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A multiplier theorem for one-sided Hardy spaces

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In this paper we give a multiplier theorem for one-sided Hardy spaces which generalizes the results given by Strömberg and Torchinsky for two-sided weights. Also we state the L^p_ω version with a Sawyer's weight ω.

1. Introduction

Many operators that are bounded on weighted Lebesgue spaces L^p_ω for weights ω on the Muckenhoupt class have one-sided versions that are bounded for the wider class of Sawyer's weights. Good examples are the one-sided singular integral operators introduced in [1]. A one-sided singular integral is a Calderón–Zygmund singular integral whose kernel K has support in $(-\infty, 0]$ or $[0, \infty)$.

The purpose of this paper is to study the one-sided version of multiplier operators. By Plancharel's theorem, a bounded measurable function $m(\xi)$ on $\mathbb{R}^n$ defines a bounded operator T_m on $L^2(\mathbb{R}^n)$ through the Fourier transform given by the expression

$$\widehat{T_m f}(\xi) = m(\xi)\hat{f}(\xi).$$

It is said that $m(\xi)$ is a multiplier on L^p if $\|T_m f\|_p \leqslant c\|f\|_p$, where the constant c does not depend on $f \in L^2 \cap L^p$, and thus T_m can be extended to a bounded operator on L^p. A classical result states that if $m(\xi)$ is a bounded C^1 function on $\mathbb{R}-\{0\}$ such that $|Dm(\xi)| \leqslant c|\xi|^{-1}$, then $m(\xi)$ is a multiplier on L^p for $1 < p < \infty$. For the $\mathbb{R}^n$ version of this result see [13, ch. IV]. The multiplier theory for weighted Lebesgue spaces L^p_ω and for weighted Hardy spaces H^p_ω, where the weight ω belongs to some Muckenhoupt class A_s, can be found in [14, ch. XI].

Given a multiplier $m(\xi)$, let $K_m \in \mathcal{S}'$ be the associated kernel, that is $\widehat{K_m} = m$ in the sense of distributions. We shall say that $m(\xi)$ is a one-sided multiplier if K_m has support in $(-\infty, 0]$ or $[0, \infty)$. In [12] the Fourier transforms of distributions with support in a half-line are characterized. For the sake of completeness we show

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in the example given in § 6 that if $m(\xi)$ can be extended to a bounded analytic function on the upper half-plane, then $\operatorname{supp}(K_m) \subset (-\infty, 0]$.

In [4] a one-sided multiplier theorem for L^p_ω with a weight ω in the Sawyer class A^+_p is obtained. In [15] a one-sided multiplier theorem for one-sided Hardy spaces $H^1_+(\omega)$ with $\omega \in A^+_1$ is given.

In this paper we prove a rather general one-sided multiplier theorem for one-sided Hardy spaces $H^p_+(\omega)$, $0 < p < \infty$, with weight ω in some Sawyer class A^+_s (theorem 3.1) that generalizes the results given in [14] for two-sided weights. Also we state the L^p_ω version in theorem 3.2. We follow the ideas of [14] and in order to overcome the difficulties that appear working with one-sided weights we use a special atomic decomposition of the one-sided Hardy spaces where for each atom of the decomposition there is another atom supported contiguously at the right of its support and with their supports having equivalent sizes [9].

2. Definitions

Let $f(x)$ be a Lebesgue measurable function defined on $\mathbb{R}$. The one-sided Hardy–Littlewood maximal functions $M^+f(x)$ and $M^-f(x)$ are defined as

$$M^+f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(t)| \, \mathrm{d}t \quad \text{and} \quad M^-f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(t)| \, \mathrm{d}t.$$

As usual, a weight ω is a measurable and non-negative function. If $E \subset \mathbb{R}$ is a Lebesgue measurable set, we denote its ω-measure by $\omega(E) = \int_E \omega(t) \, \mathrm{d}t$. A function $f(x)$ belongs to L^p_ω, $0 < p < \infty$, if

$$\|f\|_{L^p_\omega} = \left(\int_{-\infty}^{\infty} |f(x)|^p \omega(x) \, \mathrm{d}x \right)^{1/p}$$

is finite.

The class of weights ω for which M^+ is a bounded operator on L^s_ω, $1 < s < \infty$, is called the A^+_s class. This class has been characterized by Sawyer in [11], as the weights ω such that

$$\left(\int_a^b \omega(t) \, \mathrm{d}t \right) \left(\int_b^c \omega(t)^{-1/(s-1)} \, \mathrm{d}t \right)^{s-1} \leqslant C_{\omega,s}(c-a)^s$$

holds for some constant $C_{\omega,s}$ and for every $-\infty < a < b < c < \infty$. In the limit case of $s = 1$ we say that ω belongs to the class A^+_1 if $M^-\omega(x) \leqslant C_{\omega,1}\omega(x)$ a.e. By Hölder's inequality it follows that $A^+_{s_1} \subset A^+_{s_2}$ for $s_1 < s_2$. If $a \geqslant 1$ and $I = (x_0 - t_0, x_0)$, the left a-dilation of I is the interval $(x_0 - at_0, x_0)$, which we denote by aI. Given $\theta \geqslant 1$, we say that a weight ω satisfies a one-sided doubling condition D^+_θ if there exists a constant c such that $\omega(aI) \leqslant ca^\theta \omega(I)$ holds for every interval I and every $a \geqslant 1$. It is well known that if $\omega \in A^+_s$, then $\omega \in D^+_s$, and thus it can be assumed that θ satisfies $\theta \leqslant s$. Given $r > 1$ we say that a weight ω satisfies a one-sided reverse Hölder condition RH^+_r if for every $\rho > 0$ there exists a constant $c = c(\rho)$ such that

$$\left(\frac{1}{|I|} \int_I \omega(t)^r \, \mathrm{d}t \right)^{1/r} \leqslant c \frac{1}{|I^+|} \int_{I^+} \omega(t) \, \mathrm{d}t,$$

where $I=(x_0-t_0,x_0)$ and $I^+=(x_0,x_0+\rho t_0)$. It is not hard to see that if $\omega\in A_s^+$, then $\omega\in RH_r^+$ for some $r>1$. The RH_r^+ condition given here is equivalent [2, 10] to the one given in [6, lemma 5], where it is used to prove that $\omega\in A_s^+$ implies $\omega\in A_{s-\varepsilon}^+$ for some $\varepsilon>0$.

A weight ω belonging to the A_s^+ class can be equal to zero in a set with positive measure. For simplicity, we shall assume throughout that all weights are positive almost everywhere. For this case, and for $0<p<\infty$, we shall define $H_+^p(\omega)$ as in [5]. Let $\mathcal{S}$ be the space of $C^\infty(\mathbb{R})$ functions with rapidly decreasing derivatives of all orders. The topology of $\mathcal{S}$ is given by the family of norms

$$\|\varphi\|_{\alpha,\beta}=\sup_{x\in\mathbb{R}}\biggl[(1+|x|)^\beta\sum_{k=0}^{\alpha}|D^k\varphi(x)|\biggr],$$

where α is a non-negative integer and $\beta\geqslant 0$. Let $\mathcal{S}_c=\{\varphi\in\mathcal{S}:\operatorname{supp}(\varphi)\subset[c,+\infty)\}$ for any $c\in\mathbb{R}$, and let $\mathcal{S}_c'$ be its dual space. If we define $\mathcal{S}_{-\infty}^+$ as the union

$$\mathcal{S}_{-\infty}^+=\bigcup_{-\infty<c}\mathcal{S}_c,$$

we shall denote by $(\mathcal{S}_{-\infty}^+)'$ the set of linear functionals of $\mathcal{S}_{-\infty}^+$ that, restricted to $\mathcal{S}_c$, belong to $\mathcal{S}_c'$ for every $c>-\infty$. Given $\phi\in\mathcal{S}$ with $\operatorname{supp}(\phi)\subset(-\infty,0]$ and $f\in(\mathcal{S}_{-\infty}^+)'$, in [5] the maximal is defined as

$$M_1^+f(x)=\sup_{0\leqslant y-x<t}|f*\phi_t(y)|,$$

where $f*\phi_t(y)=\langle f,\phi_t(y-\cdot)\rangle$ and $\phi_t(u)=(1/t)\phi(u/t)$. For $0<p<\infty$ and $\omega\in A_s^+$, it is said that $f\in(\mathcal{S}_{-\infty}^+)'$ belongs to $H_+^p(\omega)$ if

$$\|f\|_{H_+^p(\omega)}=\biggl(\int_{-\infty}^{+\infty}M_1^+f(x)^p\omega(x)\,\mathrm{d}x\biggr)^{1/p}<\infty.$$

It is shown in [5] that the definition of $H_+^p(\omega)$ does not depend on the choice of ϕ. Note that $\|\cdot\|_{H_+^p(\omega)}$ is a norm for $1\leqslant p<\infty$ and a p-norm for $0<p<1$. In both cases $\|\cdot\|_{H_+^p(\omega)}$ gives to $H_+^p(\omega)$ a structure of complete metric space.

Let N be an integer and $1\leqslant q\leqslant\infty$. A function $a(x)$ defined on $\mathbb{R}$ is called a (q,N)-atom if there exists an interval I containing the support of $a(x)$ such that $\|a\|_q\leqslant\|\chi_I\|_q$, and it satisfies $N+1$ null-moment conditions, that is $\int_I y^k a(y)\,\mathrm{d}y=0$ for every k, $0\leqslant k\leqslant N$. Atomic decompositions for $H_+^p(\omega)$ are given in [3, 9].

We denote by $|x|\sim R$ the set $\{x\in\mathbb{R}:R/2<|x|<2R\}$. Let $\ell\geqslant 0$ and $1\leqslant q\leqslant 2$, we say that a bounded function $m(\xi)$ satisfies the condition $\mathcal{M}(q,\ell)$ if

$$\biggl(\int_{|\xi|\sim R}|D^\alpha m(\xi)|^q\,\mathrm{d}\xi\biggr)^{1/q}\leqslant cR^{1/q-\alpha}, \tag{2.1}$$

for all $R>0$ and $0\leqslant\alpha\leqslant[\ell]$, where $[\ell]$ is the integer part of ℓ, and if $\ell=[\ell]+\gamma$ with $\gamma>0$, then we also require

$$\biggl(\int_{|\xi|\sim R}|D^{[\ell]}m(\xi)-D^{[\ell]}m(\xi-z)|^q\,\mathrm{d}\xi\biggr)^{1/q}\leqslant c\biggl(\frac{|z|}{R}\biggr)^\gamma R^{1/q-[\ell]}$$

for $|z|<\frac{1}{2}R$.

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We say that an interval J 'follows' the interval I if $I = (c, d)$ and $J = (d, e)$. If $p > 1$, then its conjugate exponent will be $p' = p/(p-1)$. The c that appears in inequalities represents a positive constant not necessarily the same at each occurrence.

3. Statement of the main result

We state a multiplier theorem on $H^p_+(\omega)$ which is the one-sided version of [14, theorem 6, p. 162]. This theorem will be proved in § 5.

THEOREM 3.1. *Let* $\omega \in A^+_s \cap D^+_\theta \cap RH^+_r$, $1 \leqslant \theta \leqslant s$, $r > 1$. *Let* $1 \leqslant q \leqslant 2$, $0 < p < \infty$ *and* $m \in \mathcal{M}(q, \ell)$, *where*

$$\left.\begin{aligned} &\ell \geqslant \frac{(\theta-1)(s-p)}{p(s-1)} + \max\left(\frac{1}{q}, \frac{1}{p'} + \frac{1}{rp}\right) && \text{if } 1 < p < s, \\ &\ell > \frac{\theta}{p} - \frac{1}{q'} && \text{if } 0 < p \leqslant 1, \\ &\ell \geqslant \frac{\theta}{p} - \frac{1}{r'p} && \text{if } 0 < p \leqslant 1. \end{aligned}\right\} \tag{3.1}$$

Also assume that the associated kernel K_m, $\widehat{K_m} = m$, *has support in* $(-\infty, 0]$. *Then* m *is a multiplier on* $H^p_+(\omega)$; *that is to say that the operator* T_m *defined by*

$$\widehat{T_m f}(\xi) = m(\xi)\hat{f}(\xi)$$

for $f \in L^2 \cap H^p_+(\omega)$ *can be extended to a bounded operator in* $H^p_+(\omega)$.

It may be seen that if $s \leqslant p$ and $\omega \in A^+_s$, then $H^p_+(\omega)$ coincides with L^p_ω and the norms in both spaces are equivalent. In this case we have the following result.

THEOREM 3.2. *Let* $\omega \in A^+_s \cap RH^+_r$, $1 \leqslant q \leqslant 2$, $s \leqslant p < \infty$ *and let* $m \in M(q, \ell)$, *where* $\ell > 1/q$ *and* $\ell \geqslant \min(s/p, 1/p' + 1/pr)$. *Also assume that the associated kernel* K_m *has support in* $(-\infty, 0]$. *Then the multiplier operator associated with* m *is a bounded mapping from* L^p_ω *into itself.*

This result is the one-sided version of [14, theorem 5, p. 161] and can be proved similarly to theorem 3.1. We therefore omit the proof here.

4. Previous results and lemmas

For $m \in \mathcal{M}(q, \ell)$ with $1 \leqslant q \leqslant 2$ let $K_m \in \mathcal{S}'$ be the associated kernel, that is $\widehat{K_m} = m$. We shall assume that $\operatorname{supp}(K_m) \subset (-\infty, 0]$. The following lemma summarizes lemmas 2 and 3 in [14, pp. 157, 158]. Since no weights are involved, the proof is the same as in [14]. Because of the support of K_m, the only difference is that the atom's support dilates only to the left.

LEMMA 4.1. *Let* $m \in \mathcal{M}(q, \ell)$, $\ell > 1/q$ *and with* K_m *as above and let* $1 \leqslant p_0 < \infty$ *and* $\tilde{\ell} = \ell - \max(1/q, 1/p'_0)$. *If* a *is a* (p_0, N_0)*-atom supported in* I, *then there exists a constant* c *such that*

$$K_m * a = c \sum_{i=0}^{\infty} \frac{1}{\tilde{c}_i} b_i,$$

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where b_i are (p_0, N_0)-atoms supported in $2^{i+4}I - 2^{i+1}I$ for every $i \geqslant 1$, and b_0 is a (p_0, N_0)-atom supported in $2^3 I$. We also have that, for $i \geqslant 1$,

$$\tilde{c}_i = \begin{cases} 2^{i(N_0+2)} & \text{if } \tilde{\ell} > N_0 + 1, \\ 2^{i(\tilde{\ell}+1)}/i & \text{if } \tilde{\ell} \leqslant N_0 + 1 \text{ and } \tilde{\ell} \text{ is integer}, \\ 2^{i(\tilde{\ell}+1)} & \text{if } \tilde{\ell} \leqslant N_0 + 1 \text{ and } \tilde{\ell} \text{ is non-integer}. \end{cases}$$

The following atomic decomposition for $H^p_+(\omega)$ was given in [9].

Theorem 4.2. *Let $\omega \in A^+_s$ and $0 < p < \infty$. Then there is a integer $N(p, \omega)$ with the following property: given any $f \in H^p_+(\omega)$ and $N \geqslant N(p, \omega)$, we can find a sequence $\{\lambda_k\}$ of positive coefficients and a sequence $\{a_{k,j}(x)\}$ of (∞, N)-atoms with support contained in intervals $\{I_{k,j}\}$ respectively such that $\sum_{k,j} \lambda_k a_{k,j}$ converges unconditionally to f both in the sense of distributions and in the $H^p_+(\omega)$-norm. Moreover, $I_{k,j+2}$ follows $I_{k,j}$, $|I_{k,j+2}| \leqslant |I_{k,j}| \leqslant 4|I_{k,j+2}|$ for every j, k, and*

$$\Big\| \sum_{k,j} \lambda_k \chi_{I_{k,j}} \Big\|_{L^p_\omega} \backsim \|f\|_{H^p_+(\omega)}.$$

Remark 4.3. In the same way that the above decomposition is obtained in [9], it can be proved that

$$\Big\| \sum_{k,j} \lambda_k^\tau \chi_{I_{k,j}} \Big\|^{1/\tau}_{L^{p/\tau}_\omega} \leqslant c_\tau \|f\|_{H^p_+(\omega)} \tag{4.1}$$

holds for every $0 < \tau < \infty$. In fact, an atomic decomposition for $f \in H^p_+(\omega)$ satisfying this inequality can be obtained as in [3] by following the ideas in [14, ch. VIII]. Then, by breaking up each atom of the decomposition as in [9], we obtain (4.1).

The following lemmas will be needed throughout the paper.

Lemma 4.4. *Let $r > 1$ and ω be a weight such that $\omega^r \in A^+_s$ for some $s > 1$. Then $\omega \in RH^+_r$.*

Proof. Let $I = (x_0 - t_0, x_0)$ and $I^+ = (x_0, x_0 + \rho t_0)$ for some $\rho > 0$. From the A^+_s condition we have

$$\begin{aligned} \frac{1}{|I|}\int_I \omega^r &= \frac{1}{|I|}\int_I \omega^r \bigg(\frac{1}{|I^+|}\int_{I^+} \omega^{-r/(s-1)}\bigg)^{s-1} \bigg(\frac{1}{|I^+|}\int_{I^+} \omega^{-r/(s-1)}\bigg)^{-(s-1)} \\ &\leqslant c\bigg(\frac{1}{|I^+|}\int_{I^+} \omega^{-r/(s-1)}\bigg)^{-(s-1)} \bigg(\frac{1}{|I^+|}\int_{I^+} \omega\bigg)^{-r} \bigg(\frac{1}{|I^+|}\int_{I^+} \omega\bigg)^{r}. \end{aligned}$$

Then, by Hölder's inequality,

$$\frac{1}{|I|}\int_I \omega^r \leqslant c\bigg(\frac{1}{|I^+|}\int_{I^+} \omega\bigg)^{r}.$$

□

Lemma 4.5. *For $\omega \in A^+_s \cap D^+_\theta$ and $0 < \varepsilon < 1$ let $\omega_\varepsilon(y) = \omega(y)^\varepsilon$, then $\omega_\varepsilon \in D^+_{1+\varepsilon(\theta-1)}$.*

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Proof. First note that, applying lemma 4.4 to ω_ε with $r = 1/\varepsilon$, it follows that for any $\rho > 0$ there exists a constant c such that for every interval $J = (x - t, x)$ we have

$$\left(\frac{\omega(J)}{|J|}\right)^{\varepsilon} \leqslant c\frac{\omega_\varepsilon(J^+)}{|J^+|}, \tag{4.2}$$

where $J^+ = (x, x + \rho t)$.

Let $a \geqslant 1$ and $I = (x_0 - t_0, x_0)$. If we consider intervals $J = (x_0 - t_0, x_0 - t_0/2)$ and $J^+ = [x_0 - t_0/2, x_0)$, then $aI \subset (2aJ) \cup J^+$. By Hölder's inequality, the fact that $\omega \in D^+_\theta$ and by (4.2), we have

$$\begin{aligned}\omega_\varepsilon(aI) &\leqslant \omega_\varepsilon(2aJ) + \omega_\varepsilon(J^+)\\ &\leqslant |2aJ|^{1-\varepsilon}(\omega(2aJ))^{\varepsilon} + \omega_\varepsilon(J^+)\\ &\leqslant ca^{1+\varepsilon(\theta-1)}|J|^{1-\varepsilon}\omega(J)^{\varepsilon} + \omega_\varepsilon(J^+)\\ &\leqslant ca^{1+\varepsilon(\theta-1)}\omega_\varepsilon(J^+) + \omega_\varepsilon(J^+)\\ &\leqslant ca^{1+\varepsilon(\theta-1)}\omega_\varepsilon(J^+) \leqslant ca^{1+\varepsilon(\theta-1)}\omega_\varepsilon(I).\end{aligned}$$

□

LEMMA 4.6. *Let $\omega \in A^+_s \cap D^+_\theta$, $1 < p < \infty$ and $\rho > 0$. There exists a constant c such that, for any $a > 1$, any sequence of positive numbers λ_k and any sequence of intervals $I_k = (x_k - t_k, x_k)$, we have that*

$$\Big\|\sum \lambda_k \chi_{aI_k}\Big\|_{L^p_\omega} \leqslant ca^{\delta}\Big\|\sum \lambda_k \chi_{I^+_k}\Big\|_{L^p_\omega}$$

holds, where $I^+_k = (x_k, x_k + \rho t_k)$ and

$$\delta = \begin{cases} 1 + \dfrac{(\theta-1)(s-p)}{p(s-1)} & \text{if } 1 < p < s,\\ 1 & \text{if } s \leqslant p < \infty.\end{cases}$$

Proof. We shall consider the case $1 < p < s$. Let $\omega_\varepsilon(y) = \omega(y)^{\varepsilon}$ for

$$\varepsilon = \left(1 - \frac{p-1}{s-1}\right)\frac{1}{p}.$$

Note that $0 < \varepsilon < 1/p < 1$. For any $G \in L^{p'}_\omega$ with $\|G\|_{L^{p'}_\omega} = 1$, let $g = G\omega^{1-\varepsilon}$. Since, by lemma 4.5, $\omega_\varepsilon \in D^+_{1+\varepsilon(\theta-1)}$ and $\delta = 1 + \varepsilon(\theta - 1)$, for any $z \in I^+_k$ we have

$$\begin{aligned}\int \chi_{aI_k}(y)G(y)\omega(y)\,\mathrm{d}y &= \int \chi_{aI_k}(y)g(y)\omega_\varepsilon(y)\,\mathrm{d}y\\ &\leqslant \frac{\omega_\varepsilon(aI_k \cup I^+_k)}{\omega_\varepsilon(x_k - at_k, z)}\int_{x_k-at_k}^{z} g(y)\omega_\varepsilon(y)\,\mathrm{d}y\\ &\leqslant ca^{\delta}\omega_\varepsilon(I^+_k)M^-_{\omega_\varepsilon}g(z),\end{aligned}$$

where

$$M^-_{\omega_\varepsilon}g(z) = \sup_{h>0}\frac{1}{\omega_\varepsilon(z-h, z)}\int_{z-h}^{z} g(y)\omega_\varepsilon(y)\,\mathrm{d}y.$$

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Then

$$\int\Big(\sum\lambda_k\chi_{aI_k}(y)\Big)G(y)\omega(y)\,\mathrm{d}y \leqslant ca^\delta\int\Big(\sum\lambda_k\chi_{I_k^+}(y)\Big)M^-_{\omega_\varepsilon}g(y)\omega_\varepsilon(y)\,\mathrm{d}y \leqslant ca^\delta\Big\|\sum\lambda_k\chi_{I_k^+}\Big\|_{L^p_\omega}\Big(\int M^-_{\omega_\varepsilon}g(y)^{p'}\omega(y)^{(\varepsilon-1/p)p'}\,\mathrm{d}y\Big)^{1/p'} \leqslant ca^\delta\Big\|\sum\lambda_k\chi_{I_k^+}\Big\|_{L^p_\omega},$$

which implies the lemma for $1<p<s$. The last inequality above follows from the fact that

$$\int M^-_{\omega^\varepsilon}g(y)^{p'}\omega(y)^{(\varepsilon-1/p)p'}\,\mathrm{d}y\leqslant c\int g(y)^{p'}\omega(y)^{(\varepsilon-1/p)p'}\,\mathrm{d}y=c,$$

which is equivalent to $\omega^{(\varepsilon-1/p)p'}$ satisfying the $A^-_{p'}(\omega_\varepsilon)$ condition

$$\Big(\int_I\omega(y)^{(\varepsilon-1/p)p'}\,\mathrm{d}y\Big)\Big(\int_{I^-}(\omega(y)^{(\varepsilon-1/p)p'-\varepsilon})^{-1/(p'-1)}\omega(y)^\varepsilon\,\mathrm{d}y\Big)^{p'-1} \leqslant c\omega_\varepsilon(I\cup I^-)^{p'}, \quad (4.3)$$

where I follows I^- and $|I^-|=|I|$ [7,8]. In order to verify (4.3), we first note that

$$\Big(\varepsilon-\frac{1}{p}\Big)p'=-\frac{1}{s-1}\quad\text{and}\quad\varepsilon-\frac{1}{p'-1}\Big(\Big(\varepsilon-\frac{1}{p}\Big)p'-\varepsilon\Big)=1,$$

so the left-hand side of (4.3) is equal to

$$\Big(\int_I\omega(y)^{-1/(s-1)}\,\mathrm{d}y\Big)\Big(\int_{I^-}\omega(y)\,\mathrm{d}y\Big)^{p'-1} =\Big[\Big(\int_{I^-}\omega(y)\,\mathrm{d}y\Big)\Big(\int_I\omega(y)^{-1/(s-1)}\,\mathrm{d}y\Big)^{s-1}\Big]^{1/(s-1)}\Big(\int_{I^-}\omega(y)\,\mathrm{d}y\Big)^{(p'-1)-1/(s-1)},$$

which, as $\omega\in A^+_s$, can be estimated by a constant times

$$|I\cup I^-|^{s/(s-1)}\Big(\int_{I^-}\omega(y)\,\mathrm{d}y\Big)^{(p'-1)-1/(s-1)}.$$

Using (4.2), the latter expression is dominated by a constant times

$$|I\cup I^-|^{s/(s-1)}\Big(\frac{1}{|I|}\int_I\omega(y)^\varepsilon\,\mathrm{d}y\Big)^{((p'-1)-(1/(s-1)))/\varepsilon}|I|^{(p'-1)-1/(s-1)} =c\Big(\int_I\omega(y)^\varepsilon\,\mathrm{d}y\Big)^{p'},$$

which gives (4.3).

The case when $s\leqslant p<\infty$ can be obtained in a similar way by taking $\varepsilon=0$. □

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LEMMA 4.7. *Let b be a (p_0, N_0)-atom supported in $I = (x_0 - t_0, x_0)$. Then, for $x \leqslant x_0 - 2t_0$,*

$$M_1^+ b(x) \leqslant c\left(\frac{|I|}{x_0 - x}\right)^{N_0+2} \leqslant c(M^+\chi_I(x))^{N_0+2}.$$

Proof. We shall estimate

$$M_1^+ b(x) = \sup_{0 \leqslant y - x < t} |b * \phi_t(y)|$$

for $x \leqslant x_0 - 2t_0$, where $\phi \in \mathcal{S}$, $\operatorname{supp}(\phi) \subset (-\infty, 0]$ and $\int \phi(t)\,\mathrm{d}t \neq 0$. If $P_{N_0}(u)$ is the Taylor's polynomial of order N_0 of $\phi((y-u)/t)$ at $u = x_0$, by the null-moment conditions we have

$$\begin{aligned} |b * \phi_t(y)| &= \frac{1}{t}\left|\int_I b(u)\left(\phi\left(\frac{y-u}{t}\right) - P_{N_0}(u)\right)\mathrm{d}u\right| \\ &\leqslant c\frac{1}{t^{N_0+2}}\int_I |b(u)|\left|D^{N_0+1}\phi\left(\frac{y-\xi}{t}\right)\right||u - x_0|^{N_0+1}\,\mathrm{d}u \end{aligned} \tag{4.4}$$

with $u < \xi < x_0$. Since $\phi \in \mathcal{S}$, we have

$$\left|D^{N_0+1}\phi\left(\frac{y-\xi}{t}\right)\right| \leqslant c\frac{t^{N_0+2}}{((t^2 + |y-\xi|^2)^{1/2})^{N_0+2}} \leqslant c\frac{t^{N_0+2}}{(\xi - x)^{N_0+2}}.$$

The latter inequality holds due to the fact that $(t^2 + |y - \xi|^2)^{1/2}$ is the distance from $(\xi, 0)$ to any point of the semi-cone $0 \leqslant y - x < t$ and thus

$$0 < \frac{\xi - x}{\sqrt{2}} \leqslant (t^2 + |y-\xi|^2)^{1/2}.$$

Then, for $x \leqslant x_0 - 2t_0$, $0 < y - x < t$ and $\xi \in I$, we have

$$\left|D^{N_0+1}\phi\left(\frac{y-\xi}{t}\right)\right| \leqslant c\frac{t^{N_0+2}}{(x_0 - x)^{N_0+2}},$$

and, substituting this into (4.4), since $\int_I |b(u)|\,\mathrm{d}u \leqslant |I|$, we obtain

$$M_1^+ b(x) \leqslant c\frac{|I|^{N_0+1}}{(x_0-x)^{N_0+2}}\int_I |b(u)|\,\mathrm{d}u \leqslant c\left(\frac{|I|}{x_0 - x}\right)^{N_0+2} \leqslant c(M^+\chi_I(x))^{N_0+2}.$$

□

LEMMA 4.8. *Let $\omega \in A_s^+ \cap RH_r^+$, where $s \geqslant 1$ and $r > 1$. For $0 < p < \infty$, let $p_0 > 1$ such that $p_0 > pr'$ and N_0 is an integer satisfying $p(N_0 + 2) > s$. Then, given $\rho > 0$, there exists a constant $c > 0$ such that*

$$\|b\|^p_{H^p_+(\omega)} \leqslant c\omega(I^+)$$

holds for every (p_0, N_0)-atom b supported in $I = (x_0 - t_0, x_0)$, and $I^+ = (x_0, x_0 + \rho t_0)$.

Proof. We shall estimate $\|M_1^+ b\|_{L^p_\omega}^p$, where

$$M_1^+ b(x) = \sup_{0 \leqslant y - x < t} |b * \phi_t(y)|,$$

with $\phi \in \mathcal{S}$, $\operatorname{supp}(\phi) \subset (-\infty, 0]$ and $\int \phi(t)\,\mathrm{d}t \neq 0$. Due to the support of ϕ we get that $M_1^+ b(x) = 0$ for $x \geqslant x_0$. By lemma 4.7 and since $\omega \in A_s^+ \subset A^+_{p(N_0+2)}$, it follows that

$$\int_{-\infty}^{x_0 - 2t_0} M_1^+ b(x)^p \omega(x)\,\mathrm{d}x \leqslant c \int (M^+ \chi_I(x))^{p(N_0+2)} \omega(x)\,\mathrm{d}x \leqslant c\omega(I) \leqslant c\omega(I^+).$$

Since b is a (p_0, N_0)-atom, $\omega \in RH_r^+$ and $p_0 > pr'$, we can estimate

$$\begin{aligned}
\int_{2I} M_1^+ b(x)^p \omega(x)\,\mathrm{d}x &\leqslant c \int_{2I} Mb(x)^p \omega(x)\,\mathrm{d}x \\
&\leqslant c\biggl(\int_{2I} Mb(x)^{pr'}\,\mathrm{d}x\biggr)^{1/r'} \biggl(\int_{2I} \omega(x)^r\,\mathrm{d}x\biggr)^{1/r} \\
&\leqslant c\biggl(\biggl(\int Mb(x)^{p_0}\,\mathrm{d}x\biggr)^{pr'/p_0} |2I|^{1/(p_0/pr')'}\biggr)^{1/r'} \frac{\omega(I^+)}{|I^+|} |2I|^{1/r} \\
&\leqslant c\biggl(\biggl(\int |b(x)|^{p_0}\,\mathrm{d}x\biggr)^{pr'/p_0} |I|^{1/(p_0/pr')'}\biggr)^{1/r'} \frac{\omega(I^+)}{|I^+|} |I|^{1/r} \\
&\leqslant c(|I|^{pr'/p_0} |I|^{1/(p_0/pr')'})^{1/r'} \frac{\omega(I^+)}{|I^+|} |I|^{1/r} \\
&\leqslant c\omega(I^+).
\end{aligned}$$

The lemma follows from the estimations above. This lemma also holds for $p_0 = \infty$; moreover, in this case one can obtain $\|b\|_{H_+^p(\omega)}^p \leqslant c\omega(I)$. □

LEMMA 4.9. *Let $\omega \in A_s^+ \cap RH_r^+$, where $s \geqslant 1$ and $r > 1$. For $1 < p < \infty$, let $p_0 > 1$ be such that $p_0 > pr'$. Given $\rho > 0$, there exists a constant $c > 0$ such that, for any sequence of positive numbers λ_k and p_0-atoms a_k supported in intervals $I_k = (x_k - t_k, x_k)$, we have*

$$\Bigl\| \sum \lambda_k a_k \Bigr\|_{L^p_\omega} \leqslant c \Bigl\| \sum \lambda_k \chi_{I_k^+} \Bigr\|_{L^p_\omega},$$

where $I_k^+ = (x_k, x_k + \rho t_k)$.

Proof. Since $p_0 > pr'$, we can choose q_0 such that $p_0/r' > q_0 > p$. Note that since $p_0/q_0 > r'$ we get

$$\frac{p_0}{p_0 - q_0} = \biggl(\frac{p_0}{q_0}\biggr)' < r$$

and, therefore, $\omega \in RH^+_{p_0/(p_0 - q_0)}$.

First, we will prove that, for any $g \in L^{p'}_\omega$ with $\|g\|_{L^{p'}_\omega} = 1$,

$$\biggl| \int_{I_k} a_k(x) g(x) \omega(x)\,\mathrm{d}x \biggr| \leqslant c \int_{I_k^+} M_\omega(|g|^{q_0'})(x)^{1/q_0'} \omega(x)\,\mathrm{d}x \tag{4.5}$$

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holds, where

$$M_\omega f(x) = \sup_{x\in I} \frac{1}{\omega(I)} \int_I f(y)\omega(y)\,\mathrm{d}y.$$

By Hölder's inequality and the definition of atom, the left-hand side of the above inequality is less than

$$|I_k|^{1/p_0} \left(\int_{I_k} |g(x)|^{p_0'} \omega(x)^{p_0'}\,\mathrm{d}x \right)^{1/p_0'}.$$

This expression can be estimated using Hölder's inequality with exponents q_0'/p_0' and $(q_0'/p_0')' = q_0'/(q_0' - p_0')$ by

$$|I_k|^{1/p_0} \left(\int_{I_k} |g(x)|^{q_0'}\omega(x)\,\mathrm{d}x \right)^{1/q_0'} \left(\int_{I_k} \omega(x)^{(p_0'/q_0)(q_0'/(q_0'-p_0'))}\,\mathrm{d}x \right)^{((q_0'-p_0')/q_0')/p_0'}.$$

Since

$$\frac{p_0' q_0'}{q_0(q_0' - p_0')} = \frac{p_0}{p_0 - q_0},$$

the expression above can be written as

$$|I_k|^{1/p_0} \left(\int_{I_k} |g(x)|^{q_0'}\omega(x)\,\mathrm{d}x \right)^{1/q_0'} \left(\int_{I_k} \omega(x)^{p_0/(p_0-q_0)}\,\mathrm{d}x \right)^{((p_0-q_0)/p_0)/q_0}.$$

The fact that $\omega \in RH^+_{p_0/(p_0-q_0)}$ allows us to estimate the last expression by

$$\begin{aligned}
c|I_k|^{1/p_0} &\left(\int_{I_k} |g(x)|^{q_0'}\omega(x)\,\mathrm{d}x \right)^{1/q_0'} \left(\frac{1}{|I_k^+|} \int_{I_k^+} \omega(x)\,\mathrm{d}x \right)^{1/q_0} |I_k|^{1/q_0 - 1/p_0} \\
&\leqslant c\omega(I_k^+)^{1/q_0} \omega(I_k \cup I_k^+)^{1/q_0'} \left(\frac{1}{\omega(I_k \cup I_k^+)} \int_{I_k \cup I_k^+} |g(x)|^{q_0'}\omega(x)\,\mathrm{d}x \right)^{1/q_0'} \\
&\leqslant c\omega(I_k^+) M_\omega(|g|^{q_0'})(x)^{1/q_0'},
\end{aligned}$$

for any $x \in I_k^+$, which implies (4.5). Then, from (4.5) and since the maximal theorem for M_ω works on the real line even if ω is not doubling, by applying Hölder's inequality we have

$$\begin{aligned}
\left| \int \sum \lambda_k a_k(x) g(x) \omega(x)\,\mathrm{d}x \right| &\leqslant c \int \sum \lambda_k \chi_{I_k^+}(x) M_\omega(|g|^{q_0'})(x)^{1/q_0'} \omega(x)\,\mathrm{d}x \\
&\leqslant c \left\| \sum \lambda_k \chi_{I_k^+} \right\|_{L^p_\omega} \left(\int M_\omega(|g|^{q_0'})(x)^{p'/q_0'} \omega(x)\,\mathrm{d}x \right)^{1/p'} \\
&\leqslant c \left\| \sum \lambda_k \chi_{I_k^+} \right\|_{L^p_\omega},
\end{aligned}$$

and the lemma follows readily. □

LEMMA 4.10. *Let $\omega \in A_s^+ \cap D_\theta^+ \cap RH_r^+$, where $s \geqslant 1$, $\theta \geqslant 1$ and $r > 1$. For $1 < p < \infty$, let $p_0 > 1$ such that $p_0 > pr'$ and let N_0 be an integer such that $N_0 + 2 > \delta$, where δ is as in lemma 4.6. Given $\rho > 0$, there exists a constant $c > 0$ such that,* *for any finite sequence of* (p_0, N_0)*-atoms* b_k *supported in intervals* $I_k = (x_k - t_k, x_k)$ *and positive numbers* λ_k*, we have*

$$\Big\|\sum_{k=1}^{J}\lambda_k b_k\Big\|_{H^p_+(\omega)} \leqslant c\Big\|\sum_{k=1}^{J}\lambda_k \chi_{I_k^+}\Big\|_{L^p_\omega},$$

where $I_k^+ = (x_k, x_k + \rho t_k)$*. The constant* c *does not depend on the non-negative integer* J*.*

Proof. First, we shall estimate

$$M_1^+ b_k(x) = \sup_{0\leqslant y-x<t} |b_k * \phi_t(y)|$$

with $\phi \in \mathcal{S}$, $\operatorname{supp}(\phi) \subset (-\infty, 0]$, and $\int \phi(t)\,\mathrm{d}t \neq 0$. Since $M_1^+ b_k(x) \leqslant cMb_k(x)$, we have

$$\|\chi_{2I_k} M_1^+ b_k\|_{L^{p_0}} \leqslant c\|Mb_k\|_{L^{p_0}} \leqslant c\|b_k\|_{L^{p_0}} \leqslant c|I_k|^{1/p_0} \leqslant c|2I_k|^{1/p_0}.$$

Thus, $a_{k,0} = c^{-1}\chi_{2I_k} M_1^+ b_k$ is a p_0-atom supported in $2I_k$. If $x \leqslant x_k - 2t_k$, by lemma 4.7, we have

$$M_1^+ b_k(x) \leqslant c\bigg(\frac{|I_k|}{x_k - x}\bigg)^{N_0+2},$$

and taking into account the fact that $M_1^+ b_k(x) = 0$ for $x \geqslant x_k$, we get

$$\begin{aligned} M_1^+ b_k(x) &\leqslant ca_{k,0}(x) + c\sum_{j=1}^{\infty}\bigg(\frac{|I_k|}{x_k - x}\bigg)^{N_0+2}(\chi_{2^{j+1}I_k}(x) - \chi_{2^j I_k}(x)) \\ &\leqslant ca_{k,0}(x) + c\sum_{j=1}^{\infty} 2^{-j(N_0+2)}\chi_{2^{j+1}I_k}(x), \end{aligned} \tag{4.6}$$

for every $x \in \mathbb{R}$. Since M_1^+ is subadditive, by applying the above estimate we obtain

$$\begin{aligned} \Big\|\sum_{k=1}^{J}\lambda_k b_k\Big\|_{H^p_+(\omega)} &\leqslant \Big\|\sum_{k=1}^{J}\lambda_k M_1^+ b_k\Big\|_{L^p_\omega} \\ &\leqslant c\Big\|\sum_{k=1}^{J}\lambda_k a_{k,0}\Big\|_{L^p_\omega} + c\sum_{j=1}^{\infty}2^{-j(N_0+2)}\Big\|\sum_{k=1}^{J}\lambda_k\chi_{2^{j+1}I_k}\Big\|_{L^p_\omega}. \end{aligned} \tag{4.7}$$

Since, by lemmas 4.9 and 4.6,

$$\Big\|\sum_{k=1}^{J}\lambda_k a_{k,0}\Big\|_{L^p_\omega} \leqslant c\Big\|\sum_{k=1}^{J}\lambda_k\chi_{I_k^+}\Big\|_{L^p_\omega} \text{ and } \Big\|\sum_{k=1}^{J}\lambda_k\chi_{2^{j+1}I_k}\Big\|_{L^p_\omega} \leqslant c2^{j\delta}\Big\|\sum_{k=1}^{J}\lambda_k\chi_{I_k^+}\Big\|_{L^p_\omega},$$

and $N_0 + 2 > \delta$ by (4.7), we have

$$\Big\|\sum_{k=1}^{J}\lambda_k b_k\Big\|_{H^p_+(\omega)} \leqslant c\Big\|\sum_{k=1}^{J}\lambda_k\chi_{I_k^+}\Big\|_{L^p_\omega}$$

with a finite constant c. □

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5. Proof of the main result

In this section we prove theorem 3.1. Throughout this proof, given a bounded interval I, we denote by I^+ the interval that follows I, with $|I^+| = 2^{-6}|I|$.

Since $\omega \in A^+_{s-\varepsilon} \cap RH^+_{r+\varepsilon}$ for some $\varepsilon > 0$, we can assume that the inequalities in (3.1) are strict. Let

$$\delta = \begin{cases} 1 + \dfrac{(\theta-1)(s-p)}{p(s-1)} & \text{if } 1 < p < s, \\[2ex] \dfrac{\theta}{p} & \text{if } 0 < p \leqslant 1. \end{cases}$$

Then (3.1) can be written as $\ell > \delta - (1/q')$ and $\ell > \delta - (1/r'p)$ in both cases. We can choose $p_0 > 1$ such that

$$\ell > \delta - \frac{1}{p_0} \quad \text{and} \quad p_0 > r'p. \tag{5.1}$$

This is possible since if $\delta - \ell \leqslant 0$, then any $p_0 > \max\{1, r'p\}$ works, and if $\delta - \ell > 0$, taking into account

$$\frac{1}{\delta - \ell} > 1 \quad \text{and} \quad \frac{1}{\delta - \ell} > pr',$$

then any $p_0 > 1$ such that $1/(\delta - \ell) > p_0 > pr'$ satisfies (5.1).

By theorem 4.2, for some integer $N_0 > \max\{\delta-2, (s/p)-2\}$, given any $f \in H^p_+(\omega)$ there is an atomic decomposition $f = \sum \lambda_k a_{k,j}$, where $\lambda_k > 0$ and $a_{k,j}$ are (∞, N_0)-atoms supported in $I_{k,j}$, such that it satisfies (4.1) and $I_{k,j+2}$ follows $I_{k,j}$, with $|I_{k,j+2}| \leqslant |I_{k,j}| \leqslant 4|I_{k,j+2}|$. Also, by lemma 4.1,

$$K_m * \sum_{k,j=1}^{J} \lambda_k a_{k,j} = c \sum_{k,j=1}^{J} \lambda_k \sum_{i=0}^{\infty} \frac{1}{\tilde{c}_i} b_{k,j,i}.$$

Then, for $0 < p \leqslant 1$, we have

$$\begin{aligned} \Big\| K_m * \sum_{k,j=1}^{J} \lambda_k a_{k,j} \Big\|^p_{H^p_+(\omega)} &\leqslant c \sum_{k,j=1}^{J} \lambda_k^p \|b_{k,j,0}\|^p_{H^p_+(\omega)} + \sum_{i=1}^{\infty} \frac{c}{\tilde{c}_i^p} \sum_{k,j=1}^{J} \lambda_k^p \|b_{k,j,i}\|^p_{H^p_+(\omega)} \\ &= S_0 + S_1. \end{aligned}$$

Since $p_0 > r'p$, $p(N_0 + 2) > s$ and since the $b_{k,j,0}$ are (p_0, N_0)-atoms with support in $2^4 I_{k,j}$ and $(2^4 I_{k,j})^+ \subset I_{k,j+2}$, we can apply lemma 4.8, obtaining $\|b_{k,j,0}\|^p_{H^p_+(\omega)} \leqslant c\omega(I_{k,j+2})$. Using this estimate and (4.1) we have

$$S_0 \leqslant c \sum_{k,j=1}^{J} \lambda_k^p \omega(I_{k,j+2}) = c \Big\| \sum_{k,j=1}^{J} \lambda_k^p \chi_{I_{k,j+2}} \Big\|_{L^1_\omega} \leqslant c \Big\| \sum_{k,j} \lambda_k^p \chi_{I_{k,j}} \Big\|_{L^1_\omega} \leqslant c\|f\|^p_{H^p_+(\omega)}.$$

To estimate S_1, note that for $i \geqslant 1$ the supports of the (p_0, N_0)-atoms $b_{k,j,i}$ are contained in $2^{i+4} I_{k,j} - 2^{i+1} I_{k,j}$ and, since $(2^{i+4} I_{k,j} - 2^{i+1} I_{k,j})^+ \subset 2^{i+1} I_{k,j}$ by lemma 4.8, we have

$$\|b_{k,j,i}\|^p_{H^p_+(\omega)} \leqslant c\omega(2^{i+1} I_{k,j}) \leqslant c 2^{i\theta} \omega(I_{k,j}).$$

Using this and (4.1) we have

$$\begin{aligned} S_1 &= \sum_{i=1}^{\infty} \frac{1}{\tilde{c}_i^p} \sum_{k,j=1}^{J} \lambda_k^p \|b_{k,j,i}\|^p_{H^p_+(\omega)} \\ &\leqslant c \sum_{i=1}^{\infty} \frac{2^{i\theta}}{\tilde{c}_i^p} \sum_{k,j=1}^{J} \lambda_k^p \omega(I_{k,j}) \\ &= c \sum_{i=1}^{\infty} \biggl(\frac{2^{i\delta}}{\tilde{c}_i}\biggr)^p \biggl\| \sum_{k,j=1}^{J} \lambda_k^p \chi_{I_{k,j}} \biggr\|_{L^1_\omega} \\ &\leqslant c \sum_{i=1}^{\infty} \biggl(\frac{2^{i\delta}}{\tilde{c}_i}\biggr)^p \|f\|^p_{H^p_+(\omega)}. \end{aligned} \tag{5.2}$$

For the case $p > 1$, we have

$$\begin{aligned} \biggl\| K_m * \sum_{k,j=1}^{J} \lambda_k a_{k,j} \biggr\|_{H^p_+(\omega)} &\leqslant c \biggl\| \sum_{k,j=1}^{J} \lambda_k b_{k,j,0} \biggr\|_{H^p_+(\omega)} + \sum_{i=1}^{\infty} \frac{c}{\tilde{c}_i} \biggl\| \sum_{k,j=1}^{J} \lambda_k b_{k,j,i} \biggr\|_{H^p_+(\omega)} \\ &= S_0 + S_1. \end{aligned}$$

Recalling that $p_0 > r'p$, $N_0 + 2 > \delta$ and that the $b_{k,j,0}$ are (p_0, N_0)-atoms supported in $2^4 I_{k,j}$ and since $(2^4 I_{k,j})^+ \subset I_{k,j+2}$, we can apply lemma 4.10 to estimate S_0:

$$S_0 \leqslant c \biggl\| \sum_{k,j=1}^{J} \lambda_k \chi_{I_{k,j+2}} \biggr\|_{L^p_\omega} \leqslant c \biggl\| \sum_{k,j=1}^{\infty} \lambda_k \chi_{I_{k,j}} \biggr\|_{L^p_\omega} \leqslant c \|f\|_{H^p_+(\omega)}.$$

For $i \geqslant 1$ the $b_{k,j,i}$ are (p_0, N_0)-atoms with support in $2^{i+4} I_{k,j} - 2^{i+1} I_{k,j}$, and since $(2^{i+4} I_{k,j} - 2^{i+1} I_{k,j})^+ \subset 2^{i+1} I_{k,j}$, applying lemma 4.10 and lemma 4.6 to estimate S_1, we obtain

$$\begin{aligned} S_1 &\leqslant c \sum_{i=1}^{\infty} \frac{1}{\tilde{c}_i} \biggl\| \sum_{k,j=1}^{J} \lambda_k \chi_{2^{i+1} I_{k,j}} \biggr\|_{L^p_\omega} \leqslant c \biggl(\sum_{i=1}^{\infty} \frac{2^{i\delta}}{\tilde{c}_i} \biggr) \biggl\| \sum_{k,j=1}^{J} \lambda_k \chi_{I^+_{k,j}} \biggr\|_{L^p_\omega} \\ &\leqslant c \biggl(\sum_{i=1}^{\infty} \frac{2^{i\delta}}{\tilde{c}_i} \biggr) \biggl\| \sum_{k,j=1}^{J} \lambda_k \chi_{I_{k,j+2}} \biggr\|_{L^p_\omega} \leqslant c \biggl(\sum_{i=1}^{\infty} \frac{2^{i\delta}}{\tilde{c}_i} \biggr) \|f\|_{H^p_+(\omega)}. \end{aligned} \tag{5.3}$$

To complete the proof we shall see that the series in (5.2) and in (5.3) are convergent. Let $\tilde{\ell} = \ell - \max(1/q, 1/p_0')$ be as in lemma 4.1. Since $\ell > \delta - 1/q'$, and $\ell > \delta - 1/p_0$, in the case when $1/q > 1/p_0'$ we have

$$\tilde{\ell} + 1 = \ell - \frac{1}{q} + 1 > \delta - \frac{1}{q'} - \frac{1}{q} + 1 = \delta,$$

while if $1/q < 1/p_0'$,

$$\tilde{\ell} + 1 = \ell - \frac{1}{p_0'} + 1 > \delta - \frac{1}{p_0} - \frac{1}{p_0'} + 1 = \delta.$$

Then $\tilde{\ell} + 1 > \delta$ and $N_0 + 2 > \delta$. Thus, by lemma 4.1 we have $\tilde{c}_i \geqslant c 2^{i(\delta+\varepsilon)}$ for some $\varepsilon > 0$, and therefore the series are convergent.

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6. An example

For fixed $t \in \mathbb{R}$ and $u \geqslant 0$, let $m(\xi) = (\xi + u\mathrm{i})^{\mathrm{i}t}$. The complex power is defined with the argument determination $-\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$. So $m(\xi)$ is the boundary value of a bounded analytic function on the upper half-plane.

First we prove that $\operatorname{supp}(K_m) \subset (-\infty, 0]$ by showing that $\langle K_m, \hat{\psi} \rangle = 0$ for every $\psi \in \mathcal{S}$ with $\operatorname{supp}(\hat{\psi}) \subset (0, \infty)$. By the Fourier inversion formula we have

$$\psi(\xi) = c \int_0^\infty \hat{\psi}(s) \mathrm{e}^{\mathrm{i}\xi s} \, \mathrm{d}s.$$

For $z \in \mathbb{C}$, $\operatorname{Im}(z) \geqslant 0$, we consider

$$\psi(z) = c \int_0^\infty \hat{\psi}(s) \mathrm{e}^{\mathrm{i}zs} \, \mathrm{d}s,$$

and, since $\operatorname{supp}(\hat{\psi}) \subset (0, \infty)$, by integrating by parts twice we get

$$z^2 \psi(z) = -c \int_0^\infty D^2 \hat{\psi}(s) \mathrm{e}^{\mathrm{i}zs} \, \mathrm{d}s.$$

Thus, we have that

$$|z^2 \psi(z)| \leqslant c \int_0^\infty |D^2 \hat{\psi}(s)| |\mathrm{e}^{\mathrm{i}zs}| \, \mathrm{d}s \leqslant C < \infty.$$

The above estimation and the fact that $m(z)$ is bounded imply that

$$\lim_{R \to \infty} \int_0^\pi m(R\mathrm{e}^{\mathrm{i}\theta}) \psi(R\mathrm{e}^{\mathrm{i}\theta}) R\mathrm{i}\mathrm{e}^{\mathrm{i}\theta} \, \mathrm{d}\theta = 0,$$

and, by Cauchy's theorem, it follows that

$$\langle K_m, \hat{\psi} \rangle = \langle m, \psi \rangle = \lim_{R \to \infty} \int_{-R}^R m(\xi) \psi(\xi) \, \mathrm{d}\xi = 0.$$

Finally, for any non-negative integer α,

$$|D^\alpha m(\xi)| \leqslant (|t| + \alpha)^\alpha \exp(\pi |t|) |\xi|^{-\alpha},$$

which implies (2.1) for every α and $1 \leqslant q \leqslant 2$.

Therefore, by theorem 3.1, $m(\xi)$ is a multiplier on $H^p_+(\omega)$ for every $0 < p < \infty$ and $\omega \in A^+_s$. Moreover, if we denote by T_m the multiplier operator defined by $m(\xi)$, then

$$\|T_m f\|_{H^p_+(\omega)} \leqslant c(|t| + \alpha)^\alpha \exp(\pi |t|) \|f\|_{H^p_+(\omega)}.$$

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