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Lower central series, surface braid groups, surjections and permutations

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Abstract

Generalising previous results on classical braid groups by Artin and Lin, we determine the values of $m, n \in \mathbb{N}$ for which there exists a surjection between the n- and m-string braid groups of an orientable surface without boundary. This result is essentially based on specific properties of their lower central series, and the proof is completely combinatorial. We provide similar but partial results in the case of orientable surfaces with boundary components and of non-orientable surfaces without boundary. We give also several results about the classification of different representations of surface braid groups in symmetric groups.

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1. *Introduction*

In 1947, E. Artin published two seminal papers in the Annals of Mathematics, sometimes considered as the foundation of the theory of braid groups. The paper [4] is devoted to determining a presentation for the braid group B_n on n strings, and its interpretation in terms of automorphisms of the free group of rank n, while the subject of [5] is the study of possible homomorphisms from B_n to the symmetric group S_n on n letters. The main result of [5] is the description of all *transitive* homomorphisms (see Section 5 for the definition) between B_n and S_n. Artin considered this characterisation to be the first step in determining the group of automorphisms Aut (B_n) of B_n, for which a solution was given in [20]. In [43], Lin generalised Artin's results by characterising the homomorphisms between B_n and S_m and between B_n and B_m, for all $n > m$ (see [44] for a proof of these results and a survey of this topic). New and important results for homomorphisms between B_n and B_m with $n < m$ have been obtained recently [8, 15, 17].

The first part of this paper is devoted to studying the problem of the existence of surjective homomorphisms between (different) surface braid groups. These groups generalise both Artin's braid groups and fundamental groups of surfaces. We recall a definition and a presentation of these groups in Section 2. Given a compact, connected surface Σ, with or without boundary, orientable or non orientable, we will denote the braid group on n strands of Σ by $B_n(\Sigma)$. Further, $\mathbb{S}^2$ (resp. $\mathbb{R}P^2$) will denote the 2-sphere (resp. the real projective plane), $\mathbb{T}^2$ (resp. $\mathbb{K}^2$) will denote the 2-torus (resp. the Klein bottle), Σ_g (resp. $\Sigma_{g,b}$) will be a compact, connected, orientable surface of genus $g \geq 0$ without boundary (resp. with $b \geq 1$ boundary components), and U_g will be a compact, connected, non-orientable surface of genus $g \geq 1$ without boundary (in other words, U_g is the connected sum of g projective planes). We summarise our main results in this direction as follows.

THEOREM 1·1. *Let $m, n \in \mathbb{N}$ be such that $m \neq n$.*

(i) (a) *There is a surjective homomorphism from $B_n(\mathbb{S}^2)$ to $B_m(\mathbb{S}^2)$ if and only if $m \in \{1, 2\}$ and $n > m$.*

(b) *If $g \geq 1$, there is a surjective homomorphism from $B_n(\Sigma_g)$ to $B_m(\Sigma_g)$ if and only if $m = g = 1$.*

(ii) *Let $g \geq 1$, and let Σ be either $\Sigma_{g,b}$, where $b \geq 1$, or U_{g+1}. Suppose that one of the following conditions hold*:

(a) *$n < m$ and $n \in \{1, 2\}$;*

(b) *$n > m$ and $m \in \{1, 2\}$;*

(c) *$n > m \geq 3$ and $n \neq 4$.*

Then there is no surjective homomorphism from $B_n(\Sigma)$ to $B_m(\Sigma)$.

Remarks 1·2.

(i) Theorem 1·1 proves [16, conjecture 1·3] completely in the case where the surface is orientable and without boundary, and partially in the case where the surface is orientable with boundary, or non-orientable without boundary. The cases not covered by conditions (ii)(a)–(ii)(c) are likely to be difficult and will require new techniques. While this paper was being refereed, Chen, Kordek and Margalit gave a complete description of homomorphisms from B_n to B_m when $n < m \leq 2n$ [17]. As a straightforward consequence, implicit in [17], with these assumptions on n and m, it is not possible to surject B_n onto B_m. In order to analyse the cases that are not covered by Theorem 1·1, it thus seems interesting to try to adapt the techniques of [17] and, as suggested by an anonymous referee, to explore the maximal rank of free Abelian subgroups as the number of strands increases.

(ii) If $n = m$, every surjective homomorphism is also injective (*i.e.* $B_n(\Sigma)$ is *Hopfian*, see for instance [10, proposition 7·6]), and automorphisms of surface braid groups may be characterised in terms of (extended) mapping class groups [1, 10]).

(iii) Our proof is purely combinatorial, and makes use principally of the lower central series and torsion elements of the groups in question of the existence. The study of injective homomorphisms between surface braid groups is mainly open. The cases

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of the sphere and the projective plane are currently the subject of work in progress. For surfaces of higher genus, an approach via curve complexes ([**40**, **50**]) and cohomology seems more reasonable. Another tool that may be brought into play involves the relations with and among mapping class groups. Under some assumptions on the genus and the number of strands, homomorphisms between braid groups and mapping class groups of orientable surfaces without boundary were classified initially by Castel [**15**]. His results have recently been improved and generalised to surfaces with boundary in [**18**]. Automorphisms of surface braid groups are 'geometric', in the sense that they are induced by homeomorphisms of the punctured surface ([**38**], see also [**1**, **16**]). In another direction, there are injections between braid groups of different surfaces that are geometric in a similar sense [**48**, **49**]. More generally, and under some assumptions on the genus, homomorphisms between mapping class groups of different surfaces, possibly punctured are induced by topological *embeddings* [**3**]. We believe that an extension of these results to surface braid groups is worth considering, making use also of the homomorphisms of such groups induced by coverings [**2**, **18**, **34**].

Parts (i)(a) and (i)(b) of Theorem 1·1 will be proved in Sections 2·3 and 2·2 respectively, and part (ii) will be proved in Section 3 in the orientable case, and in Section 4 in the non-orientable case. For the case of the projective plane, in Theorem 4·6, using the knowledge of the torsion of its braid groups, we will obtain results that are slightly stronger than those of Theorem 1·1(ii), notably with respect to the case where $n < m$. As a consequence of Theorem 1·1 and some basic facts about the lower central series of surface braid groups, we give an elementary proof of [**16**, Theorem 1·2] in Corollary 2·11, and we generalise the result of this corollary to the case of orientable surfaces with boundary (Corollary 3·5), and to the non-orientable case (Corollary 4·8). In the cases of the sphere and real projective plane, the techniques are somewhat different to those used for other surfaces, since their braid groups have torsion [**21**, **51**].

Another interesting and open problem is the study of possible surjective homomorphisms between braid groups of different surfaces. One important case occurs when the domain is a braid group of a non-orientable surface U_g, and the target is a braid group of the orientable double covering Σ_{g-1}. It is known that there exists a natural injection on the level of configuration spaces that induces an injective homomorphism between $B_n(U_g)$ and $B_{2n}(\Sigma_{g-1})$ [**34**]. In Section 4, we prove the following result concerning surjections when the number of strings is the same.

PROPOSITION 1·3. *Let $n, g \geq 1$. Then there exists a surjective homomorphism of $B_n(U_g)$ onto $B_n(\Sigma_{g-1})$ if and only if $g = 1$ and $n \in \{1, 2\}$.*

Together with the existence of surjections between surface braid groups, another of our aims is to characterise homomorphisms between surface braid groups and symmetric groups following the approaches of [**39**, **43**, **44**]. In the case of B_n, this problem goes back to Artin himself [**5**], and was later studied extensively by Lin [**43**, **44**]. More recently, the existence of non-cyclic finite quotients of B_n has also been analysed [**19**]. One of the main results of [**44**] is the following.

THEOREM 1·4 ([44, theorem A]). *Let $n > m \geq 3$ and $n \neq 4$. Any homomorphism $\varphi\colon B_n \longrightarrow S_m$ is cyclic, i.e. $\varphi(B_n)$ is a cyclic group.*

This implies that if $n > m \geq 3$ and $n \neq 4$, there is no surjective homomorphism from B_n onto S_m. We shall show that a weaker version of Theorem 1·4 also holds for braid groups of compact surfaces without boundary as follows.

THEOREM 1·5. *Let $n > m \geq 2$, let $g \geq 0$, and let Σ be either Σ_g or U_{g+1}. Then there is a surjective homomorphism from $B_n(\Sigma)$ onto S_m if and only if either $m = 2$, or $(n, m) = (4, 3)$.*

If $g \geq 1$ (resp. $g = 0$) and $\Sigma = \Sigma_g$, the statement of Theorem 1·5 will be proved in Section 2·2 (resp. in Section 2·3), while in the case $g \geq 0$ and $\Sigma = U_{g+1}$, the result will be proved in Section 4.

Let $g \geq 0$, and let $n > m \geq 1$. We recall that a representation $\rho_{n,m}\colon B_n(\Sigma_g) \longrightarrow S_m$ is said to be *transitive* if the action of the image $\operatorname{Im}\left(\rho_{n,m}\right)$ of $\rho_{n,m}$ on the set $\{1, \ldots, m\}$ is transitive and is *primitive* if the only partitions of this set that are left invariant by the action of $\operatorname{Im}\left(\rho_{n,m}\right)$ are the set itself, or the partition consisting of singletons. By abuse of notation, we say that a subgroup of S_m is primitive if its action on the set $\{1, \ldots, m\}$ is primitive. Notice that a primitive representation is clearly transitive, if $m > 2$.

Inspired by Artin's characterisation of (transitive) homomorphisms between B_n and S_n, Ivanov determined all of the homomorphisms between $B_n(\Sigma_{g,b})$ and S_n, but under the stronger assumption that the homomorphisms are primitive [39, theorem 1]. We prove the following theorem for homomorphisms between $B_n(\Sigma_g)$ and S_m when $n > m$. This result may also be compared with the classification of the homomorphisms between B_n and S_m, where $n > m$, given in Theorem 5·1. As in Theorems 1·1, 1·4 and 1·5, $(n, m) = (4, 3)$ constitutes a special case, and arises from the fact that these are the only values of n and m, where $n \neq m$ and $m \geq 3$, for which there exists a surjective homomorphism from S_n to S_m.

THEOREM 1·6. *Let $n > m \geq 2$, and let $g \geq 1$. There exists a primitive representation $\rho_{n,m}\colon B_n(\Sigma_g) \longrightarrow S_m$ if and only if m is prime. This being the case, one of the following statements holds*:

- (i) *the image $\operatorname{Im}\left(\rho_{n,m}\right)$ of $\rho_{n,m}$ is generated by an m-cycle, unless $m = 2$, in which case $\operatorname{Im}\left(\rho_{n,2}\right)$ can also be equal to $\{\mathrm{Id}\}$;*
- (ii) *$n = 4$ and $m = 3$, and up to a suitable renumbering of the elements of the set $\{1, 2, 3\}$, $\rho_{4,3}(\sigma_1) = \rho_{4,3}(\sigma_3) = (1, 2)$, $\rho_{4,3}(\sigma_2) = (2, 3)$, and for all $1 \leq i \leq g$, the permutations $\rho_{4,3}(a_i)$ and $\rho_{4,3}(b_i)$ are trivial, where $\left\{\sigma_1, \sigma_2, \sigma_3, a_1, b_1, \ldots, a_g, b_g\right\}$ is the generating set of $B_4(\Sigma_g)$ given in the statement of Theorem 2·2.*

Theorem 1·6 provides a classification of primitive representations, a result that was more or less implicitly expected in [39]. In Proposition 5·4, we obtain some constraints on general (non-primitive) homomorphisms, and we answer a question of [39] by giving some examples of transitive, non-primitive, non-Abelian representations. The theory of representations in symmetric groups is a classical topic, and is interesting in its own right. In the case of surface braid groups, additional motivation for the study of this question comes from the fact that if $n > m$, representations of $B_n(\Sigma)$ in S_m factor through the metabelian quotient of $B_n(\Sigma)$ (see Proposition 5·4(iii) and Remarks 5·5 for a group presentation of this quotient). This quotient also seems to play a central rôle in the possible extension of the Bigelow–Krammer–Lawrence representation from B_n to $B_n(\Sigma)$, where Σ is an orientable surface with boundary (this is the main subject of [13]). A better understanding of the representation theory of $B_n(\Sigma)$ in symmetric groups therefore seems important in order to construct linear representations of surface braid groups, a problem that is still largely open.

The rest of this paper is divided into four sections. Sections 2 and 3 deal with the braid groups of compact, orientable surfaces without boundary and with boundary respectively, Section 4 is devoted to the braid groups of compact, non-orientable surfaces without boundary. In each of these sections, we give a presentation of the braid groups in question, we recall some known results about their lower central series and whether they are residually nilpotent or not, and we prove the relevant parts of Theorems 1·1 and 1·5. In Section 5 we explore representations of surface braid groups in symmetric groups, and we give several examples where the image is non-Abelian.

In this paper, we do not discuss the braid groups of non-orientable surfaces with boundary components. This choice is motivated by two different considerations, first that these groups have rarely been studied in the literature, and secondly, that the techniques used in the case of non-orientable surfaces without boundary apply almost verbatim to the case with boundary. This contrasts with the orientable case, where the lower central series is a stronger tool in the case without boundary than in the case with boundary.

Finally, although we show in Corollaries 2·11, 3·5 and 4·8 that it is not possible in general to surject surface braid groups onto surface pure braid groups, we do not study explicitly homomorphisms between surface pure braid groups here. We believe that this is a very interesting topic. We simply mention the main result of [16] which states that all possible surjections between pure braid groups of a given orientable surface are induced by a *forgetting map*. In contrast with the case of surface braid groups, surface pure braid groups are residually torsion free nilpotent if the surface is orientable [6, 12], and residually 2-finite otherwise [11]. A complete characterisation of quotients arising from the rational lower central series (respectively the p-linear lower central series) should give a powerful combinatorial tool to obtain constraints on homomorphisms between pure braid groups (and possibly of different surfaces). We remark that the automorphism groups of orientable surface pure braid groups are known [1]. It would be interesting to study the structure of these groups along the lines of those for classical pure braid groups, see [7] for instance.

2. *Orientable surfaces without boundary*

In Section 2·1, we start by recalling a presentation of the braid groups of compact, orientable surfaces without boundary, as well as some facts about their lower central series. In Section 2·2, we generalise certain results of [35] about the minimal number of generators of these groups, and we prove Theorems 1·1(i)(b) and 1·5 in the case where $\Sigma = \Sigma_g$, with $g \geq 1$. In Section 2·3, we prove Theorem 1·1(i)(a) and Theorem 1·5 in the case $g = 0$, which is that of the sphere.

2·1. *Presentations and the lower central series of surface braid groups*

In this paper, many of our techniques will be combinatorial and will make use of the lower central series of surface (pure) braid groups. Given a group G, recall that the *lower central series* of G is given by $\{\Gamma_i(G)\}_{i\in\mathbb{N}}$, where $G = \Gamma_1(G)$, and $\Gamma_i(G) = [G, \Gamma_{i-1}(G)]$ for all $i \geq 2$. We thus have a filtration $\Gamma_1(G) \supseteq \Gamma_2(G) \supseteq \cdots$. The group G is said to be *perfect* if $G = \Gamma_2(G)$. We shall denote the *Abelianisation* $\Gamma_1(G)/\Gamma_2(G)$ of G by G_{Ab}. If $\mathcal{P}$ is a group-theoretic property, let $\mathcal{FP}$ denote the class of groups that possess property

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$\mathcal{P}$. Following P. Hall, G is said to be *residually* $\mathcal{P}$ if for any (non-trivial) element $x \in G$, there exists a group H possessing property $\mathcal{P}$ and a surjective homomorphism $\varphi\colon\ G \longrightarrow H$ such that $\varphi(x) \neq 1$. It is well known that a group G is residually nilpotent if and only if $\bigcap_{i\geq 1} \Gamma_i(G) = \{1\}$. The lower central series of the Artin braid groups is also well known.

PROPOSITION 2·1 (see [12, 44]). *If* $n \geq 3$, $\Gamma_1(B_n)/\Gamma_2(B_n) \cong \mathbb{Z}$, *and* $\Gamma_2(B_n) = \Gamma_3(B_n)$.

Proposition 2·1 also holds trivially if $n = 2$ since $B_2 \cong \mathbb{Z}$ (and therefore $\Gamma_2(B_n) = \Gamma_3(B_n) = 1$). Using [22, 41] and the fact that P_2 is isomorphic to $\mathbb{Z}$, we see that the group P_n is residually (torsion-free) nilpotent for all $n \geq 2$.

We recall the definition of surface braid groups in terms of fundamental groups of configuration spaces [23]. Let Σ be a compact, connected surface, with or without boundary, orientable or non orientable, and let $\mathbb{F}_n(\Sigma) = \Sigma^n \setminus \Delta$, where Δ is the set of n-tuples $(x_1, \ldots, x_n)$ of elements of Σ for which $x_i = x_j$ for some $1 \leq i, j \leq n$, where $i \neq j$. The fundamental group $\pi_1(\mathbb{F}_n(\Sigma))$ is called the *pure braid group* on n strings of Σ and shall be denoted by $P_n(\Sigma)$. The symmetric group S_n acts freely on $\mathbb{F}_n(\Sigma)$ by permutation of coordinates, and the fundamental group $\pi_1(\mathbb{F}_n(\Sigma)/S_n)$ of the resulting quotient space, denoted by $B_n(\Sigma)$, is the *braid group* on n strings of Σ. Further, $\mathbb{F}_n(\Sigma)$ is a regular $n!$-fold covering of $\mathbb{F}_n(\Sigma)/S_n$, from which we obtain the following short exact sequence:

$$1 \longrightarrow P_n(\Sigma) \longrightarrow B_n(\Sigma) \longrightarrow S_n \longrightarrow 1. \qquad (2\cdot 1)$$

If Σ is the 2-disc $\mathbb{D}^2$, it is well known that $B_n(\mathbb{D}^2) \cong B_n$ and that $P_n(\mathbb{D}^2) \cong P_n$. A presentation of $B_n(\Sigma_g)$ for $g \geq 1$ is as follows.

THEOREM 2·2 ([12, theorem 6]). *Let* $g, n \in \mathbb{N}$. *Then* $B_n(\Sigma_g)$ *admits the following group presentation*:

(i) ***generators***: $a_1, b_1, \ldots, a_g, b_g, \sigma_1, \ldots, \sigma_{n-1}$;
(ii) ***relations***:

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2 \qquad (2\cdot 2)$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for all } 1 \leq i \leq n-2 \qquad (2\cdot 3)$$

$$c_i\sigma_j = \sigma_j c_i \text{ for all } j \geq 2,\ c_i = a_i \text{ or } b_i \text{ and } i = 1, \ldots, g \qquad (2\cdot 4)$$

$$c_i\sigma_1 c_i\sigma_1 = \sigma_1 c_i\sigma_1 c_i \text{ for } c_i = a_i \text{ or } b_i \text{ and } i = 1, \ldots, g \qquad (2\cdot 5)$$

$$a_i\sigma_1 b_i = \sigma_1 b_i\sigma_1 a_i\sigma_1 \text{ for } i = 1, \ldots, g \qquad (2\cdot 6)$$

$$c_i\sigma_1^{-1} c_j\sigma_1 = \sigma_1^{-1} c_j\sigma_1 c_i \text{ for } c_i = a_i \text{ or } b_i,\ c_j = a_j \text{ or } b_j \text{ and } 1 \leq j < i \leq g \qquad (2\cdot 7)$$

$$\prod_{i=1}^{g}[a_i^{-1}, b_i] = \sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1. \qquad (2\cdot 8)$$

Throughout this paper, relations (2·2) and (2·3) will be referred to as the *braid* or *Artin relations*. Observe that if we take $g = 0$ in the presentation of Theorem 2·2, we obtain the presentation of $B_n(\mathbb{S}^2)$ due to Fadell and Van Buskirk [21], the relations being the braid relations and the 'surface relation':

$$\sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1 = 1, \qquad (2\cdot 9)$$

so Theorem 2·2 is also valid in this case.

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If $g \geq 1$, the lower central series of the braid groups of Σ_g were studied in [**12**]. The statement of the following theorem contains some of the results of that paper, and provides some minor improvements, notably in the case $n = 2$.

THEOREM 2·3. *Let* $g, n \geq 1$. *Then*:

(i) $\Gamma_1(B_n(\Sigma_g))/\Gamma_2(B_n(\Sigma_g)) \cong \begin{cases} \mathbb{Z}^{2g} & \text{if } n = 1 \\ \mathbb{Z}^{2g} \oplus \mathbb{Z}_2 & \text{if } n \geq 2; \end{cases}$

(ii) (a) $\Gamma_2(B_n(\Sigma_g))/\Gamma_3(B_n(\Sigma_g)) \cong \begin{cases} \mathbb{Z}^{g(2g-1)-1} & \text{if } n = 1 \\ \mathbb{Z}_{n-1+g} & \text{if } n \geq 3; \end{cases}$

(b) *If* $n = 2$, $\Gamma_2(B_2(\mathbb{T}^2))/\Gamma_3(B_2(\mathbb{T}^2)) \cong \mathbb{Z}_2^3$, *and if* $g > 1$, $\Gamma_2(B_2(\Sigma_g))/\Gamma_3(B_2(\Sigma_g))$ *is a non-trivial quotient of* $\mathbb{Z}_2^{2g} \oplus \mathbb{Z}_{g+1}$;

(iii) $\Gamma_3(B_n(\Sigma_g)) = \Gamma_4(B_n(\Sigma_g))$ *if and only if* $n \geq 3$. *Moreover* $\Gamma_3(B_n(\Sigma_g))$ *is perfect if and only if* $n \geq 5$;

(iv) *the group* $B_n(\Sigma_g)$ *is residually nilpotent if and only if* $n \leq 2$.

Parts (i) and (ii) of Theorem 2·3 imply that the braid groups of orientable surfaces without boundary may be distinguished by their lower central series (and indeed by the first two lower central series quotients). A presentation of the group $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$ was given in [**12**, equation (10)] and may be found in Example 1. Many of the statements of this theorem were proved in [**12**, theorem 1] in the case $n \geq 3$, and may be deduced from [**42**] in the case $n = 1$. More information about the lower central series quotients of $B_1(\Sigma_g)$ may be found in [**42**]. Taking into account these papers, at the end of this section, we prove Theorem 2·3. We first give some preliminary results and properties regarding the remaining parts of the statement, notably in the case where $n = 2$. If $g = 1$, $\mathbb{T}^2$ is the 2-torus $\mathbb{T}^2$, and we have the following result for $B_2(\mathbb{T}^2)$.

THEOREM 2·4 ([**12**, theorem 3]). *The group* $B_2(\mathbb{T}^2)$ *is residually nilpotent, but is not residually torsion-free nilpotent. Further*:

$$\Gamma_2(B_2(\mathbb{T}^2))/\Gamma_3(B_2(\mathbb{T}^2)) \cong \mathbb{Z}_2^3, \text{ and } \Gamma_3(B_2(\mathbb{T}^2))/\Gamma_4(B_2(\mathbb{T}^2)) \cong \mathbb{Z}_2^5.$$

Proof. The first part of the statement is [**12**, theorem 3(a) and (c)]. To prove the second part, using ideas from [**26**], it was shown in [**12**, theorem 3(b)] that with the exception of the first term, the lower central series of $B_2(\mathbb{T}^2)$ and the free product $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ coincide. With the help of results of [**24**], for all $i \geq 2$, it follows that lower central series quotient $\Gamma_i(B_2(\mathbb{T}^2))/\Gamma_{i+1}(B_2(\mathbb{T}^2))$ is isomorphic to the direct sum of R_i copies of $\mathbb{Z}_2$, where R_i is given by an explicit formula involving the Möbius function, from which one may check that $R_2 = 3$ and $R_3 = 5$. This yields the second part of the statement.

If $g > 1$, $B_2(\Sigma_g)$ is residually nilpotent.

PROPOSITION 2·5 ([**6**, corollary 10]). *If* $g \geq 1$, *the group* $B_2(\Sigma_g)$ *is residually* 2*-finite. In particular, it is residually nilpotent.*

Remark 2·6. To prove some of our results, we will need to be sure that our residually nilpotent groups are not nilpotent, in particular that all of their lower central series quotients

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are non trivial. We claim that this is the case for the group $B_2(\Sigma_g)$ for all $g \geq 1$. If $g = 1$, the result follows from [12, theorem 3] (note that the group $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ contains a subgroup that is a free group of rank 2). So assume that $g > 1$, and suppose on the contrary that there exists $i \in \mathbb{N}$ such that $\Gamma_i(B_2(\Sigma_g)) = \{1\}$. Without loss of generality, we may suppose that i is minimal with respect to this property. Since $B_2(\Sigma_g)$ is non Abelian, it follows from Theorem 2·3(i) that $i \geq 3$. Now $\Gamma_i(B_2(\Sigma_g)) = [\Gamma_{i-1}(B_2(\Sigma_g)), B_2(\Sigma_g)] = \{1\}$, and hence $\Gamma_{i-1}(B_2(\Sigma_g))$ is contained in the centre of $B_2(\Sigma_g)$. This centre is trivial [27, 48], so $\Gamma_{i-1}(B_2(\Sigma_g)) = \{1\}$, but this contradicts the minimality of i, and so proves the result in this case.

The computation of the lower central series quotients in the case $n = 2$ and $g > 1$, namely the generalisation of Theorem 2·4 and [12, theorem 3] to surfaces of arbitrary genus, remains an open problem. The following result nevertheless gives some information about the quotient $\Gamma_2(B_2(\Sigma_g))/\Gamma_3(B_2(\Sigma_g))$.

PROPOSITION 2·7. *If $g \geq 1$, the group $\Gamma_2(B_2(\Sigma_g))/\Gamma_3(B_2(\Sigma_g))$ is non trivial, and is a quotient of $\mathbb{Z}_2^{2g} \oplus \mathbb{Z}_{g+1}$.*

Proof. If $g = 1$ then by Theorem 2·4, $\Gamma_2(B_2(\mathbb{T}^2))/\Gamma_3(B_2(\mathbb{T}^2)) \cong \mathbb{Z}_2^3$, and the result holds. So suppose that $g > 1$. In what follows we will make use freely of the Witt–Hall identities [45, theorem 5·1]. By relation (2·7), we have $1 = [c_i, \sigma_1^{-1} c_j \sigma_1]$ for $c_i = a_i$ or b_i, $c_j = a_j$ or b_j and for all $1 \leq j < i \leq g$ in $B_2(\Sigma_g)$, from which it follows that $1 = [c_i, c_j]$ in $B_2(\Sigma_g)/\Gamma_3(B_2(\Sigma_g))$. Using relation (2·5), we have $1 = [c_i, \sigma_1 c_i \sigma_1]$ in $B_2(\Sigma_g)$ for $c_i = a_i$ or b_i and for all $i = 1, \ldots, g$, which implies that $1 = [c_i, \sigma_1]^2$ in $B_2(\Sigma_g)/\Gamma_3(B_2(\Sigma_g))$. Similarly, from relation (2·6), we obtain $[b_i^{-1}, \sigma_1^{-1} a_i \sigma_1] = \sigma_1^2$ in $B_2(\Sigma_g)$ for all $i = 1, \ldots, g$, and therefore $[b_i^{-1}, a_i] = [b_i, a_i]^{-1} = \sigma_1^2$ in $B_2(\Sigma_g)/\Gamma_3(B_2(\Sigma_g))$. Since $\prod_{i=1}^{g} [a_i^{-1}, b_i] = \prod_{i=1}^{g} [b_i, a_i]$ in $B_2(\Sigma_g)/\Gamma_3(B_2(\Sigma_g))$ by relation (2·8), we see that $\sigma_1^{-2g} = \sigma_1^2$, and thus the order of σ_1^2 in $\Gamma_2(B_2(\Sigma_g))/\Gamma_3(B_2(\Sigma_g))$ divides $g + 1$. These computations imply that $\Gamma_2(B_2(\Sigma_g))/\Gamma_3(B_2(\Sigma_g))$ is an Abelian group that is generated by the commutators $[c_i, \sigma_1]$ for $c_i = a_i$ or b_i and $i = 1, \ldots, g$, which are all of order at most 2, and the commutators $[b_i, a_i]$, where $i = 1, \ldots, g$, and which are all identified to a single element σ_1^2 of order at most $g + 1$. Consequently, $\Gamma_2(B_2(\Sigma_g))/\Gamma_3(B_2(\Sigma_g))$ is a quotient of $\mathbb{Z}_2^{2g} \oplus \mathbb{Z}_{g+1}$. Remark 2·6 implies that this quotient is non trivial, which proves the result.

PROPOSITION 2·8. *If $g \geq 1$ and $n \in \{3, 4\}$, the group $\Gamma_3(B_n(\Sigma_g))$ is not perfect.*

Proof. Let $g \geq 1$ and $n \in \{3, 4\}$. Let $\pi_n\colon B_n(\Sigma_g) \longrightarrow S_n$ be the homomorphism that arises in (2·1), and for $i \geq 2$, let $\pi_{n,i}\colon \Gamma_i(B_n(\Sigma_g)) \longrightarrow \Gamma_i(S_n)$ denote the induced surjective homomorphism between the corresponding terms of the lower central series. A straightforward computation shows that $\Gamma_2(S_n) = \Gamma_3(S_n) = A_n$, where A_n is the alternating group, and that $\Gamma_3(S_n)/[\Gamma_3(S_n), \Gamma_3(S_n)]$ is isomorphic to $\mathbb{Z}_3$. Now the homomorphism $\pi_{n,3}$ induces a surjection at the level of Abelianisations, and since $\Gamma_3(S_n)/[\Gamma_3(S_n), \Gamma_3(S_n)]$ is non trivial, we conclude that $\Gamma_3(B_n(\Sigma_g))$ cannot be perfect.

Proof of Theorem 2·3. First assume that $n = 1$. We have $B_1(\Sigma_g) = \pi_1(\Sigma_g)$, which is residually free, and therefore residually (torsion free) nilpotent. Further, by [42, main theorem], $\Gamma_i(B_1(\Sigma_g))/\Gamma_{i+1}(B_1(\Sigma_g))$ is isomorphic to $\mathbb{Z}^{2g}$ if $i = 1$, to $\mathbb{Z}^{g(2g-1)-1}$ if $i = 2$, and to

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$\mathbb{Z}^{4g(g^2-1)}$ if $i=3$. In particular $\Gamma_3(B_1(\Sigma_g))$ is not perfect. Therefore all statements of the theorem pertaining to the case $n=1$ hold.

Now suppose that $n=2$. It follows in a straightforward manner from Theorem 2·2 that $\Gamma_1(B_2(\Sigma_g))/\Gamma_2(B_2(\Sigma_g)) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}_2$. Part (ii)(b) follows from Theorem 2·4 and Proposition 2·7, and parts (iii) and (iv) in the case $n=2$ are a consequence of Theorem 2·4, Proposition 2·5 and Remark 2·6.

Finally, let $n \geq 3$. Parts (i), (ii)(a), (iv), and the sufficiency of the condition in part (iii) were proved in [**12**, theorem 1]. Part (iii) in the case $n \in \{3, 4\}$ is a consequence of Proposition 2·8. This completes the proof of the theorem.

2·2. *Surjections between braid groups of orientable surfaces of non-zero genus without boundary*

With the notation of [**35**], if Γ is a finitely-generated group, let $G(\Gamma)$ denote the minimal cardinality among all generating sets of Γ. By [**35**, proposition 8], if Γ' is another finitely-generated group such that there exists a surjective homomorphism from Γ to Γ' then:

$$G(\Gamma) \geq G(\Gamma') \text{ and } G(\Gamma) \geq G(\Gamma_{\mathrm{Ab}}), \tag{2·10}$$

the second inequality following from the first by taking the homomorphism to be Abelianisation. The following proposition generalises some of the principal results of [**35**] to the case of orientable surfaces of genus $g \geq 1$.

PROPOSITION 2·9. *Let $g, m \in \mathbb{N}$. Then* $G(B_m(\Sigma_g)) = \begin{cases} 2g+m-1 & \text{if } m \in \{1,2\} \\ 2g+2 & \text{if } m \geq 3. \end{cases}$

Proof. If $m \in \{1, 2\}$, then $(B_m(\Sigma_g))_{\mathrm{Ab}} \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}_2^{m-1}$ using Theorem 2·3(i), so $G(B_m(\Sigma_g)) \geq 2g+m-1$ by (2·10). By taking the generating set of $B_m(\Sigma_g)$ given in Theorem 2·2, we see also that $G(B_m(\Sigma_g)) \leq 2g+m-1$, which proves the result in this case. So assume that $m \geq 3$. As in the proof of [**35**, proposition 4], using Theorem 2·3, we see that $\{a_1, b_1, \ldots, a_g, b_g, \sigma_1, \sigma_1 \cdots \sigma_{m-1}\}$ is a generating set for $B_m(\Sigma_g)$, and so $G(B_m(\Sigma_g)) \leq 2g+2$. Conversely, with respect to the presentation given by Theorem 2·2, let $f\colon B_m(\Sigma_g) \longrightarrow \mathbb{Z}^{2g}$ be the surjective homomorphism whose kernel is the normal closure of $\{\sigma_1, \ldots, \sigma_{m-1}\}$ in $B_m(\Sigma_g)$, and let $h\colon B_m(\Sigma_g) \longrightarrow S_m$ be the surjective homomorphism given by equation (2·1) whose kernel is $P_m(\Sigma_g)$. Using Theorem 2·2, h may also be seen to be the projection onto the quotient of $B_m(\Sigma_g)$ by the normal closure of the set $\{a_1, b_1, \ldots, a_g, b_g\}$. The map $f \times h\colon B_m(\Sigma_g) \longrightarrow \mathbb{Z}^{2g} \times S_m$ is clearly a homomorphism. To see that it is surjective, note that if $(w, \tau) \in \mathbb{Z}^{2g} \times S_m$, there exist $\alpha \in P_m(\Sigma_g)$ and β belonging to the subgroup of $B_m(\Sigma_g)$ generated by $\{\sigma_1, \ldots, \sigma_{m-1}\}$ such that $f(\alpha) = w$ and $h(\beta) = \tau$. From the description of f and h, we have $(f \times h)(\alpha\beta) = (w, \tau)$, which proves that $f \times h$ is surjective. So by (2·10), $G(B_m(\Sigma_g)) \geq G(\mathbb{Z}^{2g} \times S_m) = 2g + G(S_m) \geq 2g+2$ because $m \geq 3$. Thus $G(B_m(\Sigma_g)) = 2g+2$, and the statement then follows in this case.

COROLLARY 2·10. *Let $m \geq 3$, $n \in \{1, 2\}$ and $g \geq 1$. Then there is no surjective homomorphism from $B_n(\Sigma_g)$ to $B_m(\Sigma_g)$.*

Proof. If $m \geq 3$, $n \in \{1, 2\}$ and $g \geq 1$, the result follows from (2·10) using the fact that $G(B_m(\Sigma_g)) > G(B_n(\Sigma_g))$ by Proposition 2·9.

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Proof of Theorem 1·5 in the case where $\Sigma = \Sigma_g$ and $g \geq 1$. Let $n > m \geq 2$, and consider the map from B_n to $B_n(\Sigma_g)$ defined on the generators of B_n by sending σ_i to σ_i for all $i = 1, \ldots, n-1$. It is a homomorphism (note that by [48], it is also an embedding). Suppose first that $n > m \geq 3$ and $n \neq 4$, and let $\Phi\colon\ B_n(\Sigma_g) \longrightarrow S_m$ be a homomorphism. By Theorem 1·4, the elements $\Phi(\sigma_i)$, where $i = 1, \ldots, n-1$, are powers of a single element, and therefore commute pairwise. Using the braid relations, the fact that $\Phi(\sigma_i)$ commutes with $\Phi(\sigma_{i+1})$ for all $i = 1, \ldots, n-2$ implies that $\Phi(\sigma_1) = \cdots = \Phi(\sigma_{n-1})$. We denote this common element by σ. We see from relations (2·4) that σ commutes with $\Phi(a_j)$ and $\Phi(b_j)$ for all $j = 1, \ldots, g$. Suppose now that Φ is surjective. Then σ belongs to the centre of S_m, which is trivial since $m \geq 3$, so σ is trivial. Therefore the homomorphism Φ factors through the surjective homomorphism $\Phi'\colon\ B_n(\Sigma_g)/\langle\langle\sigma_1\rangle\rangle \longrightarrow S_m$, where $\langle\langle\sigma_1\rangle\rangle$ denotes the normal closure of σ_1 in $B_n(\Sigma_g)$. But using Theorem 2·2 (*cf.* the proof of Proposition 2·9), $B_n(\Sigma_g)/\langle\langle\sigma_1\rangle\rangle$ is isomorphic to $\mathbb{Z}^{2g}$, so is Abelian, while S_m is not. This yields a contradiction, and hence Φ is not surjective.

Conversely, if $(n, m) = (4.3)$, the map from $B_4(\Sigma_g)$ to S_3 defined by sending the elements $a_1, b_1, \ldots, a_g, b_g$ to the identity element, σ_1 and σ_3 to $(1, 2)$, and σ_2 to $(2, 3)$, extends to a well-defined, surjective homomorphism by Theorem 2·2.

We are now able to prove Theorem 1·1 for the braid groups of orientable surfaces without boundary of genus $g \geq 1$.

Proof of Theorem 1·1(i)(b). Suppose first that $m = g = 1$, and that $n \geq 2$. Since $B_1(\mathbb{T}^2) \cong \mathbb{Z}^2$, the result may be obtained by considering the surjective homomorphism $f\colon\ B_n(\mathbb{T}^2) \longrightarrow \mathbb{Z}^2$ defined in the proof of Proposition 2·9. To prove the converse, we will show that if $(g, m) \neq (1, 1)$, there is no surjective homomorphism from $B_n(\Sigma_g)$ to $B_m(\Sigma_g)$. We split the proof into the following three cases.

(i) $n < m$. If $n \in \{1, 2\}$, the result follows from Corollary 2·10. So suppose that $n \geq 3$. Theorem 2·3(ii)(a) implies that there is no surjective homomorphism from $\Gamma_2(B_n(\Sigma_g))/\Gamma_3(B_n(\Sigma_g))$ onto $\Gamma_2(B_m(\Sigma_g))/\Gamma_3(B_m(\Sigma_g))$, and hence there is no surjective homomorphism from $B_n(\Sigma_g)$ onto $B_m(\Sigma_g)$.

(ii) $n > m$, where either $g > 1$ and $m \in \{1, 2\}$, or $g = 1$ and $m = 2$. If $n \geq 3$, by Theorem 2·3(iii), $\Gamma_3(B_n(\Sigma_g))/\Gamma_4(B_n(\Sigma_g))$ is trivial, while $\Gamma_3(B_m(\Sigma_g))/\Gamma_4(B_m(\Sigma_g))$ is not, and this implies that there is no surjective homomorphism from $B_n(\Sigma_g)$ onto $B_m(\Sigma_g)$. If $n = 2$, $m = 1$ and $g > 1$, $\Gamma_2(B_2(\Sigma_g))/\Gamma_3(B_2(\Sigma_g))$ is finite by Theorem 2·3(ii)(b), and so it cannot surject onto $\Gamma_2(\pi_1(\Sigma_g))/\Gamma_3(\pi_1(\Sigma_g))$, which is a (non-trivial) free Abelian group by Theorem 2·3(ii)(a).

(iii) $n > m \geq 3$. Assume first that $n \neq 4$. There can be no surjection homomorphism from $B_n(\Sigma_g)$ onto $B_m(\Sigma_g)$, for otherwise its composition with the projection $B_m(\Sigma_g)$ onto S_m of (2·1) would yield a surjective homomorphism from $B_n(\Sigma_g)$ onto S_m, which contradicts Theorem 1·5. So assume that $n = 4$. Then $m = 3$, and there can be no surjective homomorphism from $B_4(\Sigma_g)$ to $B_3(\Sigma_g)$ because otherwise by Theorem 2·3(ii)(a), there would be a surjective homomorphism from $\Gamma_2(B_4(\Sigma_g))/\Gamma_3(B_4(\Sigma_g))$, which is isomorphic to $\mathbb{Z}_{3+g}$, onto $\Gamma_2(B_3(\Sigma_g))/\Gamma_3(B_3(\Sigma_g))$, which is isomorphic to $\mathbb{Z}_{2+g}$, but this is impossible.

COROLLARY 2·11. *Let $g \geq 1$, and let $n, m \in \mathbb{N}$. There is a surjective homomorphism of $B_n(\Sigma_g)$ onto $P_m(\Sigma_g)$ if and only if $n = m = 1$ for $g \geq 1$ and $m = 1$ for $g = 1$.*

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Proof. Let $g \geq 1$. We first prove that the conditions are sufficient. If $n = m = 1$, the result is clear since the given groups coincide with the fundamental group of the surface. If $g = m = 1$ then the result follows from Theorem 1·1(i)(b). Conversely, suppose that there exists a surjective homomorphism $\Phi\colon B_n(\Sigma_g) \longrightarrow P_m(\Sigma_g)$. Then Φ induces a surjective homomorphism of the corresponding Abelianisations, but since $(P_m(\Sigma_g))_{\mathrm{Ab}} \cong \mathbb{Z}^{2gm}$ from the presentation of $P_m(\Sigma_g)$ given in [**9**] for instance, it follows from Theorem 2·3(i) that $m = 1$. Then either $n = 1$, or $n > 1$, in which case $g = 1$ by Theorem 1·1(i)(b), and in both cases, the conclusion holds.

Remark 2·12. With the exception of the case $g = 1$, Corollary 2·11 was proved in [**16**, theorem 1·2] using different methods.

2·3. *Surjections between braid groups of the sphere*

In this section, we complete the analysis of surjections between braid groups of orientable surfaces without boundary by studying the case $g = 0$, which is that of the sphere $\mathbb{S}^2$. Theorems 1·1 and 1·5 hold also in this case, but the arguments are somewhat different. As we mentioned just after the statement of Theorem 2·2, if $n \in \mathbb{N}$, the presentation of $B_n(\mathbb{S}^2)$ in [**21**] may be obtained from the standard presentation of B_n by adding the relation (2·9), so $B_n(\mathbb{S}^2)$ is a quotient of B_n. It follows from this presentation that $B_1(\mathbb{S}^2)$ is trivial, $B_2(\mathbb{S}^2) = \mathbb{Z}_2$, $B_3(\mathbb{S}^2) = \mathbb{Z}_3 \rtimes \mathbb{Z}_4$ (with non-trivial action), and $B_n(\mathbb{S}^2)$ is an infinite group for all $n \geq 4$ [**21**, third theorem, p· 255]. The following theorem summarises some known results about the lower central series of the braid groups of the sphere.

THEOREM 2·13 ([**31**]).

(i) $\Gamma_1(B_n(\mathbb{S}^2))/\Gamma_2(B_n(\mathbb{S}^2)) \cong \mathbb{Z}_{2(n-1)}$ *for* $n \geq 2$.
(ii) $\Gamma_2(B_n(\mathbb{S}^2))) = \Gamma_3(B_n(\mathbb{S}^2))$ *for* $n \geq 2$.
(iii) $\Gamma_2(B_n(\mathbb{S}^2)$ *is perfect if and only if* $n \geq 5$.

The proofs of parts (i), (ii) and (iii) of Theorem 2·13 may be found in [**31**, Proposition 2·1, and Theorems 1·3 and 1·4 respectively]. We now prove Theorem 1·5 in the case of the sphere and Theorem 1·1(i)(a).

Proof of Theorem 1·5 *in the case where* $\Sigma = \Sigma_0$. Let $n > m \geq 3$ and $n \neq 4$, and suppose that there exists a surjective homomorphism $\Phi\colon B_n(\mathbb{S}^2) \longrightarrow S_m$. Since $B_n(\mathbb{S}^2)$ is a quotient of B_n, B_n surjects homomorphically onto $B_n(\mathbb{S}^2)$, so its composition with Φ would give rise to a surjective homomorphism from B_n to S_m, which contradicts Theorem 1·4.

Proof of Theorem 1·1(i)(a). We start by showing that the condition for the existence of a surjective homomorphism from $B_n(\mathbb{S}^2)$ to $B_m(\mathbb{S}^2)$ is sufficient. Since $B_1(\mathbb{S}^2)$ is trivial, the result is clear if $m = 1$, and if $n \geq 3$ and $m = 2$, $B_n(\mathbb{S}^2)$ surjects homomorphically onto $B_2(\mathbb{S}^2)$ since $(B_n(\mathbb{S}^2))_{\mathrm{Ab}} \cong \mathbb{Z}_{2(n-1)}$ and $B_2(\mathbb{S}^2) \cong \mathbb{Z}_2$, so there exists a surjective homomorphism from $B_n(\mathbb{S}^2)$ to $B_2(\mathbb{S}^2)$ that factors through $(B_n(\mathbb{S}^2))_{\mathrm{Ab}}$. To show that the condition is necessary, we consider the following two cases.

(i) Suppose that $n < m$. Then either $B_n(\mathbb{S}^2)$ is trivial (if $n = 1$) or $(B_n(\mathbb{S}^2))_{\mathrm{Ab}} \cong \mathbb{Z}_{2(n-1)}$ (if $n \geq 2$). So there does not exist a surjective homomorphism between $(B_n(\mathbb{S}^2))_{\mathrm{Ab}}$ and $(B_m(\mathbb{S}^2))_{\mathrm{Ab}}$, and hence there cannot exist a surjective homomorphism between $B_n(\mathbb{S}^2)$ and $B_m(\mathbb{S}^2)$.

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(ii) Now let $n > m \geq 3$. If $n \neq 4$, the fact that there does not exist a surjective homomorphism from $B_n(\Sigma_g)$ onto S_m by Theorem 1·5 implies that there does not exist a surjective homomorphism from $B_n(\mathbb{S}^2)$ onto $B_m(\mathbb{S}^2)$. The remaining case, $(n, m) = (4, 3)$, may be dealt with by studying the finite subgroups of the braid groups of $\mathbb{S}^2$ as follows. Let $\Phi\colon\ B_4(\mathbb{S}^2) \longrightarrow B_3(\mathbb{S}^2)$ be a homomorphism, and using the notation of Theorem 2·2 in $B_4(\mathbb{S}^2)$, let $\alpha_0 = \sigma_1\sigma_2\sigma_3$ and $\alpha_1 = \sigma_1\sigma_2\sigma_3^2$, and let $\Delta_4 = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$ be the half-twist braid. By [**29**, theorem 3], $B_4(\mathbb{S}^2) = \langle \alpha_0, \alpha_1 \rangle$. Now α_0 is of order 8, and the maximal torsion of $B_3(\mathbb{S}^2)$ is equal to 6, so the order of $\Phi(\alpha_0)$ is a divisor of 4 [**25**, **47**]. But the full-twist braid Δ_4^2 is the unique element of $B_4(\mathbb{S}^2)$ of order 2 [**25**]. This implies that Δ_4^2 belongs to the centre of $B_4(\mathbb{S}^2)$, and also that $\alpha_0^4 = \Delta_4^2$, from which we conclude that Δ_4^2 belongs to $\mathrm{Ker}\,(\Phi)$. Let $H = \langle \alpha_0, \Delta_4 \rangle$. By [**30**, remark, p· 234], H is isomorphic to the generalised quaternion group $\mathcal{Q}_{16}$ of order 16, where the relations are of the form $\alpha_0^4 = \Delta_4^2$ and $\Delta_4\alpha_0\Delta_4^{-1} = \alpha_0^{-1}$. Consider the restriction $\Phi|_H\colon\ H \longrightarrow \mathrm{Im}\,(\Phi|_H)$. Since Δ_4^2 belongs to $H \cap \mathrm{Ker}\,(\Phi|_H)$ and to the centre of $B_4(\mathbb{S}^2)$, we see that $\Phi|_H$ factors through the quotient $H/\langle \Delta_4^2 \rangle$. Using the relations of H in terms of its generators, this quotient is isomorphic to the dihedral group of order 8, and hence $\mathrm{Im}\,(\Phi|_H)$ is a subgroup of $B_3(\mathbb{S}^2)$ that is a quotient of this dihedral group. On the other hand, the quotients of dihedral groups are either dihedral, the trivial group, or cyclic of order 2. Further, $B_3(\mathbb{S}^2) \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_4$, the action being the non-trivial one [**21**], so $B_3(\mathbb{S}^2)$ has no dihedral subgroups. We conclude that $\mathrm{Im}\,(\Phi|_H) \subset \langle \Delta_3^2 \rangle$. Hence $\mathrm{Ker}\,(\Phi|_H)$ is either equal to H, or is a subgroup of H of index 2. If $\mathrm{Ker}\,(\Phi|_H)$ is of index 2 in H, then by analysing the images of α_0 and Δ_4 by a surjective homomorphism from H to $\mathbb{Z}_2$, we see that $\mathrm{Ker}\,(\Phi|_H)$ is equal to $\langle \alpha_0 \rangle$, to $\langle \alpha_0^2, \Delta_4 \rangle$, or to $\langle \alpha_0^2, \alpha_0\Delta_4 \rangle$. So if either $\mathrm{Ker}\,(\Phi|_H)$ is equal to H, or is a subgroup of H of index 2, we conclude from these possibilities that $\alpha_0^2 \in \mathrm{Ker}\,(\Phi)$. It follows again from the fact that Δ_3^2 is the unique element of $B_3(\mathbb{S}^2)$ of order 2 that $\Phi(\alpha_0) \in \langle \Delta_3^2 \rangle$, and so is central in $B_3(\mathbb{S}^2)$. Since $B_4(\mathbb{S}^2) = \langle \alpha_0, \alpha_1 \rangle$, we conclude that $\mathrm{Im}\,(\Phi)$ is cyclic, and hence Φ cannot be surjective.

Remark 2·14. It follows from Theorem 1·1(i)(a) that there is no surjective homomorphism from $B_4(\mathbb{S}^2)$ to $B_3(\mathbb{S}^2)$. However, the maps from $B_4(\mathbb{S}^2)$ to S_3 defined by sending the generators σ_1 and σ_3 to $(1, 2)$ and σ_2 to $(2, 3)$ and from B_4 to B_3 defined by sending the generators σ_1 and σ_3 to σ_1 and σ_2 to σ_2, extend to well-defined, surjective homomorphisms.

3. *Surjections between braid groups of orientable surfaces with boundary*

Let $\Sigma_{g,b}$ be a compact, connected orientable surface of genus g with $b \geq 0$ boundary components. A presentation for $B_n(\Sigma_{g,b})$ may be found in [**13**, proposition 3·1], and in the case $b = 1$, a presentation for $B_n(\Sigma_{g,1})$ may be obtained from that of $B_n(\Sigma_g)$ given in Theorem 2·2 by deleting relation (2·8). The case $b = 0$ was dealt with in Section 2, so we shall assume henceforth that $b \geq 1$. The following two results generalise those of Theorem 2·3 to the braid groups of $\Sigma_{g,b}$.

THEOREM 3·1 ([**12**, Theorem 2]). *Let* $g, b \geq 1$, *and let* $n \geq 3$. *Then*:

(i) $\Gamma_1(B_n(\Sigma_{g,b}))/\Gamma_2(B_n(\Sigma_{g,b})) \cong \mathbb{Z}^{2g+b-1} \oplus \mathbb{Z}_2$;

(ii) $\Gamma_2(B_n(\Sigma_{g,b}))/\Gamma_3(B_n(\Sigma_{g,b})) \cong \mathbb{Z}$;

(iii) $\Gamma_3(B_n(\Sigma_{g,b})) = \Gamma_4(B_n(\Sigma_{g,b}))$. *Moreover* $\Gamma_3(B_n(\Sigma_{g,b}))$ *is perfect for* $n \geq 5$;

(iv) $B_n(\Sigma_{g,b})$ *is not residually nilpotent.*

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The following proposition treats the case $n = 2$.

PROPOSITION 3·2. *Let* $g, b \geq 1$.

(i) *The group* $B_2(\Sigma_{g,b})$ *is residually* 2*-finite and therefore residually nilpotent, but is not nilpotent.*
(ii) $\Gamma_1(B_2(\Sigma_{g,b}))/\Gamma_2(B_2(\Sigma_{g,b})) \cong \mathbb{Z}^{2g+b-1} \oplus \mathbb{Z}_2$.
(iii) *The group* $\Gamma_2(B_2(\Sigma_{g,b}))/\Gamma_3(B_2(\Sigma_{g,b}))$ *is a non-trivial quotient of* $\mathbb{Z}_2^{2g+b-1} \oplus \mathbb{Z}$.

Presentations for $B_n(\Sigma_{g,b})/\Gamma_3(B_n(\Sigma_{g,b}))$ were exhibited in [**12**, eq· (10)] for $b = 1$, and in [**13**, proposition 3·13] for $b \geq 1$.

Proof of Proposition 3·2*.* Let $n = 2$, and consider the short exact sequence (2·1), where we take $\Sigma = \Sigma_{g,b}$. Since $S_2 \cong \mathbb{Z}_2$ and $P_2(\Sigma_{g,b})$ is residually torsion free nilpotent [**12**, theorem 4], and therefore 2-finite, the hypotheses of [**36**, lemma 1·5] are fulfilled, so $B_2(\Sigma_{g,b})$ is residually nilpotent. To see that it is not nilpotent, suppose on the contrary that there exists $i \in \mathbb{N}$ such that $\Gamma_i(B_2(\Sigma_{g,b})) = \{1\}$. Without loss of generality, we may suppose that i is minimal with respect to this property. Since $B_2(\Sigma_{g,b})$ is non Abelian, it follows from Theorem 3·1(i) that $i \geq 3$. Now $\Gamma_i(B_2(\Sigma_{g,b})) = [\Gamma_{i-1}(B_2(\Sigma_{g,b})), B_2(\Sigma_{g,b})] = \{1\}$, and hence $\Gamma_{i-1}(B_2(\Sigma_{g,b}))$ is contained in the centre of $B_2(\Sigma_{g,b})$. This centre is trivial [**27**, **48**], so $\Gamma_{i-1}(B_2(\Sigma_{g,b})) = \{1\}$, but this contradicts the minimality of i. Part (i) follows.

For part (ii), we just give the proof in the case $b = 1$. The general case may be obtained in a similar manner using the presentation of $B_2(\Sigma_{g,b})$ given in [**13**]. As we mentioned above, a presentation of $B_2(\Sigma_{g,1})$ may be obtained by deleting relation (2·8) from the presentation of Theorem 2·2. Thus the proof given in Proposition 2·7 for Σ_g is also valid in the case of $\Sigma_{g,b}$, except that we can no longer conclude that σ_1^2 is of finite order, so the second factor in the direct product decomposition of $B_2(\Sigma_{g,1})/\Gamma_3(B_2(\Sigma_{g,1}))$ is $\mathbb{Z}$. Part (iii) is a consequence of part (i).

Remark 3·3. Theorem 3·1 (in the case $n \geq 3$) and Proposition 3·2 (in the case $n = 2$) generalise Theorem 2·3. If $n = 1$, $B_1(\Sigma_{g,b})$ is a free group of rank $2g + b - 1$, and its lower central series is well known, see [**42**] for instance. Note that in particular $B_1(\Sigma_{g,b})$ is residually nilpotent. It follows from Theorem 3·1(iv) and Proposition 3·2(i) that $B_n(\Sigma_{g,b})$ is residually nilpotent if and only if $n \leq 2$. As in the proof of Proposition 2·8, we see that $\Gamma_3(B_n(\Sigma_{g,b}))$ is not perfect if $n \in \{3, 4\}$. Hence using Theorem 3·1(iii), $\Gamma_3(B_n(\Sigma_{g,b}))$ is perfect if and only if $n \geq 5$.

We now prove Theorem 1·1(ii) in the orientable case and Corollary 3·5. We first require the following result.

LEMMA 3·4. *There is no surjective homomorphism from* $B_2(\Sigma_{1,1})$ *onto* $\pi_1(\Sigma_{1,1})$.

Proof. To prove the result, suppose on the contrary that there exists a surjective homomorphism $\varphi\colon\ B_2(\Sigma_{1,1}) \longrightarrow \pi_1(\Sigma_{1,1})$. Let $\alpha = a_1\sigma_1$, $\beta = b_1\sigma_1$. Then α, β, σ_1 generate $B_2(\Sigma_{1,1})$, and the defining relations of Theorem 2·2 become:

$$\alpha^2 = \sigma_1\alpha^2\sigma_1^{-1} \tag{3·1}$$

$$\beta^2 = \sigma_1\beta^2\sigma_1^{-1} \tag{3·2}$$

$$\alpha\beta\sigma_1^{-1} = \sigma_1\beta\alpha. \tag{3·3}$$

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Now $\pi_1(\Sigma_{1,1})$ is a free group of rank 2, and so if $u, v \in \pi_1(\Sigma_{1,1})$, the relation $u^2 = v^2$ implies that $u = v$. Applying this to relations (3·1) and (3·2), we deduce that $\varphi(\sigma_1)$ is central in $\pi_1(\Sigma_{1,1})$, and so $\varphi(\sigma_1) = 1$. Since φ is surjective, it follows that $\pi_1(\Sigma_{1,1}) = \langle \varphi(\alpha), \varphi(\beta) \rangle$. Relation (3·3) implies that $\varphi(\alpha)$ and $\varphi(\beta)$ commute. Consequently, $\langle \varphi(\alpha), \varphi(\beta) \rangle$ is cyclic, and this contradicts the assumption that φ is surjective.

In contrast with the case of Σ_g, Theorem 3·1 implies that the lower central series does not distinguish the number of strings for braid groups of orientable surfaces with boundary if $n \geq 3$. Nevertheless, we are able to show that in certain cases, there does not exist a surjective homomorphism between $B_n(\Sigma_{g,b})$ and $B_m(\Sigma_{g,b})$.

Proof of Theorem 1·1(ii) in the orientable case. We consider in turn the three cases given in the statement.

- (i) Let $n < m$ and $n \in \{1, 2\}$. The arguments used in Proposition 2·9 apply verbatim to the case with boundary. In particular $G(B_1(\Sigma_{g,b})) = 2g + b - 1$, $G(B_2(\Sigma_{g,b})) = 2g + b$, and $G(B_m(\Sigma_{g,b})) = 2g + b + 1$ for all $m \geq 3$. It follows that there does not exist a surjective homomorphism in this case.
- (ii) Suppose that $n > m$ and $m \in \{1, 2\}$. First let $n \geq 3$. Then $B_m(\Sigma_{g,b})$ is residually nilpotent by Proposition 3·2 and Remark 3·3. Since $B_n(\Sigma_{g,b})$ is not residually nilpotent by Theorem 3·1(iv), it cannot surject homomorphically onto $B_m(\Sigma_{g,b})$. So suppose that $n = 2$ and $m = 1$. The case $(g, b) = (1, 1)$ was dealt with in Lemma 3·4, so we may assume that $(g, b) \neq (1, 1)$, in which case $2g + b \geq 4$. By Proposition 3·2(iii), the Abelian group $\Gamma_2(B_2(\Sigma_{g,b}))/\Gamma_3(B_2(\Sigma_{g,b}))$ is of rank at most 1. On the other hand, $\Gamma_2(\pi_1(\Sigma_{g,b}))/\Gamma_3(\pi_1(\Sigma_{g,b}))$ is free Abelian of rank $(2g + b - 1)(2g + b - 2)/2$ [**42**], and this rank is strictly greater than 1. Thus $B_2(\Sigma_{g,b})$ cannot surject homomorphically onto $B_1(\Sigma_{g,b})$.
- (iii) Suppose that $n > m \geq 3$ and $n \neq 4$. Using the presentation of $B_m(\Sigma_{g,b})$ given in [**13**, proposition 3·1], the proof of Theorem 1·5 goes through in this case, the only difference being that $B_n(\Sigma_{g,b})/\langle\langle \sigma_1 \rangle\rangle$ is isomorphic to $\mathbb{Z}^{2g+b-1}$. The result then follows by an argument similar to that given in case (iii) of the proof of Theorem 1·1(i)(b) in Section 2·2.

The following result is the analogue of Corollary 2·11 in the case where the surface has boundary.

COROLLARY 3·5. *Let $g \geq 1$, and let $n, m \in \mathbb{N}$. Then there exists a surjective homomorphism of $B_n(\Sigma_{g,b})$ onto $P_m(\Sigma_{g,b})$ if and only if $n = m = 1$.*

Proof. Let $g \geq 1$. The proof is similar to that of Corollary 2·11. If $n = m = 1$, the result is clear, so suppose that there exists a surjective homomorphism $\Phi\colon B_n(\Sigma_{g,b}) \longrightarrow P_m(\Sigma_{g,b})$, where $(n, m) \neq (1, 1)$. Then Φ induces a surjective homomorphism of the corresponding Abelianisations, but since $(P_m(\Sigma_{g,b}))_{\text{Ab}}$ is isomorphic to $\mathbb{Z}^{(2g+b-1)m}$ using a presentation of $P_m(\Sigma_{g,b})$ (see [**9**] for instance), it follows from Theorem 3·1(i), Proposition 3·2(ii) and the fact that $(B_1(\Sigma_{g,b}))_{\text{Ab}}$ is isomorphic to $\mathbb{Z}^{2g+b-1}$ that $m = 1$. By Theorem 1·1(ii)(b), we conclude that $n = 1$.

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4. *Surjections between braid groups of non-orientable surfaces*

We start this section by recalling a presentation of the braid groups of compact, non-orientable surfaces without boundary.

THEOREM 4·1 ([9]). *Let $g \geq 1$, let $n \geq 2$, and let U_g be a compact, connected non-orientable surface without boundary of genus g. Then $B_n(U_g)$ admits the following group presentation*:

(i) ***generators***: $\rho_1, \ldots, \rho_g, \sigma_1, \ldots, \sigma_{n-1}$;
(ii) ***relations***:

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2 \tag{4·1}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for all } 1 \leq i \leq n-2 \tag{4·2}$$

$$\rho_i\sigma_j = \sigma_j\rho_i \text{ for all } j \geq 2 \text{ and } i = 1, \ldots, g \tag{4·3}$$

$$\rho_i\sigma_1\rho_i\sigma_1 = \sigma_1^{-1}\rho_i\sigma_1\rho_i \text{ for } i = 1, \ldots, g \tag{4·4}$$

$$\rho_r\sigma_1^{-1}\rho_s\sigma_1 = \sigma_1^{-1}\rho_s\sigma_1\rho_r \text{ for } 1 \leq s < r \leq g \tag{4·5}$$

$$\prod_{i=1}^{g} \rho_i^{-2} = \sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \cdots \sigma_1. \tag{4·6}$$

The presentation of $B_n(U_g)$ of [9, theorem A·3] is slightly different from that given in Theorem 4·1, but one can obtain the first presentation from the second by replacing each generator a_i in [9, theorem A·3] by ρ_i^{-1} in Theorem 4·1 for all $i = 1, \ldots, g$.

Remark 4·2. Notice that [9, theorem A·3] was stated for $g > 1$, but the presentation is also valid if $g = 1$, in which case the relation (4·5) does not exist. This may be seen by showing that the map from $B_n(U_1)$ to itself that sends the generator σ_i (resp. ρ_1) of [9, theorem A·3] to the generator σ_i (resp. ρ_1^{-1}) of [51] for all $1 \leq i \leq n-1$ is well defined, and that it is an isomorphism. The presentation also holds if $n = 1$. In particular, $B_1(U_g)$ is a one-relator group, and the results of [42] apply.

The following theorem summarises some of the known results about the lower central series of braid groups of non-orientable surfaces without boundary [33, 37, 46], and is the analogue of Theorems 2·3 and 3·1. One may consult [11] for the case of pure braid groups.

THEOREM 4·3 ([33, 37]). *Let $g \geq 1$. Then*:

(i) $\Gamma_1(B_n(U_g))/\Gamma_2(B_n(U_g)) = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ *for all* $n \geq 2$;
(ii) $\Gamma_2(B_n(U_g)) = \Gamma_3(B_n(U_g))$ *for all* $n \geq 3$;
(iii) $\Gamma_2(B_n(U_g))$ *is perfect if and only if* $n \geq 5$;
(iv) $B_n(U_g)$ *is residually nilpotent if and only if* $n \leq 2$.

Proof. If $g = 1$, the four statements were proved in [33, theorem 1 and proposition 6]. So in the rest of the proof, we assume that $g \geq 2$.

(i) The statement follows in a straightforward manner using the presentation of Theorem 4·1.
(ii) If $n \geq 3$, the fact that $\Gamma_2(B_n(U_g)) = \Gamma_3(B_n(U_g))$ is a consequence of the proof of [37, proposition 5·21] (resp. of [37, theorem 6·1]) if $g = 2$ (resp. if $g \geq 3$).

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(iii) The 'if' part is a consequence of [**37**, theorem 1·4]. The proof of the 'only if' part is similar to that of Proposition 2·8, and is left to the reader.

(iv) This follows from [**37**, theorem 1·4].

Proof of Theorem 1·5 *in the case where* $\Sigma = U_g$*, and* $g \geq 1$. Let $n > m \geq 3$, where $n \neq 4$. Let $\Phi\colon\ B_n(U_g) \longrightarrow S_m$. As in the proof of the orientable case, we see that $\Phi(\sigma_1) = \cdots = \Phi(\sigma_{n-1})$. We denote this common element by σ. From relations (4·3), σ commutes with $\Phi(\rho_j)$ for all $j = 1, \ldots, g$, so σ belongs to the centre of S_m, and we conclude once more that σ is trivial, and hence the homomorphism Φ factors through a surjective homomorphism $\Phi'\colon\ B_n(U_g)/\langle\!\langle \sigma_1 \rangle\!\rangle \longrightarrow S_m$, where $\langle\!\langle \sigma_1 \rangle\!\rangle$ denotes the normal closure of σ_1 in $B_n(U_g)$. But $B_n(U_g)/\langle\!\langle \sigma_1 \rangle\!\rangle$ is Abelian by relations (4·5), which yields a contradiction because an Abelian group cannot surject homomorphically onto a non-Abelian group. So there is no surjective homomorphism from $B_n(U_g)$ onto S_m.

As in the case of orientable surfaces, we may obtain more information about the lower central series of $B_2(U_g)$.

PROPOSITION 4·4. *Let* $g \geq 1$. *Then the group* $B_2(U_g)$ *is residually 2-finite, and so is residually nilpotent. Moreover, the group* $\Gamma_2(B_2(U_g))/\Gamma_3(B_2(U_g))$ *is a non-trivial quotient of* $\mathbb{Z}_2^g$.

Proof. The case $g = 1$ is straightforward because $B_2(\mathbb{R}P^2)$ is isomorphic to the generalised quaternion group of order 16 [**51**, theorem, p· 94]. In particular, the quotient $\Gamma_2(B_2(\mathbb{R}P^2))/\Gamma_3(B_2(\mathbb{R}P^2))$ is isomorphic to $\mathbb{Z}_2$. If $g \geq 2$, the residual nilpotence of $B_2(U_g)$ follows by arguing as in the proof of Proposition 3·2(i), using the fact that $P_2(U_g)$ is residually 2-finite [**11**, theorem 1·1]. Note that $B_2(U_2)$ is not nilpotent, since otherwise $P_2(U_2)$ would be nilpotent, but we know from [**37**, theorem 5·4] that this is not the case. If $g \geq 3$, the centre of the group $B_2(U_g)$ is trivial [**48**, proposition 1·6], and as in Remark 2·6, we can prove that the group $\Gamma_2(B_2(U_g))/\Gamma_3(B_2(U_g))$ is non trivial. To see that this group is a quotient of $\mathbb{Z}_2^g$, observe that for all $1 \leq s < r \leq g$, we have $[\rho_r, \sigma_1^{-1}\rho_s\sigma_1] = 1$ in $B_2(U_g)$ by relation (4·5), so $[\rho_r, \rho_s] = 1$ in $B_2(U_g)/\Gamma_3(B_2(U_g))$. Thus $\Gamma_2(B_2(U_g))/\Gamma_3(B_2(U_g))$ is generated by the commutators of the form $[\rho_i, \sigma_1]$, where $1 \leq i \leq g$. Since $\sigma_1^2 = 1$ in $B_2(U_g)/\Gamma_3(B_2(U_g))$ by relation (4·4), these commutators are of order at most 2.

At this point, we may prove Proposition 1·3 concerning the existence of a surjective homomorphism between $B_n(U_g)$ onto $B_n(\Sigma_{g-1})$.

Proof of Proposition 1·3. First suppose that $g = 1$, in which case $U_1 = \mathbb{R}P^2$ and $\Sigma_0 = \mathbb{S}^2$. If $n = 1$, $B_1(\mathbb{S}^2)$ is trivial, and so there is clearly a surjection of $B_1(\mathbb{R}P^2)$ onto $B_1(\mathbb{S}^2)$, and if $n = 2$ then $B_2(\mathbb{R}P^2)$ is isomorphic to the generalised quaternion group of order 16 [**51**, theorem, p· 94], and $B_2(\mathbb{S}^2) \cong \mathbb{Z}_2$ [**21**, third theorem, p· 255], and $B_2(\mathbb{R}P^2)$ surjects homomorphically onto $B_2(\mathbb{S}^2)$. Finally, if $n \geq 3$, $(B_n(\mathbb{R}P^2))_{\mathrm{Ab}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ by Theorem 4·3(i), and $(B_n(S^2))_{\mathrm{Ab}} \cong \mathbb{Z}_{2(n-1)}$ by Theorem 2·13(i), which implies that there is no surjective homomorphism from $B_n(\mathbb{R}P^2)$ onto $B_n(\mathbb{S}^2)$. This proves the result in the case $g = 1$.

Now assume that $g \geq 2$. If $n \geq 2$, $(B_n(U_g))_{\mathrm{Ab}} \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ by Theorem 4·3(i) and $(B_n(\Sigma_{g-1}))_{\mathrm{Ab}} = \mathbb{Z}^{2(g-1)} \oplus \mathbb{Z}_2$ by Theorem 2·3(i) while if $n = 1$, $(B_1(U_g))_{\mathrm{Ab}} \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$ and $(B_n(\Sigma_{g-1}))_{\mathrm{Ab}} = \mathbb{Z}^{2(g-1)}$. Therefore it is not possible to surject $B_n(U_g)$ onto $B_n(\Sigma_{g-1})$ in this case.

To prove Theorem 1·1 in the non-orientable case, we will require the following lemma for the Klein bottle.

LEMMA 4·5. *Let $\mathbb{K}^2$ be the Klein bottle. If x and y are elements of $\pi_1(\mathbb{K}^2)$, then $xyxy = y^{-1}xyx$ if and only if $y = 1$.*

Proof. If $y = 1$ then the relation clearly holds. Conversely, suppose that there exist $x, y \in \pi_1(\mathbb{K}^2)$ that satisfy the relation. Recall that $\pi_1(\mathbb{K}^2)$ is isomorphic to the semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}$, where the action is given by multiplication by -1. With respect to this decomposition, let $x = (a, b)$ and $y = (c, d)$. Substituting these elements into the given relation, the second coordinate yields $2b + 2d = 2b$, so $d = 0$, and computing the first coordinate, we obtain $a + (-1)^b c + (-1)^b (a + (-1)^b c) = -c + a + (-1)^b (c + a)$. Therefore $-c = c$, so $c = 0$ and hence y is the trivial element of $\pi_1(\mathbb{K}^2)$.

We now prove Theorem 1·1(ii) in the non-orientable case, where $g > 1$.

Proof of Theorem 1·1(ii), where $\Sigma = U_g$, and $g > 1$. We study the three cases of the statement of Theorem 1·1(ii) separately.

(i) Let $n < m$ and $n \in \{1, 2\}$. Using Theorem 4·3 and the fact that the quotient $\Gamma_1(B_1(U_g))/\Gamma_2(B_1(U_g))$ is isomorphic to $\mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$, the arguments used in the proof of Proposition 2·9 also apply to the non-orientable case. In particular, $G(B_1(U_g)) = g$, $G(B_2(U_g)) = g + 1$ and $G(B_m(U_g)) = g + 2$ for all $m \geq 3$. It follows that there is no surjective homomorphism in this case.

(ii) Suppose that $n > m$ and $m \in \{1, 2\}$. If $n \geq 3$, we have $\Gamma_2(B_n(U_g)) = \Gamma_3(B_n(U_g))$ by Theorem 4·3(ii). On the other hand, $B_m(U_g)$ is residually nilpotent if $m = 1$ (and is in fact residually 2-finite, see for instance [**11**, proof of theorem 4·5]), or if $m = 2$ by Proposition 3·2. So $B_n(U_g)$ cannot surject homomorphically onto $B_m(U_g)$ if $n \geq 3$ and $m \in \{1, 2\}$. So assume that $n = 2$ and $m = 1$. If $g > 2$, the result follows in a similar manner by noting that $\Gamma_2(B_2(U_g))/\Gamma_3(B_2(U_g))$ is finite by Proposition 4·4, but that $\Gamma_2(\pi_1(U_g))/\Gamma_3(\pi_1(U_g))$ is infinite [**42**]. So suppose that $g = 2$, and assume that there exists a surjective homomorphism $\Phi\colon B_2(U_2) \longrightarrow \pi_1(U_2)$. Applying Φ to relation (4·4) with $i = 1$, we have that $\Phi(\rho_1)\Phi(\sigma_1)\Phi(\rho_1)\Phi(\sigma_1) = \Phi(\sigma_1)^{-1}\Phi(\rho_1)\Phi(\sigma_1)\Phi(\rho_1)$. The relation given in the statement of Lemma 4·5 is therefore satisfied if we take $x = \Phi(\rho_1)$ and $y = \Phi(\sigma_1)$, and thus $\Phi(\sigma_1) = 1$. It follows from relation (4·5) that $\Phi(\rho_1)$ and $\Phi(\rho_2)$ commute. We conclude that the image of Φ is an Abelian subgroup of $\pi_1(U_2)$, and since this latter group is non Abelian, Φ cannot be surjective.

(iii) If $n > m \geq 3$ and $n \neq 4$, it suffices to argue as in the proof of part (iii) of Theorem 1·1(i)(b) given in Section 2·2, and apply Theorem 1·5 in the non-orientable case.

The following theorem provides some results in the case where $g = 1$.

THEOREM 4·6.

(i) *Suppose that one of the following conditions holds*:
   (a) *$n < m$ and $n \in \{1, 2\}$*;
   (b) *$n > m \geq 2$.*

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Then there is no surjective homomorphism from $B_n(\mathbb{R}P^2)$ *to* $B_m(\mathbb{R}P^2)$.

(ii) *Let* $m, n \geq 2$, *let* $n' = 2\lfloor n/2 \rfloor$ *and let* $m' = 2\lfloor m/2 \rfloor$, *where* $\lfloor x \rfloor$ *denotes the greatest integer less than or equal to* x. *Set* $n' = 2^l s$ *and* $m' = 2^k r$, *where* $l, k \in \mathbb{N}$, *and* s, r *are odd integers. If* $l > k$ *then the image of any homomorphism* $f\colon B_n(\mathbb{R}P^2) \longrightarrow B_m(\mathbb{R}P^2)$ *is finite cyclic. In particular, there is no surjective homomorphism from* $B_n(\mathbb{R}P^2)$ *to* $B_m(\mathbb{R}P^2)$ *in this case.*

Remarks 4·7.

(i) If $n \geq 2$ then $B_n(\mathbb{R}P^2)_{\text{Ab}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ by Theorem 4·3(i), and since $B_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$, we see that $B_n(\mathbb{R}P^2)$ surjects homomorphically onto $B_1(\mathbb{R}P^2)$ via Abelianisation.

(ii) Comparing the statement of Theorem 1·1(ii) with that of Theorem 4·6(i), if $n > 2$, then by the previous remark, there is a surjective homomorphism from $B_n(\Sigma)$ to $B_2(\Sigma)$ if $\Sigma = \mathbb{R}P^2$, which is not the case if $\Sigma = \Sigma_{g,b}$ or U_{g+1}, where $b, g \geq 1$. Further, if $n = 4$ and $m = 3$, we do not know whether there exists a surjective homomorphism from $B_4(\Sigma)$ to $B_3(\Sigma)$ if $\Sigma = \Sigma_{g,b}$ or U_{g+1}, where $b, g \geq 1$, but Theorem 4·6(i)(b) shows that there does not exist such a homomorphism if $\Sigma = \mathbb{R}P^2$.

Proof of Theorem 4·6.

(i) (a) If $n < m$ and $n \in \{1, 2\}$, the conclusion follows from the fact that $B_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$, $B_2(\mathbb{R}P^2)$ is isomorphic to the dicyclic group of order 16, and if $m \geq 3$, $B_m(\mathbb{R}P^2)$ is infinite [**51**, theorem, p·94].

(b) Assume that $n > m \geq 2$. If $m = 2$, the result is a consequence of the fact that $B_2(\mathbb{R}P^2)$ is residually nilpotent, while $B_n(\mathbb{R}P^2)$ is not for all $n \geq 3$ by Theorem 4·3. Now suppose that $m \geq 3$. If $n \neq 4$, the result follows as in the proof of part (iii) of Theorem 1·1(ii) by applying Theorem 1·5 in the non-orientable case. We defer the proof of the case $n = 4$ and $m = 3$ to part (ii).

(ii) Let $m, n \geq 2$, let $n' = 2\lfloor n/2 \rfloor$ and let $m' = 2\lfloor m/2 \rfloor$, and let $n' = 2^l s$ and $m' = 2^k r$, where $l, k \in \mathbb{N}$, and s, r are odd integers. Let $\varphi\colon B_n(\mathbb{R}P^2) \longrightarrow B_m(\mathbb{R}P^2)$ be a homomorphism. Consider the elements $a = \rho_n \sigma_{n-1} \cdots \sigma_1$ and $b = \rho_{n-1}\sigma_{n-2} \cdots \sigma_1$ of $B_n(\mathbb{R}P^2)$, where we use Van Buskirk's presentation of $B_n(\mathbb{R}P^2)$ [**51**, p· 83]. By [**28**, proposition 26], a (resp. b) is of order $4n$ (resp. $4(n-1)$). Let $x = a$ and $x' = b$ (resp. $x = b$ and $x' = a$) if n is even (resp. is odd). Then x is of order $4n'$, which in terms of the notation introduced in the statement, is equal to $2^{l+2}s$. Observe also that from the proof of [**35**, theorem 6], $B_n(\mathbb{R}P^2) = \langle x, x' \rangle$. By [**28**, theorem 4], the (maximal) torsion of $B_n(\mathbb{R}P^2)$ (resp. of $B_m(\mathbb{R}P^2)$) is $4n$ and $4(n-1)$ (resp. $4m$ and $4(m-1)$), and so the maximal torsion in $B_n(\mathbb{R}P^2)$ that is a power of 2 is equal to 2^{l+2} in $B_n(\mathbb{R}P^2)$, and is realised by x^s, and the maximal torsion in $B_m(\mathbb{R}P^2)$ that is a power of 2 is equal to 2^{k+2}. It follows that the order of $f(x^s)$ is a divisor of 2^{k+2}, in particular $f(x^{2^{k+2}s}) = 1$ in $B_m(\mathbb{R}P^2)$. Now $l \geq k+1$ by hypothesis, and so $1 = (f(x^{2^{k+2}s}))^{2^{l-k-1}} = f(x^{2^{l+1}s})$. Since x is of order $2^{l+2}s$, $x^{2^{l+1}s}$ is of order 2, so is equal to the full twist braid Δ_n^2 of $B_n(\mathbb{R}P^2)$, using the fact that Δ_n^2 is the unique element of $B_n(\mathbb{R}P^2)$ of order 2 [**28**, proposition 23]. We conclude that $\Delta_n^2 \in \text{Ker}\,(f)$. Now let $H = \langle x, y \rangle$, where $y = \Delta_n$ (resp. $y = \Delta_n a^{-1}$) if n is even (resp. n is odd). By [**34**, proposition 15], H is isomorphic to the dicyclic group $\text{Dic}_{4n'}$ of order $4n'$, and the generators satisfy the relations $x^{n'} = y^2$ and $yxy^{-1} = x^{-1}$. Using once more the fact that Δ_n^2 is the unique element of $B_n(\mathbb{R}P^2)$ of order 2, we have $\Delta_n^2 \in \text{Ker}\,(\varphi) \cap H$.

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Further, Δ_n^2 is central in $B_n(\mathbb{R}P^2)$ [**47**, proposition 6·1], and hence the restriction $f|_H : H \longrightarrow f(H)$ of f to H factors through the quotient $H/\langle \Delta_n^2 \rangle$. But using the relations of H, this quotient is isomorphic to the dihedral group of order $2n'$, so $f(H)$ is a subgroup of $B_m(\mathbb{R}P^2)$ that is a quotient of $H/\langle \Delta_n^2 \rangle$. Now the quotients of dihedral groups are either the trivial group, cyclic of order 2 or dihedral, and since the braid groups of $\mathbb{R}P^2$ do not have dihedral subgroups [**34**, theorem 5], it follows that $f(H)$ is either trivial or cyclic of order 2, so $\text{Ker}\,(f|_H)$ is either equal to H, or is a subgroup of H of index 2. If $\text{Ker}\,(f|_H)$ is of index 2 in H, then by analysing the images of x and y by a surjective homomorphism from H to $\mathbb{Z}_2$, we see that either $\text{Ker}\,(f|_H) = \langle x \rangle$, or if n' is even, additionally $\text{Ker}\,(f|_H) = \langle x^2, y \rangle$, or $\text{Ker}\,(f|_H) = \langle x^2, xy \rangle$. So if either $\text{Ker}\,(f|_H)$ is equal to H, or is an subgroup of H of index 2, we conclude from these possibilities that $x^2 \in \text{Ker}\,(f)$. It follows again from the fact that Δ_m^2 is the unique element of $B_m(\mathbb{R}P^2)$ of order 2 that $f(x) \in \langle \Delta_m^2 \rangle$. Since $f(x')$ is of finite order and $f(x)$ is central in $B_m(\mathbb{R}P^2)$, using the fact mentioned above in the first paragraph that $B_n(\mathbb{R}P^2) = \langle x, x' \rangle$, we see that the image of f is finite cyclic as required. In particular, in the outstanding case of the proof of part (i)(b), where $n = 4$ and $m = 3$, there is no surjective homomorphism from $B_4(\mathbb{R}P^2)$ to $B_3(\mathbb{R}P^2)$.

COROLLARY 4·8. *Let $g \geq 1$, and let $m, n \in \mathbb{N}$. Then there exists a surjective homomorphism of $B_n(U_g)$ onto $P_m(U_g)$ if and only if either $g = m = 1$ or $n = m = 1$.*

Proof. If $n = m = 1$ and $g \geq 1$, the result is clear, and if $g = m = 1$, the result follows from Remarks 4·7(i). Conversely, suppose that there exists a surjective homomorphism $\Phi\colon B_n(U_g) \longrightarrow P_m(U_g)$. Then Φ induces a surjective homomorphism of the corresponding Abelianisations, but since $(P_m(U_g))_{\text{Ab}} \cong \mathbb{Z}^{(g-1)m} \oplus \mathbb{Z}_2^m$ using a presentation of $P_m(U_g)$ (see [**32**, theorem 3] for instance), it follows from Theorem 4·3(i) and the fact that $(B_1(U_g))_{\text{Ab}} \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$ that $m = 1$. So either $n = 1$ or $g = 1$, and thus the conclusion holds, or else $n > 1$ and $g > 1$, in which case we obtain a contradiction using Theorem 1·1(ii)(b).

5. *Surjections between braid groups of orientable surfaces and symmetric groups*

In this section, we start by recalling Theorem 5·1 due to Ivanov [**39**], concerning transitive representations of B_n and S_m, where $n > m \geq 2$ (the definitions of primitive and transitive representations were given in Section 1). We then prove Theorem 1·6 that generalises Theorem 5·1 to braid groups of compact, orientable surfaces. We shall assume that the surfaces are without boundary, but the results extend easily to the case with boundary. In [**39**], Ivanov gave some transitive, imprimitive representations of $B_n(\Sigma_g)$ in S_n, where $g \geq 1$ and $n \geq 3$. These representations have the property that their images are Abelian subgroups of S_n. We shall construct some transitive, imprimitive representations of $B_n(\Sigma_g)$ in S_m whose images are non Abelian, so they are different from those of Ivanov.

The following result is a variant of Theorem 1·4 for transitive representations.

THEOREM 5·1 ([**39**, lemma 3]). *Let $n > m \geq 2$, and let $\rho\colon B_n \longrightarrow S_m$ be a transitive representation. Then one of the following statements holds*:

(i) *$\rho(\sigma_1) = \cdots = \rho(\sigma_{n-1})$, and this permutation is an m-cycle*;

(ii) *if $n = 4$ and $m = 3$, up to a suitable renumbering of the elements of the set* $\{1, 2, 3\}$, *$\rho(\sigma_1) = \rho(\sigma_3) = (1, 2)$ and $\rho(\sigma_2) = (2, 3)$.*

We can give an alternative proof of Theorem 5·1 using Theorem 1·4.

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Proof of Theorem 5·1. Suppose that $n > m \geq 2$. If $m = 2$ then Im (ρ) is either trivial, which contradicts the transitivity hypothesis, or is equal to S_2, and statement (i) holds. So suppose that $n > m \geq 3$, and assume that $n \neq 4$. Arguing as in the first part of the proof of Theorem 1·5 in Section 2·2, it follows that $\rho(\sigma_1) = \cdots = \rho(\sigma_{n-1})$, and the fact that ρ is transitive implies that the permutation $\rho(\sigma_1)$ of S_m is an m-cycle, so once more statement (i) holds. Finally, assume that $(n, m) = (4, 3)$. We claim that $\rho(\sigma_1)$ and $\rho(\sigma_2)$ have the same cycle type. To see this, first note that if one of $\rho(\sigma_1)$ or $\rho(\sigma_2)$ is equal to the identity permutation then the Artin relations imply that the other is also equal to the identity, which proves the claim in this case. So suppose that one of these two elements is a transposition and the other is a 3-cycle. Then $\rho(\sigma_1\sigma_2\sigma_1)$ and $\rho(\sigma_2\sigma_1\sigma_2)$ have opposite signatures, which yields a contradiction using the Artin relations, and proves the claim. It follows in a similar manner that $\rho(\sigma_2)$ and $\rho(\sigma_3)$ have the same cycle type, hence $\rho(\sigma_1)$, $\rho(\sigma_2)$ and $\rho(\sigma_3)$ all have the same cycle type. By the transitivity hypothesis, they cannot be equal to the identity permutation, and they cannot be equal to the same transposition. So we are reduced to analysing the following two cases:

(i) $\rho(\sigma_1)$, $\rho(\sigma_2)$ and $\rho(\sigma_3)$ are transpositions. Since σ_1 and σ_3 commute, it follows that $\rho(\sigma_1) = \rho(\sigma_3)$, and the fact that $\rho(\sigma_1)$, $\rho(\sigma_2)$ and $\rho(\sigma_3)$ do not coincide implies that the condition given in part (ii) is satisfied;

(ii) $\rho(\sigma_1)$, $\rho(\sigma_2)$ and $\rho(\sigma_3)$ are 3-cycles. By the Artin relations, it follows that $(\rho(\sigma_1))^{-1} \neq \rho(\sigma_2)$, so $\rho(\sigma_1) = \rho(\sigma_2)$. In a similar fashion, $\rho(\sigma_2) = \rho(\sigma_3)$, and thus the condition given in part (i) is satisfied.

We now recall the following result of [**14**] about the structure of the centraliser $C_{S_m}(u)$ of a permutation u in S_m. Note that $C_{S_m}(u)$ is equal to the centraliser $C_{S_m}(\langle u\rangle)$ of the subgroup $\langle u\rangle$ in S_m. If $k \in \mathbb{N}$, let C_k denote the cyclic group of order k.

PROPOSITION 5·2 ([**14**, lemma 1·1]). *Let $u \in S_m$ be a permutation whose cycle type is equal to $(1)^{\ell_1}(2)^{\ell_2}\ldots(m)^{\ell_m}$, and let $I(u) = \{k \in \{1, 2, \ldots, m\} \mid \ell_k > 0\}$, so that we have $\Sigma_{k\in I(u)} k\ell_k = m$. Then the centraliser $C_{S_m}(u)$ of u in S_m is isomorphic to $\prod_{k\in I(u)} C_k^{\ell_k} \rtimes S_{\ell_k} = \prod_{k\in I(u)} C_k \wr S_{\ell_k}$.*

In the semi-direct product $\prod_{k\in I(u)} C_k^{\ell_k} \rtimes S_{\ell_k}$ given in the statement of Proposition 5·2, the action of an element τ of S_{ℓ_k} is given by indexing the copies of C_k by $\{1, \ldots, \ell_k\}$, and by sending a given element of C_k to the corresponding element of $C_{\tau(k)}$. Further, the partition associated with the cycle decomposition of u is left invariant by the elements of $C_{S_m}(u)$.

Let $n > m \geq 1$, and let $\rho_{n,m}\colon\ B_n(\Sigma_g) \longrightarrow S_m$ be a representation. Considering B_n to be a subgroup of $B_n(\Sigma_g)$ induced by the inclusion of a topological disc in Σ_g, by abuse of notation, we also denote the restriction of $\rho_{n,m}$ to B_n by $\rho_{n,m}$. We now prove Theorem 1·6 that generalises Theorem 5·1.

Proof of Theorem 1·6. Suppose that $n > m \geq 2$. If $m = 2$ then Im $(\rho_{n,2})$ is either equal to $\{\mathrm{Id}\}$ or is isomorphic to $\mathbb{Z}_2$, and statement (i) of the theorem holds. So assume that $n > m \geq 3$, and suppose additionally that $n \neq 4$. Since $m \geq 3$, Im $(\rho_{n,m}) \neq \{\mathrm{Id}\}$. By considering the composition of $\rho_{n,m}$ with the inclusion of B_n in $B_n(\Sigma_g)$, we see as in the proof of Theorem 1·5 in Section 2·2 that $\rho_{n,m}(\sigma_1) = \cdots = \rho_{n,m}(\sigma_{n-1})$. We denote this common element of S_m by σ. Relation (2·4) implies that σ commutes with $\rho_{n,m}(a_i)$ and with $\rho_{n,m}(b_i)$ for all $1 \leq i \leq g$. So if σ is an m-cycle then $\rho_{n,m}(a_i)$ and $\rho_{n,m}(b_i)$ are powers of σ for all $1 \leq i \leq g$, in which case

Im $(\rho_{n,m})$ is generated by σ, and statement (i) of the theorem holds. So assume that σ is not an m-cycle. Then the decomposition of σ as a product of disjoint cycles gives rise to a partition of the set $\{1, \ldots, m\}$ that is different from the set $\{1, \ldots, m\}$ itself and that is invariant under the action of σ. Since $\rho_{n,m}(a_i)$ and $\rho_{n,m}(b_i)$ commute with σ for all $1 \leq i \leq g$, they also leave this partition invariant, and it follows from the hypothesis that $\rho_{n,m}$ is primitive that σ is the identity permutation. Thus $\rho_{n,m}$ factors through the quotient $B_n(\Sigma_g)/\langle\langle\{\sigma_1, \ldots, \sigma_{n-1}\}\rangle\rangle$ of $B_n(\Sigma_g)$ by the normal closure $\langle\langle\{\sigma_1, \ldots, \sigma_{n-1}\}\rangle\rangle$ of $\{\sigma_1, \ldots, \sigma_{n-1}\}$ in $B_n(\Sigma_g)$, and induces a homomorphism $\overline{\rho}_{n,m}\colon B_n(\Sigma_g)/\langle\langle\{\sigma_1, \ldots, \sigma_{n-1}\}\rangle\rangle \longrightarrow S_m$. But from the proof of Proposition 2·9, $B_n(\Sigma_g)/\langle\langle\{\sigma_1, \ldots, \sigma_{n-1}\}\rangle\rangle$ is isomorphic to $\mathbb{Z}^{2g}$. So Im $(\overline{\rho}_{n,m}) =$ Im $(\rho_{n,m})$ is non trivial and Abelian, and $\overline{\rho}_{n,m}$ is primitive. Since Im $(\overline{\rho}_{n,m})$ is Abelian, any non-trivial element $u \in$ Im $(\overline{\rho}_{n,m})$ commutes with all of the elements of Im $(\overline{\rho}_{n,m})$, from which we see that Im $(\overline{\rho}_{n,m})$ is contained in the centraliser of u in S_m. If u is an m-cycle then Im $(\overline{\rho}_{n,m})$ coincides with $C_{S_m}(u)$, which is equal to $\langle u\rangle$, and thus part (i) of the statement holds. So assume that Im $(\overline{\rho}_{n,m})$ contains no m-cycle. Then the cycle decomposition of u contains a non-trivial cycle of length strictly less than m, so by Proposition 5·2, $C_{S_m}(u)$ is imprimitive. But since Im $(\overline{\rho}_{n,m}) \subset C_{S_m}(u)$, this implies that $\overline{\rho}_{n,m}$ is also imprimitive, which yields a contradiction. This argument also implies that m has to be prime, and that u is an m-cycle. This completes the proof of the case $n > m \geq 3$ and $n \neq 4$.

Finally, let $n = 4$ and $m = 3$. Suppose first that the restriction of the representation $\rho_{4,3}\colon B_4(\Sigma_g) \longrightarrow S_3$ to B_4 is intransitive. Thus $\rho_{4,3}(B_4)$ is equal to a subgroup of S_3 of order 1 or 2, and in either case, it follows that $\rho_{4,3}(\sigma_1) = \rho_{4,3}(\sigma_2) = \rho_{4,3}(\sigma_3)$ using the Artin relations (2·2) and (2·3). We denote this element by σ. As in the discussion of the previous paragraph of the case where σ is not an m-cycle for $n > m \geq 3$ and $n \neq 4$, we obtain a contradiction. Therefore the restriction of the representation $\rho_{4,3}$ to B_4 is transitive, and by Theorem 5·1, we just have to consider the following two cases.

(i) $\rho_{4,3}(B_4)$ is generated by a 3-cycle, and $\rho_{4,3}(\sigma_1) = \rho_{4,3}(\sigma_2) = \rho_{4,3}(\sigma_3)$. Using once more relation (2·4) of Theorem 2·2, we see that $\rho_{4,3}(a_i)$ and $\rho_{4,3}(b_i)$ commute with the 3-cycle $\rho_{4,3}(\sigma_1)$ for all $1 \leq i \leq g$, so they are powers of $\rho_{4,3}(\sigma_1)$. Thus Im $(\rho_{4,3}) = \langle\rho_{4,3}(\sigma_1)\rangle$, and hence part (i) of the statement holds.

(ii) Up to a suitable renumbering of the elements of the set $\{1, 2, 3\}$, $\rho_{4,3}(\sigma_1) = \rho_{4,3}(\sigma_3) = (1, 2)$ and $\rho_{4,3}(\sigma_2) = (2, 3)$. Relation (2·4) of Theorem 2·2 implies once more that $\rho_{4,3}(a_i)$ and $\rho_{4,3}(b_i)$ commute with the elements $\rho_{4,3}(\sigma_2)$ and $\rho_{4,3}(\sigma_3)$ for all $1 \leq i \leq g$. Since these transpositions generate S_3, it follows that the permutations $\rho_{4,3}(a_i)$ and $\rho_{4,3}(b_i)$ belong to the centre of S_3 for all $1 \leq i \leq g$, so are trivial. Hence part (ii) of the statement holds.

Remark 5·3. Using the methods of the proof of Theorem 1·6 and the presentation given by [**13**, proposition 3·1], the statement of Theorem 1·6 also holds if the surface has boundary.

We may obtain some information about an arbitrary representation $\rho_{n,m}\colon B_n(\Sigma_g) \longrightarrow S_m$ in a more general setting.

PROPOSITION 5·4. *Let $g \geq 1$, let $n > m \geq 2$, and assume that $(n, m) \neq (4, 3)$. Suppose that $\rho_{n,m}\colon B_n(\Sigma_g) \longrightarrow S_m$ is a homomorphism, and let $\rho_{n,m}(B_n)$ be the image of the subgroup B_n of $B_n(\Sigma_g)$ under $\rho_{n,m}$.*

(i) *The subgroup $\rho_{n,m}(B_n)$ of S_m is cyclic, and therefore $\rho_{n,m}(\sigma_1) = \cdots = \rho_{n,m}(\sigma_{n-1})$.*

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(ii) *The subgroup Im* $(\rho_{n,m})$ *is contained in the centraliser* $C_{S_m}(\rho_{n,m}(B_n))$ *of* $\rho_{n,m}(B_n)$ *in* S_m*. This centraliser is described by Proposition 5·2.*

(iii) *There is an inclusion* $\Gamma_3(B_n(\Sigma_g)) \subset Ker\,(\rho_{n,m})$*, so the homomorphism* $\rho_{n,m}$ *factors through the quotient* $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$*, and the subgroup Im* $(\rho_{n,m})$ *is nilpotent of nilpotency degree at most* 2*.*

Proof. Parts (i) and (ii) follow from the group presentation of $B_n(\Sigma_g)$ and by repeating the arguments given for instance in the proof of Theorem 1·6. If $n \geq 3$, the first statement of part (iii) is a consequence of part (i) and the fact that the subgroup $\Gamma_3(B_n(\Sigma_g))$ is isomorphic to the normal closure of the element $\sigma_1\sigma_2^{-1}$ in $B_n(\Sigma_g)$ [**12**, proof of theorem 1(c)], which is contained in Ker $(\rho_{n,m})$ using part (i). The second statement of part (iii) then follows.

In [**39**, p· 317], Ivanov gave some transitive, imprimitive representations of $B_n(\Sigma_g)$ in S_n for $g \geq 1$, $n \geq 3$, and he commented that 'I do not know to what extent these examples exhaust the imprimitive representations'. All of the examples he proposed are representations whose images are Abelian. We now describe some imprimitive representations $\rho_{n,m}\colon\ B_n(\Sigma_g) \longrightarrow S_m$ whose images are non Abelian, so are different from those of Ivanov.

Example 1.

(i) By [**12**, equation (10)], if $g \geq 1$ and $n \geq 3$, $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$ admits the following presentation:

generators: $a_1, b_1, \ldots, a_g, b_g$ and σ.

relations: $\sigma^{2(n-1+g)} = 1$, and the elements of $\{a_1, b_1, \ldots, a_g, b_g, \sigma\}$ commute pairwise, except for the pairs $(a_i, b_i)_{i=1,\ldots,g}$, for which $[a_1, b_1] = \cdots = [a_g, b_g] = \sigma^2$.

Let $n > 2$ be even, and let $g = 1$. From the above presentation, we have:

$$B_n(\mathbb{T}^2)/\Gamma_3(B_n(\mathbb{T}^2)) = \langle a_1, b_1, \sigma \mid [a_1, \sigma] = [b_1, \sigma] = 1,\ [a_1, b_1] = \sigma^2,\ \sigma^{2n} = 1\rangle.$$

We define a map $\theta\colon\ B_n(\mathbb{T}^2)/\Gamma_3(B_n(\mathbb{T}^2)) \longrightarrow S_8$ on the generators of the group $B_n(\mathbb{T}^2)/\Gamma_3(B_n(\mathbb{T}^2))$ by:

$$\theta(a_1) = (1, 3)(2, 4),\ \theta(b_1) = (1, 5)(2, 6)(3, 7)(4, 8) \text{ and}$$
$$\theta(\sigma) = (1, 2, 3, 4)(5, 6, 7, 8).$$

It is straightforward to check that θ respects the relations of $B_n(\mathbb{T}^2)/\Gamma_3(B_n(\mathbb{T}^2))$, the equality $(\theta(\sigma))^{2n} = 1$ being a consequence of the fact that $2n$ is divisible by 4, so θ is a homomorphism. If $p\colon\ B_n(\mathbb{T}^2) \longrightarrow B_n(\mathbb{T}^2)/\Gamma_3(B_n(\mathbb{T}^2))$ is the canonical projection, then the representation $\theta \circ p\colon\ B_n(\mathbb{T}^2) \longrightarrow S_8$ is transitive, and it is imprimitive since the non-trivial partition $\{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\}$ is preserved by the subgroup Im $(\theta \circ p)$ of S_8. This is perhaps the simplest example of an imprimitive representation $\rho_{n,m}\colon\ B_n(\Sigma_g) \longrightarrow S_m$ whose image is non Abelian. In particular, if we take $n = 8$, we obtain a transitive, imprimitive representation of $B_8(\mathbb{T}^2)$ in S_8 whose image is non Abelian, so it is not included in the examples of [**39**].

(ii) If $g + n$ is odd and $m = 2^{g+2}$, Example 1 may be generalised to construct a homomorphism $\theta_g\colon\ B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g)) \longrightarrow S_m$ such that Im (θ_g) is non Abelian. Composing θ_g with the projection $p_g\colon\ B_n(\Sigma_g) \longrightarrow B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$, we thus obtain a homomorphism $\theta_g \circ p_g\colon\ B_n(\Sigma_g) \longrightarrow S_m$ such that Im $(\theta_g \circ p_g)$ is non

Abelian. To do so, first let us denote the image by θ_g of the element $\sigma \in B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$ given in Example 1 by $\overline{\sigma} \in S_m$. By Proposition 5·4, $\operatorname{Im}\left(\theta_g\right)$ is contained in the centraliser of $\overline{\sigma}$ in S_m, which is described in Proposition 5·2. Our strategy is to make use of the structure of this centraliser to construct imprimitive representations whose images are non Abelian. In Example 1, the image of θ is isomorphic to $(\mathbb{Z}_4 \oplus \mathbb{Z}_4) \rtimes \mathbb{Z}_2$, and is the centraliser of $\overline{\sigma} = (1_4, 1_4; 0_2)$ given by Proposition 5·2. In the general case, $n+g$ is odd, $m = 2^{g+2}$, and the centraliser of $\overline{\sigma}$ is isomorphic to $\mathbb{Z}_4^{2^g} \rtimes \mathbb{Z}_{2^g}$. We now give two examples of this construction, one in the case where g is odd, and in the other in the case where g is even.

(a) Suppose that $g = 3$, so $m = 32$, and $n \geq 4$ is even. Consider $\mathbb{Z}_4^8 \rtimes S_8$, which we interpret as a subgroup of S_{32}. Define the homomorphism:

$$\theta_3 \colon\ B_n(\Sigma_3)/\Gamma_3(B_n(\Sigma_3)) \longrightarrow S_{32}$$

by:

$$\theta_3(a_1) = (2, 0, 2, 0, 2, 0, 2, 0) \qquad \theta_3(a_2) = (2, 2, 0, 0, 2, 2, 0, 0)$$
$$\theta_3(a_3) = (2, 2, 2, 2, 0, 0, 0, 0) \qquad \theta_3(\sigma) = (1, 1, 1, 1, 1, 1, 1, 1),$$

regarded as elements of S_{32}, where each factor 1 denotes the cyclic permutation of length 4 associated to the four integers corresponding to these four positions, 2 denotes the square of this cyclic permutation, and 0 denotes the identity permutation associated to these four integers. Finally, let $\theta_3(b_1) = (1,2)(3,4)(5,6)(7,8)$, $\theta_3(b_2) = (1,3)(2,4)(5,7)(6,8)$, $\theta_3(b_3) = (1,5)(2,6)(3,7)(4,8)$, all regarded as elements of $S_8 \subset \mathbb{Z}_4^8 \rtimes S_8$. In terms of explicit elements of S_{32}, we have:

$$\begin{aligned}
\theta_3(a_1) =& (1,3)(2,4)(9,11)(10,12)(17,19)(18,20)(25,27)(26,28)\\
\theta_3(a_2) =& (1,3)(2,4)(5,7)(6,8)(17,19)(18,20)(21,23)(22,24)\\
\theta_3(a_3) =& (1,3)(2,4)(5,7)(6,8)(9,11)(10,12)(13,15)(14,16)\\
\theta_3(b_1) =& (1,5)(2,6)(3,7)(4,8)(9,13)(10,14)(11,15)(12,16)(17,21)\cdot\\
& (18,22)(19,23)(20,24)(25,29)(26,30)(27,31)(28,32)\\
\theta_3(b_2) =& (1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16)(17,25)\cdot\\
& (18,26)(19,27)(20,28)(21,29)(22,30)(23,31)(24,32)\\
\theta_3(b_3) =& (1,17)(2,18)(3,19)(4,20)(5,21)(6,22)(7,23)(8,24)(9,25)\cdot\\
& (10,26)(11,27)(12,28)(13,29)(14,30)(15,31)(16,32)\\
\theta_3(\sigma) =& (1,2,3,4)(5,6,7,8)(9,10,11,12)(13,14,15,16)(17,18,19,20)\cdot\\
& (21,22,23,24)(25,26,27,28)(29,30,31,32).
\end{aligned}$$

Using these expressions, we may check that $[\theta_3(a_i), \theta_3(a_j)] = [\theta_3(b_i), \theta_3(b_j)] = [\theta_3(a_i), \theta_3(b_j)]$ for all $1 \leq i < j \leq 3$, and $[\theta_3(a_l), \theta_3(b_l)] = (\theta_3(\sigma))^2$ for all $1 \leq l \leq 3$, so θ_3 is a homomorphism.

(b) Now suppose that $g = 2$, so $m = 16$, and $n \geq 3$ is odd. Consider the subgroup $\mathbb{Z}_4^4 \rtimes S_4$, which we interpret as a subgroup of the symmetric group S_{16}. We define a homomorphism $\theta_{2,1} \colon\ B_n(\Sigma_2)/\Gamma_3(B_n(\Sigma_2)) \longrightarrow S_{16}$ as follows. Let $\theta_{2,1}(a_1) = (2, 0, 2, 0)$ and $\theta_{2,1}(a_2) = (2, 2, 0, 0)$ in S_{16}, where as

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in the previous example, each factor 1 denotes the cyclic permutation of length 4 associated to the integers corresponding to the these four positions, 2 is the square of this cyclic permutation, and 0 is the identity permutation associated to these four integers. We also take $\theta_{2,1}(b_1) = (1, 2)(3, 4) \in S_4$, $\theta_{2,1}(b_2) = (1, 3)(2, 4) \in S_4$, and $\theta_{2,1}(\sigma) = (1, 1, 1, 1) \in S_{16}$. Since $n + 2$ is odd, $\theta_{2,1}$ defines a homomorphism. In S_{16}, the elements are given explicitly by:

$$\begin{aligned}
\theta_{2,1}(a_1) &= (1, 3)(2, 4)(9, 11)(10, 12)\\
\theta_{2,1}(a_2) &= (1, 3)(2, 4)(5, 7)(6, 8)\\
\theta_{2,1}(b_1) &= (1, 5)(2, 6)(3, 7)(4, 8)(9, 13)(10, 14)(11, 15)(12, 16)\\
\theta_{2,1}(b_2) &= (1, 9)(2, 10)(3, 11)(4, 12)(5, 13)(6, 14)(7, 15)(8, 16)\\
\theta_{2,1}(\sigma) &= (1, 2, 3, 4)(5, 6, 7, 8)(9, 10, 11, 12)(13, 14, 15, 16).
\end{aligned}$$

Example 2. Let $n > 2$, and consider the group $\mathbb{Z}_{2n}^n \rtimes S_n$ seen as subgroup of S_{2n^2}. Let $\theta\colon\ B_n(\mathbb{T}^2)/\,\Gamma_3(B_n(\mathbb{T}^2)) \longrightarrow S_{2n^2}$ be the homomorphism defined by $\theta(a_1) = (a, a+2, a+4, \ldots, a-2) \in \mathbb{Z}_{2n}^n$ for a any element of $\mathbb{Z}_{2n}$, $\theta(b_1) = (1, 2, \ldots, n) \in S_n$, and $\theta(\sigma) = (1_{2n}, \ldots, 1_{2n}) \in \mathbb{Z}_{2n}^n$. It follows that $\theta(\sigma)$ is of order $2n$, $\theta(\sigma)$ commutes with $\theta(a_1)$ and $\theta(b_1)$, and that $\theta([a_1, b_1]) = \theta(\sigma)^2$. The image of θ is the subgroup generated by $\{\theta(a_1), \theta(b_1), \theta(\sigma)\}$ and it is non Abelian.

We conclude this paper with the following remarks.

Remarks 5·5.

(i) The construction of Example 2 also enables us to obtain an example of a homomorphism $\theta\colon\ B_n(\mathbb{T}^2)/\,\Gamma_3(B_n(\mathbb{T}^2)) \longrightarrow S_m$, where $n > m$, and the order of $\theta(\sigma)$ is equal to $2n$. First note that if $l \geq 3$ and l divides n, then it follows from the presentation given at the beginning of Example 1 that the map $\tau_l\colon\ B_n(\mathbb{T}^2)/\,\Gamma_3(B_n(\mathbb{T}^2)) \longrightarrow B_l(\mathbb{T}^2)/\,\Gamma_3(B_l(\mathbb{T}^2))$ defined by sending the generators a_1, b_1 and σ of the quotient $B_n(\mathbb{T}^2)/\,\Gamma_3(B_n(\mathbb{T}^2))$ to the generators a_1, b_1 and σ respectively of $B_l(\mathbb{T}^2)/\,\Gamma_3(B_l(\mathbb{T}^2))$ extends to a (well-defined) surjective homomorphism. Let $n = 3 \cdot 5 \cdot 7 \cdot 11 = 1155$. For $l = 3, 5, 7, 11$, let $\theta_l\colon\ B_l(\mathbb{T}^2)/\,\Gamma_3(B_l(\mathbb{T}^2)) \longrightarrow S_{2l^2}$ be the homomorphism given as in Example 2, and let $\theta\colon\ B_n(\mathbb{T}^2)/\,\Gamma_3(B_n(\mathbb{T}^2)) \longrightarrow S_{408}$ be defined by $\theta(x) = (\theta_3 \circ \tau_3(x), \theta_5 \circ \tau_5(x), \theta_7 \circ \tau_7(x), \theta_{11} \circ \tau_{11}(x)) \in S_{18} \times S_{50} \times S_{98} \times S_{242}$ for all $x \in B_n(\mathbb{T}^2)/\,\Gamma_3(B_n(\mathbb{T}^2))$. Interpreting $S_{18} \times S_{50} \times S_{98} \times S_{242}$ as a subgroup of S_{408}, we may thus take $m = 408$, and the element $\theta(\sigma)$ is of order 2310.

(ii) Let $g \geq 1$, and let G be the group that admits the following presentation:
generators: $a_1, b_1, \ldots, a_g, b_g$ and σ.
relations: $\sigma^{2(1+g)} = 1$, and the elements of $\{a_1, b_1, \ldots, a_g, b_g, \sigma\}$ commute pairwise, except for the pairs $(a_i, b_i)_{i=1,\ldots,g}$, for which $[a_1, b_1] = \cdots = [a_g, b_g] = \sigma^2$.

Observe that this is the group obtained by taking $n = 2$ in the presentation of the quotient $B_n(\Sigma_g)/\,\Gamma_3(B_n(\Sigma_g))$ given in Example 1 (we suspect that $B_2(\Sigma_g)/\,\Gamma_3(B_2(\Sigma_g))$ is not isomorphic to G in this case). Using the presentation of $B_2(\Sigma_g)$ given by Theorem 2·2, the map $\rho\colon\ B_2(\Sigma_g) \longrightarrow G$ given by sending the generators a_i, b_i and σ_1 of $B_2(\Sigma_g)$ to the generators a_i, b_i and σ respectively of G

for all $1 \leq i \leq g$ may be seen to extend to a well-defined surjective homomorphism. To check that relation (2·8) is respected by ρ, note that in G:

$$\prod_{i=1}^{g}[a_i^{-1}, b_i] = \prod_{i=1}^{g} a_i^{-1}[b_i, a_i]a_i = \prod_{i=1}^{g} a_i^{-1}\sigma^{-2}a_i = \sigma^{-2g} = \sigma^2 \text{ since } \sigma^{2g+2} = 1.$$

Hence ρ induces a surjective homomorphism $\overline{\rho}\colon B_2(\Sigma_g)/\Gamma_3(B_2(\Sigma_g)) \longrightarrow G/\Gamma_3(G)$. Using the presentation of G and the fact that $\Gamma_2(G)$ is the normal closure in G of the commutators of the generators of G, we see that $\Gamma_2(G) = \langle\sigma^2\rangle$, and thus $\Gamma_3(G)$ is trivial. Therefore $\overline{\rho}$ is a surjective homomorphism from $B_2(\Sigma_g)/\Gamma_3(B_2(\Sigma_g))$ to G. Observe that $\overline{\rho}$ is not an isomorphism if $g = 1$ because $\Gamma_2(B_2(\mathbb{T}^2))/\Gamma_3(B_2(\mathbb{T}^2)) \cong \mathbb{Z}_2^3$ by Theorem 2·3(ii)(b), and $\Gamma_2(G)/\Gamma_3(G) = \langle\sigma^2\rangle \cong \mathbb{Z}_2$. The construction of Example 1 may be applied to G if g is odd, and composing with $\overline{\rho}$, shows that it may also be extended to the case $n = 2$ to yield a representation of $B_2(\Sigma_g)$ in S_{2g+2} whose image is non Abelian.

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