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Packing Loose Hamilton Cycles

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A subset C of edges in a k-uniform hypergraph H is a *loose Hamilton cycle* if C covers all the vertices of H and there exists a cyclic ordering of these vertices such that the edges in C are segments of that order and such that every two consecutive edges share exactly one vertex. The binomial random k-uniform hypergraph $H^k_{n,p}$ has vertex set $[n]$ and an edge set E obtained by adding each k-tuple $e \in \binom{[n]}{k}$ to E with probability p, independently at random.

Here we consider the problem of finding edge-disjoint loose Hamilton cycles covering all but $o(|E|)$ edges, referred to as the *packing problem*. While it is known that the threshold probability of the appearance of a loose Hamilton cycle in $H^k_{n,p}$ is

$$p = \Theta\left(\frac{\log n}{n^{k-1}}\right),$$

the best known bounds for the packing problem are around $p = \text{polylog}(n)/n$. Here we make substantial progress and prove the following asymptotically (up to a polylog(n) factor) best possible result: for $p \geqslant \log^C n/n^{k-1}$, a random k-uniform hypergraph $H^k_{n,p}$ with high probability contains

$$N := (1-o(1))\frac{\binom{n}{k}p}{n/(k-1)}$$

edge-disjoint loose Hamilton cycles.

Our proof utilizes and modifies the idea of 'online sprinkling' recently introduced by Vu and the first author.

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1. Introduction

For $k \in \mathbb{N}$, the binomial random k-uniform hypergraph $H^k_{n,p}$ consists of a vertex set

$$V(H^k_{n,p}) = [n] := \{1,\dots,n\}$$

and an edge set $E(H^k_{n,p})$ which is obtained by adding each k-tuple $e \in \binom{[n]}{k}$ to $E(H^k_{n,p})$ independently with probability p. Note that $H^2_{n,p}$ is the standard binomial random graph $G_{n,p}$.

The binomial random hypergraph model has introduced many natural problems analogous to those for the binomial random graph model. However, in the hypergraph setting, completely new

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techniques are often required. In this paper we utilize and modify a new approach due to Vu and the first author [5] to deal with the following problem (we discuss the precise definition of Hamilton cycles in hypergraphs below).

Problem 1.1. *Given a hypergraph H with m edges, is it possible to find a collection of edge-disjoint Hamilton cycles in H which covers all but $o(m)$ edges of H?*

The problem of packing Hamilton cycles in random graphs is well studied and is in fact completely solved (see *e.g.* [7], [9], [10] and the references therein). Moreover, in a recent paper [4], an asymptotically optimal solution for the *directed* random graph model is also given. Therefore, it is somewhat surprising that for the hypergraph case so little is known.

For a hypergraph $H = H^k_{n,p}$, there is more flexibility in the definition of a Hamilton cycle. Letting $1 \leqslant \ell \leqslant k-1$, a subset C of edges in H is a *type-ℓ Hamilton cycle* if C covers all the vertices of H and there exists a cyclic ordering of these vertices such that the edges in C are segments of that order and such that every two consecutive edges share exactly ℓ vertices. In this work, we study *loose Hamilton cycles* (or *loose cycles* for brevity) which are type-ℓ Hamilton cycles with $\ell = 1$. For a loose cycle, let $e_1, e_2, \ldots$ indicate its edges in an ordering induced by the ordering of the vertices. It follows from the definition that the sets $e_{i-1} \setminus e_i$ are disjoint sets of size $k-1$ that cover the entire vertex set. Therefore, $k-1$ divides n is a necessary condition for the existence of a loose cycle. Extending a result of Frieze [6], Dudek and Frieze proved the following in [1].

Theorem 1.2. *Assume that $2(k-1)$ divides n and $p = \omega \log n / n^{k-1}$, where $\omega = \omega_n$ is any function tending to infinity with n, then w.h.p.*[1] *there exists a loose Hamilton cycle in $H^k_{n,p}$.*

In [2], Dudek, Frieze, Loh and Speiss extended the result to $(k-1)|n$ (for a shorter proof see [3]). However, the packing problem for Hamilton cycles seems to be more difficult. In the paper of Frieze and Krivelevich [8], even after restricting the range of ℓ, some effort is required to show that an edge-disjoint packing exists for $p \gg \log^2 n/n$. This bound does not address the dependence on k and is significantly larger than the threshold at which a Hamilton cycle appears.

In what follows, we close the gap for the case of loose cycles by showing that up to a polylog term, the above lower bound in p for the appearance of a loose cycle also guarantees the existence of an asymptotically optimal packing. In particular, we prove the following.

Theorem 1.3. *Let $k \geqslant 3$ be an integer. Assume that $(k-1)|n$ and*

$$p \geqslant \frac{\log^{2k+2} n}{n^{k-1}}.$$

Then w.h.p. $H^k_{n,p}$ contains

$$\frac{\binom{n}{k} p}{n/(k-1)}(1+o(1))$$

edge-disjoint loose Hamilton cycles.

[1] A sequence of events $(\mathcal{E}_n)$ is said to occur with high probability (w.h.p.) if $\lim_{n\to\infty} \mathbb{P}(\mathcal{E}_n) = 1$. In this paper, all asymptotic notations assume that the parameter n tends to infinity.

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Remark. The number of packed loose cycles

$$\frac{\binom{n}{k}p}{n/(k-1)}(1+o(1))$$

is optimal in the sense that in expectation, there are $\binom{n}{k}p$ edges and there are $n/(k-1)$ edges in each loose cycle.

The proof of the main theorem builds on the idea of 'online sprinkling' introduced by the first author and Vu in [5]. Their idea is to embed a desired structure using a randomized algorithm while simultaneously exposing the random hypergraph. This general strategy embeds a particular structure in every round such that the remaining unexposed parts of the hypergraph are sufficiently random to iterate the algorithm. In their work, the structure is a perfect matching. For the problem of loose cycles, we implement 'online sprinkling' in a different way. In every step of our algorithm, the use of this sprinkling generates a long loose path that includes the majority of the vertices. To complete this embedding into a cycle, we construct an auxiliary hypergraph (as in another work of the first author [3]) and apply Theorem 1.2.

Our proof technique does not apply to type-l Hamilton cycles for l larger than 1. The reason is that on a subset of roughly αn vertices we expose all the possible k-tuples with the 'correct' probability

$$p = \frac{\text{polylog}(n)}{(\alpha n)^{k-\ell}}$$

(that is, the threshold for appearance of a type-ℓ cycle) in order to close only one cycle. This procedure is too wasteful, and fortunately for $\ell = 1$ we only need to pay a polylog(n) factor, but for larger ℓ we lose a polynomial factor. One way to overcome this problem may be to find a way to re-use the edges we exposed in order to close other paths. It would be very interesting to see such a result.

Conjecture 1.4. *Let $k > l \geqslant 2$. There exists a constant C such that if $(k-l)|n$ and*

$$p \geqslant \frac{\log^C n}{n^{k-l}},$$

then w.h.p. $H^k_{n,p}$ contains

$$\frac{\binom{n}{k}p}{n/(k-l)}(1+o(1))$$

edge-disjoint type-l Hamilton cycles.

2. Proof of Theorem 1.3

2.1. Outline

We begin by working under the assumption that

$$\frac{\log^{2k+2} n}{n^{k-1}} \leqslant p \leqslant \frac{2\log^{2k+2} n}{n^{k-1}}.$$

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We show how to remove this restriction at the end of the proof. Fix $\varepsilon > 0$; our goal is to show that w.h.p. there exist at least

$$(1-2\varepsilon)\frac{\binom{n}{k}p}{n/(k-1)}$$

edge-disjoint loose cycles in $H = H^k_{n,p}$ when n is sufficiently large. The proof of the result is algorithmic in nature. We introduce a randomized algorithm in Section 2.2 that will pack each loose Hamilton cycle one at a time as edges of the hypergraph are exposed. Our algorithm consists of

$$N = (1-\varepsilon)\frac{\binom{n}{k}p}{n/(k-1)} = O(\text{polylog}(n))$$

rounds, each of which produces one loose Hamilton cycle. Each round is further divided into several steps, where at each such time step, except the last one, the algorithm tries to extend a current loose path by one edge. In particular, the algorithm repeatedly tries to colour certain k-tuples with an appropriate edge probability, independently at random, until at least one k-tuple is coloured (it is allowed to expose the same k-tuple twice). A successful colouring also indicates a successful exposure of an edge in the hypergraph. The colouring is simply a tool to expose edges while labelling the successfully exposed edge with the time step. A round is terminated by closing the path to create a cycle. This is done by exposing and identifying a Hamilton cycle in an appropriately defined auxiliary hypergraph.

In Section 2.3, the analysis will show that w.h.p. the algorithm outputs the correct number of cycles and that the cycles are edge-disjoint. As the exposure of the underlying hypergraph is concurrent with the running of our algorithm and we allow multiple exposures of any k-tuple (meaning a k-tuple that was not successfully coloured can be considered again at the next step), it is also necessary to show that our exposure process generates a hypergraph that can be embedded in H. We use a coupling lemma to show that such an embedding exists if the random weight that every k-tuple accumulates during the algorithm is bounded by p. We then use a concentration result to guarantee the latter.

2.2. The algorithm

Let $\omega_n \leqslant \log n$ be a sequence tending to infinity with n to be chosen later. The following is a randomized algorithm which will generate

$$(1-2\varepsilon)\frac{\binom{n}{k}p}{n/(k-1)}$$

loose cycles w.h.p. for sufficiently large n. We divide our algorithm into

$$N = (1-\varepsilon)\frac{\binom{n}{k}p}{n/(k-1)}$$

independent rounds. In each round $1 \leqslant i \leqslant N$ we perform $K+1$ steps, where $K = \Theta(n)$ is a deterministic number to be chosen later. We denote step j in round i by step (i,j), where $1 \leqslant j \leqslant K+1$. One successful round will generate a loose Hamilton cycle.

For each round $i = 1,\ldots,N$, we proceed as follows.

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Generating the long path

Step 1. We randomly assign colour $(i,1)$ to each k-tuple of all the $\binom{n}{k}$ tuples independently with probability n^{-k}. We repeat this colouring procedure until at least one k-tuple is coloured or at most ω_n times. If we reach ω_n and no k-tuple has been successfully coloured, the entire round fails and we move to the next round. Otherwise, let T_1^i be the number of times we perform this colouring procedure; then T_1^i is a random variable taking values between 1 and ω_n. At time T_1^i, among the k-tuples that are coloured, we choose one uniformly at random and call it edge E_1^i. This is the first edge in our path. Choose an arbitrary ordering on E_1^i, say $E_1^i = (v_1^i, \dots, v_k^i)$.

Step $j(2 \leqslant j \leqslant K)$**.** At this point, we have already obtained $j-1$ (ordered) edges

$$E_s^i = (v^i_{(s-1)(k-1)+1}, \dots, v^i_{s(k-1)+1}) \quad (1 \leqslant s \leqslant j-1),$$

forming a loose path. Let R_j^i be the collection of all k-tuples whose intersection with the union of the above edges is exactly the last vertex $v^i_{(j-1)(k-1)+1}$. Randomly assign colour (i,j) to the k-tuples in R_j^i independently with probability q, where q is a deterministic number to be chosen later. We repeat this colouring procedure until at least one k-tuple is coloured or at most A_j times, where A_j is a deterministic number to be chosen later. If we reach A_j and no k-tuple has been successfully coloured, the entire round fails and we move to the next round. Otherwise, let T_j^i be the number of times we perform this colouring procedure; then T_j^i is a random variable taking values between 1 and A_j. At time T_j^i, among the k-tuples that are coloured, we choose one uniformly at random and call it edge E_j^i. Randomly choose an order on the last $k-1$ vertices of E_j^i, say

$$E_j^i = (v^i_{(j-1)(k-1)+1}, \dots, v^i_{j(k-1)+1}).$$

Closing the path

Step $K+1$**.** We will choose K so that the number of the remaining vertices is

$$n - K(k-1) - 1 = \alpha n, \tag{2.1}$$

where

$$\alpha = \frac{1}{\omega_n^3 \log n}(1+o(1))$$

with the $o(1)$ being chosen so that $2(k-1)|(\alpha n+1)$. Let V^i be the collection of the remaining vertices. Let v_0 be a dummy vertex which will later be appropriately replaced by either the first point v_1^i or the last point $v^i_{K(k-1)+1}$ of the long path.

Let R^i_{K+1} be the collection of all k-tuples e in $[n]$ such that e contains at least $k-1$ vertices in V^i, and $e \setminus V^i$, if not empty, is either $\{v_1^i\}$ or $\{v^i_{K(k-1)+1}\}$. Each k-tuple in R^i_{K+1} is assigned the colour $(i, K+1)$ randomly and independently with probability $r = \omega_n \log n / \alpha^{k-1} n^{k-1}$.

Having coloured the k-tuples in R^i_{K+1}, we generate an auxiliary k-uniform hypergraph G^i as follows. The vertex set is $V^i \cup \{v_0\}$. A k-tuple e' is included as an edge of G^i if one of the following holds:

- $e' \subset V^i$ and e' has colour $(i, K+1)$,

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- e' contains v_0 and either

$$\{v_1^i\} \cup e' \setminus \{v_0\} \quad \text{or} \quad \{v_{K(k-1)+1}^i\} \cup e' \setminus \{v_0\}$$

(or both) has colour $(i, K+1)$.

Thus, in the random hypergraph G^i, each k-tuple appears independently with probability at least r. We look for loose Hamilton cycles in G^i with v_0 at the intersection of two consecutive edges, one of which inherits the colour from an edge in R_{K+1}^i that contains v_1^i and the other inherits the colour from an edge in R_{K+1}^i that contains $v_{K(k-1)+1}^i$. If there exists at least one such cycle, we choose one at random. Otherwise, we repeat the colouring procedure in R_{K+1}^i (and thus obtain a new G^i) until at least one such cycle appears or at most ω_n times. Let T_{K+1}^i be the number of times we perform this colouring procedure. If we have repeated the procedure ω_n times and found no such cycle, the entire round fails and we move to the next round. Assume that the round does not fail at this step and we obtain a cycle as desired. That is, v_0 is contained in two consecutive edges e' and e'' such that

$$E_{n/(k-1)}^i := \{v_1^i\} \cup e' \setminus \{v_0\} \quad \text{and} \quad E_{K+1}^i := \{v_{K(k-1)+1}^i\} \cup e'' \setminus \{v_0\}$$

have colour $(i, K+1)$. Replace the edges e' and e'' of the cycle by $E_{n/(k-1)}^i$ and E_{K+1}^i respectively and insert the long path obtained from step 1 to step K between the two edges to form a loose Hamilton cycle on $[n]$. Return this loose cycle and move to the next round.

2.3. Properties of the algorithm

In this section we verify some properties of the above algorithm which will complete the proof of Theorem 1.3 for

$$\frac{\log^{2k+2} n}{n^{k-1}} \leqslant p \leqslant \frac{2\log^{2k+2} n}{n^{k-1}}.$$

At the end of this section, we will show how to remove this restriction.

For

$$\frac{\log^{2k+2} n}{n^{k-1}} \leqslant p \leqslant \frac{2\log^{2k+2} n}{n^{k-1}},$$

we choose the parameters mentioned in the above algorithm as follows. Let

$$\omega_n = \log^{1/6k} n, \quad \alpha = \frac{1}{\omega_n^3 \log n}(1+o(1)), \quad K = \frac{n - \alpha n - 1}{k-1}, \tag{2.2}$$

where we choose the $o(1)$ so that $2(k-1)|(\alpha n + 1)$. Further, let

$$r = \omega_n \log n / \alpha^{k-1} n^{k-1}, \quad q = \frac{1}{n^{k-1} \log n}, \quad A_j = \frac{2 \log n}{q\binom{n-(j-1)(k-1)-1}{k-1}} \tag{2.3}$$

for all $2 \leqslant j \leqslant K$. We ignore the rounding to integer values for many of these parameters as this makes no essential difference.

Lemma 2.1. *There are $N(1+o(1))$ successes among the N rounds w.h.p.*

Proof. First, we claim that each round succeeds w.h.p. Indeed, consider round 1 (say); step 1 fails if there is no coloured k-tuple after ω_n colouring procedures on all k-tuples. This happens

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with probability

$$(1-n^{-k})^{\omega_n\binom{n}{k}} \leqslant \exp\left(-n^{-k}\binom{n}{k}\omega_n\right) = o(1).$$

Similarly, the probability of failure in one of the steps from 2 to K is bounded from above by

$$\sum_{j=2}^{K}(1-q)^{A_j\binom{n-(j-1)(k-1)-1}{k-1}} \leqslant \sum_{j=2}^{K}\exp\left(-qA_j\binom{n-(j-1)(k-1)-1}{k-1}\right) = o(1) \qquad (2.4)$$

by the choices of q and A_j in (2.3).

Finally consider step $K+1$. By Theorem 1.2, w.h.p. there exists at least one loose cycle in G^1. By symmetry of vertices, with probability $1/(k-1)+o(1)$ there exists one loose cycle in G^1 with v_0 at the intersection of two consecutive edges. Conditioning on the appearance of such a cycle, with probability at least $1/2$, one of the two consecutive edges inherits the colour from an edge containing v_1^1 and the other edge v_{K+1}^1. In other words, with probability at least $1/2k+o(1)$, there exists a loose cycle with the prescribed properties in the algorithm. Since we repeat the experiment ω_n times, with probability at least

$$1-\left(1-\frac{1}{2k}+o(1)\right)^{\omega_n} = 1-o(1),$$

there exists one loose cycle with the above property.

In summary, each round succeeds w.h.p. Let f_n^{-1} be an upper bound for the probability of failure in one round where $f_n \to \infty$ with n. Then the expected number of failures is N/f_n. By Markov's inequality, with probability at least $1-f_n^{-1/2}$, the number of failures is at most $N/\sqrt{f_n}$. In that event, the number of successful rounds is at least $N-N/\sqrt{f_n} = N(1-o(1))$. □

Lemma 2.2. *The loose Hamilton cycles obtained from the successful rounds are edge-disjoint w.h.p.*

Proof. By the description of the algorithm, in each round, a k-tuple is considered at step 1 and at most one more step between step 2 and step $K+1$. Therefore, the probability that a k-tuple is coloured in a given round is bounded by

$$p' := \omega_n n^{-k} + \max\{qA_j, r\omega_n\} = O(\mathrm{polylog}(n)/n^{k-1}).$$

Since there are

$$N = (1-\varepsilon)\frac{\binom{n}{k}p}{n/(k-1)} = O(\mathrm{polylog}(n))$$

rounds, by applying the union bound we obtain that the probability of having an edge appearing in two loose cycles is bounded from above by

$$\binom{n}{k}\binom{N}{2}p'^2 = o(1).$$ □

Let H' be the hypergraph on $[n]$ consisting of all the k-tuples that are coloured in at least one step of the algorithm. For each k-tuple e, we define the random weight Q_e that e accumulates

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during the algorithm by

$$Q_e = 1 - \prod_{i=1}^{N}\left((1-n^{-k})^{T_1^i}(1-r)^{T_{K+1}^i \mathbf{1}_{e\in R_{K+1}^i}} \prod_{j=2}^{K}(1-q)^{T_j^i \mathbf{1}_{e\in R_j^i}}\right)$$
$$\leqslant \sum_{i=1}^{N}\left(n^{-k}T_1^i + \sum_{j=2}^{K} qT_j^i \mathbf{1}_{e\in R_j^i} + rT_{K+1}^i \mathbf{1}_{e\in R_{K+1}^i}\right) := \hat{Q}_e. \tag{2.5}$$

We will show later that $Q_e \leqslant p$ holds for all e w.h.p. We next demonstrate that when this holds, H' can be embedded into $H_{n,p}^k$. Intuitively, no k-tuple has accrued so much probability mass that its chance of being an edge is greater than p.

Lemma 2.3. *There exists a coupling of H' and $H = H_{n,p}^k$ such that whenever $Q_e \leqslant p$ holds for all e, we have $H' \subset H$.*

Proof. We introduce independent random variables, U_e, uniform on $[0,1]$ for each k-tuple e. Let H be the random hypergraph in which a k-tuple e is an edge if $U_e \leqslant p$. Observe that H is distributed as $H_{n,p}^k$.

Next, we construct a copy H'' of H'. Note that the algorithm in Section 2.2 consists of a series of queries, each of which asks whether a certain k-tuple will be assigned a colour with a particular probability of success. For notational convenience, we enumerate these queries by $1,2,\dots$ in the order that they are made. To construct H'', we will use the U_e's and some independent coin flips to answer these queries. We will recursively define certain partial sums, $\{S_e(t)\}_{t=0,1,\dots,}$, which keep track of the query probabilities.

Start the algorithm with $S_e(0) = 0$ for all k-tuples e. Assume that the algorithm is going to make the tth query and the $S_e(t-1)$ are already defined for all e. Assume that the tth query asks whether the k-tuple e_t receives a certain colour with probability q_t of success. We set $S_e(t) = S_e(t-1)$ for $e \neq e_t$ and set $S_e(t) = S_e(t-1) + q_t(1 - S_e(t-1))$ for $e = e_t$. Consider two cases, $U_{e_t} < S_{e_t}(t-1)$ and $U_{e_t} \geqslant S_{e_t}(t-1)$. In the former, toss a q_t-coin independent of all previous random variables to decide the result of the tth query. In the latter, the tth query returns success if and only if $U_{e_t} \in [S_{e_t}(t-1), S_{e_t}(t))$. Note that in either case, conditioned on the previous queries, the tth query returns success with probability q_t.

A k-tuple e is said to be an edge in H'' if it is successfully coloured in at least one query during the algorithm. Observe that

- H'' has the same distribution as H',
- at the last query of the algorithm, $S_e = Q_e$ for all e,
- a k-tuple e is coloured if and only if $U_e \leqslant S_e(t)$ for some t, or equivalently, $U_e \leqslant Q_e$.

Therefore, whenever $Q_e \leqslant p$ for all e, we have $H'' \subset H$ as desired. □

To show that w.h.p. $Q_e \leqslant p$ for all e, we will show that w.h.p. $\hat{Q}_e \leqslant p$ for all e. We first show that it holds in expectation.

Lemma 2.4. *For sufficiently large n, we have $\mathbb{E}(\hat{Q}_e) \leqslant p(1-\varepsilon/2)$ for every k-tuple e.*

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Proof. Since the N rounds are identically distributed, by linearity of expectation, we have

$$\frac{\mathbb{E}(\hat{Q}_e)}{N} = \mathbb{E}\left(n^{-k}T_1^1 + \sum_{j=2}^{K} qT_j^1 \mathbf{1}_{e\in R_j^1} + rT_{K+1}^1 \mathbf{1}_{e\in R_{K+1}^1}\right). \tag{2.6}$$

By symmetry, we have $\mathbb{E}(\hat{Q}_e) = \mathbb{E}(\hat{Q}_{e'})$ for all k-tuples e and e'. Thus,

$$\frac{\mathbb{E}(\hat{Q}_e)}{N} = \binom{n}{k}^{-1} \sum_{e'} \frac{\mathbb{E}(\hat{Q}_{e'})}{N},$$

where the sum runs over all k-tuples e'.

Using equation (2.6), the bounds $T_1^1 \leqslant \omega_n$ and $T_{K+1}^1 \leqslant \omega_n$, the fact that $\sum_{e'} \mathbf{1}_{e'\in R_j^1} = |R_j^1|$ for all $2 \leqslant j \leqslant K+1$, and the bound $|R_{K+1}^1| = O(\alpha^k n^k)$, we get

$$\begin{aligned}\frac{\mathbb{E}(\hat{Q}_e)}{N} &= \binom{n}{k}^{-1} \sum_{e'} \mathbb{E}\left(n^{-k}T_1^1 + \sum_{j=2}^{K} qT_j^1 \mathbf{1}_{e'\in R_j^1} + rT_{K+1}^1 \mathbf{1}_{e'\in R_{K+1}^1}\right) \\ &= O(\omega_n n^{-k} + \omega_n \alpha^k r) + \binom{n}{k}^{-1} \mathbb{E}\sum_{j=2}^{K} qT_j^1 |R_j^1|.\end{aligned}$$

Since T_j^1 is simply a truncated geometric random variable, we have

$$\mathbb{E}T_j^1 \leqslant \frac{1}{q|R_j^1|}.$$

Thus,

$$\begin{aligned}\frac{\mathbb{E}(\hat{Q}_e)}{N} &\leqslant O(\omega_n n^{-k} + \omega_n \alpha^k r) + \binom{n}{k}^{-1} K \\ &= o(p/N) + \frac{p(1-\varepsilon)}{N} \leqslant \frac{p}{N}(1-\varepsilon/2)\end{aligned}$$

as desired. □

Finally, we show that w.h.p. $\hat{Q}_e \leqslant p$ for all e. We make use of McDiarmid's concentration inequality [11].

Theorem 2.5. *Let $X_1, \ldots, X_t$ be independent random variables, with $a_k \leqslant X_k \leqslant b_k$ for each k. Let $S_t = \sum_{k=1}^{t} X_k$ and let $\mu = \mathbb{E}[S_t]$. Then for each $\lambda \geqslant 0$,*

$$\mathbb{P}[|S_t - \mu| \geqslant \lambda] \leqslant 2e^{-2\lambda^2/\Sigma(b_k-a_k)^2}.$$

Note that $\hat{Q}_e$ is the sum of N independent random variables, each of which is bounded by

$$n^{-k}\omega_n + r\omega_n + q \max_{2\leqslant j\leqslant K}\{A_j\} \leqslant 2\omega_n^2 \log n/\alpha^{k-1}n^{k-1}.$$

We know that $\mathbb{E}\hat{Q}_e \leqslant (1-\varepsilon/2)p$. Thus, by Theorem 2.5,

$$\begin{aligned}\mathbb{P}(\hat{Q}_e > p) \leqslant \mathbb{P}(|\hat{Q}_e - \mathbb{E}\hat{Q}_e| \geqslant \varepsilon p/3) &\leqslant 2\exp\left(-\frac{\varepsilon^2 p^2 \alpha^{2k-2} n^{2k-2}}{18N\omega_n^4 \log^2 n}\right) \\ &\leqslant n^{-\omega_n}\end{aligned}$$

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since we set $\omega_n = \log^{1/6k} n$ in (2.2). We take a union bound over all k-tuples of vertices to yield the claim and complete the proof of the theorem for the case

$$\frac{\log^{2k+2} n}{n^{k-1}} \leqslant p \leqslant \frac{2\log^{2k+2} n}{n^{k-1}}.$$

In the general case when p can be greater than

$$2\frac{\log^{2k+2} n}{n^{k-1}},$$

let

$$M = \left\lfloor \frac{pn^{k-1}}{\log^{2k+2} n} \right\rfloor.$$

Let $H = H^k_{n,p}$. Each edge in H is assigned a number from 1 to M uniformly and independently at random. Let H_i be the graph consisting of all edges assigned number i. Then H_i has the same distribution as $H^k_{n,p/M}$. We have shown that each H_i contains

$$\frac{p\binom{n}{k}}{Mn/(k-1)}(1+o(1))$$

disjoint loose cycles w.h.p. By the same argument with Markov's inequality as in the proof of Lemma 2.1, w.h.p. there are $M(1+o(1))$ graphs among the H_i's having the aforementioned property. Since the H_i are edge-disjoint, one can add up the number of loose cycles in each H_i and obtain the desired number of loose cycles in H.

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