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ON SOME WEIGHTED AVERAGE VALUES OF L-FUNCTIONS

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Abstract

Let $q \geq 2$ and $N \geq 1$ be integers. W. Zhang recently proved that for any fixed $\varepsilon > 0$ and $q^{\varepsilon} \leq N \leq q^{1/2-\varepsilon}$,

$$\sum_{\chi \neq \chi_0} \left| \sum_{n=1}^{N} \chi(n) \right|^2 |L(1, \chi)|^2 = (1 + o(1))\alpha_q q N,$$

where the sum is taken over all nonprincipal characters χ modulo q, $L(1, \chi)$ denotes the L-functions corresponding to χ, and $\alpha_q = q^{o(1)}$ is some explicit function of q. Here we improve this result and show that the same asymptotic formula holds in the essentially full range $q^{\varepsilon} \leq N \leq q^{1-\varepsilon}$.

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1. Introduction

For integers $q \geq 2$ and $N \geq 1$, we consider the average value

$$S(q; N) = \sum_{\chi \neq \chi_0} \left| \sum_{n=1}^{N} \chi(n) \right|^2 |L(1, \chi)|^2$$

taken over all nonprincipal characters χ modulo an integer $q \geq 2$, with L-functions $L(1, \chi)$ corresponding to χ, weighted by incomplete character sums.

Zhang [2] has given an asymptotic formula for $S(q; N)$ which is nontrivial when $q^{\varepsilon} \leq N \leq q^{1/2-\varepsilon}$, for any fixed $\varepsilon > 0$ and sufficiently large q.

In this article, we improve the error term of Zhang's formula, thereby making it nontrivial in the range $q^{\varepsilon} \leq N \leq q^{1-\varepsilon}$.

More precisely, let

$$\alpha_q = (\beta_q + \gamma_q) \frac{\varphi(q)^2}{q^2},$$

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where

$$\beta_q = \frac{\pi^2}{6} \prod_{\substack{p \mid q \\ p \text{ prime}}} \left(1 - \frac{1}{p^2}\right),$$

$$\gamma_q = \frac{\pi^2}{3\zeta(3)} \prod_{\substack{p \mid q \\ p \text{ prime}}} \left(1 - \frac{1}{p^2+p+1}\right) \sum_{\substack{m,n=1 \\ \gcd(nm(n+m),q)=1}} \frac{1}{nm(n+m)},$$

$\zeta(s)$ is the Riemann zeta-function, and $\varphi(q)$ denotes the Euler function.

It is shown in [2] that

$$S(q, N) = \alpha_q q N + O(\varphi(q) 2^{\omega(q)} (\log q)^2 + N^3 (\log q)^2), \tag{1}$$

where $\omega(q)$ is the number of prime divisors of q.

Since

$$2^{\omega(q)} \le \tau(q) = q^{o(1)} \quad \text{and} \quad \varphi(q) = q^{1+o(1)}, \tag{2}$$

where $\tau(q)$ is the number of positive integer divisors of q (see [1, Theorems 317 and 328]), we deduce that $\alpha_q = q^{o(1)}$; so the error term in (1) is of the form $O(q^{1+o(1)} + N^3 q^{o(1)})$.

In particular, the asymptotic formula (1) is nontrivial if $q^{\varepsilon} \le N \le q^{1/2-\varepsilon}$ for any fixed $\varepsilon > 0$ and q is large enough.

Here we present a more accurate estimate of a certain sum which arises in [2]; this allows us to essentially replace N^3 in (1) by $N^2 q^{o(1)}$, rendering the formula nontrivial in the range $q^{\varepsilon} \le N \le q^{1-\varepsilon}$.

2. Main result

THEOREM. *Let q, N be integers with $q > N \ge 1$. Then*

$$S(q, N) = \alpha_q q N + O(\varphi(q) 2^{\omega(q)} (\log q)^2 + N^2 q^{o(1)})$$

as $q \to \infty$.

PROOF. In [2] it was shown that that

$$S(q, N) = M_1 + M_2 + O(N^2 (\log q)^2),$$

where

$$M_1 = \varphi(q) \sum_{m,n=1}^{N} \sum_{\substack{u,v=1 \\ mu=nv}}^{q^2} \frac{1}{uv} \quad \text{and} \quad M_2 = \varphi(q) \sum_{m,n=1}^{N} \sum_{\substack{u,v=1 \\ mu \equiv nv \pmod q \\ mu \ne nv}}^{q^2} \frac{1}{uv}.$$

Furthermore, Zhang [2] has showed that

$$M_1 = \alpha_q q N + O(\varphi(q) 2^{\omega(q)} (\log q)^2).$$

Thus, it remains to show that

$$M_2 \le N^2 q^{o(1)}. \tag{3}$$

Let

$$J = \lfloor 2 \log q \rfloor.$$

Then, on changing the order of summation, we obtain

$$\begin{aligned}
M_2 &= \varphi(q) \sum_{u,v=1}^{q^2} \frac{1}{uv} \sum_{\substack{m,n=1 \\ mu \equiv nv \,(\mathrm{mod}\ q) \\ mu \ne nv}}^{N} 1 \\
&\le \varphi(q) \sum_{i,j=0}^{J} \sum_{e^i \le u < e^{i+1}} \frac{1}{u} \sum_{e^j \le v < e^{j+1}} \frac{1}{v} \sum_{\substack{m,n=1 \\ mu \equiv nv \,(\mathrm{mod}\ q) \\ mu \ne nv}}^{N} 1 \\
&\le 2\varphi(q) \sum_{0 \le i \le j \le J} \sum_{e^i \le u < e^{i+1}} \frac{1}{u} \sum_{e^j \le v < e^{j+1}} \frac{1}{v} \sum_{\substack{m,n=1 \\ mu \equiv nv \,(\mathrm{mod}\ q) \\ mu \ne nv}}^{N} 1 \\
&\le 2\varphi(q) \sum_{0 \le i \le j \le J} e^{-i-j} \sum_{e^i \le u < e^{i+1}} \sum_{e^j \le v < e^{j+1}} \sum_{\substack{m,n=1 \\ mu \equiv nv \,(\mathrm{mod}\ q) \\ mu \ne nv}}^{N} 1.
\end{aligned}$$

Therefore

$$M_2 \le 2\varphi(q) \sum_{0 \le i \le j \le J} e^{-i-j} T_{i,j}, \tag{4}$$

where $T_{i,j}$ is the number of solutions (m, n, u, v) to the congruence

$$mu \equiv nv \ (\mathrm{mod}\ q), \quad 1 \le m, n \le N, \quad e^i \le u < e^{i+1}, \quad e^j \le v < e^{j+1},$$

with $mu \ne nv$.

For a solution (m, n, u, v), if we write $mu = nv + kq$ with some integer k, then we see that

$$1 \le |k| \le q^{-1} \max\{mu, nv\} \le q^{-1} N \max\{e^{i+1}, e^{j+1}\} = e^{j+1} N/q.$$

Thus, there are $O(e^j N/q)$ possible values for k. Clearly, there are at most e^{i+1} possible values for u and N possible values m. Hence the product $nv = mu - kq$ can take at most $e^{i+j+2} N^2/q$ possible values and these are all of size $O(Nq^2) = O(q^3)$.

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Therefore, from the bound on the divisor function given in (2), we see that if m, u and k are fixed, then n and v can take at most $q^{o(1)}$ possible values. Hence

$$T_{i,j} \le e^{i+j} N^2 q^{-1+o(1)},$$

which, after substitution in (4), gives

$$M_2 \le J^2 \varphi(q) N^2 q^{-1+o(1)}.$$

The bound (3) then follows. □

3. Final remarks

As has already been mentioned, our result is nontrivial for $q^{\varepsilon} \le N \le q^{1-\varepsilon}$. However, the author sees no reason why an appropriate asymptotic formula cannot hold for even larger values of N, up to $q/2$, say. It would be interesting to clarify this issue.

References

[1] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers* (Oxford University Press, Oxford, 1979).

[2] W. Zhang, 'On the mean value of L-functions with the weight of character sums', *J. Number Theory* **128** (2008), 2459–2466.

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