THE EQUIVARIANT CHEEGER–MÜLLER THEOREM ON LOCALLY SYMMETRIC SPACES

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Abstract   In this paper, we provide a concrete interpretation of equivariant Reidemeister torsion, and demonstrate that Bismut–Zhang's equivariant Cheeger–Müller theorem simplifies considerably when applied to locally symmetric spaces. In a companion paper, this allows us to extend recent results on torsion cohomology growth and torsion cohomology comparison for arithmetic locally symmetric spaces to an equivariant setting.

Introduction

The contents of this introduction are as follows.

- In § 0.1, we exhibit two numerical examples of torsion in the cohomology of (arithmetic) locally symmetric spaces, and describe how the Cheeger–Müller theorem explains both examples. The desire to prove equivariant analogs of the phenomena underlying these examples necessitates the analysis of equivariant Reidemeister torsion undertaken in this paper.

- In § 0.2, we describe how an equivariant analog of the Cheeger–Müller theorem, together with the trace formula comparison of [10], can in principle be used to prove equivariant analogs of the phenomenon discussed in § 0.1.

- In § 0.3, we describe difficulties which must be resolved in order for the equivariant Cheeger–Müller theorem to be used effectively for implementing the strategy outlined in § 0.2.

- In § 0.4, we state the main results of this paper, which resolve the difficulties highlighted in § 0.3.

- In § 0.5, we outline the contents of this paper.

- In § 0.6, we consolidate notation to be used regularly throughout.

0.1. The Cheeger–Müller theorem applied to number theory

Bergeron and Venkatesh [1] and Calegari and Venkatesh [5] have recently applied the Cheeger–Müller theorem to prove striking results about torsion in the cohomology of arithmetic locally symmetric spaces.

We present two numerical examples of torsion in the cohomology of arithmetic hyperbolic 3-manifolds exhibiting behavior suggested by their results. We then explain how the Cheeger–Müller theorem sheds light on both examples.

This section serves to motivate equivariant analogs of the results from [1, 5]; these are resolved in the case of "cyclic base change for quaternion algebras" in [10] with the help of the results proven in this paper.

0.1.1. Growth of torsion in cohomology. In the ring of integers $\mathbb{Z}[\sqrt{-2}]$, let $\mathfrak{p}$ be a prime ideal of residue degree 1 and norm 4969. Consider the congruence subgroup

$$\Gamma_0(\mathfrak{p}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{PGL}_2(\mathbb{Z}[\sqrt{-2}]) : \mathfrak{p} \text{ divides } c \right\}.$$

Sengün [18] has computed

$$\Gamma_0(\mathfrak{p})^{\mathrm{ab}} \cong (\mathbb{Z}/27287333293706982259194583 99)$$
$$\oplus (\mathbb{Z}/11452555958474009403487881957882603818 71) \oplus \cdots,$$

where the above two large integers are prime. Furthermore, the data computed in [18] suggest that this behavior among congruence subgroups of $\mathbf{PGL}_2(\mathbb{Z}[\sqrt{-2}])$ is in no way anomalous.

To highlight a connection with the next section, we note in passing that $\mathbf{PGL}_2(\mathbb{Z}[\sqrt{-2}])$ acts almost freely on hyperbolic 3-space $\mathbb{H}^3$, and we can make the identification

$$\Gamma_0(\mathfrak{p})^{\mathrm{ab}} = H_1(M, \mathbb{Z}), M = \Gamma_0(\mathfrak{p}) \backslash \mathbb{H}^3.$$

M is a cusped, hyperbolic 3-orbifold.

0.1.2. A relationship between cohomology groups on two incommensurable hyperbolic 3-manifolds. Let F be the cubic field $\mathbb{Q}(\theta), \theta^3 - \theta + 1 = 0$, which has one real place ∞. Let $\mathfrak{p}_5$ and $\mathfrak{p}_7$ denote the unique prime ideals of O_F of norm 5 and 7.

Let D denote the quaternion algebra over F ramified at $\mathfrak{p}_5, \infty$, and let D' denote the quaternion algebra over F ramified at $\mathfrak{p}_7, \infty$. We let $\Gamma(\mathfrak{n})$ denote the norm 1 units of $D^\times$ lying in the Eichler order of D of level $\mathfrak{n}$ and $\Gamma'(\mathfrak{n})$ the norm 1 units lying in the Eichler order of D' of level $\mathfrak{n}$. We consider the compact hyperbolic 3-orbifolds

$$W(\mathfrak{n}) = \Gamma(\mathfrak{n}) \backslash \mathbb{H}^3, W'(\mathfrak{n}) = \Gamma'(\mathfrak{n}) \backslash \mathbb{H}^3.$$

Letting $\mathfrak{n}_{5049}$ denote the unique ideal of O_F of norm 5049, one can compute (see [5, § 1.2]) that, if S is divisible by all prime numbers less than 40, then

$$H_1(W(\mathfrak{p}_7 \mathfrak{n}_{5049}), \mathbb{Z}[S^{-1}])$$
$$= (\mathbb{Z}/43)^4 \oplus (\mathbb{Z}/61)^2 \oplus (\mathbb{Z}/127) \oplus (\mathbb{Z}/139)^2 \oplus (\mathbb{Z}/181) \oplus (\mathbb{Z}[S^{-1}])^{81} \oplus (\mathbb{Z}/67)^2$$
$$H_1(W'(\mathfrak{p}_5 \mathfrak{n}_{5049}), \mathbb{Z}[S^{-1}])$$
$$= (\mathbb{Z}/43)^4 \oplus (\mathbb{Z}/61)^2 \oplus (\mathbb{Z}/127) \oplus (\mathbb{Z}/139)^2 \oplus (\mathbb{Z}/181) \oplus (\mathbb{Z}[S^{-1}])^{113}.$$

This is striking because the groups $\Gamma(\mathfrak{p}_7\mathfrak{n}_{5049})$ and $\Gamma'(\mathfrak{p}_5\mathfrak{n}_{5049})$ are incommensurable; there is no natural map between $W(\mathfrak{p}_7\mathfrak{n}_{5049})$ and $W'(\mathfrak{p}_5\mathfrak{n}_{5049})$.

0.1.3. The Cheeger–Müller theorem: a common thread. Ray and Singer [17] made the amazing discovery that Reidemeister torsion $RT(M, F)$ (see [6, §1] or §1.2 for the definition) of orthogonally flat local systems $F \to M$ of real vector spaces can be computed by analytic means. Cheeger [6] and Müller [13] independently later proved that, for a compact Riemannian manifold M,

$$RT(M, F) = \tau(M, F),$$

where $\tau(M, F)$ is the analytic torsion of F (see [6, §3] for the definition). Analytic torsion is an invariant of the spectrum of the Laplace operators attached to $F \to M$. More generally, Müller later proved in [14] that, for *unimodular* local systems of real or complex vector spaces $L \to M$ equipped with a metric on L,

$$RT(M, L) = \tau(M, L). \tag{1}$$

It is an insight of Calegari and Venkatesh [5] (see (b) below) and Bergeron and Venkatesh [1] (see (a) below) that the Cheeger–Müller theorem can be used to great effect in number theory in at least two different ways. For these applications, it is crucial to observe that, if $L \to M$ is a unimodular local system of free abelian groups equipped with a metric on $L_{\mathbb{R}} \to M$, then

$$RT(M, L) = \frac{|H^1(M, L)_{\mathrm{tors}}||H^3(M, L)_{\mathrm{tors}}|\cdots}{|H^0(M, L)_{\mathrm{tors}}||H^2(M, L)_{\mathrm{tors}}|\cdots} \times \frac{R^0(M, L)R^2(M, L)\cdots}{R^1(M, L)R^3(M, L)\cdots}, \tag{2}$$

where $R^i(M, L)$ equals the volume $H^i(M, L)_{\mathrm{free}}$ (see §1.2.2, example (3)).

(a) Let $\mathbf{G}$ be an algebraic group over $\mathbb{Q}$. The real group $\mathbf{G}(\mathbb{R})$ is (essentially) the isometry group of a symmetric space X of non-compact type. Fix a rational representation $\rho : \mathbf{G} \to \mathbf{GL}(V)$; let $\mathcal{O} \subset V$ be a $\mathbb{Z}$-lattice and $\mathbf{G}(\mathbb{Z})$ its stabilizer. As explained in [10, §2], for appropriate sequences of spaces $Y_n = \Gamma_n \backslash X$ with $\Gamma_n \subset \mathbf{G}(\mathbb{Q})$ an arithmetic subgroup, e.g., a subgroup of $\mathbf{G}(\mathbb{Z})$ cut out by finitely many congruence conditions, the representation ρ gives rise to a compatible sequence $L_n \to Y_n$ of local systems of free $\mathbb{Z}$-modules. Bergeron and Venkatesh are able to show that, for appropriate representations ρ, provided the injectivity radius of Y_n approaches infinity, the spectral invariants $\log \tau(Y_n, L_n)/\mathrm{vol}(Y_n)$ converge to a spectral invariant $\tau^{(2)}(X, \rho_{\mathbb{R}})$ of the pair $(X, \rho_{\mathbb{R}})$. By finding explicit examples (X, ρ) where $\tau^{(2)}(X, \rho_{\mathbb{R}}) \neq 0$, they prove that

$$\log RT(Y_n, L_n)/\mathrm{vol}(Y_n) = \log \tau(Y_n, L_n)/\mathrm{vol}(Y_n) \to \tau^{(2)}(X, \rho_{\mathbb{R}}) \neq 0 \tag{3}$$

as a consequence of the Cheeger–Müller theorem. Restricting consideration to representations ρ for which $\log R^\bullet(Y_n, L_n) = 0$ provides examples for which torsion in the cohomology groups of $L_n \to Y_n$ grows exponentially with the volume.
Hyperbolic 3-space $\mathbb{H}^3$ happens to satisfy $\log \tau(\mathbb{H}^3, \rho_{\mathbb{R}}) \neq 0$ for every ρ. Though the proof of Bergeron and Venkatesh does not apply to the trivial local system

(see [1, § 4], especially the discussion of "strongly acyclic" representations), the fact that $H_1(\Gamma_0(\mathfrak{p})\backslash\mathbb{H}^3, \mathbb{Z})_{\text{tors}}$ is so large when the norm of $\mathfrak{p}$ equals 4969 is a shadow of (3), which is still expected to be true.

(b) Though the hyperbolic 3-orbifolds $M = W(\mathfrak{p}_7\mathfrak{n}_{5049})$ and $N = W'(\mathfrak{p}_5\mathfrak{n}_{5049})$ are incommensurable, they have very closely related length spectra. Using the trace formula, one can show that correspondingly the spectra of the Laplace operators of M and N are very closely related; the definitive generalization of this spectral comparison was proven by Jacquet and Langlands in [8].

Calegari and Venkatesh use this fact to relate $\tau(M, \underline{\mathbb{Z}})$ and $\tau(N, \underline{\mathbb{Z}})$. In conjunction with the Cheeger–Müller theorem, this implies a very close relationship between the sizes of $H_1(M, \mathbb{Z})_{\text{tors}}$ and $H_1(N, \mathbb{Z})_{\text{tors}}$. Proving generalizations of the second computational example in this manner forms the content of their book [5].

0.2. Motivation: cohomology growth and cohomology relationships in the equivariant setting

Let D be any quaternion algebra over a number field F with a real places and b complex places. Let E/F be a cyclic Galois extension of prime degree p with Galois group $\Gamma_{E/F} = \langle\sigma\rangle$. Let Γ be an appropriately chosen congruence subgroup of $D^\times/F^\times$, let Γ' be a carefully chosen Galois-stable "matching" subgroup of $D_E^\times/E^\times$, and let L and $\mathcal{L}$ be "matching local systems" (see [10, § 4.3]). The way in which Γ and Γ' must be related is the subject of [10, § 5], [10, § 6]. The local systems $\mathcal{L}, L$ are described in [10, § 7.1]. Notably, they are local systems of free O_N-modules, for an appropriate finite extension N/F, for which $L_N, \mathcal{L}_N$ are acyclic. Let $M = \Gamma\backslash(\mathbb{H}^2)^a \times (\mathbb{H}^3)^b$, and let $\mathcal{M} = \Gamma'\backslash(\mathbb{H}^2)^{ap} \times (\mathbb{H}^3)^{bp}$. The local system $\mathcal{L} \to \mathcal{M}$ is equivariant for the natural $\Gamma_{E/F}$ action.

In the companion paper [10], we use trace formula methods to prove an identity of the shape

$$\tau_\sigma(\mathcal{M}, \mathcal{L}_\iota) = \tau(M, L_\iota)^p \tag{4}$$

for each embedding $\iota : N \hookrightarrow \mathbb{C}$. In this equation, $\tau_\sigma(\mathcal{M}, \mathcal{L}_\iota)$ denotes equivariant analytic torsion (see Definition 1.3); $\tau_\sigma(\mathcal{M}, \mathcal{L}_\iota)$ is a spectral invariant of $\mathcal{L}_\iota$ together with the action of $\langle\sigma\rangle$. One hopes that, by an equivariant analog of the Cheeger–Müller theorem, (4) will yield

$$RT_\sigma(\mathcal{M}, \mathcal{L}_\iota) = RT(M, L_\iota)^p, \tag{5}$$

where $RT_\sigma(M, L_\iota)$ denotes equivariant Reidemeister torsion (see Definition 1.14). Let us assume the validity of (5). Combining (4) and (5) with known results on the growth of $\tau(M, L_\iota)$ (see [1]) would yield growth results for $RT_\sigma(\mathcal{M}, \mathcal{L}_\iota)$. An equivariant analog of (2) could be directly combined with (5) to yield a numerical comparison of torsion cohomology.

0.3. Difficulties with the equivariant Cheeger–Müller theorem

Lott and Rothenberg in [11] and Lück in [12] have proven (see Theorem 1.23) that

$$\tau_\sigma(\mathcal{M}, \mathcal{L}_\iota) = RT_\sigma(\mathcal{M}, \mathcal{L}_\iota)$$

for *unitarily flat* equivariant local systems $\mathcal{L}_\iota \to \mathcal{M}$. However, other than the trivial local system, the most natural local systems $\mathcal{L}$ on spaces such as $\Gamma'\backslash(\mathbb{H}^2)^{ap} \times (\mathbb{H}^3)^{bp}$ are not unitarily flat. Those which are unitarily flat are not acyclic.

In the interest of generalizing the torsion growth theorem (a) and the numerical torsion comparison (b) discussed in §0.1.3, we are very fortunate to have the Bismut–Zhang formula available; this is discussed at length in §4. Bismut and Zhang in [4] prove that

$$\log \tau_\sigma(\mathcal{M}, \mathcal{L}_\iota) - \log RT_\sigma(\mathcal{M}, \mathcal{L}_\iota) = E(\mathcal{M}, \mathcal{L}),$$

where $E(\mathcal{M}, \mathcal{L})$ is an error term which localizes to the fixed point set $\mathcal{M}_\sigma$. The expression for E from [4] is unfortunately not explicit enough for the aforementioned growth and numerical comparison applications.

In addition to the question of whether $E(\mathcal{M}, \mathcal{L}) = 0$, there is a further issue of interpreting what $RT_\sigma(\mathcal{M}, \mathcal{L})$ is. For the applications sketched in §0.2, we require an analog of (2) giving a concrete interpretation for Reidemeister torsion in terms of sizes of torsion cohomology groups.

The main purpose of this paper is to resolve these two issues.

0.4. Statement of main results

We quickly set some notation to state our first main result. Let p be prime. Let $P(x) = x^{p-1} + x^{p-2} + \cdots + 1$, the p-cyclotomic polynomial.

Let R be a commutative ring. For any R-module A acted on R-linearly by $\langle \sigma \rangle \cong \mathbb{Z}/p\mathbb{Z}$ and any polynomial $h \in R[x]$, let

$$A^{h(\sigma)} := \{a \in A : h(\sigma) \cdot a = 0\}.$$

Corollary 3.8 (Concrete interpretation of twisted Reidemeister torsion)**.** *Let $\mathcal{L} \to M$ be a rationally acyclic, metrized, unimodular local system $\mathcal{L} \to \mathcal{M}$ of free abelian groups acted on isometrically by $\langle \sigma \rangle \cong \mathbb{Z}/p\mathbb{Z}$. Suppose that the fixed point set $\mathcal{M}_\sigma$ has Euler characteristic 0. Then*

$$\log RT_\sigma(\mathcal{M}, \mathcal{L}) = -\left(\log \left|H^*(\mathcal{M}, \mathcal{L})[p^{-1}]^{\sigma-1}\right| - \frac{1}{p-1}\log \left|H^*(\mathcal{M}, \mathcal{L})[p^{-1}]^{P(\sigma)}\right|\right)$$
$$+ O\left(\log|H^*(\mathcal{M}, \mathcal{L})[p^\infty]| + \log|H^*(\mathcal{M}, \mathcal{L}_{\mathbb{F}_p})| + \log|H^*(M, \mathcal{L}_{\mathbb{F}_p})|\right).$$

This is proven by relating the "naive twisted Reidemeister torsion" $NRT_\sigma(C^\bullet(\mathcal{M}, \mathcal{L}; K))$ (see Definition 3.1) of the cochain complex $C^\bullet(\mathcal{M}, \mathcal{L}; K)$ to the twisted Reidemeister torsion of $(\mathcal{M}, \mathcal{L})$ (see Proposition 3.7) and then using a spectral sequence argument to relate NRT_σ to the cohomology of $H^\bullet(\mathcal{M}, \mathcal{L})$ (see Proposition 3.4) in a "triangulation independent" manner.

The next theorem describes circumstances under which twisted Reidemeister torsion equals twisted analytic torsion.

Corollary 5.5 (τ_σ often equals RT_σ)**.** *Let $\mathcal{L} \to \mathcal{M}$ be an equivariant, metrized, rationally acyclic local system of free abelian groups over a locally symmetric space $\mathcal{M}$ acted on equivariantly and isometrically by $\langle \sigma \rangle \cong \mathbb{Z}/p\mathbb{Z}$. Suppose further that the restriction to the fixed point set $\mathcal{L}|_{\mathcal{M}_\sigma} = L^{\otimes p}$ for $L \to \mathcal{M}_\sigma$ self-dual, and that $\mathcal{M}_\sigma$ is odd dimensional.*

Then

$$\log \tau_\sigma(\mathcal{M}, \mathcal{L}) = \log RT_\sigma(\mathcal{M}, \mathcal{L}).$$

This theorem is proven in two steps.

- Proposition 4.5 compares the error terms of two different applications of the Bismut–Zhang formula (31), one for $\mathcal{L} \to \mathcal{M}$ and the other for $L^{\boxtimes p} \to \mathcal{M}_\sigma^p$, to prove that

$$\log RT_\sigma(\mathcal{M}, \mathcal{L}) - \log \tau_\sigma(\mathcal{M}, \mathcal{L}) = \log RT_\sigma(\mathcal{M}_\sigma^p, L^{\boxtimes p}) - \log \tau_\sigma(\mathcal{M}_\sigma^p, L^{\boxtimes p}).$$

- Section 5 proves that the latter difference

$$\log RT_\sigma(\mathcal{M}_\sigma^p, L^{\boxtimes p}) - \log \tau_\sigma(\mathcal{M}_\sigma^p, L^{\boxtimes p})$$

is zero by separately proving that

$$\log \tau_\sigma(\mathcal{M}_\sigma^p, L^{\boxtimes p}) = p \log \tau(\mathcal{M}_\sigma, L)$$

in Proposition 5.1,

$$\log RT_\sigma(\mathcal{M}_\sigma^p, L^{\boxtimes p}) = p \log RT(\mathcal{M}_\sigma, L)$$

in Theorem 5.4, and then applying the untwisted Cheeger–Müller theorem to conclude.

0.5. Outline

- In §1, we recall the definition of twisted analytic torsion and twisted Reidemeister torsion of an (equivariant) metrized local system $\mathcal{L} \to \mathcal{M}$ of O_N-modules over a compact Riemannian manifold, where O_N is the ring of integers of a number field N. In §1.6, we prove a "compatibility with restriction of scalars" property of twisted Reidemeister torsion for equivariant local systems of O_N-modules, N a number field. This property is used to relate our main results, which concern comparisons of Reidemeister torsion for matching local systems $\mathcal{L} \to \mathcal{M}$ and $L \to M$ of O_N-modules over different manifolds, to a relationship between sizes of cohomology groups (see [10, §7.2]).
  In §1.7, we state Lück's version of the equivariant Cheeger–Müller theorem from [12]. This theorem, along with the untwisted Cheeger–Müller theorem for unimodular local systems proven in [14], can be used in conjunction with the spectral comparison theorem proven in [10, §4.4] to yield numerical cohomology comparisons.

- In §2, we discuss a version of the twisted Cheeger–Müller theorem valid for a finite, rationally acyclic metrized chain complex $A^\bullet$ of free abelian groups acted on isometrically by σ with $\sigma^p = 1$. This calculation naturally leads to the definition of "naive equivariant Reidemeister torsion" (see Definition 3.1).

- In §3, in the case where $A^\bullet = C^\bullet(\mathcal{M}, \mathcal{L}; K)$ is the group of $\mathcal{L}$-valued cochains on $\mathcal{M}$ with respect to a fixed equivariant triangulation, we carry through a spectral sequence argument to relate the naive Reidemeister torsion, up to a controlled error, to a quantity which is patently related to the structure of $H^*(\mathcal{M}, \mathcal{L})$ as a σ-module and which is independent of the triangulation K. Finally, in §3.3, for $\mathcal{L} \to \mathcal{M}$ an equivariant, unimodular local system over a locally symmetric space, we relate the naive twisted Reidemeister torsion of $C^\bullet(\mathcal{M}, \mathcal{L}; K)$ to its twisted Reidemeister torsion.

- In §4, we recall the statement of the equivariant Cheeger–Müller theorem, proven by Bismut and Zhang. Let $\mathcal{L} \to \mathcal{M}$ be a σ-equivariant metrized local system. The Bismut–Zhang formula enables us to prove Proposition 4.5, which shows that the difference between the twisted Reidemeister torsion and the twisted analytic torsion of a σ-equivariant local system $\mathcal{L} \to \mathcal{M}$ equals the same difference for an $L^{\boxtimes p} \to (\mathcal{M}_\sigma)^p$, where σ acts on the latter by cyclic shift. This constitutes progress, since the twisted analytic torsion and twisted Reidemeister torsion of a product are individually computed in §5.

- In §5, we study the equivariant analytic torsion and equivariant Reidemeister torsion for a product. We directly calculate both the equivariant analytic torsion of a metrized local system and the "naive Reidemeister torsion" (see Definition 3.1) of the local system $L_\rho^{\boxtimes n} \to \mathcal{M} = M^n$ over the compact Riemannian manifold M^n with respect to the cyclic shift σ. In §5.3, we show that, for many unimodular metrized local systems $L \to M$ which are not necessarily unitarily flat, the conclusion of Lück's variant of the twisted Cheeger–Müller theorem continues to hold for $L^{\boxtimes p} \to M^p$: its twisted Reidemeister torsion often equals its twisted analytic torsion.

0.6. Notation used throughout

This section compiles a list of frequently used notation. The descriptions given are consistent with the most common usage of the corresponding symbols. The reader should be warned, however, that, within a given chapter or section, the symbols below might carry a slightly different meaning; such local changes of notation will be made clear as necessary.

- $L \to M$ denotes a local system of projective O_F, F, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{R}$, or $\mathbb{C}$-modules, depending on the context.

- $\mathcal{L} \to \mathcal{M}$ denotes a local system equivariant for the action of a finite group Γ, usually $\Gamma = \langle \sigma \rangle$ with $\sigma^p = 1$.

- $C^\bullet(M, L; K)$ denotes the complex of L-valued cochains on M with respect to a triangulation K.

- Let f be a Morse function on a smooth closed manifold M, and let X be a weakly gradient-like vector field with respect to f. Let Φ_X denote the flow generated by X. For every critical point p of X, we let $W^u(p)$ denote the unstable (ascending) manifold of p and $W^s(p)$ denote the stable (descending) manifold of p. These are respectively defined as

$$W^u(p) = \{m \in M : \lim_{t \to -\infty} \Phi_{X,t}(m) = p\}$$
$$W^s(p) = \{m \in M : \lim_{t \to \infty} \Phi_{X,t}(m) = p\}.$$

See [15, §2.4] for further discussion.

- $\mathrm{MS}(X, L)$ denotes the Morse–Smale complex associated with a vector field X on M which is weakly gradient like with respect to a fixed Morse function f and which satisfies Morse–Smale transversality.

- $RT(M, L)$ denotes the Reidemeister torsion of the cochain complex $C^\bullet(M, L; K)$ for a local system $L \to M$ provided the implicit volume forms and triangulation are understood. $RT_\sigma(\mathcal{M}, \mathcal{L})$ denotes the twisted Reidemeister torsion (evaluated at σ) of the complex $C^\bullet(M, L; K)$ for a $\langle\sigma\rangle$-equivariant local system $\mathcal{L} \to \mathcal{M}$ of free abelian groups.

- $RT(X, L)$ denotes the Reidemeister torsion of the Morse–Smale complex $\mathrm{MS}(X, L)$ for a vector field X, satisfying Morse–Smale transversality, and a local system $L \to M$, provided the Morse function f and the implicit volume forms are understood. $RT_\sigma(X, \mathcal{L})$ denotes the twisted Reidemeister torsion of the Morse–Smale complex whenever $\mathcal{L} \to \mathcal{M}$ is a $\langle\sigma\rangle$-equivariant local system.

- For an R-module A acted on R-linearly by $\langle\sigma\rangle$, we let $A^{\sigma-1} := \{a \in A : (\sigma - 1) \cdot a = 0\}$ and $A^{P(\sigma)} = \{a \in A : P(\sigma) \cdot a = 0\}$, where $P(\sigma)$ denotes the p-cyclotomic polynomial $P(x) = x^{p-1} + x^{p-2} + \cdots + 1$. Sometimes, we denote these by $A[\sigma - 1]$ and $A[P(\sigma)]$ as well.

- For an R-module A acted on R-linearly by $\langle\sigma\rangle$, we define $A' := A/(A[\sigma - 1] \oplus A[P(\sigma)])$. Similarly, if $A^\bullet$ is a complex of R-modules acted on R-linearly by $\langle\sigma\rangle$, we define $A'^\bullet := A^\bullet/(A^\bullet[\sigma - 1] \oplus A^\bullet[P(\sigma)])$.

- For a finite abelian group B and a rational prime p, we let $B[p^\infty]$ and $B[p^{-1}]$ respectively denote the p-primary subgroup of B and the prime to p subgroup of B. These are canonically isomorphic to $B \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ and $B \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}]$ respectively.

- $\sum^*, \prod^*, \otimes^*$ respectively denote alternating sum, product, and tensor product. The alternating tensor product of R-modules $M_1, \ldots, M_n$ means $M_1 \otimes_R M_2^* \otimes_R M_3 \otimes \cdots$

- For a projective R-module M, we let $\det(M)$ denote $\wedge_R^{\mathrm{top}} M$. For a complex $M^\bullet := 0 \to M_0 \to M_1 \to \cdots$ of projective R-modules, its determinant is defined to be $\det(M) := \otimes^* \det(M_i)$.

1. The equivariant Cheeger–Müller theorem

- In §1.1, we define equivariant analytic torsion of a metrized local system $\mathcal{L} \to \mathcal{M}$ of $\mathbb{C}$-vector spaces acted on equivariantly by isometries by a finite group Γ.

- In §1.2, we define the Reidemeister torsion of a complex $A^\bullet$ of K-vector spaces equipped with volume forms on the chain groups $A^\bullet$ and the cohomology groups $H^*(A^\bullet)$, where K is any field.

- In §1.3, we define the twisted Reidemeister torsion of an abstract metrized complex of $\mathbb{C}$-vector spaces.

- In §1.4, we define the Morse–Smale complex associated to a local system $\mathcal{L} \to X$ together with a Morse function $f : X \to \mathbb{R}$.

- In §§1.5 and 1.6, we explain how auxiliary volume forms associated with a $\langle\sigma\rangle$-equivariant unimodular local system of N-vector spaces $\mathcal{L} \to \mathcal{M}$, for any field N, allows us to define an $N^\times$-valued version of equivariant Reidemeister torsion. In particular, for N a number field, we prove in §1.6.2 a "norm compatibility" between the equivariant Reidemeister torsion of the geometric complex $C^\bullet(\mathcal{M}, \mathcal{L}; K)$, where $\mathcal{L}$

is a local system of N vector spaces, and the same geometric complex $C^\bullet(\mathcal{M}, \mathcal{L}_\mathbb{Q}; K)$, where $\mathcal{L}_\mathbb{Q}$ denotes $\mathcal{L}$ viewed as a local system of $\mathbb{Q}$-vector spaces.

- In §1.7, we finally state the twisted Cheeger–Müller theorem due to Lück in [12] and the untwisted Cheeger–Müller theorem due to Müller in [13].

1.1. Definition of twisted analytic torsion

Let $\mathcal{M}$ be a compact Riemannian manifold.

Definition 1.1. A *metrized local system* $\mathcal{L} \to \mathcal{M}$ is a local system of free abelian groups equipped with a metric on $\mathcal{L}_\mathbb{R}$. The metric on $\mathcal{L}_\mathbb{R}$ is *not* required to be compatible with the flat structure on $\mathcal{L}$, which is to say that parallel transport need not be unitary.

Definition 1.2. If the parallel transport on $\mathcal{L}_\mathbb{R}$ induced by the flat structure is unitary, we call $\mathcal{L} \to \mathcal{M}$ *unitarily flat*.

Let Γ be a group of finite order acting equivariantly on a metrized local system $\mathcal{L} \to \mathcal{M}$ by isometries. Let $\Delta_{j,\mathcal{L}}$ denote the j-form Laplacian acting on $\Omega^j(\mathcal{M}, \mathcal{L})$.

Note that the λ-eigenspace of $\Delta_{j,\mathcal{L}}$, call it $E_{j,\mathcal{L},\lambda}$, is preserved under pullback by Γ because Γ acts by isometries; each $E_{j,\mathcal{L},\lambda}$ is a representation of Γ. Let $\mathsf{Rep}(\Gamma)$ denote the representation ring of the finite group Γ. We can form the $\mathsf{Rep}(\Gamma)_\mathbb{C}$-valued equivariant zeta functions

$$\zeta_{j,\mathcal{L},\Gamma}(s) = \sum \lambda^{-s}[E_{j,\mathcal{L},\lambda}] \in \mathsf{Rep}(\Gamma)_\mathbb{C},$$

where the sum ranges over the non-zero eigenvalues of $\Delta_{j,\mathcal{L}}$. All of the functions $\zeta_{j,\mathcal{L},\sigma}$ admit meromorphic continuation to the entire complex plane and are holomorphic in a neighborhood of $s = 0$ (see [12, Lemma 1.13]). Form the linear combination

$$Z_{\mathcal{L},\Gamma}(s) = \frac{1}{2} \sum_j (-1)^j j \cdot \zeta_{j,\mathcal{L},\Gamma}(s).$$

Definition 1.3. The *twisted (or equivariant) analytic torsion* τ_Γ is defined by

$$\tau_\Gamma(\mathcal{L}) = Z'_{\mathcal{L},\Gamma}(0) \in \mathsf{Rep}(\Gamma)_\mathbb{C}.$$

If $\Gamma = \langle \sigma \rangle$ is a cyclic group, we will often denote

$$\tau_\sigma(\mathcal{L}) := \mathrm{tr}\{\sigma \mid \tau_\Gamma(\mathcal{L})\}.$$

Remarks.

- If $\mathcal{L} = \mathbb{Z}_\mathcal{M}$ is the trivial local system, then we will sometimes label our generating functions with a subscript $\mathcal{M}$ instead of a subscript $\mathbb{Z}_\mathcal{M}$.

- If $\sigma = 1$, we recover the usual untwisted torsion from this definition.

1.2. Definition of Reidemeister torsion

Let K be any field. We recall the definition of the determinant of a K-vector space or a complex of K-vector spaces.

Definition 1.4. Let V be a K-vector space of finite dimension n. We define its determinant to be $\det(V) := \wedge^n V$. If $A^\bullet = 0 \to A^0 \to \cdots \to A^n \to 0$ is a finite complex of finite-dimensional K-vector spaces, we define

$$\det(A^\bullet) := \bigotimes{}^* \det(A^\bullet) := \bigotimes_{i=0}^{n} \det(A^i)^{(-1)^i},$$

where the superscript -1 denotes K-dual. In words, this is the "alternating tensor product" of the determinants of the constituents of the complex $A^\bullet$.

Let $(A^\bullet, d^\bullet)$ be a bounded complex of finite-dimensional K-vector spaces. Let $B^\bullet, Z^\bullet, H^\bullet$ be the associated complexes of coboundaries, cocycles, and cohomology. Suppose that A^i, B^i, Z^i, and H^i have respective dimensions a_i, b_i, z_i, h_i. Suppose further that each group $A^\bullet, H^*(A^\bullet)$ is equipped with a volume form, i.e., there are given volume forms $\mu_i \in \left(\wedge^{h_i} H^i(A^\bullet) \right)^*$, $\omega_i \in \left(\wedge^{a_i} A^i \right)^*$. There is a canonical isomorphism

$$\det(A^\bullet) \otimes_K \det(H^i(A^\bullet))^{-1} \cong K \tag{6}$$

(see [9, §1, Proposition 1]). Let the section $s_{A^\bullet}$ of $\det(A^\bullet)^{-1} \otimes_K \det(H^*(A^\bullet))$ be the preimage of 1 under the isomorphism of (6).

Definition 1.5 (Reidemeister torsion through volume forms)**.** The *Reidemeister torsion* $RT(A^\bullet, \omega, \mu) \in K^\times$ is the value of $\bigotimes^* \omega_i \otimes \bigotimes^* \mu_i^{-1}$ evaluated on the section $s_{A^\bullet}$ of $\det(A^\bullet)^{-1} \otimes_K \det(H^*(A^\bullet))$.

The Reidemeister torsion of $A^\bullet$ can be computed as follows. Choose $\rho_i \in \left(\wedge^{b_i} A^i \right)^*$ with $\rho_i|_{\wedge^{b_i} B^i} \neq 0$ and $\sigma_i \in \left(\wedge^{h_i} A^i \right)^*$ with $\sigma|_{\wedge^{h_i} Z^i} = \pi^*(\mu_i)$, where π denotes the projection $Z^\bullet \to H^\bullet$. Then

$$\rho_i \wedge d_{i+1}^*(\rho_{i+1}) \wedge \sigma_i = m_i \omega_i$$

for some $m_i \in K^\times$. Then

$$RT(A^\bullet, \omega, \mu) = \prod{}^* m_i \in K^\times$$

(cf. [6, §1]).

1.2.1. Norms versus volume forms for complexes of $\mathbb{C}$-vector spaces.
Let $A^\bullet$ be a finite complex of finite $\mathbb{C}$-vector spaces. We introduce some terminology.

Definition 1.6 (Norms and volume forms)**.** Let V be a $\mathbb{C}$-vector space of finite dimension n. By a *norm on V*, we mean a non-degenerate norm on the one-dimensional complex vector space $\wedge^n V$. A *volume form* denotes a non-zero element of $(\wedge^n V)^*$. The absolute value of a volume form on V is a norm on V.

Definition 1.7 (Reidemeister torsion through norms)**.** Suppose that each A^i is endowed with a norm α_i and that each $H^i(A^\bullet)$ is endowed with a norm β_i. We can choose $\alpha_i' \in \left(\wedge^{\text{top}} A^i \right)^*$ and $\beta_i' \in \left(\wedge^{\text{top}} H^i(A^\bullet) \right)^*$ for which $|\alpha_i'| = \alpha_i$, $|\beta_i'| = \beta_i$. We define

$$RT(A^\bullet, \alpha, \beta) = |RT(A^\bullet, \alpha', \beta')|.$$

Remark 1.8. The choice of α_i', β_i' is only ambiguous up to a complex number of absolute value 1, so the absolute value $|RT(A^\bullet, \alpha', \beta')|$ is independent of all choices.

1.2.2. Some useful examples and properties of Reidemeister torsion.

(1) Any finite-free abelian group B gives rise to a canonical norm on $\det(B_\mathbb{C})$, namely that which assigns $e_1 \wedge \cdots \wedge e_r$ norm 1 for any basis $e_1, \ldots, e_r$ of B. For any complex $A^\bullet$ of finite-free abelian groups, we let $\alpha_\mathbb{Z}$ and $\beta_\mathbb{Z}$ denote these norms arising from the integral structure in this manner. One computes that

$$RT(A^\bullet, \alpha_\mathbb{Z}, \beta_\mathbb{Z}) = \prod {}^* |H^i(A^\bullet)_{\text{tors}}|$$

(see [6, § 1]).

(2) If β' is a different choice of norms on $H^i(A_\mathbb{C}^\bullet)$ with $\beta_i' = k_i \beta_i$, then

$$RT(A^\bullet, \alpha, \beta') = RT(A^\bullet, \alpha, \beta) \times \prod {}^* k_i. \tag{7}$$

(3) Suppose that $L \to M$ is a metrized local system of free $\mathbb{Z}$-modules over a compact Riemannian manifold (M, g). Let $A^\bullet$ denote the group of L-valued cochains on M with respect to a fixed choice of triangulation; the cohomology of this chain complex computes $H^\bullet(M, L)$. The Riemannian metric induces metrics on each $H^i(A^\bullet) \cong H^i(M, L)$ via Hodge theory, and hence norms $\beta_{i,g}$ on the $\det(H^i(A^\bullet))$. In such geometric situations, we define

$$RT(A^\bullet) := RT(A^\bullet, \alpha_\mathbb{Z}, \beta_g).$$

Note that $\beta_{i,g} = \text{vol}(H^i(M, L_\mathbb{R})/H^i(M, L)) \times \beta_{i,\mathbb{Z}}$. So, by (7),

$$RT(A^\bullet) = \prod {}^* |H^i(M, L)_{\text{tors}}| \times \prod {}^* \text{vol}(H^i(M, L_\mathbb{R})/H^i(M, L)).$$

(4) In our imminent discussion of Reidemeister torsion, it will be necessary to think of Reidemeister torsion in terms of (Hermitian) metrics on $A^\bullet$. Any such metric h induces a collection of norms α_h, where α_h^i is the norm that h induces on $A^i \otimes \mathbb{C}$. Suppose that $A^\bullet = C^\bullet(M, L; K)$ for some triangulation K of the manifold M from (3). The group of cochains is generated by "indicator cochains" $\mathbf{1}_{C,s}$, i.e., those which assign some cell C a global section s of $L|_C$. Assume that $L \to M$ is unitarily flat. We define a metric $h_\mathbb{Z}$ on $A^\bullet$ by insisting that

$$h_\mathbb{Z}(\mathbf{1}_{C,s}, \mathbf{1}_{C,s}) = ||s||^2 \quad \text{and} \quad h_\mathbb{Z}(\mathbf{1}_{C,s}, \mathbf{1}_{C',s'}) = 0 \quad \text{if } C, C' \text{ are distinct},$$

where, for a section $s \in L(C)$, we define $||s|| = ||s_x||$ for any $x \in C$. Because L is unitarily flat, this is well defined. Provided that $||e_1 \wedge \cdots \wedge e_n|| = 1$ for every integral basis $e_1, \ldots, e_n$ of $L(C)$ and every cell C of the triangulation K,

$$RT(A^\bullet, \alpha_{h_\mathbb{Z}}, \beta) = RT(A^\bullet, \alpha_\mathbb{Z}, \beta).$$

1.3. Definition of twisted Reidemeister torsion for an abstract metrized complex $A^\bullet$

Let $A^\bullet$ be a finite complex of finite-dimensional $\mathbb{C}$-vector spaces together with metrics h_i on $A^\bullet$ and induced metrics g_i on each $H^i(A^\bullet)$.

Definition 1.9. Let Γ be a finite group acting on $A^\bullet$ by isometries. The *equivariant Reidemeister torsion of* $A^\bullet$ is defined by the formula

$$\log RT_\Gamma(A^\bullet, h, g) := \sum_{\pi \in \widehat{\Gamma}} \frac{1}{\dim \pi} \log RT(A^\bullet[\pi], h|_{A^\bullet[\pi]}, g|_{H^i(A^\bullet[\pi])}) \cdot \pi \in \text{Rep}(\Gamma)_\mathbb{C},$$

where $A^\bullet[\pi]$ denotes the π-isotypic subcomplex of $A^\bullet$ and $\text{Rep}(\Gamma)$ denotes the representation ring of Γ.

Remark 1.10. The representation ring $\text{Rep}(\Gamma)$ is free as a $\mathbb{Z}$-module with basis given by the isomorphism classes of irreducible complex representations of Γ. The map assigning to each representation its character defines an isomorphism between $\text{Rep}(\Gamma)_\mathbb{C}$ and the complex-valued class functions on Γ. We can and will use this isomorphism to evaluate elements of $\text{Rep}(\Gamma)_\mathbb{C}$ on conjugacy classes of Γ.

Let N be a number field with ring of integers O_N.

Definition 1.11. Let $\mathcal{L} \to \mathcal{M}$ be a local system of O_N-modules, and let $\iota : N \hookrightarrow \mathbb{C}$ be an embedding. Suppose that $\mathcal{L}_\iota = \mathcal{L} \otimes_\iota \mathbb{C}$ is endowed with a unitarily flat metric. Suppose that the finite group Γ acts equivariantly on $\mathcal{L}_\iota \to \mathcal{M}$ by isometries. Let K be a fixed Γ-equivariant triangulation with cochain group $C^\bullet(\mathcal{M}, \mathcal{L}; K)$. We define

$$\log RT_\Gamma(\mathcal{M}, \mathcal{L}_\iota; K) := \log RT_\Gamma(C^\bullet(\mathcal{M}, \mathcal{L}_\iota; K)), \tag{8}$$

where the metrics implicit on the right-hand side of (8) are those induced on $C^\bullet(\mathcal{M}, \mathcal{L}_\iota; K)$ and $H^*(C^\bullet(\mathcal{M}, \mathcal{L}_\iota; K))$ by $\mathcal{L}_\iota$ (see example (4) from §1.2.2). For notational shorthand, if the group Γ is understood – often $\Gamma = \langle \sigma \rangle$ – we define

$$RT_\sigma(\mathcal{M}, \mathcal{L}_\iota; K) := RT_\Gamma(\mathcal{M}, \mathcal{L}_\iota; K)(\sigma).$$

If $\mathcal{M}$, the triangulation, and the complex embedding ι are all understood, we denote this by $RT_\sigma(\mathcal{L})$.

1.4. The Morse–Smale complex and the definition of twisted Reidemeister torsion

Definition 1.12. Let $f : M \to \mathbb{R}$ be a Morse function on a compact manifold M. Let X be a vector field on M. We say that X is a *weakly gradient-like vector field associated to f* if the critical points of X equal the critical points of f and if $X_p(f) > 0$ for all non-critical points $p \in M$. That is, f increases along the flow of X.

Definition 1.13. Let X be a weakly gradient-like vector field associated to a Morse function $f : \mathcal{M} \to \mathbb{R}$. We say that X satisfies *Morse–Smale transversality* if, for every pair p, q of critical points of X, the ascending manifold $W^u(p)$ of p and the descending manifold $W^s(q)$ of q intersect transversely.

Let X be a weakly gradient-like vector field associated to a Morse function f. Such a vector field X satisfying Morse–Smale transversality gives rise to a Morse–Smale

complex which computes the cohomology of any local system $\mathcal{L} \to \mathcal{M}$ of finite projective R-modules, where R is any ring. The following formulation is taken from [3, § 1 (c)], and is repeated here for convenience. Define a chain complex

$$\mathrm{MS}^i(X, \mathcal{L}) = \bigoplus_{x \in \mathrm{Crit}(X), \mathrm{ind}(x) = i} R[W^u(x)] \otimes_R \mathcal{L}_x,$$

where $R[W^u(x)]$ denotes the free rank-1 R-module with basis $[W^u(x)]$ and $\mathrm{ind}(x)$ denotes the Morse index of the critical point x. Because X satisfies Morse–Smale transversality, the set of flow lines $\Gamma(x, y)$ from x to y for each pair of critical points x, y is finite and empty unless $\mathrm{ind}(y) = \mathrm{ind}(x) + 1$.

Consider a pair of critical points x, y with $\mathrm{ind}(y) = \mathrm{ind}(x) + 1$. Fix an orientation on X and on each unstable manifold $W^u(x)$; this determines an orientation on each $W^s(x)$. Because $W^u(x)$ and $W^s(y)$ intersect transversally, we can "flow their orientations" along any integral curve γ from x to y to obtain a well-defined number $n_\gamma(x, y) = \pm 1$, where the sign is $+1$ if the flowed orientations agree and -1 if they are opposite. We define a boundary map

$$\delta : \mathrm{MS}^i(X, \mathcal{L}) \to \mathrm{MS}^{i+1}(X, \mathcal{L})$$
$$\delta(W^u(x) \otimes a) = \sum_{y \in \mathrm{Crit}(X), \mathrm{ind}(y) = i+1} \sum_{\gamma \in \Gamma(x, y)} n_\gamma(x, y) \cdot W^u(y) \otimes PT_\gamma(a),$$

where $PT_\gamma(a)$ denotes the parallel transport of a along the integral curve γ. Some comments are in order.

- The complex $\mathrm{MS}^\bullet(X, \mathcal{L})$ computes the cohomology of $\mathcal{L} \to \mathcal{M}$.
- Suppose that f is an invariant Morse function, that X is an invariant weakly gradient-like vector field, and that $R = \mathbb{R}$. Let $\mathcal{L} \to \mathcal{M}$ be a Γ-equivariant, metrized local system. Assign the vector space $\bigoplus_{\mathrm{ind}(x)=i} \mathbb{R}[W^u(x)]$ the combinatorial metric where the $[W^u(x)]$ form an orthonormal basis. Then Γ acts on $\mathrm{MS}^\bullet(X, \mathcal{L})$ by isometries, and so the equivariant Reidemeister torsion of this complex makes sense.

Definition 1.14. Let $\mathcal{L}$ be a metrized local system. We define $RT_\Gamma(X, \mathcal{L})$ to be the Γ-equivariant Reidemeister torsion of the metrized Γ-complex $\mathrm{MS}^\bullet(X, \mathcal{L})$.

1.5. Reidemeister torsion of the σ-isotypic pieces of $\mathrm{MS}^\bullet(X, \mathcal{L})$

In §§ 1.5.1 and 1.6.1, we specialize Definition 1.5 for Reidemeister torsion to isotypic pieces of geometric complexes associated with local systems $\mathcal{L} \to X$-equivariant for a cyclic group action. In Lemma 1.21, the Reidemeister torsion of these isotypic pieces is related to the equivariant Reidemeister torsion of the complexified local system $\mathcal{L}_\iota \to X$ for embeddings $\iota : N \hookrightarrow \mathbb{C}$. The local systems we consider will all be *rationally acyclic*.

Definition 1.15. A local system $\mathcal{L} \to X$ of N-vector spaces is *acyclic* if $H^*(X, \mathcal{L}) = 0$. A local system $\mathcal{L}' \to X$ of O_N-modules is *rationally acyclic* (or $\mathbb{Q}$-acyclic or N-acyclic) if $\mathcal{L}'_N$, or equivalently $\mathcal{L}_\mathbb{Q}$, is acyclic.

1.5.1. Reidemeister torsion of the cochain complex MS$^\bullet(X, \mathcal{L})[\sigma - 1]$ for local systems of rationally acyclic N-vector spaces. Let $\mathcal{L} \to \mathcal{M}$ be an equivariant local system of N-vector spaces acted on N-linearly by $\langle \sigma \rangle$ with $\sigma^p = 1$. Assume that $\mathcal{L}$ is rationally acyclic. Let $A^\bullet = \mathrm{MS}^\bullet(X, \mathcal{L})$ for a vector field X which satisfies Morse–Smale transversality and is weakly gradient like with respect to a σ-invariant Morse function f on $\mathcal{M}$. By a *volume form* on a local system $\mathcal{L} \to \mathcal{M}$ of N-vector spaces, we mean a global section of $\det(\mathcal{L})^*$.

Suppose that we are given volume forms ω on $\mathcal{L} \to \mathcal{M}$, $\omega_{\sigma-1}$ on $\mathcal{L}[\sigma-1] \to \mathcal{M}_\sigma$, and $\omega_{P(\sigma)}$ on $\mathcal{L}[P(\sigma)] \to \mathcal{M}_\sigma$. These give rise to volume forms on the chain groups $A^\bullet[\sigma - 1]$ and $A^\bullet[P(\sigma)]$. Indeed, let $\mathcal{C}$ be a set of representatives for the orbits of σ acting on the critical points of X. Then $A^\bullet[\sigma - 1]$ has a "geometric" N-basis given as follows.

- Suppose that $x \in \mathcal{C}$ is not fixed by σ. We let $\mathcal{O}_x := \bigoplus_j W(\sigma^j \cdot x) \otimes \mathcal{L}_{\sigma^j \cdot x}$, where $\mathcal{L}_x$ denotes the fiber of $\mathcal{L}$ over x. Then a basis for $\mathcal{O}_x[\sigma - 1]$ is given by $\{\sum_j \sigma^j W^u(x) \otimes \sigma^j e_k^x\}$, where $e_1^x, \ldots, e_r^x$ runs through a basis of $\mathcal{L}_x$.

- If $x \in \mathcal{C}$ is fixed by σ, then let $f_1^x, \ldots, f_s^x$ be a basis for $\mathcal{L}_x[\sigma - 1]$. A basis for $\mathcal{O}_x[\sigma - 1] = W^u(x) \otimes \mathcal{L}_x[\sigma - 1]$ is given by $\{W^u(x) \otimes f_k^x\}$.

Then a volume form $\theta_{\sigma-1}^i$ is given by

$$\theta_{\sigma-1}^i \left(\bigwedge_{x \text{ not fixed}} \bigwedge_k \left(\sum_j W^u(\sigma^j x) \otimes \sigma^j e_k^x \right) \wedge \bigwedge_{x \text{ fixed}} \bigwedge_k W^u(x) \otimes f_k^x \right)$$

$$= \prod_{x \text{ not fixed}} \omega(e_1^x \wedge \cdots \wedge e_r^x) \prod_{x \text{ fixed}} \omega_{\sigma-1}(f_1^x \wedge \cdots \wedge f_s^x),$$

where x ranges over all critical points of $\mathcal{C}$ of index i. Using the volume form $\omega_{P(\sigma)}$, we can construct a collection of geometric volume forms $\theta_{P(\sigma)}^i$ in a completely analogous manner. By the definition of Reidemeister torsion, it then follows that

$$RT(A^\bullet[\sigma - 1], \theta_{\sigma-1}), \quad RT(A^\bullet[P(\sigma)], \theta_{P(\sigma)}) \in N^\times.$$

1.6. Reidemeister torsion of the σ-isotypic pieces of $C^\bullet(\mathcal{M}, \mathcal{L}; K)$

1.6.1. Reidemeister torsion of the cochain complex $C^\bullet(\mathcal{M}, \mathcal{L}; K)[\sigma - 1]$ for local systems of rationally acyclic N-vector spaces. Let $\mathcal{L} \to \mathcal{M}$ be an equivariant local system of N-vector spaces acted on N-linearly by $\langle \sigma \rangle$ with $\sigma^p = 1$. Assume that $\mathcal{L}$ is rationally acyclic. Let $A^\bullet = C^\bullet(\mathcal{M}, \mathcal{L}; K)$ for a σ-equivariant triangulation of $\mathcal{M}$. By a *volume form* on a local system $\mathcal{L} \to \mathcal{M}$ of N-vector spaces, we mean a global section of $\det(L)^*$.

Suppose that we are given volume forms ω on $\mathcal{L} \to \mathcal{M}$, $\omega_{\sigma-1}$ on $\mathcal{L}[\sigma-1] \to \mathcal{M}_\sigma$, and $\omega_{P(\sigma)}$ on $\mathcal{L}[P(\sigma)] \to \mathcal{M}_\sigma$. These give rise to volume forms on the chain groups $A^\bullet[\sigma - 1]$ and $A^\bullet[P(\sigma)]$. Indeed, let $\mathcal{C} = \{C\}$ be a set of representatives for the orbits of σ acting on the i-cells of the triangulation K. Then $A^\bullet[\sigma - 1]$ has a "geometric" N-basis given as follows.

- Suppose that C is not fixed by σ. We let $\mathcal{O}_C := \bigoplus \sigma^j \cdot C \otimes \mathcal{L}(C)$, where $\mathcal{L}(C)$ denotes the N-vector space of sections of $\mathcal{L}$ over C. Then a basis for $\mathcal{O}_C[\sigma - 1]$ is given by $\{\sum_j \sigma^j C \otimes \sigma^j e_k^C\}$, where $e_1^C, \ldots, e_r^C$ runs through a basis of $\mathcal{L}(C)$.

- If C is fixed by σ, then let $f_1^C, \ldots, f_s^C$ be a basis for $L[\sigma - 1](C)$. A basis for $\mathcal{O}_C[\sigma - 1] = C \otimes \mathcal{L}[\sigma - 1](C)$ is given by $\{C \otimes f_k^C\}$.

Then a volume form $\theta_{\sigma-1}^i$ is given by

$$
\theta_{\sigma-1}^i \left(\bigwedge_{C \text{ not fixed}} \bigwedge_k \left(\sum_j \sigma^j C \otimes \sigma^j e_k^C \right) \wedge \bigwedge_{C \text{ fixed}} \bigwedge_k C \otimes f_k^C \right)
$$
$$
= \prod_{C \text{ not fixed}} \omega(e_1^C \wedge \cdots \wedge e_r^C) \prod_{C \text{ fixed}} \omega_{\sigma-1}(f_1^C \wedge \cdots \wedge f_s^C).
$$

Using the volume form $\omega_{P(\sigma)}$, we can construct a collection of geometric volume forms $\theta_{P(\sigma)}^i$ in a completely analogous manner. By the definition of Reidemeister torsion, it then follows that

$$
RT(A^\bullet[\sigma - 1], \theta_{\sigma-1}), \quad RT(A^\bullet[P(\sigma)], \theta_{P(\sigma)}) \in N^\times.
$$

Remark 1.16 (The relative merits of the Morse–Smale complex)**.** §1.6 is nearly identical to §1.5. We see fit to mention the following.

- The Bismut–Zhang theorem (see §4.1) is expressed in the language of Morse–Smale complexes.

- §1.5 is a special case of §1.6 where we take the decomposition of $\mathcal{M}$ into (the closures of) unstable cells $W^u(x)$ as our cell decomposition.

- The complex $\mathrm{MS}^\bullet(X, \mathcal{L})$ appears to be a slight enrichment of $C^\bullet(M, \mathcal{L}; \{W^u(x)\})$ in that each unstable manifold is endowed with a distinguished point which we were used to define the volume forms on $\mathrm{MS}(X, \mathcal{L})[\sigma - 1]$, $\mathrm{MS}(X, \mathcal{L})[P(\sigma)]$. However, as long as $\mathcal{L} \to \mathcal{M}, \mathcal{L}[\sigma - 1] \to \mathcal{M}_\sigma$ and $\mathcal{L}[P(\sigma)] \to \mathcal{M}_\sigma$ are unimodular, the aforementioned volume forms are "independent of choice of basepoint". The Morse–Smale perspective proves more convenient in [3, 4], where the authors prove Cheeger–Müller type theorems for local systems $\mathcal{L}$ not assumed to be unimodular.

1.6.2. Reidemeister torsion and norm compatibility.   Let N be a number field, and let $\pi : \mathrm{Spec}\, O_N \to \mathrm{Spec}\, \mathbb{Z}$ be the natural map. For any O_N or N-module P, we let $\pi_* P$ denote its restriction of scalars. To describe the compatibility between the Reidemeister torsion of the complex $P^\bullet$ and of the complex $\pi_* P^\bullet$ requires the notion of the norm of a projective O_N-module. We refer to [16, II, §4] for the required foundations.

Lemma 1.17. *Let P be a projective O_N-module of finite rank d. There is a canonical isomorphism*

$$
\det{}_{\mathbb{Z}}(\pi_* P) = \mathrm{Norm}_{O_N/\mathbb{Z}}(\det{}_{O_N}(P)) \otimes (\det{}_{\mathbb{Z}} O_N)^{\otimes d}.
$$

Proof. See [16, II, §4].   □

Lemma 1.18. *Let $P^\bullet$ be a finite complex N-vector spaces with P^i of dimension d^i. Suppose that $P^\bullet$ is equipped with a collection of volume forms $\omega_\bullet$ and that the cohomology spaces $H^*(P^\bullet)$ are equipped with volume forms $\mu_\bullet$. Suppose that $RT(P^\bullet, \omega_\bullet, \mu_\bullet) = f \in N^\times$. Then $RT(\pi_* P^\bullet, \mathrm{Norm}_{N/\mathbb{Q}} \omega_\bullet, \mathrm{Norm}_{N/\mathbb{Q}} \mu_\bullet) = \mathrm{Norm}_{N/\mathbb{Q}} f$.*

Remark 1.19. In accordance with Lemma 1.17, $\mathrm{Norm}_{N/\mathbb{Q}}\omega_\bullet$, $\mathrm{Norm}_{N/\mathbb{Q}}\mu_\bullet$ are not volume forms, but rather $\mathrm{Norm}_{N/\mathbb{Q}}\omega_\bullet \otimes \gamma_0^{\otimes \dim_N P^\bullet}$ and $\mathrm{Norm}_{N/\mathbb{Q}}\mu_\bullet \otimes \gamma_0^{\otimes \dim_N H^*(P^\bullet)}$ are, for $\gamma_0 \in (\wedge_{\mathbb{Q}}^{\mathrm{top}} N)^*$. However, as will be clear from the proof to follow, the Reidemeister torsion of the resulting complex is independent of choice of γ_0; this is why γ_0 is omitted from the notation.

Proof. Let $d_i = \dim_N P^i$, $h_i = \dim_N H^i(P^\bullet)$. By Lemma 1.17,

$$
\det_{\mathbb{Q}}(\pi_* P^\bullet) \otimes \det_{\mathbb{Q}}(\pi_* H^*(P^\bullet))^{-1}
$$
$$
= \mathrm{Norm}_{N/\mathbb{Q}}(\det_N P^\bullet) \otimes \mathrm{Norm}_{N/\mathbb{Q}}(\det_N H^*(P^\bullet))^{-1} \otimes (\det_{\mathbb{Q}} N)^{\otimes \sum^* d_i - \sum^* h_i}
$$
$$
= \mathrm{Norm}_{N/\mathbb{Q}}(\det_N P^\bullet) \otimes \mathrm{Norm}_{N/\mathbb{Q}}(\det_N H^*(P^\bullet))^{-1}.
$$

The last equality follows because $\sum^* d_i = \sum^* h_i$, both equaling the Euler characteristic of the complex $P^\bullet$. The complex $P^\bullet$ gives rise to the section $s_{P^\bullet} = f \cdot \omega_\bullet \otimes \mu_\bullet^{-1}$ of $\det_N(P^\bullet) \otimes \det_N(H^*(P^\bullet))^{-1}$, where $f = RT(P^\bullet, \omega_\bullet, \mu_\bullet)$. Therefore,

$$
\mathrm{Norm}_{N/\mathbb{Q}} s_{P^\bullet} = \mathrm{Norm}_{N/\mathbb{Q}}(f \cdot \omega_\bullet \otimes \mu_\bullet^{-1})
$$
$$
= \mathrm{Norm}_{N/\mathbb{Q}} f \cdot (\mathrm{Norm}_{N/\mathbb{Q}}\omega_\bullet) \otimes (\mathrm{Norm}_{N/\mathbb{Q}}\mu_\bullet)^{-1}.
$$

Unraveling the isomorphism from Lemma 1.17 (see [16, II, §4.2]), we readily find that $\mathrm{Norm}_{N/\mathbb{Q}} s_{P^\bullet} = s_{\pi_* P^\bullet}$. The lemma follows. $\qquad\square$

1.6.3. Applying norm compatibility to the geometric complexes $C^\bullet(\mathcal{M}, \mathcal{L}; K), \mathrm{MS}(X, \mathcal{L})$ **for local systems of** O_N**-modules.** Suppose that $\mathcal{L} \to \mathcal{M}$ is an equivariant local system of projective O_N-modules for which $\det(\mathcal{L})^*$ is trivial and $\det(\mathcal{L}[\sigma-1])^*, \det(\mathcal{L}[P(\sigma)])^* \to \mathcal{M}_\sigma$ are trivial with bases $\omega, \omega_{\sigma-1}, \omega_{P(\sigma)}$. Suppose further that $\mathcal{L}_N$ is acyclic. Let $A^\bullet = C^\bullet(\mathcal{M}, \mathcal{L}; K)$ or $\mathrm{MS}(X, \mathcal{L})$, as defined in §1.5 and §1.6. As before, we construct volume forms $\theta_{\sigma-1}^i, \theta_{P(\sigma)-1}^i$, now generators of the *free* O_N-modules $(\wedge_{O_N}^{\mathrm{top}} A^i[\sigma-1])^*$ and $(\wedge_{O_N}^{\mathrm{top}} A^i[P(\sigma)])^*$ respectively. Let $\gamma_0 \in (\wedge_{\mathbb{Z}}^{\mathrm{top}} O_N)^*$.

As a corollary to Lemma 1.18, we obtain the main result of this section.

Corollary 1.20. *Let* $A^\bullet = C^\bullet(\mathcal{M}, \mathcal{L}; K)$ *or* $\mathrm{MS}(X, \mathcal{L})$. *If* $RT(A^\bullet[\sigma-1], \theta_{\sigma-1}) = f$, $RT(A^\bullet[P(\sigma)], \theta_{P(\sigma)}) = f' \in N^\times$, *then*

$$
\mathrm{Norm}_{N/\mathbb{Q}} f = \pm \prod{}^* |H^i(A^\bullet[\sigma-1])|, \quad \mathrm{Norm}_{N/\mathbb{Q}} f' = \pm \prod{}^* |H^i(A^\bullet[P(\sigma)])|.
$$

Proof. As explained in item (1) of §1.2.2, the Reidemeister torsion for any finite, rationally acyclic complex $C^\bullet$ of $\mathbb{Z}$-modules with volume forms α_i given by a generator of $(\wedge_{\mathbb{Z}}^{\mathrm{top}} C^i)^*$ satisfies

$$
RT(C^\bullet, \alpha) = \pm \prod{}^* |H^i(C^\bullet)|.
$$

The result follows immediately by Lemma 1.18, with $C^\bullet = \mathrm{MS}^\bullet(X, \mathcal{L})[\sigma-1]$, $\mathrm{MS}^\bullet(X, \mathcal{L})[P(\sigma)]$. $\qquad\square$

1.6.4. Twisted Reidemeister torsion of $\mathrm{MS}^\bullet(X, \mathcal{L}), C^\bullet(\mathcal{M}, \mathcal{L}; K)$ **and compatibility with base change to** $\mathbb{C}$**.** Let N be a number field. Let $\iota : N \hookrightarrow \mathbb{C}$ be a

complex embedding. Let $\mathcal{L} \to M$ be an equivariant local system of O_N-modules acted on by $\langle \sigma \rangle$ with $\sigma^p = 1$. Suppose that there are global volume forms ω on $\det(\mathcal{L}) \to \mathcal{M}$, $\omega_{\sigma-1}$ on $\mathcal{L}[\sigma - 1] \to \mathcal{M}_\sigma$, and $\omega_{P(\sigma)}$ on $\mathcal{L}[P(\sigma)] \to \mathcal{M}_\sigma$.

Suppose further that there is a metric h on $\mathcal{L}_\iota := \mathcal{L} \otimes_\iota \mathbb{C}$ which induces the norm $|\omega_\iota|$ and which also induces the norms $|\omega_{\sigma-1,\iota}|, |\omega_{P(\sigma),\iota}|$ on $\mathcal{L}[\sigma - 1] \otimes_\iota \mathbb{C} \to \mathcal{M}_\sigma$, $\mathcal{L}[P(\sigma)] \otimes_\iota \mathbb{C} \to \mathcal{M}_\sigma$. Suppose further that σ acts isometrically on $\mathcal{L} \to \mathcal{M}$ with respect to h.

Lemma 1.21. *Let* $A^\bullet = C^\bullet(\mathcal{M}, \mathcal{L}; K)$ *or* $\mathrm{MS}(X, \mathcal{L})$. *Suppose that* $RT(A^\bullet[\sigma - 1], \theta_{\sigma-1}) = f$, $RT(A^\bullet[P(\sigma)], \theta_{P(\sigma)}) = f' \in N^\times$. *Then*

$$\log RT_\sigma(\mathcal{M}, \mathcal{L}_\iota, h) = \log |\iota(f)| - \frac{1}{p-1} \log |\iota(f')|.$$

Furthermore, if the volume forms $\omega_{\sigma-1}, \omega_{P(\sigma)}$ *giving rise to* $\theta_{\sigma-1}, \theta_{P(\sigma)}$ *are generators of* $\det_{O_N}(\mathcal{L}[\sigma - 1])^*$ *and* $\det_{O_N}(\mathcal{L}_{P(\sigma)})^*$, *then*

$$|\mathrm{Norm}_{N/\mathbb{Q}}(f)| = \prod {}^* |H^i(A^\bullet[\sigma - 1])|,$$
$$|\mathrm{Norm}_{N/\mathbb{Q}}(f')| = \prod {}^* |H^i(A^\bullet[P(\sigma)])|.$$

Proof. For any bounded chain complex of finite-dimensional N-vector spaces $B^\bullet$, volume forms θ on $B^\bullet$ give rise to volume forms θ_ι on the base changed complex $B_\iota^\bullet$. According to the recipe described after Definition 1.5 for computing Reidemeister torsion, if g is the Reidemeister torsion of $(B^\bullet, \theta)$, then $\iota(g)$ is the Reidemeister torsion of $(B_\iota^\bullet, \theta_\iota)$. By the hypothesis that the norms $|\omega_\iota|, |\omega_{\sigma-1,\iota}|$, and $|\omega_{P(\sigma),\iota}|$ are all induced by the metric h, the first part of the proposition follows from this observation. The second part follows by Corollary 1.20. $\square$

1.7. Statements of two variants of the Cheeger–Müller theorem

We recall the statements of one variant of the untwisted Cheeger–Müller theorem and one of its twisted counterpart. These are one crucial input into the main theorem [10, Theorem 7.4].

Theorem 1.22 [14]**.** *Let* M *be a closed manifold, and let* $L \to M$ *be a metrized, unimodular local system of* $\mathbb{C}$*-vector spaces. Then*

$$\tau(M, L) = RT(M, L).$$

Theorem 1.23 [12, Theorem 4.5]**.** *Let* $\mathcal{M}$ *be a closed manifold, and let* $\mathcal{L} \to \mathcal{M}$ *be a* $\langle \sigma \rangle$*-equivariant local system of* $\mathbb{C}$*-vector spaces. Suppose that* $\mathcal{L}$ *is equipped with a metric with respect to which it is* unitarily flat *and for which a finite cyclic group* $\langle \sigma \rangle$ *acts equivariantly by isometries. Then*

$$\tau_\sigma(\mathcal{M}, \mathcal{L}) = RT_\sigma(\mathcal{M}, \mathcal{L}).$$

See Definition 1.3 and the subsequent remarks for the definition of twisted and untwisted analytic torsion, respectively denoted τ_σ and τ. See Definition 1.14 for the definition of twisted Reidemeister torsion, denoted RT_σ, which defines Reidemeister torsion, denoted RT, when $\langle \sigma \rangle = 1$.

Remark 1.24 (History of the Cheeger–Müller theorem)**.** The remarkable discovery that the untwisted Cheeger–Müller theorem might be true was made by Ray and Singer in [17]. The untwisted Cheeger–Müller theorem was proven independently by Cheeger in [6] and Müller in [13] for orthogonal local systems. Müller later generalized this result to arbitrary unimodular local systems in [14]. A more general and difficult to state variant for arbitrary local systems was proven by Bismut and Zhang in [3].

The twisted Cheeger–Müller theorem for the trivial local system was first proved by Lott and Rothenberg in [11]. It was generalized to equivariant orthogonal local systems by Lück in [12]. The ultimate version we will use was proven by Bismut and Zhang in [4]; we apply the Bismut–Zhang variant of the twisted Cheeger–Müller theorem to equivariant unimodular local systems. We discuss the Bismut–Zhang variant at length in § 4.

2. Equivariant Cheeger–Müller theorem for finite chain complexes

The equivariant Cheeger–Müller theorem 1.23 will be crucial in what follows. To motivate it, we derive a version for finite-dimensional chain complexes.

- In § 2.1, we recall the statement of the untwisted Cheeger–Müller theorem for finite chain complexes.
- In § 2.2, we define the twisted analytic torsion of a chain complex $A^\bullet$ of free abelian groups for which $A^\bullet_\mathbb{R}$ is metrized and acted on isometrically by a finite group Γ.
- In § 2.3, we derive a homological expression for the twisted analytic torsion of a complex $A^\bullet$ as above.

2.1. Statement of the untwisted Cheeger–Müller theorem for finite chain complexes

Let $(A^\bullet, d)$ be a chain complex of finite-free abelian groups, equipped with metrics on $A^\bullet_\mathbb{R}$, such that $\mathrm{vol}(A^i_\mathbb{R}/A^i) = 1$ for each i. This complex has a Laplace operator

$$\Delta_j = d_{j-1}d^*_{j-1} + d^*_j d_j.$$

Associated to Δ_j are spectral generating functions

$$\zeta_{j,A^\bullet}(s) = \sum_{\lambda \text{ eigenvalue of } \Delta_j} \lambda^{-s}, \; Z_{A^\bullet}(s) = \frac{1}{2}\sum (-1)^j j \zeta_{j,A^\bullet}(s).$$

Definition 2.1. The *(untwisted) analytic torsion* of the metrized complex $A^\bullet$ is defined to be

$$\tau(A^\bullet) = \exp(-Z'_{A^\bullet}(0)). \tag{9}$$

There is a Hodge decomposition for any finite chain complex which allows us to represent each cohomology class of $A^\bullet_\mathbb{R}$ uniquely by a *harmonic cochain*, defined to be an element of the kernel of $\Delta_\bullet$. Thus, $H^i(A^\bullet_\mathbb{R})$ inherits a metric from the space of harmonic cochains. Define the regulators to be

$$R^i(A^\bullet) = \mathrm{vol}(H^i(A^\bullet_\mathbb{R})/\mathrm{im}\{H^i(A^\bullet) \to H^i(A^\bullet_\mathbb{R})\}).$$

Lemma 2.2. *There is an equality*

$$\sum{}^* \log R^i(A^\bullet) - \log |H^i(A^\bullet)_{\text{tors}}| = \log \tau(A^\bullet). \tag{10}$$

Proof. See [1, § 2] or [6, § 1]. □

Remark 2.3. Suppose that the complex $A^\bullet$ arises as the group of L-valued cochains – associated to a particular triangulation – for a metrized local system $L \to M$ over a compact, smooth, Riemannian manifold M. The content of the de Rham version of untwisted Cheeger–Müller theorem (see [13]) is that the identity (10) "passes to a de Rham limit" under successively finer triangulations of M.

2.2. Definition of twisted analytic torsion for finite chain complexes

Let $A^\bullet$ be a complex of metrized, finite-free $\mathbb{Z}$-modules acted on isometrically by a finite group Γ. We can form the equivariant zeta functions

$$\zeta_{j,A^\bullet,\Gamma}(s) = \sum_{\lambda \text{ eigenvalue of } \Delta_j} \lambda^{-s}[E_\lambda] \in \text{Rep}(\Gamma)_{\mathbb{C}},$$

where $[E_\lambda]$ is the λ-eigenspace of the combinatorial Laplacian Δ_j, thought of as a representation of Γ.

Definition 2.4. The *equivariant (or twisted) analytic torsion* $\tau_\Gamma(A^\bullet)$ of the finite, metrized chain complex $A^\bullet$, acted on isometrically by a finite group Γ, is defined to be

$$\tau_\Gamma(A^\bullet) := Z'_{A^\bullet,\Gamma}(0), \quad \text{where } Z_{A^\bullet,\Gamma}(s) = \frac{1}{2} \sum (-1)^j j \zeta_{j,A^\bullet,\Gamma}. \tag{11}$$

As a notational shorthand, we define

$$\tau_\sigma(A^\bullet) := \text{tr}\{\sigma \mid \tau_\Gamma(A^\bullet)\}. \tag{12}$$

2.3. A homological expression for the twisted analytic torsion $\tau_\sigma(A^\bullet)$

Suppose that the metrized complex $A^\bullet$ from above is endowed with an involution σ acting by isometries. Of course, we are more generally interested in $\sigma^p = 1$, but all difficulties are already present for $p = 2$, the necessary changes are clear, and the notation is simpler.

We let $D^i = A^{i,+} \oplus A^{i,-}$. Unfortunately, it is usually not the case that $D^i = A^i$.

Lemma 2.5. *Assume that* $\text{vol}(A^i) = 1$ *for every* i. *Then, letting* $A'^\bullet := A^\bullet/(A^\bullet[\sigma - 1] \oplus A^\bullet[\sigma + 1])$,

$$\tau_\sigma(A^\bullet) = \frac{\prod^* |H^i(A^\bullet[\sigma-1])_{\text{tors}}|^{-1} \prod^* R(A^i[\sigma-1])}{\prod^* |H^i(A^\bullet[\sigma+1])_{\text{tors}}|^{-1} \prod^* R(A^i[\sigma+1])} \times \prod{}^* |H^i(A'^\bullet)|^{-1}. \tag{13}$$

Proof. To derive a combinatorial expression for $\tau_\sigma(A^\bullet)$, we split the complex into the ± 1 eigenspaces of σ. Let $A^{\bullet,+} = (A^\bullet)^{\sigma-1}$ and $A^{\bullet,-} = (A^\bullet)^{\sigma+1}$. Note that

$$
\begin{aligned}
\zeta_{j,\sigma}(s) &= \sum \mathrm{tr}\{\sigma | A^i(\lambda)\}\lambda^{-s} \\
&= \sum \dim A^{i,+}(\lambda)\lambda^{-s} - \sum \dim A^{i,-}(\lambda)\lambda^{-s} \\
&= \zeta_{j,A^{+,\bullet}}(s) - \zeta_{j,A^{-,\bullet}}(s).
\end{aligned}
$$

So, by the computation of [1, §2] of untwisted analytic torsion for finite chain complexes, namely (10), we arrive at the identity

$$
\tau_\sigma(A^\bullet) = \frac{\prod{}^*|H^i(A^{+,\bullet})_{\mathrm{tors}}|^{-1} \prod{}^* R(A^{+,i})}{\prod{}^*|H^i(A^{-,\bullet})_{\mathrm{tors}}|^{-1} \prod{}^* R(A^{-,i})} \times \prod{}^* \mathrm{vol}(D^i)^{-1}. \tag{14}
$$

Because $\mathrm{vol}(A^i) = 1$,

$$
\mathrm{vol}(D^i) = [A^i : D^i].
$$

This allows us to re-express the product of volumes homologically:

$$
\begin{aligned}
\prod{}^* \mathrm{vol}(D_i)^{-1} &= \prod{}^* [A^i : D^i]^{-1} \\
&= \prod{}^* |H^i(A'^\bullet)|^{-1}. \tag{15}
\end{aligned}
$$

Equation (15) makes sense because $A^i_{\mathbb{Q}} = D^i_{\mathbb{Q}}$ and $H^i(A'^\bullet)$ is finite for every i. Substituting (15) back into (14) proves the lemma. $\square$

3. A "triangulation independent" estimate of twisted Reidemeister torsion

In Lemma 2.5, we derived a homological expression for the twisted analytic torsion of a finite metrized complex $A^\bullet$ of free abelian groups acted on isometrically by $\mathbb{Z}/2\mathbb{Z} = \langle \sigma \rangle$. This computation specializes to the following when $A^\bullet$ is $\mathbb{Q}$-acyclic:

$$
\tau_\sigma(A^\bullet) = \frac{\prod{}^*|H^i(A^{+,\bullet})_{\mathrm{tors}}|^{-1}}{\prod{}^*|H^i(A^{-,\bullet})_{\mathrm{tors}}|^{-1}} \times \prod{}^* |H^i(A'^\bullet)|^{-1}, \tag{16}
$$

where we recall that $A'^\bullet = A^\bullet / (A^{+,\bullet} \oplus A^{-,\bullet})$. The expression on the right-hand side of (16) will come up often enough that we see fit to give it a definition.

Definition 3.1. Let C be any bounded complex of torsion-free $\mathbb{Z}[\sigma]$-modules, where $\sigma^p = 1$ for some prime p, and assume that $C_{\mathbb{Q}}$ is acyclic. We define the *naive equivariant Reidemeister torsion of C*, denoted NRT_σ, as

$$
\begin{aligned}
\log NRT_\sigma(C^\bullet) &= \left\{ \sum{}^* \log |H^i(C^{\sigma-1})| - \frac{1}{p-1}\log|H^i(C^{P(\sigma)})| \right\} + \left\{ \sum{}^* \log |C'_i| \right\} \\
&= \left\{ \sum{}^* \log |H^i(C^{\sigma-1})| - \frac{1}{p-1}\log|H^i(C^{P(\sigma)})| \right\} + \left\{ \sum{}^* \log |H^i(C')| \right\},
\end{aligned}
$$

where $C' := C/(C^{\sigma-1} \oplus C^{P(\sigma)})$.

We use the word "naive" because $NRT_\sigma(A^\bullet)$ does not always equal $RT_\sigma(A^\bullet)$. Nonetheless, it is concrete, and we can directly relate it to $RT_\sigma(A^\bullet)$ in geometric situations where $A^\bullet$ arises as the group of $\mathcal{L}$-valued cochains on $\mathcal{M}$ for some $\langle\sigma\rangle$-equivariant local system $\mathcal{L} \to \mathcal{M}$ or as the Morse–Smale complex of $\mathcal{L}$ for some gradient vector field X on $\mathcal{M}$.

The goal of this section is to prove that approximately

$$NRT_\sigma(A^\bullet) \sim \prod {}^* \frac{|H^i(A^\bullet)^{\sigma-1}_{\text{tors}}|}{|H^i(A^\bullet)^{\sigma+1}_{\text{tors}}|} \quad \text{for } \sigma^2 = 1 \quad (\text{and an analog for } \sigma^p = 1), \qquad (17)$$

where $\sim$ denotes equality up to a "controlled" power of 2. On the right-hand side of (17), σ acts on the cohomology groups of $A^\bullet$, *not* on $A^\bullet$ itself as in the definition of naive equivariant Reidemeister torsion. Equation (17) will be important for two reasons.

- The twisted Cheeger–Müller theorem relates analytic torsion to Reidemeister torsion, *not directly* to sizes of cohomology groups. In the interest of proving cohomology growth theorems following [1], it is important to concretely understand the relationship between the twisted Reidemeister torsion $RT_\sigma(\mathcal{M}, \mathcal{L})$ and the cohomology $H^*(\mathcal{M}, \mathcal{L})$.

- The numerical torsion functoriality theorems proven in [10, §7.2] concern $RT_\sigma(\mathcal{M}, \mathcal{L})$. In order to prove numerical versions of torsion functoriality, in the spirit of [5], we need to directly relate RT_σ to sizes of cohomology groups.

To relate $RT_\sigma(A^\bullet)$ to the sizes of cohomology groups $H^*(A^\bullet)$, we do the following.

- In §3.1, we separate the expression for NRT_σ into a 2-power torsion part and prime to 2 torsion parts.

- In §3.2, we use a spectral sequence argument to relate $NRT_\sigma(A^\bullet)$ to $\prod {}^* \frac{|H^i(A^\bullet)^{\sigma-1}_{\text{tors}}|}{|H^i(A^\bullet)^{\sigma+1}_{\text{tors}}|}$ in the case where $A^\bullet$ is the complex of $\mathcal{L}$-valued cochains of $\mathcal{M}$ for an equivariant local system $\mathcal{L} \to \mathcal{M}$ of free abelian groups adapted to an equivariant triangulation.

- In §3.3, we will compare the naive twisted Reidemeister torsion of the cochain complex $C^\bullet(\mathcal{M}, \mathcal{L}; K)$, acted on isometrically by $\mathbb{Z}/p\mathbb{Z}$, to its actual twisted Reidemeister torsion. We obtain a homological expression for the difference between these two quantities, which is often zero. This is made precise in Proposition 3.7, the main result of §3.

3.1. Adapting $NRT_\sigma(A^\bullet)$ to the prime 2

We decompose the product (16) defining $NRT_\sigma(A^\bullet)$ as follows:

$$\log NRT_\sigma(A^\bullet) = -\left(\sum {}^* \log\left|H^i(A^\bullet)[2^{-1}]^{\sigma-1}\right| - \log\left|H^i(A^\bullet)[2^{-1}]^{\sigma+1}\right|\right) \qquad (18)$$

$$- \left(\sum {}^* \log\left|H^i(A^{+,\bullet})[2^\infty]\right| - \log\left|H^i(A^{-,\bullet})[2^\infty]\right|\right) \qquad (19)$$

$$- \left(\sum {}^* \log\left|H^i(A'^\bullet)\right|\right), \qquad (20)$$

where $A'^\bullet := A^\bullet/(A^\bullet[\sigma-1] \oplus A^\bullet[\sigma+1])$.

Lemma 3.2.

$$|(19)| + |(20)| \leqslant \log\left|H^*(A^\bullet)[2^\infty]\right| + 2\log\left|H^*(A'^\bullet)\right|. \qquad (21)$$

Proof. We first bound (19):

$$|(19)| \leqslant \sum \log \left| H^i(A^{+,\bullet})[2^\infty] \right| + \log \left| H^i(A^{-,\bullet})[2^\infty] \right|$$
$$= \log \left| H^*(A^\bullet[\sigma-1] \oplus A^\bullet[\sigma+1])[2^\infty] \right|.$$

By using the long exact sequence associated to $0 \to A^\bullet[\sigma-1] \oplus A^\bullet[\sigma+1] \to A^\bullet \to A'^\bullet \to 0$, we obtain the further bound

$$|(19)| \leqslant \log \left| H^*(A^\bullet)[2^\infty] \right| + \log \left| H^*(A'^\bullet) \right|.$$

The lemma follows. $\square$

3.2. Further estimates when $A^\bullet$ arises geometrically

A key observation is that, when $A^\bullet$ arises as the group of cochains of a triangulation or the Morse–Smale complex for a gradient vector field, then $H^i(A'^\bullet)$ (the $'$ notation is defined in §0.6) has a nice homological interpretation. In this section, we give an interpretation to the second summand $H^i(A'^\bullet)$ occurring on the right-hand side of (21) from Lemma 2.5 in the following geometric situation: $A^\bullet$ is the complex of $\mathcal{L}$-valued cochains on $\mathcal{M}$ corresponding to some equivariant triangulation of $\mathcal{M}$ and some equivariant, metrized local system $\mathcal{L} \to \mathcal{M}$ of free abelian groups.

Suppose that the compact manifold $\mathcal{M}$ is acted on isometrically by $\mathbb{Z}/2\mathbb{Z} = \langle \sigma \rangle$. Let $\mathcal{L} \to \mathcal{M}$ be an equivariant, metrized local system. Assume that $\mathcal{L}$ is $\mathbb{Q}$-acyclic. Let $A^\bullet$ be the group of cochains arising from an equivariant triangulation of $\mathcal{M}$, one which extends a triangulation on the fixed point set $\mathcal{M}_\sigma = M$.

Proposition 3.3. *We can identify*

$$H^*(A'^\bullet) = H_c^*((\mathcal{M}-M)/\langle\sigma\rangle, \mathcal{L}_{\mathbb{F}_2}),$$

where, abusing notation, $\mathcal{L}$ denotes the unique descent of the local system $\mathcal{L}|_{\mathcal{M}-M}$ to the quotient space $(\mathcal{M}-M)/\langle\sigma\rangle$.

Proof. Let $D^\bullet = A^\bullet[\sigma-1] \oplus A^\bullet[\sigma+1]$ so that $A'^\bullet = A^\bullet/D^\bullet$. Let $\overline{D}^\bullet = \mathrm{im}(D^\bullet \to A_{\mathbb{F}_2}^\bullet)$. We can identify $\overline{D}^\bullet = A_{\mathbb{F}_2}^\bullet[\sigma-1] + \mathrm{im}(\sigma-1)$. Therefore,

$$A'^\bullet = \sigma\text{-coinvariants of } A_{\mathbb{F}_2}^\bullet/A_{\mathbb{F}_2}^\bullet[\sigma-1]. \tag{22}$$

There is an exact sequence of complexes

$$0 \to A_{\mathbb{F}_2}^\bullet[\sigma-1] \to A_{\mathbb{F}_2}^\bullet \to A_{\mathbb{F}_2}^\bullet/A_{\mathbb{F}_2}^\bullet[\sigma-1] \to 0.$$

Because σ acts freely on the rightmost term, taking coinvariants is exact. Therefore, we can identify coinvariants of the rightmost term with the quotient of coinvariant groups

$$A'^\bullet \cong A_{\mathbb{F}_2,\langle\sigma\rangle}^\bullet/(A_{\mathbb{F}_2}^\bullet[\sigma-1])_{\langle\sigma\rangle}. \tag{23}$$

Since taking coinvariants has the effect of identifying σ-equivalent cells, (23) implies that

$$H^*(A'^\bullet) \cong H^*(\mathcal{M}/\langle\sigma\rangle, M, \mathcal{L}_{\mathbb{F}_2}), \tag{24}$$

where $\mathcal{M}/\langle\sigma\rangle$ denotes the point set topological quotient. Since the fixed point set $M \subset \mathcal{M}/\langle\sigma\rangle$ is a Euclidean neighborhood retract, the right-hand side of (24) can be identified with

$$H_c^*(\mathcal{M}/\langle\sigma\rangle - M, \mathcal{L}_{\mathbb{F}_2}) = H_c^*((\mathcal{M} - M)/\langle\sigma\rangle, \mathcal{L}_{\mathbb{F}_2}), \tag{25}$$

as desired. □

Next, we carry through a spectral sequence argument to estimate the cohomology $H_c^*((\mathcal{M} - M)/\langle\sigma\rangle, \mathcal{L}_{\mathbb{F}_2})$ in terms of $H^*(M, \mathcal{L}_{\mathbb{F}_2})$ and $H^*(\mathcal{M}, \mathcal{L}_{\mathbb{F}_2})$.

Proposition 3.4.

$$\log NRT_\sigma(\mathcal{L}) = -(\log |H^*(\mathcal{M}, \mathcal{L})[2^{-1}]^{\sigma-1}| - \log |H^*(\mathcal{M}, \mathcal{L})[2^{-1}]^{\sigma+1}|)$$
$$+ O\left(\log |H^*(\mathcal{M}, \mathcal{L})[2^\infty]| + \log |H^*(\mathcal{M}, \mathcal{L}_{\mathbb{F}_2})| + \log |H^*(M, \mathcal{L}_{\mathbb{F}_2})|\right). \tag{26}$$

Proof. By the long exact sequence for cohomology, relative to the pair $(\mathcal{M}, M)$, we obtain that

$$\log |H_c^*(\mathcal{M} - M, \mathcal{L}_{\mathbb{F}_2})| \leqslant \log |H^*(\mathcal{M}, \mathcal{L}_{\mathbb{F}_2})| + \log |H^*(M, \mathcal{L}_{\mathbb{F}_2})|. \tag{27}$$

We can relate the left-hand side of (27) to the cohomology of the quotient using a spectral sequence argument. Consider the fibration

$$\mathcal{M} - M \longrightarrow (\mathcal{M} - M) \times_\sigma E\langle\sigma\rangle$$
$$\pi \downarrow$$
$$B\langle\sigma\rangle$$

together with the sheaf $\mathcal{L}_{\mathbb{F}_2} \to (\mathcal{M} - M)/\langle\sigma\rangle$. The corresponding Serre spectral sequence has E_2 page

$$E_2^{p,q} = H^p(\langle\sigma\rangle, H^q(\mathcal{M} - M, \mathcal{L}_{\mathbb{F}_2})) \implies H^{p+q}((\mathcal{M} - M)/\langle\sigma\rangle, \mathcal{L}_{\mathbb{F}_2}).$$

The entries on the E_2 page are all finite abelian groups. As we turn the page, the orders of the groups appearing can only decrease. Thus,

$$\log |H^*((\mathcal{M} - M)/\langle\sigma\rangle, \mathcal{L}_{\mathbb{F}_2})| \leqslant \sum_{p,q \leqslant \dim \mathcal{M}} \log |E_2^{p,q}|.$$

But fortunately, the cohomology of cyclic groups is well understood. In particular,

$$E_2^{p,q} = \begin{cases} \text{quotient of } H^q(\mathcal{M} - M, \mathcal{L}_{\mathbb{F}_2})^\sigma & \text{if } p \text{ is even} \\ H^1(\langle\sigma\rangle, H^q(\mathcal{M} - M, \mathcal{L}_{\mathbb{F}_2})) & \text{if } p \text{ is odd.} \end{cases}$$

Accordingly, there is a crude upper bound

$$\log |H^*(A'^\bullet)| = \log |H_c^i((\mathcal{M} - M)/\langle\sigma\rangle, \mathcal{L}_{\mathbb{F}_2})|$$
$$\leqslant \sum_{p,q, \leqslant \dim \mathcal{M}} \log |E_2^{p,q}|$$
$$\leqslant \sum_{p,q \leqslant \dim \mathcal{M}} \log |H^q(\mathcal{M} - M, \mathcal{L}_{\mathbb{F}_2})|$$
$$= \dim \mathcal{M} \cdot \log |H^*(\mathcal{M} - M, \mathcal{L}_{\mathbb{F}_2})|. \tag{28}$$

Combining (27) and (28) with the preliminary calculations of the previous section, we obtain (26). □

Remark 3.5. The restriction to involutions σ was made for *notational convenience only*. A completely analogous argument can be carried out if $\sigma^p = 1$, and the exact same estimate

$$
\log NRT_\sigma(\mathcal{L}) = -\left(\log \left| H^*(\mathcal{M}, \mathcal{L})[p^{-1}]^{\sigma - 1} \right| - \frac{1}{p-1} \log \left| H^*(\mathcal{M}, \mathcal{L})[p^{-1}]^{P(\sigma)} \right| \right)
$$
$$
+ O\left(\log |H^*(\mathcal{M}, \mathcal{L})[p^\infty]| + \log |H^*(\mathcal{M}, \mathcal{L}_{\mathbb{F}_p})| + \log |H^*(M, \mathcal{L}_{\mathbb{F}_p})| \right) \quad (26)_p
$$

is obtained.

3.3. Comparing NRT_σ with RT_σ on locally symmetric spaces

Let $\mathcal{L} \to \mathcal{M}$ be a unimodular metrized local system of free abelian groups *over a locally symmetric space* $\mathcal{M}$ which is acted on equivariantly by an isometry σ with $\sigma^p = 1$.

Definition 3.6. If B is a finite-free abelian group of rank n together with a Hermitian metric h on $B_\mathbb{C}$, we let the volume of B denote the norm $||e_1 \wedge \cdots \wedge e_n||_h$ for any basis $e_1, \ldots, e_n$ of M. The vector $e_1 \wedge \cdots \wedge e_n$ in $\wedge^n B_\mathbb{C}$ is independent of basis, up to sign, and so its norm is well defined.

Proposition 3.7. *Suppose that $A^\bullet$ is one of the following complexes.*

- $A^\bullet = C^\bullet(\mathcal{L}, \mathcal{M}; K)$ *is the group of $\mathcal{L}$-valued cochains – with respect to a fixed triangulation K of $\mathcal{M}$ extending a triangulation on $\mathcal{M}_\sigma$ – of a rationally acyclic, metrized, unimodular local system $\mathcal{L} \to \mathcal{M}$ of free abelian groups acted on isometrically by $\langle \sigma \rangle \cong \mathbb{Z}/p\mathbb{Z}$.*

- $A^\bullet = \mathrm{MS}(X, \mathcal{L})$, *the Morse–Smale complex for $\mathcal{L}$ as above and some gradient vector field X on $\mathcal{M}$.*

Let $A'^\bullet$ be as defined in § 0.6. Then

$$
\log RT_\sigma(A^\bullet, h) = \log NRT_\sigma(A^\bullet) - \sum{}^* \log |H^i(A'^\bullet)| + e \cdot \chi(\mathcal{M}_\sigma) \quad (29)
$$

for some constant e depending only on $\mathcal{L}$.

Proof. We prove the proposition for $A^\bullet = C^\bullet(\mathcal{L}, \mathcal{M}; K)$, the proof for $A^\bullet = \mathrm{MS}(X, \mathcal{L})$ being identical. We would like to evaluate

$$
\log RT_\sigma(A^\bullet, h) = \frac{1}{2} \sum_{\pi \in \widehat{\mathbb{Z}/p\mathbb{Z}}} \log RT(A^\bullet[\pi], h|_{A^\bullet[\pi]}) \cdot \mathrm{tr}\, \pi(\sigma).
$$

We have omitted the "g" factor from the definition of $RT_\sigma(A^\bullet, h, g)$ because our complex is assumed rationally acyclic. Because $A_\mathbb{C}^\bullet[P(\sigma)] = \oplus_{\chi \neq \mathbf{1}} A_\mathbb{C}^\bullet[\chi]$ (orthogonal direct sum), for every non-trivial character χ of $\mathbb{Z}/p\mathbb{Z}$,

$$
\log RT(A^\bullet[\chi], h|_{A^\bullet[\chi]}) = \frac{1}{p-1} \log RT(A^\bullet[P(\sigma)], h|_{A^\bullet[P(\sigma)]}).
$$

Thus,

$$
\begin{aligned}
\log RT_\sigma(A^\bullet, h) &= \sum_{\pi \in \widehat{\mathbb{Z}/p\mathbb{Z}}} \log RT(A^\bullet[\pi], h|_{A^\bullet[\pi]}) \cdot \operatorname{tr} \pi(\sigma) \\
&= \log RT(A^\bullet[\sigma-1], h|_{A^\bullet[\sigma-1]}) + \frac{\zeta + \cdots + \zeta^{p-1}}{(p-1)} \\
&\quad \times \log RT(A^\bullet[P(\sigma)], h|_{A^\bullet[P(\sigma)]}) \\
&= \log RT(A^\bullet[\sigma-1], h|_{A^\bullet[\sigma-1]}) - \frac{1}{(p-1)} \log RT(A^\bullet[P(\sigma)], h|_{A^\bullet[P(\sigma)]}).
\end{aligned}
$$

We try to understand $A^\bullet[\sigma-1]$, $A^\bullet[P(\sigma)]$ by decomposing the action of σ on $A^\bullet$ into orbits. Let $\mathcal{C}$ be a set of representatives for the orbits of the i-cells under the action of σ. For each $C \in \mathcal{C}$, we let $e_1^C, \ldots, e_r^C$ be a basis for the global sections of $\mathcal{L}$ restricted to C (in particular, $r = \operatorname{rank}(\mathcal{L})$ is fixed).

- Suppose first that C is not fixed by the action of σ. The h-volume of $\mathcal{O}_C[\sigma-1]$, where $\mathcal{O}_C$ is the metrized abelian group $\mathcal{O}_C = \bigoplus \sigma^i \cdot C \otimes L(\sigma^i \cdot C)$, equals $\sqrt{p} \times \|e_1^C \wedge \cdots \wedge e_r^C\|_h = \sqrt{p}$, because the local system is unimodular and compatible with the metric h, and σ acts by isometries. Similarly, it is readily seen that the h-volume of $\mathcal{O}_C[P(\sigma)]$ equals $\sqrt{p}^{p-1}$.

- On the other hand, suppose that C is fixed by the σ-action. Then $C \subset \mathcal{M}_\sigma$. Let $\mathcal{O}_C := C \otimes L(C)$. Furthermore, $c = \operatorname{vol}_h(C \otimes L(C)[\sigma-1])$, $d = \operatorname{vol}_h(C \otimes L(C)[P(\sigma)])$ are independent of C. Indeed, this follows because $L[\sigma-1]$ and $L[P(\sigma)]$ are unimodular flat bundles *over the fixed point set* $\mathcal{M}_\sigma$, and the volume form is induced by a particular invariant metric, which is a constant multiple of h.

Because the $\mathcal{O}_C$ are mutually orthogonal for distinct representatives $C \in \mathcal{C}$, the above facts imply that the ratios of the norms $\operatorname{vol}_h(A^i[\sigma-1])/\alpha_{\sigma-1,\mathbb{Z}}^i$ and $\{\operatorname{vol}_h(A^i[P(\sigma)])/\alpha_{P(\sigma),\mathbb{Z}}^i\}^{\frac{1}{p-1}}$ (see §1.2.2 for a discussion of $\alpha_\mathbb{Z}$) are almost exactly equal. Their ratio is $(c/d^{\frac{1}{p-1}})^{\#i\text{-cells of } \mathcal{M}_\sigma}$. Let $e = \frac{1}{p-1}\log d - \log c$. Taking the alternating product over all i, it follows that

$$
\begin{aligned}
&\log RT_\sigma(A^\bullet, h)(\sigma) \\
&= \log RT(A^\bullet[\sigma-1], h|_{A^\bullet[\sigma-1]}) - \frac{1}{(p-1)} \log RT(A^\bullet[P(\sigma)], h|_{A^\bullet[P(\sigma)]}) \\
&= \log RT(A^\bullet[\sigma-1], \alpha_{\sigma-1,\mathbb{Z}}) - \frac{1}{(p-1)} \log RT(A^\bullet[P(\sigma)], \alpha_{P(\sigma),\mathbb{Z}}) + e \cdot \chi(\mathcal{M}_\sigma) \\
&= \sum{}^* \left\{ \log |H^i(A^\bullet[\sigma-1])| - \frac{1}{p-1} \log |H^i(A^\bullet[P(\sigma)])| \right\} + e \cdot \chi(\mathcal{M}_\sigma) \\
&= \log NRT_\sigma(A^\bullet) - \sum{}^* \log |H^i(A'^\bullet)| + e \cdot \chi(\mathcal{M}_\sigma). \qquad \square
\end{aligned}
$$

Corollary 3.8. *Let $\mathcal{L} \to M$ be a rationally acyclic, metrized, unimodular local system $\mathcal{L} \to \mathcal{M}$ of free abelian groups acted on isometrically by $\langle \sigma \rangle \cong \mathbb{Z}/p\mathbb{Z}$. Suppose that the*

fixed point set $\mathcal{M}_\sigma$ *has Euler characteristic* 0. *Then*

$$\log RT_\sigma(\mathcal{M}, \mathcal{L}) = -\sum_i{}^* \left(\log \left| H^i(\mathcal{M}, \mathcal{L})[p^{-1}]^{\sigma-1} \right| - \frac{1}{p-1} \log \left| H^i(\mathcal{M}, \mathcal{L})[p^{-1}]^{P(\sigma)} \right| \right)$$
$$+ O\left(\log |H^*(\mathcal{M}, \mathcal{L})[p^\infty]| + \log |H^*(\mathcal{M}, \mathcal{L}_{\mathbb{F}_p})| + \log |H^*(M, \mathcal{L}_{\mathbb{F}_p})| \right).$$

Proof. This follows immediately by combining the analog for p of Proposition 3.4 (see $(26)_p$) with Proposition 3.7. □

4. Bismut–Zhang's equivariant Cheeger–Müller theorem

Bismut and Zhang in [3, Theorem 0.2] relate equivariant Reidemeister torsion to equivariant analytic torsion, even for non-unitarily flat local systems. The difference between equivariant analytic torsion and equivariant Reidemeister torsion localizes to the fixed point set of the group action in a very controlled way, and we are able to compute this difference for unimodular local systems over certain locally symmetric spaces.

A model case for understanding both equivariant analytic torsion and equivariant Reidemeister torsion is that of products. Let $L \to M$ be any metrized local system. Then $L^{\boxtimes p} \to M^p$ is a metrized local system, equivariant with respect to the cyclic shift. In §5, we compute both the equivariant analytic torsion and equivariant Reidemeister torsion of the product local system $L^{\boxtimes p} \to M^p$. This case is of considerable importance because, after going through some contortions, the Bismut–Zhang formula (stated in Theorem 4.1) will allow us to directly relate the difference between analytic and Reidemeister torsion for general local systems to the corresponding difference for product local systems.

Let $\mathcal{L} \to \mathcal{M}$ be an equivariant, metrized local system over Riemannian manifold $\mathcal{M}$, where $\langle\sigma\rangle$ of prime order p acts compatibly on $\mathcal{M}$ and $\mathcal{L}$ by isometries. Let

$$E(\mathcal{M}, \mathcal{L}) =: \log \tau_\sigma(\mathcal{L}) - \log RT_\sigma(\mathcal{L}). \tag{30}$$

In all situations that will concern us, the restriction of $\mathcal{L}$ to the fixed point set $\mathcal{M}_\sigma$ is isomorphic, as a metrized, equivariant, local system, to $L^{\otimes p}$ for an appropriate local system over the fixed point set $\mathcal{M}_\sigma$. Our general strategy for understanding $E(\mathcal{M}, \mathcal{L})$ is to prove that it equals $E(\mathcal{M}_\sigma^p, L^{\otimes p})$. That such a comparison might be possible is suggested by the explicit form of the error term in the Bismut–Zhang theorem, which is local to a germ of the fixed point set $\mathcal{M}_\sigma$.

- In §4.1, we set up notation and state the version of the equivariant Cheeger–Müller theorem proven by Bismut and Zhang.

- In §4.2, we relate the normal bundles of the inclusions $\mathcal{M}_\sigma \subset \mathcal{M}_\sigma^p$ and $\mathcal{M}_\sigma \subset \mathcal{M}$, so as to allow a comparison of Morse functions on their respective normal neighborhoods.

- In §4.3, we compare Morse functions, connections, and local systems on $\mathcal{L} \to \mathcal{M}$ and $L^{\boxtimes p} \to \mathcal{M}_\sigma^p$. This allows us to conclude that $E(\mathcal{M}, \mathcal{L})$ equals $E(\mathcal{M}_\sigma^p, L^{\otimes p})$.

4.1. Statement of the Bismut–Zhang formula

Notational setup.

- $\mathcal{L} \to \mathcal{M}$ denotes a metrized local system, with metric $h^{\mathcal{L}}$, with covariant derivative $\nabla^{\mathcal{L}}$ for the canonical flat structure on $\mathcal{L}$.

- Γ denotes a finite group acting on $\mathcal{L} \to \mathcal{M}$ equivariantly by isometries.
- f denotes a Γ-equivariant Morse function on $\mathcal{M}$. Let X be a Γ-invariant, weakly gradient-like vector field associated to f, i.e., X has non-degenerate critical points equal to the critical points of f and $X(f) > 0$ away from the critical points of f. Examples of these are provided by classical gradient vector fields $X = \text{grad}_{g_0}(f)$ for Γ-invariant metrics g_0.
- $\sigma \in \Gamma$ acts on the normal bundle of $\mathcal{M}_\sigma$ in $\mathcal{M}$ by isometries and so induces an eigenbundle decomposition $N = \oplus N(\beta_j)$, where the eigenvalues of σ acting on $N(\beta_j)$ are $e^{\pm i\beta_j}$, $\beta_j \in (0, \pi]$.

Bismut and Zhang prove the following.

Theorem 4.1 [3, Theorem 0.2]. *Let f be a Γ-equivariant Morse function on $\mathcal{M}$ with associated weakly gradient-like vector field X. Assume further that X satisfies Morse–Smale transversality. With notation as above,*

$$2[\log RT_\Gamma(\mathcal{L}, f) - \log \tau_\Gamma(\mathcal{L})](\sigma) \tag{31}$$
$$= -\int_{\mathcal{M}_\sigma} \theta_\sigma(\mathcal{L}, h^{\mathcal{L}}) \wedge X^* \psi(T\mathcal{M}_\sigma, \nabla^{T\mathcal{M}_\sigma})$$
$$- \frac{1}{4} \sum_{x \in \text{Crit}(f) \cap \mathcal{M}_\sigma} (-1)^{\text{ind}(f|_{\mathcal{M}_\sigma}, x)} \sum_j \text{ind}(f, N(\beta_j); x) \cdot C_j \cdot \text{tr}[\sigma|_{\mathcal{L}_x}].$$

In this formula, the following hold.

- $\text{ind}(f, N(\beta_j); x)$ *denotes the index of the Hessian of f acting on $N(\beta_j)$.*
- $C_j = \Gamma/\Gamma'(\beta_j/2\pi) + \Gamma/\Gamma'(1 - \beta_j/2\pi) - 2\Gamma'(1)$. *These numbers are related to the equivariant torsion of odd-dimensional spheres* [11, § 11].
- ψ *is a current on $T\mathcal{M}_\sigma$ whose restriction to $T\mathcal{M}_\sigma - 0$ transgresses the Euler class for $T\mathcal{M}_\sigma$. This is the very same current used by Bismut and Cheeger in* [2].
- $\theta_\sigma(\mathcal{L}, h^{\mathcal{L}}) = \text{tr}(\sigma \cdot \omega(\mathcal{L}, h^{\mathcal{L}}))$, *where $\omega(\mathcal{L}, h^{\mathcal{L}}) = (h_\mathcal{L})^{-1} \circ \nabla^{\text{Hom}(\mathcal{L}, \check{\mathcal{L}})}(h^L)$.*
  The connection $\nabla^{\mathcal{L}}$ induces a connection $\nabla^{\text{Hom}(\mathcal{L}, \check{\mathcal{L}})}$ on $\text{Hom}(\mathcal{L}, \check{\mathcal{L}})$, and we view the metric tensor $h^{\mathcal{L}}$ as a global section of $\text{Hom}(\mathcal{L}, \check{\mathcal{L}})$. Then $\nabla^{\mathcal{L}}(h^{\mathcal{L}})$ can be viewed as a $\text{Hom}(\mathcal{L}, \check{\mathcal{L}})$-valued 1-form, which after composing with $(h^{\mathcal{L}})^{-1}$ becomes a $\text{Hom}(\mathcal{L}, \mathcal{L})$-valued 1-form.

For our later purposes, it will be crucial for us to know that the right-hand side of (31) collapses significantly because the closed 1-form $\theta_\sigma(\mathcal{L}, h^{\mathcal{L}})$ appearing in the above integral over $\mathcal{M}_\sigma$ often vanishes.

Lemma 4.2. *Suppose that the local system $\det(\mathcal{L}) \to \mathcal{M}$ is unitarily flat. Then*

$$\theta_\sigma(\mathcal{L}, h^{\mathcal{L}}) = 0.$$

In particular, $\theta_\sigma(\mathcal{L}, h^{\mathcal{L}}) = 0$ whenever $\det(\mathcal{L})$ is the trivial local system.

Proof. Recall that

$$\theta_\sigma(\mathcal{L}, h^{\mathcal{L}}) = \text{tr}\{\sigma \cdot \omega(\mathcal{L}, h^{\mathcal{L}})\} \quad \text{where} \quad \omega(\mathcal{L}, h^{\mathcal{L}}) = (h^{\mathcal{L}})^{-1} \nabla^{\mathcal{L}}(h^{\mathcal{L}}).$$

Because trace is the infinitesimal determinant, we readily check that

$$\mathrm{tr}\{\sigma \cdot \omega(\mathcal{L}, h^{\mathcal{L}})\} = \sigma \cdot \omega(\det(\mathcal{L}), h^{\det(\mathcal{L})}), \quad \text{where } \omega(\det(\mathcal{L}), h^{\det(\mathcal{L})})$$
$$:= (h^{\det(\mathcal{L})})^{-1} \nabla^{\det(\mathcal{L})}(h^{\det(\mathcal{L})}).$$

But the form $\omega(\det(\mathcal{L}), h^{\det(\mathcal{L})})$ measures the obstruction to the bundle $\det(\mathcal{L}) \to \mathcal{M}$ being unitarily flat relative to the pair $\nabla^{\det(\mathcal{L})}, h^{\det(\mathcal{L})}$. Therefore, if $\det(\mathcal{L}) \to M$ is unitarily flat, $\omega(\det(\mathcal{L}), h^{\det(\mathcal{L})}) = 0$, and the result follows. $\qquad\square$

4.2. Intrinsic identification of the normal bundle

We aim to compare the normal bundles of the inclusions $\mathcal{M}_\sigma \xrightarrow{i_1} \mathcal{M}$ and $\mathcal{M}_\sigma \xrightarrow{i_2} \mathcal{M}_\sigma^p$, with a view to comparing the error terms of two different applications of the Bismut–Zhang theorem 4.1, one on a local system over $\mathcal{M}$ and the other on a local system over $\mathcal{M}_\sigma^p$.

Let $\mathcal{M}$ denote a Galois-stable locally symmetric space associated to the group $R_{E/F}\mathbf{G}$ for a cyclic degree p Galois extension E/F.

Proposition 4.3. *The normal bundles of the inclusions*

$$i_1^* N(\mathcal{M}_\sigma \subset \mathcal{M}_\sigma^p) \to \mathcal{M}_\sigma \quad \text{and} \quad i_2^* N(\mathcal{M}_\sigma \subset \mathcal{M}) \to \mathcal{M}_\sigma$$

are isometric.

Proof. For ease of notation, we will assume that F is imaginary quadratic (see Remark 4.4).

Fix a complex embedding ι of F. For the cyclic degree p-extension E/F, we have the symmetric space

$$S = \prod_{v|\iota} \mathbb{H}_v^3,$$

which is the universal cover of $\mathcal{M}$. Correspondingly, there is a decomposition of the tangent bundle

$$TS = \bigoplus_{v|\iota} T\mathbb{H}_v^3.$$

This decomposition is Galois invariant, and the subbundles $T\mathbb{H}_v$ are individually invariant under $\mathbf{G}(E_{\mathbb{R}})$. Fix a Galois-stable path component M^0 of either $M_\sigma \times \cdots \times M_\sigma$ or $\mathcal{M}$. Let M_σ^0 denote the union of those path components of M_σ contained in M^0.

There is a Galois-equivariant covering $S \to M^0$. The above decomposition of TS descends to a Galois-equivariant decomposition of the tangent bundle of M^0:

$$TM^0 = V_1 \oplus \cdots \oplus V_p.$$

The restriction V of the tangent bundle TM^0 to M_σ^0 carries an action of $\Gamma_{E/F} = \langle \sigma \rangle$. The compositions

$$\phi_i := V^\sigma \hookrightarrow V \xrightarrow{\pi_i} V_i$$

are all $p^{-1/2}$ times an isometry. Indeed, any invariant vector must have $\sigma^i \cdot 1$-component $\sigma^i \cdot v$ for some vector v. This vector has norm $p^{1/2}||v||$, and its projection has norm $||v||$. Thus,

$$\phi = \oplus(p^{1/2}\phi_i)^{-1} : V = \oplus V_i|_{M_\sigma^0} \to (V^\sigma)^{\oplus p}$$

is an isometry which identifies the normal bundle of $M_\sigma^0 \subset M^0$ with the orthogonal complement $\Delta(V^\sigma)^\perp$ of the diagonal in $(V^\sigma)^{\oplus p}$. This intrinsic description of the normal bundle shows that the normal bundle of the inclusions $i_1 : \mathcal{M}_\sigma \subset \mathcal{M}_\sigma^p$ and $i_2 : \mathcal{M}_\sigma \subset \mathcal{M}$ are Γ-equivariantly isometrically isomorphic. $\qquad\square$

Remark 4.4. Every symmetric space S admits a decomposition as a product $G_1/K_1 \times \cdots \times G_n/K_n$. The $K_1 \times \cdots \times K_n$ representation $\mathfrak{g}_1/\mathfrak{k}_1 \times \cdots \times \mathfrak{g}_n/\mathfrak{k}_n$, isomorphic to the tangent space at $(eK_1, \ldots, eK_n)$, as a $K_1 \times \cdots \times K_n$ representation, is multiplicity free. Thus, it uniquely decomposes into a direct sum of irreducible subrepresentations and so induces a corresponding canonical decomposition of the tangent bundle TS. If a finite order automorphism σ of the group $G_1 \times \cdots \times G_n$ preserves $K_1 \times \cdots \times K_n$ and normalizes a discrete group Γ of isometries of S, then this decomposition descends to give a σ-stable decomposition of $T(\Gamma \backslash S)$. The above identification of normal bundles carries through in this generality.

4.3. Le mariage de la carpe et du lapin

The error term in the Bismut–Zhang theorem sees very little of the space $\mathcal{M}$, only a Γ-invariant Morse function f on an arbitrarily small neighborhood of the fixed point set $\mathcal{M}_\sigma$. The previous section allows us to relate the Morse function f in a neighborhood of $\mathcal{M}_\sigma \subset \mathcal{M}$ to a different Γ-invariant Morse function f' on $\mathcal{M}_\sigma \subset (\mathcal{M}_\sigma)^p$. The goal of this section will be to relate all parts of the error terms in two different applications of the Bismut–Zhang formula, the first on $\mathcal{M}$ and the second on $\mathcal{M}_\sigma^p$. In the notation of (30) from the introduction to §4, we will prove that

$$E(\mathcal{M}, \mathcal{L}) = E(\mathcal{M}_\sigma^p, L^{\boxtimes p}) \tag{32}$$

for an appropriate local system $L \to \mathcal{M}_\sigma$ (to be stated more precisely in Proposition 4.5). This represents progress because, as we will see in §5, both the twisted analytic torsion and the twisted Reidemeister torsion in the case of the product $(M_\sigma)^p$ can be explicitly computed, thus enabling us to understand the left-hand side of (32).

4.3.1. Transport of structure. In this section, we will provide all of the necessary ingredients for comparing the error terms in two applications of the Bismut–Zhang formula; in §4.5, this will enable us to prove that

$$\log RT_\Gamma(\mathcal{M}, \mathcal{L}, X_1) - \log \tau_\Gamma(\mathcal{M}, \mathcal{L}) = \log RT_\Gamma(\mathcal{M}_\sigma^p, L^{\boxtimes p}, X_2) - \log \tau_\Gamma(\mathcal{M}_\sigma^p, L^{\boxtimes p})$$

for some special choices of gradient vector fields X_1, X_2.

Let $N_1 \to M_\sigma \leftarrow N_2$ denote the normal bundles of $i_1 : M_\sigma \subset M, i_2 : M_\sigma \subset (M_\sigma)^p$ respectively. The previous section produces an explicit isometric isomorphism $N_1 \xrightarrow{\Phi} N_2$.

Comparison of Morse functions. Let f_1 be a Γ-invariant Morse function on a small exponential neighborhood of $M_\sigma \subset M$ of radius r. Because σ acts by isometries on M, this neighborhood is Γ-invariant. We construct a matching function f_2 on the exponential neighborhood of $M_\sigma \subset (M_\sigma)^p$ of radius r by

$$f_2(\exp_p(\Phi(Y))) := f_1(\exp_p(Y))$$

for any $Y \in B_p(r) \subset (N_1)_p$. f_2 is a Γ-invariant Morse function on this exponential neighborhood. Furthermore, because

$$\exp_p(Y) \xrightarrow{\Phi} \exp_p(\phi(Y))$$

is a Γ-equivariant diffeomorphism, all of the critical points and indices of the two Morse functions are equal. For example,

$$\mathrm{ind}(f_1, N_1(\beta_j); x) = \mathrm{ind}(f_2, N_2(\beta_j); x)$$

because both can be computed by exponentiating $N(\beta_j)(x)$ to form a submanifold Π a neighborhood of x and computing the Morse index of $f|_{\Pi}$. But, clearly, the indices of $f_1|_{\Pi}$ and $f_2|_{\Phi(\Pi)}$ are equal, as they are related by a diffeomorphism.

Remark. It is easily seen that f_2 can be extended to a Γ-invariant Morse function on $M_\sigma \times \cdots \times M_\sigma$. Indeed, a generic function on the quotient $(M_\sigma^p - \Delta(M_\sigma))/\langle \sigma \rangle$ is a Morse function. Extend f_2, which descends to a Morse function on this quotient, by a generic bump function.

Comparison of local systems. All metrized local systems on $\mathcal{L} \to \mathcal{M}$ that we will encounter in our applications will have the property that $\mathcal{L}|_{\mathcal{M}_\sigma} = L^{\otimes p}$ for a local system $L \to \mathcal{M}_\sigma$, endowed with the obvious tensor product metric. Thus, for each $x \in \mathcal{M}_\sigma$,

$$\mathrm{tr}(\sigma|\mathcal{L}_x) = \mathrm{tr}(\sigma|L_x^{\boxtimes p}).$$

4.3.2. Concluding the comparison. Combining all of the comparisons of §4.3.1, we can prove that the error term in the Bismut–Zhang formula arising from the equivariant unimodular metrized local system $\mathcal{L} \to \mathcal{M}$ exactly equals the error term arising from the local system $L^{\boxtimes p} \to \mathcal{M}_\sigma^p$.

Proposition 4.5. *Let $\mathcal{L} \to \mathcal{M}$ and $L \to \mathcal{M}_\sigma$ be matching unimodular local systems, and let f_1 and f_2 be invariant Morse functions which match in the sense described above. Then there are invariant gradient vector fields X_1 associated to f_1 and X_2 associated to f_2 satisfying Morse–Smale transversality for which*

$$\begin{aligned}
&[\log RT_\Gamma(\mathcal{L} \to \mathcal{M}, X_1) - \log \tau_\Gamma(\mathcal{L} \to \mathcal{M})](\sigma) \\
&= [\log RT_\Gamma(L^{\boxtimes p} \to \mathcal{M}_\sigma^p, f_2) - \log \tau_\Gamma(L^{\boxtimes p} \to \mathcal{M}_\sigma^p)](\sigma).
\end{aligned} \tag{33}$$

Proof. For any invariant Morse function f_1 defined on an exponential neighborhood N_1 of $\mathcal{M}_\sigma \subset \mathcal{M}$, we have constructed an invariant Morse function f_2 which matches it on a Γ-equivariantly diffeomorphic tubular neighborhood N_2 of $\mathcal{M}_\sigma \subset \mathcal{M}_\sigma^p$. For these matching Morse functions, the discrete parts of the error term of the Bismut–Zhang formula (31) match:

$$\begin{aligned}
&\sum_{x \in \mathrm{Crit}(f_1) \cap \mathcal{M}_\sigma} (-1)^{\mathrm{ind}(f_1|_{\mathcal{M}_\sigma}, x)} \sum_j \mathrm{ind}(f, N(\beta_j); x) \cdot C_j \cdot \mathrm{tr}[\sigma|\mathcal{L}_x] \\
&= \sum_{x \in \mathrm{Crit}(f_2) \cap \mathcal{M}_\sigma} (-1)^{\mathrm{ind}(f_2|_{\mathcal{M}_\sigma}, x)} \sum_j \mathrm{ind}(f, N(\beta_j); x) \cdot C_j \cdot \mathrm{tr}[\sigma|L_x^{\boxtimes p}].
\end{aligned} \tag{34}$$

Furthermore, because $\mathcal{L}$ is unimodular by assumption, Lemma 4.2 implies that the continuous part of the error term of the Bismut–Zhang formula is zero, no matter which choices of gradient vector fields X_1, X_2 we ultimately make. However, in order for this error term to compute the correct quantity, we need to do the following:

(a) extend f_1 from N_1 to an invariant Morse function on $\mathcal{M}$ and f_2 from N_2 to an invariant Morse function on $\mathcal{M}_\sigma^p$;

(b) find Γ-invariant, weakly gradient-like vector fields X_1, X_2 associated to f_1, f_2 which satisfy Morse–Smale transversality.

The first item (a) can be readily accomplished. Indeed, $\langle \sigma \rangle$ acts freely on both $\mathcal{M} - \mathcal{M}_\sigma$ and $\mathcal{M}_\sigma^p - \mathcal{M}_\sigma$, and so $(\mathcal{M} - \mathcal{M}_\sigma)/\langle \sigma \rangle$ and $(\mathcal{M}_\sigma^p - \mathcal{M}_\sigma)/\langle \sigma \rangle$ are manifolds. The Morse functions f_1 on $(N_1 - \mathcal{M}_\sigma)/\langle \sigma \rangle$ and f_2 on $(N_2 - \mathcal{M}_\sigma)/\langle \sigma \rangle$ can be extended randomly to Morse functions on $(\mathcal{M} - \mathcal{M}_\sigma)/\langle \sigma \rangle$ and $(\mathcal{M}_\sigma^p - \mathcal{M}_\sigma)/\langle \sigma \rangle$ by genericity of Morse functions. Their pullbacks can be glued with the original f_1 and f_2 to give invariant Morse functions, which we continue to call f_1, f_2.

The second item (b) is more delicate. Fix a metric g_1 on $\mathcal{M}$, such as the group invariant metric. We make a somewhat special choice of f_1; we require that the Hessian $d^2 f_1(x)$ is negative definite on N_x, where N denotes the normal bundle to $\mathcal{M}_\sigma \subset \mathcal{M}$ at all critical points on $\mathcal{M}_\sigma$. This can be accomplished by letting f_1, in exponential coordinates, equal $f_1(x, v) = f(x) + g_0(v)$ for any Morse function f on $\mathcal{M}_\sigma$ and any metric g_0 on the normal bundle.

By [3, Theorem 1.8], for any such choice of f_1, there is a metric $\tilde{g}_1$ equal to g_1 on a neighborhood of all critical points of f_1 on $\mathcal{M}_\sigma$ for which

$$X_1 := \operatorname{grad}_{\tilde{g}_1}(f_1) \text{ satisfies Morse–Smale transversality.}$$

Because $\tilde{g}_1$ equals g_1 in a neighborhood of the critical points in $\mathcal{M}_\sigma$, the equality of the error terms from (34) is preserved.

Let f_2 be the Morse function matching f_1. We can play the same game by modifying a fixed metric g_2 on $\mathcal{M}_\sigma^p$, such as the group invariant metric, to a new metric $\tilde{g}_2$ for which

$$X_2 := \operatorname{grad}_{\tilde{g}_2}(f_2) \text{ satisfies Morse–Smale transversality.}$$

Finally, the assumptions of the Bismut–Zhang theorem are satisfied, and we can conclude that

$$\begin{aligned}
&[\log RT_\Gamma(\mathcal{L} \to \mathcal{M}, X_1) - \log \tau_\Gamma(\mathcal{L} \to \mathcal{M}, \tilde{g}_1)](\sigma) \\
&= [\log RT_\Gamma(L^{\boxtimes p} \to \mathcal{M}_\sigma^p, f_2) - \log \tau_\Gamma(L^{\boxtimes p} \to \mathcal{M}_\sigma^p, \tilde{g}_2)](\sigma).
\end{aligned} \tag{35}$$

Note that the analytic torsion terms $\log \tau_\sigma(\mathcal{L} \to \mathcal{M}, \tilde{g}_1)$ and $\log \tau_\sigma(L^{\boxtimes p} \to \mathcal{M}_\sigma^p, \tilde{g}_2)$ appear to depend on the metrics $\tilde{g}_1$ and $\tilde{g}_2$ over which we have no control. Such a dependence would be devastating because, ultimately, we can only calculate information concerning comparisons or growth of twisted analytic torsion relative to *group invariant metrics*. That being said, we are rescued by the anomaly formula of [3, Theorem 0.1], which states that

$$\log \tau_\sigma (\mathcal{L} \to \mathcal{M}, g_1) - \log \tau_\sigma (\mathcal{L} \to \mathcal{M}, \tilde{g}_1)$$
$$= \int_{\mathcal{M}_\sigma} A \wedge e(T\mathcal{M}_\sigma, \nabla^{T\mathcal{M}_\sigma}_{\tilde{g}_1}) - \int_{\mathcal{M}_\sigma} \theta_\sigma (\mathcal{L}, h^{\mathcal{L}}) \wedge B, \qquad (36)$$

where A, B are some differential forms that we will not specify, $e(T\mathcal{M}_\sigma, \nabla^{T\mathcal{M}_\sigma}_{\tilde{g}_1})$ is the Euler form of $T\mathcal{M}_\sigma$ relative to the Levi-Civita connection of $T\mathcal{M}_\sigma$ associated to the metric $\tilde{g}_1$, and $\theta_\sigma(\mathcal{L}, h^{\mathcal{L}})$ is the closed 1-form from Lemma 4.2. According to Lemma 4.2, the form $\theta_\sigma(\mathcal{L}, h^{\mathcal{L}})$ vanishes identically because $\mathcal{L} \to \mathcal{M}$ is unimodular. Also, because $\mathcal{M}_\sigma$ is an odd-dimensional manifold in our case, the Euler form is identically zero. It follows that

$$\log \tau_\sigma (\mathcal{L} \to \mathcal{M}, g_1) - \log \tau_\sigma (\mathcal{L} \to \mathcal{M}, \tilde{g}_1) = 0.$$

The proposition now follows. □

5. Calculations on a product

Let K be a triangulation of a Riemannian manifold M, and let $L \to M$ be a metrized local system of free abelian groups. Let K' be an equivariant refinement of K^p which extends a triangulation of the diagonal M inside the product M^p. In this section, we aim to prove that the difference

$$[\log RT_\Gamma (L^{\boxtimes p} \to M^p, K') - \log \tau_\Gamma (L^{\boxtimes p} \to M^p)](\sigma)$$

is small, where the group $\Gamma = \langle \sigma \rangle$ acts by cyclic permutation. We will separately relate the twisted analytic and twisted Reidemeister torsions appearing in the above equation to their untwisted counterparts $\log RT(L \to M)$ and $\log \tau(L \to M)$ respectively.

- In § 5.1, we prove that

$$\log \tau_\sigma (M^p, L^{\boxtimes p}) = p \log \tau(M, L),$$

 where σ denotes the cyclic shift.

- In § 5.2, we prove that

$$\log NRT_\sigma (M^p, L^{\boxtimes p}) = p \log RT(M, L).$$

 We prove this using the close relationship between NRT_σ and analytic torsion (see (16) and Definition 3.1).

- In § 5.3, we prove that the difference

$$\log RT_\sigma (M^p, L^{\boxtimes p}) - \log \tau_\sigma (M^p, L^{\boxtimes p})$$

 is often zero. This is a direct consequence of Lück's variant of the equivariant Cheeger–Müller theorem when $L \to M$ is *unitarily flat*, but we prove that this difference is zero for arbitrary local systems L over odd-dimensional locally symmetric spaces M for which $L_{\mathbb{F}_p}$ is self-dual.

5.1. Twisted analytic torsion of a product

Let M be a compact Riemannian manifold together with a metrized local system $L \to M$. We next compute the equivariant torsion of $L^{\boxtimes n} \to M^n$ with respect to a cyclic shift σ.

Proposition 5.1. *If $L^{\boxtimes n} \to M^n$ is equipped with its product metric, then*

$$\log \tau_\sigma(L^{\boxtimes n}) = n \left[\log \tau(L) - \log(n) Z_L(0) \right]. \tag{37}$$

Proof. Let $\pi_i : M^n \to M$ denote the ith coordinate projection. Let Δ_j be the j-form Laplacian acting on $\Omega^j(M^n, L^{\boxtimes n})$. We let $E_{\lambda,j}$ denote the λ-eigenspace of the j-form Laplacian of M. Note that the λ-eigenspace of Δ_{j,M^n} is spanned by the image of

$$\bigoplus E_{\lambda_1, i_1} \otimes \cdots \otimes E_{\lambda_n, i_n} \to \Omega_j(M^n, L^{\boxtimes n})$$
$$\omega_1 \otimes \cdots \otimes \omega_n \mapsto \pi_1^* \omega_1 \wedge \cdots \wedge \pi_n^* \omega_n,$$

where the sum ranges over all $\lambda_1 + \cdots + \lambda_n = \lambda$, and $i_1 + \cdots + i_n = j$.

The isometry σ acts as a signed permutation on the vectors $\pi_1^* \omega_1 \wedge \cdots \wedge \pi_n^* \omega_n$, where ω_k runs over a basis of E_{λ_k, i_k}. Thus, such a basis vector contributes to the trace exactly when $\omega_1 = \cdots = \omega_n$. So, in particular, j must be divisible by n. Say $j = an$ and $\lambda = n\lambda'$. Then the trace of σ acting on the image of $E_{\lambda', a} \otimes \cdots \otimes E_{\lambda', a}$ is readily seen to be

$$(-1)^{a^2(n-1)} \dim E_{\lambda', a}.$$

It follows that

$$\sum_j (-1)^j j \zeta_{j,\sigma}(s) = \sum_a (-1)^{na} na (-1)^{a^2(n-1)} \zeta_a(s) n^{-s}$$
$$= n \sum_a (-1)^{n(a^2-a)-a^2} a \zeta_a(s) n^{-s}$$
$$= n \sum_a (-1)^a a \zeta_a(s) n^{-s}.$$

Differentiating at $s = 0$ gives the result. $\qquad \square$

Remark 5.2. The $\log(n) Z_L(0)$ summand in (37) arises because the diagonal $\Delta(M^n) \subset M^n$ has two possible metrics: the induced metric from the product metric on M^n and the given metric on M. The above calculations were made with respect to the given metric on M. If we had instead chosen the metric induced on the diagonal, the second summand in (37) would not appear.

5.2. Twisted Reidemeister torsion on a product

Let C be a finite, $\mathbb{Q}$-acyclic complex acted on by $\mathbb{Z}/p\mathbb{Z} = \langle \sigma \rangle$. For convenience, we recall the definition of naive Reidemeister torsion from Definition 3.1.

$$\log NRT_\sigma(C^\bullet) := \left\{ \sum{}^* \log |H^i(C^{\sigma-1})| - \frac{1}{p-1} \log |H^i(C^{P(\sigma)})| \right\} + \left\{ \sum{}^* \log |H^i(C')| \right\}, \tag{38}$$

where $C' := C/(C[\sigma - 1] \oplus C[P(\sigma)])$.

Lemma 5.3. *Suppose that $C = A^{\otimes p}$, with σ acting by cyclic permutation. Then*

$$\log NRT_\sigma(A^{\otimes p}) = p \log RT(A).$$

Proof. By Lemma 2.5, we have that

$$\log NRT_\sigma(A^{\otimes p}) = \log \tau_\sigma(A^{\otimes p}).$$

By a computation identical to that of §5.1, we see that

$$\log \tau_\sigma(A^{\otimes p}) = p \log \tau(A) = p \log RT(A),$$

and we are done. $\square$

5.3. Proof that $\log RT_\sigma(L^{\boxtimes p} \to M^p, X) - \log \tau_\sigma(L^{\boxtimes p} \to M^p)$ is often 0

Theorem 5.4. *Suppose that M is an odd-dimensional, compact manifold equipped with a triangulation K_0. Let $L \to M$ be a metrized local system of free abelian groups with $L_{\mathbb{F}_p}$ self-dual. Let X be a vector field on M^p which is weakly gradient like for a Morse function f on M^p and which satisfies Morse–Smale transversality. Then*

$$\log RT_\sigma(L^{\boxtimes p} \to M^p, X) - \log \tau_\sigma(L^{\boxtimes p} \to M^p) = 0.$$

Proof. For shorthand, let $C = (A^\bullet)^{\otimes p}$, where $A = C^\bullet(M, L; K_0)$; then $C^\bullet(M^p, L^{\boxtimes p}; K_0^p) \cong A^{\otimes p}$. We also let $D = MS^\bullet(X, L^{\boxtimes p})$. By Proposition 3.7, there is an equality

$$\log RT_\sigma(L^{\boxtimes p} \to M^p, K) - \log \tau_\sigma(L^{\boxtimes p} \to M^p)$$
$$= \log NRT_\sigma(D) - \log \tau_\sigma(L^{\boxtimes p} \to M^p) - \sum{}^* \log |H^i(D')| + e \cdot \chi(M).$$

By Lemma A.1, NRT_σ is invariant under equivariant chain homotopy (see also [11, §5, Proposition 8]). However, by the main result of [7], the triangulation of M^p by unstable manifolds of X and the product triangulation K_0^p – both smooth triangulations equivariant for the cyclic shift on M^p – admit a common smooth equivariant refinement. Therefore, the complexes C and D are equivariantly chain homotopic, from which it follows that $NRT_\sigma(D) = NRT_\sigma(C)$. The latter expression thus equals

$$= \log NRT_\sigma(C) - \log \tau_\sigma(L^{\boxtimes p} \to M^p) - \sum{}^* \log |H^i(D')| + e \cdot \chi(M)$$
$$= p \log RT(A^\bullet) - p \log \tau(L \to M) - \sum{}^* \log |H^i(D')| + 0$$
$$= 0 - \sum{}^* \log |H^i(D')|$$
$$= - \sum{}^* \log |H^i(D')|.$$

The transition from the first line to the second line follows by Lemma 5.3 and because the odd-dimensional compact manifold M has $\chi(M) = 0$; the transition from the second line to the third line follows by the untwisted Cheeger–Müller theorem. But, as observed in Proposition 3.3,

$$H^i(D') = H_c^i((M^p - M)/\langle \sigma \rangle, L_{\mathbb{F}_p}^{\boxtimes p}).$$

We thus continue:

$$
\begin{aligned}
&= -\sum\nolimits^* \log |H^i(D')| \\
&= -\log p \cdot \chi_c((M^p - M)/\langle\sigma\rangle, L^{\boxtimes p}_{\mathbb{F}_p}) \\
&= -\log p \cdot p \cdot \chi_c(M^p - M, L^{\boxtimes p}_{\mathbb{F}_p}) \\
&= -\log p \cdot p \cdot [\chi(M^p, L^{\boxtimes p}_{\mathbb{F}_p}) - \chi(M, L^{\boxtimes p}_{\mathbb{F}_p}|_M)] \\
&= -\log p \cdot p \cdot [\chi(M, L_{\mathbb{F}_p})^p - \chi(M, L^{\otimes p}_{\mathbb{F}_p})].
\end{aligned}
$$

Since M is an odd-dimensional manifold and $L_{\mathbb{F}_p}$ is a self-dual local system on M, Poincaré duality implies that both Euler characteristics appearing in the final line above vanish. We conclude that

$$
\log RT_\sigma(L^{\boxtimes p} \to M^p, X) - \log \tau_\sigma(L^{\boxtimes p} \to M^p) = 0
$$

exactly. □

Corollary 5.5. *Let $\mathcal{L} \to \mathcal{M}$ be an equivariant, metrized, rationally acyclic local system of free abelian groups over a locally symmetric space $\mathcal{M}$ acted on equivariantly and isometrically by $\langle\sigma\rangle \cong \mathbb{Z}/p\mathbb{Z}$. Suppose further that the restriction to the fixed point set $\mathcal{L}|_{\mathcal{M}_\sigma} = L^{\otimes p}$ for $L \to \mathcal{M}_\sigma$ self-dual and that $\mathcal{M}_\sigma$ is odd dimensional. It follows that*

$$
\log \tau_\sigma(\mathcal{M}, \mathcal{L}) = \log RT_\sigma(\mathcal{M}, \mathcal{L}).
$$

Proof. This follows immediately by combining Theorem 5.4 with Proposition 4.5. □

Example 5.6. Let $\mathbf{G}$ be the adjoint group of the units of a quaternion algebra over a number field F. Let $K \subset \mathbf{G}(F_\mathbb{R})$ be a maximal compact subgroup. Let $\rho : \mathbf{G} \to \mathbf{GL}(V)$ be an algebraic representation defined over F. Fix a lattice $\mathcal{O} \subset V$, and let $U_0 \subset \mathbf{G}(\mathbb{A}^{\mathrm{fin}}_F)$ be its stabilizer, a compact open subgroup. Let $U \subset \mathbb{A}^{\mathrm{fin}}_F$ be any compact open subgroup contained in U_0. To the representation ρ is associated a local system of O_F-modules $L_\rho \to M_U = \mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F)/KU$ (see [10, §2]).

In this case, every $L_{\rho,\mathbb{F}_p}$ is self-dual. Furthermore, M_U is odd dimensional precisely when the number of complex places of F is odd. In such cases, $L_\rho \to M_U$ satisfies the requirements of Theorem 5.4, from which it follows that

$$
\log RT_\sigma(L^{\boxtimes p} \to M^p, X) = \log \tau_\sigma(L^{\boxtimes p} \to M^p).
$$

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Appendix A. Naive Reidemeister torsion of a tensor product complex

A.1. Robustness properties of NRT_σ

The goal of this section is to show that NRT_σ, as defined in Definition 3.1, is very well behaved in two respects:

(a) if $C \to E$ is a chain homotopy of $\mathbb{Z}[\sigma]$-complexes of free $\mathbb{Z}$-modules, then $NRT_\sigma(C) = NRT_\sigma(E)$;

(b) the naive Reidemeister torsion is additive for tensor complexes, i.e., if $C_0 = A^{\otimes p}, C_1 = B^{\otimes p}$ for complexes A, B of finite-free $\mathbb{Z}$-modules with σ acting on C_0, C_1 by cyclic permutation, then

$$\log NRT_\sigma(C_0 \oplus C_1) = \log NRT_\sigma(C_0) + \log NRT_\sigma(C_1).$$

We now outline a proof of these two properties.

Lemma A.1 ((a) Chain homotopy invariance)**.** *NRT is invariant under $\mathbb{Z}[\sigma]$-equivariant chain homotopy.*

Proof. Suppose that $f : C \to E$ and $g : E \to C$ are inverse up to chain homotopy of $\mathbb{Z}[\sigma]$-modules. Then $f^{\sigma-1} : C^{\sigma-1} \to E^{\sigma-1}$ and $g^{\sigma-1} : E^{\sigma-1} \to C^{\sigma-1}$ are chain homotopy inverses. Likewise, $f^{P(\sigma)} : C^{P(\sigma)} \to E^{P(\sigma)}$ and $g^{P(\sigma)} : E^{P(\sigma)} \to C^{P(\sigma)}$ are chain homotopy inverses. Therefore,

$$f^{\sigma-1} : H^*(C^{\sigma-1}) \xrightarrow{\sim} H^*(E^{\sigma-1})$$
$$f^{P(\sigma)} : H^*(C^{P(\sigma)}) \xrightarrow{\sim} H^*(E^{P(\sigma)})$$

are isomorphisms. By the five lemma,

$$f : H^*(C/(C^{\sigma-1} \oplus C^{P(\sigma)})) \xrightarrow{f} H^*(E/(E^{\sigma-1} \oplus E^{P(\sigma)}))$$

is an isomorphism too. The result follows. $\qquad\square$

Lemma A.2 ((b) Additivity)**.** *Let C_0 and C_1 be bounded chain complexes of finite-free abelian groups. Suppose that either $\chi(C_0) = 0$ or $\chi(C_1) = 0$. Then*

$$\log NRT_\sigma((C_0 \oplus C_1)^{\otimes p}) = \log NRT_\sigma(C_0^{\otimes p}) + \log NRT_\sigma(C_1^{\otimes p}).$$

Proof. We will actually prove this equality for each curly brace occurring in the definition of NRT_σ individually.

- The equality

$$(p-1)[(C_0 \oplus C_1)^{\otimes p}]^{\sigma-1} - [(C_0 \oplus C_1)^{\otimes p}]^{P(\sigma)} = (p-1)[C_0^{\otimes p}]^{\sigma-1} - [C_0^{\otimes p}]^{P(\sigma)}$$
$$+ (p-1)[C_1^{\otimes p}]^{\sigma-1} - [C_1^{\otimes p}]^{P(\sigma)}.$$

in the Grothendieck group of $\mathbb{Z}$-modules can be checked after making a finite flat base change to $R = \mathbb{Z}[\mu_p]$. We can expand

$$(C_0 \oplus C_1)^{\otimes p} = \oplus_\epsilon C_{\epsilon_1} \otimes \cdots \otimes C_{\epsilon_p} =: \oplus_\epsilon C_\epsilon,$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_p)$ runs over all binary sequences of length p. σ acts by cyclic permutation on $(C_0 \oplus C_1)^{\otimes p}$ and so permutes the above summands in an evident manner. Consider a σ-orbit $\mathcal{O}$ of summands

$$C_\mathcal{O} = C_\epsilon \oplus \cdots \oplus C_{\sigma^{p-1}\epsilon},$$

where not all ϵ_i are equal. Clearly, for every pth root of unity ζ, the group

$$C_\mathcal{O}^{\sigma-\zeta} \cong C_\epsilon$$

by projection. Thus,

$$(p-1)C_\mathcal{O}^{\sigma-1} - C_\mathcal{O}^{P(\sigma)} = (p-1)C_\mathcal{O}^{\sigma-1} - \oplus_{\zeta \in \mu_p} C_\mathcal{O}^{\sigma-\zeta}$$
$$= (p-1)C_\epsilon - \oplus_{\zeta \in \mu_p} C_\epsilon$$
$$= 0$$

in the Grothendieck group of R-modules. Additivity for the first curly-braced term follows.

- For any complex E which is $\mathbb{Z}[\sigma]$-free,

$$\sum {}^* \log |E/(E^{\sigma-1} \oplus E^{P(\sigma)})| = \log p \sum {}^* \mathrm{rank}_{\mathbb{Z}[\sigma]}(E_i).$$

But the complex

$$C_\mathcal{O} = C_\epsilon \oplus \cdots \oplus C_{\sigma^{p-1}\epsilon},$$

where not all ϵ_i are equal, is free over $\mathbb{Z}[\sigma]$ of rank $\mathrm{rank}_{\mathbb{Z}} C_\epsilon$. Thus,

$$\sum {}^* \log |C_\mathcal{O}/(C_\mathcal{O}^{\sigma-1} \oplus C_\mathcal{O}^{P(\sigma)})| = \log p \sum {}^* \mathrm{rank}_{\mathbb{Z}} C_\epsilon$$
$$= \log p \; \chi(C_\epsilon)$$
$$= \log p \prod \chi(C_{\epsilon_i})$$
$$= 0.$$

Putting these two calculations together, we see that indeed

$$\log NRT_\sigma((C_0 \oplus C_1)^{\otimes p}) = \log NRT_\sigma(C_0^{\otimes p}) + \log NRT_\sigma(C_1^{\otimes p}). \qquad \square$$

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