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SURFACE GROUPS WITHIN BAUMSLAG DOUBLES

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Dedicated to Colin Maclachlan on the occasion of his 70th birthday

Abstract A conjecture of Gromov states that a one-ended word-hyperbolic group must contain a subgroup that is isomorphic to the fundamental group of a closed hyperbolic surface. Recent papers by Gordon and Wilton and by Kim and Wilton give sufficient conditions for hyperbolic surface groups to be embedded in a hyperbolic Baumslag double G. Using Nielsen cancellation methods based on techniques from previous work by the second author, we prove that a hyperbolic orientable surface group of genus 2 is embedded in a hyperbolic Baumslag double if and only if the amalgamated word W is a commutator: that is, $W = [U, V]$ for some elements $U, V \in F$. Furthermore, a hyperbolic Baumslag double G contains a non-orientable surface group of genus 4 if and only if $W = X^2Y^2$ for some $X, Y \in F$. G can contain no non-orientable surface group of smaller genus.

Keywords: hyperbolic group; orientable surface group; quadratic word

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1. Introduction

A *Baumslag double* is an amalgamated product of the form

$$G = F \underset{\{W=\bar{W}\}}{\star} \bar{F},$$

where F is a finitely generated free group, $\bar{F}$ is an isomorphic copy, W is a non-trivial word in F and $\bar{W}$ is its copy in $\bar{F}$. An orientable surface group of genus 2 is a Baumslag double and, in fact, Baumslag doubles were introduced in [**2**] to prove that surface groups are residually free. If G is a Baumslag double and if the identified word W is not a proper power in F, it follows from the combination theorems of Juhasz and Rosenberger [**7**], Kharlampovich and Myasnikov [**8**] and Bestvina and Feighn [**3**] that the group G is hyperbolic. In fact, the Baumslag double G is hyperbolic if and only if W is not a proper power in F because W is a proper power in F if and only if $\bar{W}$ is a proper power in $\bar{F}$.

An open conjecture of Gromov [**6**] states that a one-ended word-hyperbolic group must contain a subgroup that is isomorphic to the fundamental group of a closed hyperbolic surface. Recent work by Gordon and Wilton [**6**] and by Kim and Wilton [**9**] gives sufficient conditions for hyperbolic surface groups to be embedded in a Baumslag double G. The work of Gordon and Wilton uses group cohomology and 3-manifold theory, while that of Kim and Wilton proceeds by realizing a Baumslag double as the fundamental group of a non-positively curved square complex.

In this paper, we use Nielsen cancellation methods based on the techniques in [**12**] to prove that a hyperbolic orientable surface group of genus 2 is embedded in a hyperbolic Baumslag double if and only if the amalgamated word W is a commutator: that is, $W = [U, V]$ for some elements $U, V \in F$. Since an orientable surface group of genus 2 contains surface groups of all finite genus, it follows that G contains hyperbolic surface groups of all finite genus if and only if W is a commutator in F. Furthermore, a Baumslag double G contains a non-orientable surface group of genus 4 if and only if $W = X^2Y^2$ for some $X, Y \in F$.

2. Main result

As mentioned in § 1 it follows from the combination theorems of Juhasz and Rosenberger [**7**], Kharlampovich and Myasnikov [**8**] and Bestvina and Feighn [**3**] that the Baumslag double

$$G = F \underset{\{W=\bar{W}\}}{\star} \bar{F}$$

is hyperbolic if and only if the identified word W is not a proper power in F. We call such a group a *hyperbolic Baumslag double*. Here we assume that W is a reduced word in the free group F. Our main result is the following.

Theorem 2.1. *Let*

$$G = F \underset{\{W=\bar{W}\}}{\star} \bar{F}$$

be a hyperbolic Baumslag double. Then G contains a hyperbolic orientable surface group of genus 2 if and only if W is a commutator: that is, $W = [U, V]$ for some elements $U, V \in F$. Furthermore, a Baumslag double G contains a non-orientable surface group of genus 4 if and only if $W = X^2Y^2$ for some $X, Y \in F$.

Since an orientable surface group of genus 2 contains an orientable surface group of any finite genus as a subgroup, we immediately get the following corollary.

Corollary 2.2. *Let*

$$G = F \underset{\{W=\bar{W}\}}{\star} \bar{F}$$

be a hyperbolic Baumslag double. Then G contains orientable surface groups of all finite genus if and only if W is a commutator.

Before giving the proof we recall some material about surface groups and cyclically pinched one-relator groups.

A *surface group* is the fundamental group of a compact orientable or non-orientable surface. If the genus of the surface is g, then we say that the corresponding surface group also has genus g.

An orientable surface group S_g of genus $g \geqslant 1$ has a one-relator presentation of the form

$$S_g = \langle a_1, b_1, \dots, a_g, b_g; [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle,$$

while a non-orientable surface group T_g of genus $g \geqslant 1$ also has a one-relator presentation, which now has the form

$$T_g = \langle a_1, a_2, \dots, a_g; a_1^2 a_2^2 \cdots a_g^2 = 1 \rangle.$$

Much of combinatorial group theory originally arose out of the theory of one-relator groups and the concepts and ideas surrounding the Freiheitssatz or Independence Theorem of Magnus (see [**11**] or [**10**]). Going backwards, the ideas of the Freiheitssatz were motivated by the topological properties of surface groups [**1**].

The algebraic generalization of the one-relator presentation type of a surface group presentation leads to *cyclically pinched one-relator groups*. These groups have the same general form as a surface group and have proved to be quite amenable to study. In particular, a *cyclically pinched one-relator group* is a one-relator group of the following form:

$$G = \langle a_1, \dots, a_p, a_{p+1}, \dots, a_n; U = V \rangle,$$

where $1 \neq U = U(a_1, \dots, a_p)$ is a cyclically reduced, non-primitive (i.e. not part of a free basis) word in the free group F_1 on $a_1, \dots, a_p$ and where $1 \neq V = V(a_{p+1}, \dots, a_n)$ is a cyclically reduced, non-primitive word in the free group F_2 on $a_{p+1}, \dots, a_n$.

Clearly, such a group is the free product of the free groups on $a_1, \dots, a_p$ and $a_{p+1}, \dots, a_n$, respectively, amalgamated over the cyclic subgroups generated by U and V. Cyclically pinched one-relator groups have been shown to be extremely similar to surface groups [**1**].

A cyclically pinched one-relator group is hyperbolic if either U or V is not a proper power in its respective free group factor [**3, 7, 8**].

In [**2**], Baumslag introduced a double of a free group, now called a *Baumslag double*, in order to prove that orientable surface groups are residually free. In that paper he also proved that if neither U nor V is a proper power, then a cyclically pinched one-relator group is 2-free: that is, any 2-generator subgroup must be free. This was generalized by Rosenberger, who proved the following result [**12**, Theorem 3.3, p. 335] using Nielsen cancellation methods. This result is one of the bases for the proof of Theorem 2.1.

Theorem 2.3 (Rosenberger [12]). *Let G be a cyclically pinched one-relator group of the form*

$$G = \langle a_1, \dots, a_p, a_{p+1}, \dots, a_n; W = V \rangle,$$

where $1 \neq W = W(a_1, \dots, a_p)$ is a cyclically reduced, non-primitive (i.e. not part of a free basis) word in the free group F_1 on $a_1, \dots, a_p$ and where $1 \neq V = V(a_{p+1}, \dots, a_n)$ is a cyclically reduced, non-primitive word in the free group F_2 on $a_{p+1}, \dots, a_n$. Suppose that neither W nor V is a proper power in its respective free group factor. Then we have the following.

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(a) *Every subgroup $H \subset G$ of rank 3 is free of rank 3.*

(b) *Let $H \subset G$ be a subgroup of rank 4. One of the following two cases then occurs.*

(i) *H is free of rank 4.*

(ii) *If $\{x_1, x_2, x_3, x_4\}$ is a generating system of H, then there is a Nielsen transformation from $\{x_1, x_2, x_3, x_4\}$ to a system $\{y_1, \dots, y_n\}$ with $y_1, y_2 \in zF_1z^{-1}$ and $y_3, y_4 \in zF_2z^{-1}$ for some $z \in G$. Moreover, there is a one-relator presentation for H on the generating system $\{x_1, x_2, x_3, x_4\}$.*

Before presenting the proof we need two other ideas concerning Nielsen cancellation in free products with amalgamation. A word $w \in F$, where F is a free group on $x_1, \dots, x_n$, is *regular* if there exists no automorphism $\alpha : F \mapsto F$ such that $\alpha(w) = w'$, when written as a word in $x_1, \dots, x_n$, contains fewer of the generators than w itself does. An ordered set $U = \{u_1, \dots, u_n\} \subset F$ is regular if there exists no Nielsen transformation from U to a system $U' = \{u'_1, \dots, u'_n\}$ in which one of the elements equals 1. This type of regularity is extended to free products with amalgamation in the following way. Suppose that G is a free product with amalgamation with factors H_1 and H_2 such that $G = H_1 \star_A H_2$. An ordered set $U = \{u_1, \dots, u_n\} \subset G$ is then regular if there exists no Nielsen transformation from U to a system $U' = \{u'_1, \dots, u'_n\}$ in which one of the elements is conjugate to an element of A.

The other crucial result for the proof of Theorem 2.1 is the following technical theorem [**5**, Theorem 5.3]. Recall that if F is a free group on $X = \{x_1, \dots, x_n\}$, then a reduced word $w = w(x_1, \dots, x_n)$ is a *quadratic word* if each x_i, which appears in w as x_i or x_i^{-1}, appears exactly twice. For example, the surface group word of genus 2, $[x_1, x_2][x_3, x_4] = x_1^{-1}x_2^{-1}x_1x_2x_3^{-1}x_4^{-1}x_3x_4$, is a quadratic word.

Theorem 2.4 (Fine *et al.* [5]). *Suppose that $G = H_1 \star_A H_2$ with $H_1 \neq A \neq H_2$ and A malnormal in both H_1 and H_2. Let F be a free group of rank n with $1 \leqslant n \leqslant 4$ and let $1 \neq w = w(x_1, \dots, x_n)$ be a regular quadratic word on the ordered basis $X = \{x_1, \dots, x_n\}$. Furthermore, let $\phi : F \mapsto G$ be a homomorphism such that $U = \phi(X)$ is regular in G and $\phi(w) = 1$. Then the pair (w, U) is Nielsen equivalent to a pair $(w', U') = (\alpha(w), \alpha^{-1}(U))$ with $\alpha : F \mapsto F$ an automorphism such that*

(1) *$w' = w_1w_2$, where w_1, w_2 are also quadratic in F,*

(2) *for $i = 1, 2$ we have that $\phi(\alpha^{-1}(w_i))$ is conjugate to an element of A and*

(3) *for $i = 1, 2$ there is a $\nu_i \in \{1, 2\}$ and a $g_i \in G$ with $\phi(\alpha^{-1}(x_j)) \in g_iH_{\nu_i}g_i^{-1}$ for each x_j that occurs in w_i.*

We now give the proof of Theorem 2.1.

Proof of Theorem 2.1. Suppose that

$$G = F \underset{\{W=\bar{W}\}}{\star} \bar{F}$$

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is a hyperbolic Baumslag double, where F is a free group on $X = \{x_1, \dots, x_n\}$. Since we are assuming hyperbolicity, we have that W and hence $\bar{W}$ are not proper powers. Furthermore, if W were either trivial or primitive in F, then G would be a free group, so G could not contain a surface group that is either orientable or non-orientable. Furthermore, if G contains a surface group, then G cannot be a free group and hence W is neither trivial nor primitive. Therefore, we may assume that the amalgamated word W is neither trivial nor primitive.

We consider the orientable case first. Suppose that $W = [u, v]$ in F. Then

$$W(\bar{W})^{-1} = [u, v]([\bar{u}, \bar{v}])^{-1} = [u, v][\bar{v}, \bar{u}] = 1.$$

Consider the subgroup $H = \langle u, v, \bar{u}, \bar{v} \rangle$ of G. We can see that H cannot be a free group by applying Theorems 2.3 and 2.4 to the equation $[u, v][\bar{u}, \bar{v}] = 1$ in G and from the fact that G does not contain a free abelian group of rank 2. Hence H is a one-relator group of rank 4 by Theorem 2.3. We show that a defining relation is precisely $[u, v][\bar{v}, \bar{u}] = 1$.

Let G be a cyclically pinched one-relator group of the form

$$G = \langle a_1, \dots, a_p, a_{p+1}, \dots, a_n; W = V \rangle$$

as in Theorem 2.3. Let $H = \langle x_1, x_2, x_3, x_4 \rangle$ be a rank 4 subgroup of G. Within the proof of Theorem 2.3 [**12**, Theorem 3.3, pp. 335–340] it is shown that if H is not free, then not only is H a one-relator group but a method is described showing how to obtain a defining relation for H [**12**, p. 340]. This is done in the following manner. If there is a Nielsen transformation from $\{x_1, x_2, x_3, x_4\}$ to a system where one element is conjugate to an element in the amalgamated subgroup, then H is free of rank 4. Now assume that H is not free of rank 4. Then, by the statement of Theorem 2.3, G is a one-relator group, and we may assume, possibly after a Nielsen transformation and a conjugation, that x_1, x_2 are in F_1, the free group on $a_1, \dots, a_p$, that x_3, x_4 are in F_2, the free group on $a_{p+1}, \dots, a_n$, and that W is in $\langle x_1, x_2 \rangle$ or V is in $\langle x_3, x_4 \rangle$. Let W be in $\langle x_1, x_2 \rangle$. We consider the subgroup $K = \langle V, x_3, x_4 \rangle$ in F_2. (Recall that $W = V$ in G.) K cannot be a free group of rank 3 because otherwise H is free of rank 4. Hence K is a one-relator group in V, x_3, x_4 and therefore H is a one-relator group in x_1, x_2, x_3, x_4. The relation is obtained as follows. Take the relation for K and replace V by W as a word in x_1, x_2.

If we apply this to the hyperbolic Baumslag double with $W = [u, v]$, we must consider the situation where we have a free group $F = \langle a, b;\ \rangle$ of rank 2 generated by a system $\{r, s, [a, b]\}$. However, if $F = \langle a, b;\ \rangle$ is generated by a system $\{r, s, [a, b]\}$, it follows from [**12**, Lemma 3.17] (see also [**13**, Hilfsatz 5]) that there is a free Nielsen transformation $\mathcal{T}$ from $\{r, s, [a, b]\}$ to $\{a, b, [a, b]\}$, where $[a, b]$ is not replaced. Not replaced means that in all the elementary Nielsen transformations of which $\mathcal{T}$ is composed, the commutator $[a, b]$ either remains unchanged, is changed to $([a, b])^{-1}$ or is put in a different location in the respective triple (see [**12**, pp. 335–340] for more details). In the hyperbolic Baumslag double the transformations are identical in the other factor. Therefore, $[u, v] = [\bar{u}, \bar{v}]$ must be a defining relation for H. It follows that H is an orientable surface group of genus 2 and G contains such a subgroup.

Conversely, let H be a subgroup of G that is an orientable surface group of genus 2. Hence H has a presentation

$$H = \langle x_1, x_2, x_3, x_4; [x_1, x_2][x_3, x_4] = 1 \rangle.$$

Consider the system $\{x_1, x_2, x_3, x_4\} \subset G$ and apply Nielsen cancellations within the amalgamated free product G with respect to the quadratic word $v = [x_1, x_2][x_3, x_4]$.

The system $\{x_1, x_2, x_3, x_4\}$ is regular. That is, there is no Nielsen transformation from $\{x_1, x_2, x_3, x_4\}$ to a system that contains an element that is conjugate in G to a power of W or $\bar{W}$. If the system $\{x_1, \dots, x_n\}$ was not regular, then H would have to be a free group from [**12**, Lemma 3.1]. Now we apply Theorem 2.4 to $X = \{x_1, x_2, x_3, x_4\}$ and $w = [x_1, x_2][x_3, x_4]$. Then $w' = \alpha(w) = w_1 w_2$, with w_1 and w_2 both quadratic words as described in Theorem 2.4. Since w is a product of commutators in the hyperbolic Baumslag double G and α is an automorphism, it follows that w' is alternating in the same way as w: that is, each x_i occurs in w' exactly once as x_i and exactly once as x_i^{-1}. Since both w_1 and w_2 are quadratic and F is a non-abelian free group, this implies, up to conjugation and renaming, that $w_1 = [x, y]$ for some $x, y \in F$. Recall that a free group word $[a, b][c, d]$ is not Nielsen equivalent to a word $r^2 s^2 t^2 p^2$, otherwise an orientable surface group of genus 2 would be isomorphic to a non-orientable surface group of genus 4. That this cannot happen is clear from abelianization. Since the amalgamated subgroup $A = \langle W \rangle$ is cyclic and w_1 is conjugate to an element of A, it follows that $[x, y]$ is conjugate to W^n for some non-zero $n \in \mathbb{Z}$. However, since a commutator in a free group is never a proper power (see [**10**, p. 52] or [**4**]) this implies that $[x, y]$ is conjugate to W or W^{-1}. Since a conjugate of a commutator is also a commutator, it follows that $W = [U, V]$ for some elements $U, V \in F$, proving the theorem in the orientable case.

The proof for the non-orientable case is almost identical except that when we get $w = w_1 w_2$ we must have $w_1 = x^2 y^2$ for some $x, y \in F$. We must also use the analogous argument that if a free group $F = \langle a, b; \ \rangle$ is generated by a system $\{r, s, a^2 b^2\}$, then there is a free Nielsen transformation from $\{r, s, a^2 b^2\}$ to $\{a, b, a^2 b^2\}$ where $a^2 b^2$ is not replaced. As in the orientable case, this follows from Lemma 3.17 in [**12**] and the remark immediately after that lemma. □

We note that a hyperbolic Baumslag double can never contain an orientable surface group of genus 1 (that is, a free abelian group of rank 2) and can never contain a non-orientable surface group of genus less than or equal to 3 by Theorem 2.4.

References

1. P. Ackermann, B. Fine and G. Rosenberger, Surface groups: motivating examples in combinatorial group theory, in *Groups St Andrews 2005*, London Mathematical Society Lecture Notes Series, Volume 339, pp. 96–130 (Cambridge University Press, 2007).
2. G. Baumslag, On generalised free products, *Math. Z.* **78** (1962), 423–438.
3. M. Bestvina and M. Feighn, A combination theorem for negatively curved groups, *J. Diff. Geom.* **35** (1992), 85–101.
4. L. Comerford, C. Edmonds and G. Rosenberger, Commutators as powers in free products, *Proc. Am. Math. Soc.* **122** (1994), 47–52.

5. B. Fine, G. Rosenberger and M. Stille, Nielsen transformations and applications: a survey, in *Groups Korea 94* (De Gruyter, Berlin, 1995).
6. C. Gordon and H. Wilton, Surface subgroups of doubles of free groups, in preparation.
7. A. Juhasz and G. Rosenberger, On the combinatorial curvature of groups of F-type and other one-relator products of cyclics, *Contemp. Math.* **169** (1994), 373–384.
8. O. Kharlampovich and A. Myasnikov, Hyperbolic groups and free constructions, *Trans. Am. Math. Soc.* **350**(2) (1998), 571–613.
9. S. Kim and H. Wilton, Surface subgroups of doubles of free groups, in preparation.
10. R. Lyndon and P. E. Schupp, *Combinatorial group theory* (Springer, 1977).
11. W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory* (Wiley Interscience, 1966).
12. G. Rosenberger, On one-relaor groups that are free products of two free groups with cyclic amalgamation, in *Groups St Andrews 1981*, pp. 328–344 (Cambridge University Press, 1981).
13. H. Zieschang, Über die Nielsensche Küerzungmethode in freien Produkten mit Amalgam, *Invent. Math.* **10** (1970), 4–37.