.2.1. Assume that $k_0 \geqslant \frac{k+1}{2}$. This holds automatically in the semistable $(k_0 = k/2)$ and potentially crystalline ($k_0 = \frac{k+1}{2}$) cases. In the crystalline case $\alpha^* = \varepsilon^{-1}(p)\alpha$ is a root of $1 - \bar{a}_p X + \varepsilon^{-1}(p)p^{k-1}X^2$ and using the functional equation for p-adic L-functions one can reduce the study of $L_{p,\alpha}(f, \omega^{k_0}, s)$ at $s = \frac{k-1}{2}$ to the study of $L_{p,\alpha^*}(f^*, \omega^{k_0+1}, s)$ at $s = \frac{k+1}{2}$.

Lemma 4.2.2. *Assume that $L_{p,\alpha}(f, \omega^{k_0}, s)$ has a trivial zero on the right of the central point (i.e. $k_0 \geqslant k/2$). Then $D_\alpha = \mathbf{D}_{\mathrm{cris}}(V_f)^{\varphi = p^{-1}}$ is a one-dimensional L-vector space which satisfies one of the conditions (a), (b) of Proposition 2.1.2.*

Proof. From 4.1.2 it follows that $\alpha p^{-k_0} = p^{-1}$ is an eigenvalue of φ acting on $\mathbf{D}_{\mathrm{cris}}(V_f)$. Thus $\dim_L D_\alpha \geqslant 1$. If $\dim_L D_\alpha = 2$ then V_f will be crystalline and φ will act on $\mathbf{D}_{\mathrm{cris}}(V_f)$ as multiplication by p^{-1}. This contradicts the weak admissibility of $\mathbf{D}_{\mathrm{cris}}(V_f)$. Finally D_α satisfies (a) in the crystalline and potentially crystalline cases and (b) in the semistable case. □

4.2.3. The ℓ-invariant of modular forms. From Lemma 4.2.2 it follows that if $L_{p,\alpha}(f, \omega^{k_0}, s)$ has a trivial zero at $k_0 \geqslant k/2$, the ℓ-invariant $\ell(V_f, D_\alpha)$ is well defined. To simplify notation we will denote it by $\ell_\alpha(f)$. The general definition of the ℓ-invariant can be made more explicit in the case of modular forms.

- *The semistable case.* Let $\{e_\alpha, e_\beta\}$ denote the basis of $\mathbf{D}_{\mathrm{st}}(W_f)$ as in 4.1.3. In [7, Proposition 2.3.7] it is proved that

$$\ell_\alpha(f) = \mathscr{L}_{\mathrm{FM}}(f) \tag{32}$$

where $\mathscr{L}_{\mathrm{FM}}(f)$ is the Fontaine–Mazur invariant [54] which is defined as the unique element of L such that

$$e_\beta + \mathscr{L}_{\mathrm{FM}}(f)\, e_\alpha \in \mathrm{Fil}^{k-1}\mathbf{D}_{\mathrm{st}}(W_f).$$

- *The crystalline and potentially crystalline cases.* The (φ, Γ)-module $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_f) \cap \big(D_\alpha \otimes_L \mathscr{R}_L[1/t]\big)$ is isomorphic to $\mathscr{R}_L(\delta)$ with $\delta(x) = |x| x^{\frac{k+1}{2}}$ and the exact sequence (14) takes the form

$$0 \to \mathscr{R}_L(\delta) \to \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_f) \to \mathscr{R}_L(\delta') \to 0$$

for some character $\delta' : \mathbb{Q}_p^* \to L^*$. Since $\dim_L H^1(\mathscr{R}_L(\delta)) = 2$ we have $H^1(\mathbb{Q}_p, V_f) \simeq H^1(\mathscr{R}_L(\delta))$. Therefore $H^1_{D_\alpha}(V_f) = H^1_{f,\{p\}}(\mathbb{Q}, V_f)$ and $\mathscr{L}_\alpha(f) = \mathscr{L}(V_f, D_\alpha)$ is the slope of the image of the localization map $H^1_{f,\{p\}}(\mathbb{Q}, V_f) \to H^1(\mathbb{Q}_p, V_f)$ under the canonical decomposition (6)

$$H^1(\mathbb{Q}_p, V_f) \simeq H^1_f(\mathscr{R}_L(\delta)) \times H^1_c(\mathscr{R}_L(\delta)).$$

The formula (22) gives

$$\ell_\alpha(f) = -\mathscr{L}_\alpha(f). \tag{33}$$

4.3. The main result

4.3.1. Kato's Euler systems. Using the theory of modular units, Kato [46] constructed an element $\mathbf{z}_{\mathrm{Kato}} \in H^1_{\mathrm{Iw},S}(W_{f^*})$ which is closely related to the complex

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and the p-adic L-functions via the Bloch–Kato exponential map. The CM case was considered before by Rubin [72]. Set

$$\mathbf{z}_{\mathrm{Kato}}(j) = \mathrm{Tw}_j^{\varepsilon}(\mathbf{z}_{\mathrm{Kato}}) \in H^1_{\mathrm{Iw},S}(W_{f^*}(j))$$

and denote by $\mathbf{z}_{\mathrm{Kato}}(j)_0 = \mathrm{pr}_0(\mathbf{z}_{\mathrm{Kato}}(j))$ the projection of $\mathbf{z}_{\mathrm{Kato}}(j)$ on $H^1_S(W_{f^*}(j))$. The following statements are direct analogues of properties (1), (2) of cyclotomic units from § 3.3.1:

(1) *Relation to the complex L-function.* One has

$$\exp^*_{W_{f^*}(j)}\left(\mathrm{loc}_p\left(\mathbf{z}_{\mathrm{Kato}}(j)_0\right)\right) = \Gamma(k-j)^{-1} E_p\left(f, p^{k-j}\right) \widetilde{L}(f, k-j)\, \omega_j^*, \quad 1 \leqslant j \leqslant k-1, \tag{34}$$

for some canonical basis ω_j^* of $\mathrm{Fil}^0 \mathbf{D}_{\mathrm{cris}}(W_f^*(j))$ [46, Theorem 12.5]. Note that $\omega_{j+1}^* = \omega_j^* \otimes e_1$ where $e_1 = \varepsilon^{-1} \otimes t$.

(2) *Relation to the p-adic L-function.* Fix a generator d_α of D_α. Let $\mathfrak{L}^{(\alpha),\varepsilon}_{W_{f^*}(k),1}$ denote the large logarithmic map $\mathfrak{L}^{(\varepsilon)}_{W_{f^*}(k),1,\eta}$ associated with $\eta = d_\alpha \otimes e_{\frac{k+1}{2}} \in \mathbf{D}_{\mathrm{cris}}(W_f)$. Then

$$\mathfrak{L}^{(\alpha),\varepsilon}_{W_{f^*}(k),1}\left(\mathrm{loc}_p(\mathbf{z}_{\mathrm{Kato}}(k))\right) = \mathbf{M}(\lambda_{f,\alpha}) \left[d_\alpha \otimes e_{\frac{k+1}{2}}, \omega_k^*\right]_{W_f} \tag{35}$$

[46, Theorem 16.2]. We can now prove the main result of this paper.

Theorem 4.3.2. *Let f be a newform on $\Gamma_0(N)$ of character ε and weight k and let p be an odd prime. Assume that the p-adic L-function $L_{p,\alpha}\left(f, \omega^{k_0}, s\right)$ has a trivial zero at $s = k_0 \geqslant k/2$. Then*

$$L'_{p,\alpha}(f, \omega^{k_0}, k_0) = \ell_\alpha(f) \left(1 - \frac{\varepsilon(p)}{p}\right) \widetilde{L}(f, k_0)\,.$$

Proof. To simplify notation set $\mathbf{z} = \mathbf{z}_{\mathrm{Kato}}(k - k_0)$. By Lemma 1.3.5 one has

$$\mathfrak{L}^{(\alpha),\varepsilon}_{V_f^*(1),1-k_0}(\mathrm{loc}_p(\mathbf{z})) = \mathrm{Tw}_{k_0}\left(\mathfrak{L}^{(\alpha),\varepsilon}_{W_{f^*}(k),1}(\mathrm{loc}_p(\mathbf{z}_{\mathrm{Kato}}(k)))\right).$$

Let $\mu_{\mathbf{z}}$ be the distribution defined by $\mathbf{M}(\mu_{\mathbf{z}}) = \mathfrak{L}^{(\alpha),\varepsilon}_{V_f^*(1),1-k_0}(\mathrm{loc}_p(\mathbf{z}))$. Then (35) gives

$$\mathbf{M}(\mu_{\mathbf{z}}) = \mathrm{Tw}_{k_0}\left(\mathbf{M}(\lambda_{f,\alpha})\right) [d_\alpha \otimes e_{k_0}, \omega_k^*]_{W_f} = \mathrm{Tw}_{k_0}\left(\mathbf{M}(\lambda_{f,\alpha})\right) [d_\alpha, \omega_{k_0-1}^*]_{V_f}$$

and from (12) and (28) it follows that

$$L_p(\mu_{\mathbf{z}}, s) = L_{p,\alpha}\left(f, \omega^{k_0}, s + k_0\right) \left[d_\alpha, \omega_{k_0-1}^*\right]_{V_f}.$$

Now, applying Proposition 2.2.2 we obtain

$$L'_{p,\alpha}\left(f, \omega^{k_0}, k_0\right) \left[d_\alpha, \omega_{k_0-1}^*\right]_{V_f} = \ell_\alpha(f)\, \Gamma(k_0) \left(1 - \frac{1}{p}\right)^{-1} \left[d_\alpha, \exp^*_{V_f^*(1)}(\mathrm{loc}_p(\mathbf{z}_0))\right]_{V_f}. \tag{36}$$

On the other hand, for $j = k - k_0$ the formula (34) gives

$$\exp^*_{V_f^*(1)}(\mathrm{loc}_p(\mathbf{z}_0)) = \Gamma(k_0)^{-1} E_p\left(f, p^{k_0}\right) \widetilde{L}(f, k_0)\, \omega_{k_0-1}^*. \tag{37}$$

Since $E_p\left(f, p^{k_0}\right) = \left(1 - \frac{1}{p}\right)\left(1 - \frac{\varepsilon(p)}{p}\right)$ and $\left[d_\alpha, \omega^*_{k_0-1}\right]_{V_f} \neq 0$, from (36) and (37) we obtain that

$$L'_{p,\alpha}\left(f, \omega^{k_0}, k_0\right) = \ell_\alpha(f)\left(1 - \frac{\varepsilon(p)}{p}\right)\widetilde{L}(f, k_0)$$

and the theorem is proved. □

Corollaries 4.3.3. (1) *In the semistable case, k is even and $\varepsilon(p) = 0$. Theorem 4.3.2 together with (32) gives the Mazur–Tate–Teitelbaum conjecture*

$$L'_{p,\alpha}(f, \omega^{k/2}, k/2) = \mathscr{L}_{\mathrm{FM}}(f)\widetilde{L}(f, k/2)$$

and our proof can be seen as a revisiting of the Kato–Kurihara–Tsuji approach using the theory of (φ, Γ)-modules.

(2) *In the crystalline and potentially crystalline cases, Theorem 4.3.2 reads*

$$L'_{p,\alpha}\left(f, \omega^{\frac{k+1}{2}}, \frac{k+1}{2}\right) = -\mathscr{L}_\alpha(f)\left(1 - \frac{\varepsilon(p)}{p}\right)\widetilde{L}\left(f, \frac{k+1}{2}\right)$$

(see (33)).

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