Euro. Jnl of Applied Mathematics (2004), *vol.* 15, *pp.* 781–789.
doi:10.1017/S0956792504005583 Printed in the United Kingdom

Explicit solutions for Hele–Shaw corner flows

I. MARKINA and A. VASIL'EV

Departamento de Matemática, Universidad Técnica Federico Santa María,
Casilla 110-V, Valparaíso, Chile
email: {irina.markina,alexander.vasiliev}@mat.utfsm.cl

(*Received 4 September 2002; revised 29 October 2003*)

We consider two-dimensional bubbles in a corner flow in a Hele–Shaw cell of a viscous incompressible fluid that occupies the complement to a bubble. We discuss the governing equations, some basic properties of the free interface of the bubbles, their geometry, and construct explicit solutions that present asymmetric long bubbles analogous to the famous Saffman–Taylor fingers in a wedge of arbitrary angle $\alpha \in (0, 2\pi)$.

1 Introduction

We consider Hele-Shaw flow in a domain bounded by non-parallel walls, a corner flow, using Polubarinova-Kochina and Galin's approach [5, 7, 8]. This model is motivated by Saffman and Taylor's famous fingering in a channel [10], and later works by Ben Amar *et al.* [2, 3, 4] on Hele–Shaw flows in a wedge.

We construct explicit solutions, finding relevant conformal maps, for one-phase corner Hele–Shaw flows, i.e. a flow in the Hele-Shaw cell between non-parallel walls that form a corner of angle α. Primary, we are concerned with the deformation of two-dimensional bubbles in a corner flow in which there is a replacement of two immiscible fluids one of which is viscous and the other is effectively inviscid. We shall give self-similar (homothetic) drop-shaped solutions in a corner that include Ben Amar's [2], as well as those constructed in Arnéodo *et al.* [1] and Thomé *et al.* [11] as particular cases. Tu [12] also did an analysis of viscous fingering in corners applying the hodograph method for the complex velocity potential. In the symmetric case this leads to Ben Amar's [2] solution given in terms of hypergeometric functions, whereas in the non-symmetric case no explicit solution was given.

2 Mathematical model

We suppose that the viscous fluid occupies a simply connected domain $\Omega(t)$ in the phase z-plane whose boundary $\Gamma(t)$ at an instant t consists of two walls $\Gamma_1(t)$ and $\Gamma_2(t)$ of the corner and a free interface $\Gamma_3(t)$ between them. The inviscid fluid (or air) fills the complement of $\Omega(t)$. The simplifying assumption of constant pressure at the interface between the fluids means that the surface tension effect is neglected. The bubble is assumed to originate at the vertex as in Figure 1 and the bubble-wall angles are $\beta \in (0, \alpha/2)$. We

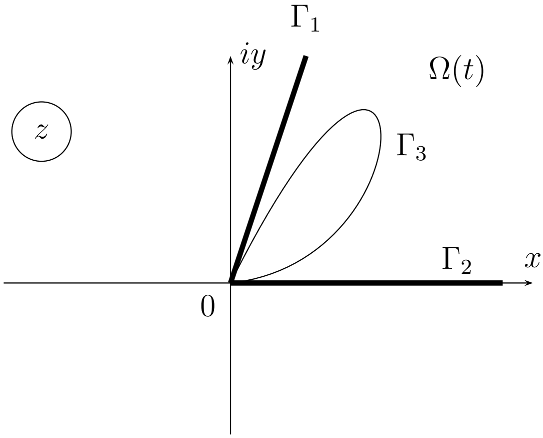

FIGURE 1. $\Omega(t)$ is the phase domain within an infinite corner and the homogeneous sink/source at ∞.

let the positive real axis x contain one of the walls and fix the angle between walls as $\alpha \in (0, 2\pi)$.

In this model the field equation for the fluid pressure $p(z,t) \equiv p(x,y,t)$ is simply

$$\Delta p = 0 \quad \text{in the flow region } \Omega(t), \tag{2.1}$$

and the fluid velocity $\mathbf{V}$ averaged across the gap is $\mathbf{V} = -\nabla p$. The free boundary conditions

$$p\Big|_{\Gamma_3} = 0, \quad \frac{\partial p}{\partial t}\Big|_{\Gamma_3} = (\nabla p)^2 \tag{2.2}$$

are imposed on the free boundary $\Gamma_3 \equiv \Gamma_3(t)$. This implies that the normal velocity v_n of the free boundary Γ_3 outwards from $\Omega(t)$ is expressed by

$$\frac{\partial p}{\partial n}\Big|_{\Gamma_3} = -v_n. \tag{2.3}$$

On the walls $\Gamma_1 \equiv \Gamma_1(t)$ and $\Gamma_2 \equiv \Gamma_2(t)$ the boundary conditions are

$$\frac{\partial p}{\partial n}\Big|_{\Gamma_1 \cup \Gamma_2} = 0. \tag{2.4}$$

We suppose that the motion is driven by a homogeneous source/sink at infinity. Since the angle between the walls at the infinity is also α, the pressure behaves near infinity as

$$p \sim \frac{-Q}{\alpha} \log |z|, \quad \text{as } |z| \to \infty,$$

where Q corresponds to the constant strength of the source ($Q < 0$) or sink ($Q > 0$). Finally, we assume that $\Gamma_3(0)$ is a given analytic curve.

Let us consider an auxiliary parametric complex ζ-plane, $\zeta = \xi + i\eta$. We set $D = \{\zeta : |\zeta| > 1, 0 < \arg \zeta < \pi\}$, $D_3 = \{z : z = e^{i\theta}, \theta \in (0, \pi)\}$, $D_1 = \{z : z = -r, r > 1\}$,

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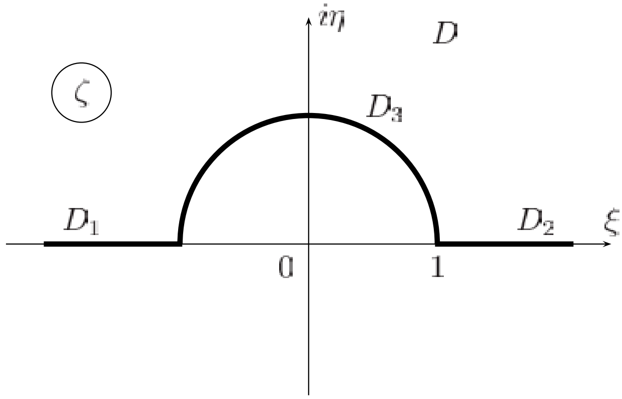

FIGURE 2. The parametric domain D.

$D_2 = \{z : z = r, r > 1\}$, $\partial D = D_1 \cup D_2 \cup D_3$. Construct a conformal univalent time-dependent map $z = f(\zeta, t)$, $f : D \to \Omega(t)$, such that being continued onto ∂D, $f(\infty, t) \equiv \infty$, and the circular arc D_3 of ∂D is mapped onto Γ_3 (see Figure 2).

This map has an expansion $f(\zeta, t) = \zeta^{\alpha/\pi} \sum_{k=0}^{\infty} a_k(t)\zeta^{-k}$ about infinity, and $a_0(t) > 0$. The function f parameterizes the boundary of the domain $\Omega(t)$ by $\Gamma_j = \{z : z = f(\zeta, t),\ \zeta \in D_j\}$, $j = 1, 2, 3$.

We will use the notations $\dot{f} = \partial f/\partial t$, $f' = \partial f/\partial \zeta$. The free surface condition is expressed in terms of the function f [5, 7, 8] by

$$\mathrm{Re}\,(\dot{f}\ \overline{\zeta f'}) = \frac{Q}{\pi}, \quad \text{for } \zeta \in D_3, \tag{2.5}$$

and the wall conditions imply that

$$\mathrm{Im}\,(\dot{f}e^{i\alpha}) = 0 \quad \text{for } \zeta \in D_1; \quad \mathrm{Im}\,(\dot{f}) = 0 \quad \text{for } \zeta \in D_2. \tag{2.6}$$

Let us make some comments about the geometry of the bubbles. We have studied [6, 9, 13] the geometric properties of the moving interface that are preserved during the time of existence of solutions to the Hele–Shaw problem in the stable case (well-posed problems). Here we note that all considerations of the mentioned works can be applied in our case. In particular, for the advancing fluid in the corner, the interface is star-shaped during the evolution if the initial interface is.

3 Explicit solutions

We are going to construct an analogue of the Saffman–Taylor fingers for the corner flows (*self-dilating drops* whose interface contains the vertex). Analytical solutions have been discovered first in the case $\alpha = \pi/2$ in Thomé *et al.* [11] and then for general angles in Ben Amar [2, 3] and Ben Amar *et al.* [4]. We give generalization that, in fact, presents possible self-similar solutions, and in particular, we obtain exact solutions for non-symmetric drops.

To simplify matters, we replace the angles α, β by their ratios: $\alpha \to \alpha\pi$, $\beta \to \beta\pi/2$. Let us analyze the auxiliary mapping $f(\zeta, t)$. In the case of self-dilating solutions the phase domain $\Omega(t)$ is a dilation of an initial domain $\Omega(0)$. Then the solution $f(\zeta, t)$ to the equations (2.5)–(2.6) is represented as $f(\zeta, t) = G(t)F(\zeta)$. Since Q does not depend on t, the equation (2.5) implies that $G(t) = C\sqrt{t}$, where C is a constant. Reducing the mapping f to a regular function we represent it as

$$f(\zeta, t) = \sqrt{t}\zeta^{\alpha} g(\zeta),$$

where $g(\zeta)$ is an analytic function which is regular at infinity.

The boundary Γ_3 starts and ends at the origin under the same bubble-wall angles $\beta \in (0, \alpha)$, and forms a self-similar drop-shaped bubble. Therefore, the function $g(\zeta)$ can be represented as

$$g(\zeta) = \left(1 - \frac{1}{\zeta^2}\right)^{\beta} h(\zeta),$$

where $h(\zeta)$ is a regular function in the closure of D. We differentiate equation (2.5) with respect to θ, taking into account $\zeta = e^{i\theta}$, $\theta \in (0, \pi)$. Then (2.5) is reduced to

$$\operatorname{Im}\left[(2\alpha + 1)\frac{\zeta g'(\zeta)}{g(\zeta)} + \frac{\zeta^2 g''(\zeta)}{g(\zeta)}\right] = 0, \quad \zeta = e^{i\theta},$$

or in terms of the function h we have $\operatorname{Im} G(\zeta) = 0$, where

$$G(\zeta) = \frac{2\beta(2\alpha+1)}{\zeta^2 - 1} + \frac{4\beta(\beta-1)}{(\zeta^2-1)^2} - \frac{6\beta}{\zeta^2-1} + \left((2\alpha+1) + \frac{4\beta}{\zeta^2-1}\right)\frac{\zeta h'(\zeta)}{h(\zeta)} + \frac{\zeta^2 h''(\zeta)}{h(\zeta)}.$$

Equations (2.6) imply that $\operatorname{Im} G(\zeta) = 0$ is satisfied on the whole boundary $D_1 \cup D_2 \cup D_3$. The function $h(\zeta)$ is regular at ± 1, therefore

$$G(\zeta) \sim \frac{1}{(\zeta^2 - 1)^2}, \quad \text{as } \zeta \to \pm 1.$$

Taking into account the regularity of $h(\zeta)$ near infinity we propose that the function G has the form

$$G(\zeta) = \frac{4\beta(\beta-1)\zeta^2}{(\zeta^2-1)^2},$$

although other forms may be possible. Our intention is to obtain a complex differential equation for which we can construct explicit solutions. We have the differential equation

$$\frac{4\beta(\alpha-\beta)}{\zeta^2-1} + \left((2\alpha+1) + \frac{4\beta}{\zeta^2-1}\right)\frac{\zeta h'(\zeta)}{h(\zeta)} + \frac{\zeta^2 h''(\zeta)}{h(\zeta)} = 0.$$

Changing variables $w = 1/\zeta^2$, $Y(w) \equiv h(1/\sqrt{w})$ we come to the hypergeometric equation

$$(1-w)wY'' + (1 - \alpha - (1 + 2\beta - \alpha)w)\,Y' - \beta(\beta - \alpha)Y = 0. \tag{3.1}$$

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Its general solution can be given in terms of the Gauss hypergeometric function $\mathbf{F} \equiv {}_2\mathbf{F}_1$. We thus have two linearly independent solutions

$$h_1(\zeta) = \mathbf{F}\left(\beta-\alpha, \beta, 1-\alpha; \frac{1}{\zeta^2}\right), \quad h_2(\zeta) = \frac{1}{\zeta^{2\alpha}}\mathbf{F}\left(\beta, \beta+\alpha, 1+\alpha; \frac{1}{\zeta^2}\right).$$

Finally, we find $f(\zeta, t)$ in the form

$$f(\zeta, t) = \sqrt{t}\zeta^{\alpha}\left(1-\frac{1}{\zeta^2}\right)^{\beta}(C_1 h_1(\zeta) + C_2 h_2(\zeta)), \tag{3.2}$$

for real constants C_1, C_2 and we choose the branch so that $f(r) > 0$ and $h(r) > 0$ for $r > 1$. Since the primitive

$$\int \operatorname{Im}\left(|f|^2 G(e^{i\theta})\Big|_{h=C_1h_1+C_2h_2}\right) d\theta = \operatorname{Re} \dot{f}(e^{i\theta}, t)\overline{e^{i\theta} f'(e^{i\theta}, t)}$$

is constant, we can choose C_1, C_2 such that it is exactly $Q/\pi > 0$ and $f(\zeta, t)$ satisfies the equation (2.5) in the arc $\{e^{i\theta}, \theta \in (0, \pi)\}$. By construction we have that the function f maps the rays $(-\infty, -1]$ and $[1, \infty)$ onto the walls Γ_1 and Γ_2 respectively. In order to check the univalence of f we note that given a positive Q and f of the form (3.2), we choose the constants C_1, C_2 as mentioned above. The function f is starlike with respect to the origin because $Q > 0$ and, hence, univalent. If the constant C_2 vanishes, then the equality $f(-\bar{\zeta}, t) = e^{i\alpha\pi}\overline{f(\zeta, t)}$ is easily verified. This means that the solution is symmetric with respect to the bisectrix of the phase angle, say the ray $z = re^{i\alpha/2}$, $r > 0$.

In Figures 3 and 4 we present asymmetric drops in angles $\pi/3$ and $2\pi/3$ (a), (c), as well as symmetric case (b).

In the case $\alpha = 1/2$ the hypergeometric functions are reduced to a simpler form:

$$h_1(\zeta) = \frac{1}{2}\left(\left(1+\frac{1}{\zeta}\right)^{1-2\beta} + \left(1-\frac{1}{\zeta}\right)^{1-2\beta}\right),$$

$$h_2(\zeta) = \frac{1}{2(1-2\beta)}\left(\left(1+\frac{1}{\zeta}\right)^{1-2\beta} - \left(1-\frac{1}{\zeta}\right)^{1-2\beta}\right),$$

and we have

$$f(\zeta, t) = \sqrt{\frac{t}{\zeta}}\left(A(\zeta+1)^{1-\beta}(\zeta-1)^{\beta} + B(\zeta-1)^{1-\beta}(\zeta+1)^{\beta}\right), \tag{3.3}$$

where $\beta \in (0, 1/2)$, $Q = 4AB(1-2\beta)\sin(\frac{\pi}{2}(1-2\beta))$, $A, B > 0$. We remark here that the map $f(\zeta, t)$ becomes non-univalent for other choices of A, B, β.

The map $f(\zeta, t)$ obviously satisfies the equations (2.5), (2.6). It maps D onto $\Omega(t)$ that is complement of a bubble for any time t. The boundary Γ_3 starts and ends at the origin under the same bubble-wall angle $\pi\beta/2$, and forms a self-similar drop-shaped

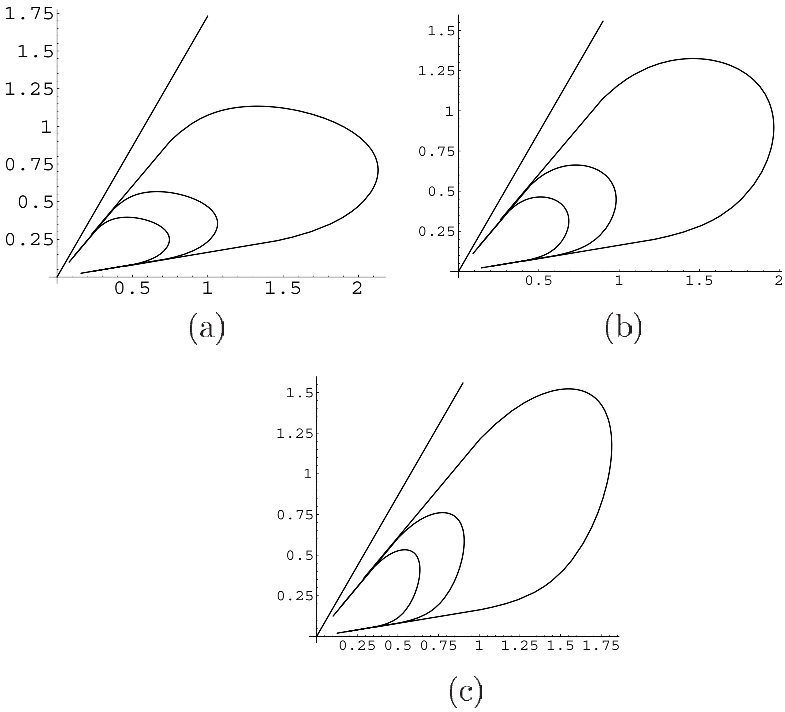

FIGURE 3. Finger dynamics in the wedge angle $\pi/3$ and the bubble-wall angles $\pi/20$: (a) $C_1 = 1$, $C_2 = 0.9$; (b) $C_1 = 1$, $C_2 = 0$; (c) $C_1 = 1$, $C_2 = -1$.

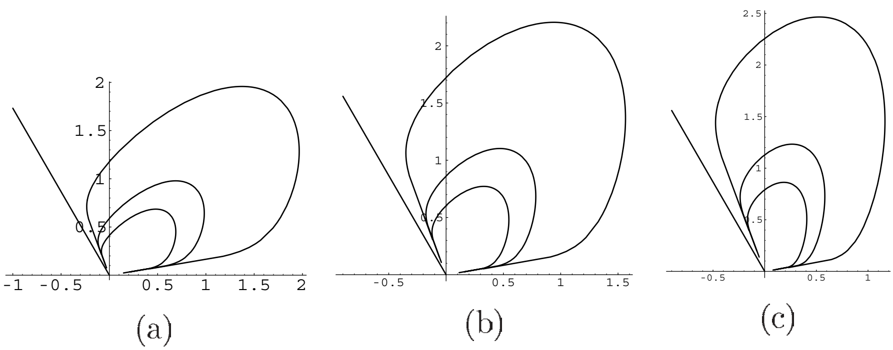

FIGURE 4. Finger dynamics in the wedge angle $2\pi/3$ and the bubble-wall angle $\pi/20$: (a) $C_1 = 1$, $C_2 = 0.9$; (b) $C_1 = 1$, $C_2 = 0$; (c) $C_1 = 1$, $C_2 = -1$.

bubble. If $A = B$, then the bubble is symmetric with respect to the bisectrix of the corner (Figures 3 (b), 4 (b) and 5) and the solution is known [1, 11]. If $A \neq B$, then we have non-symmetric dynamics (Figures 3 (a), (c), 4 (a), (c), 6 and 7). It is interesting that even the bubble-wall angles are the same, we have a two-parameter $(A/B, \beta)$ continuum of possible developments of fingers.

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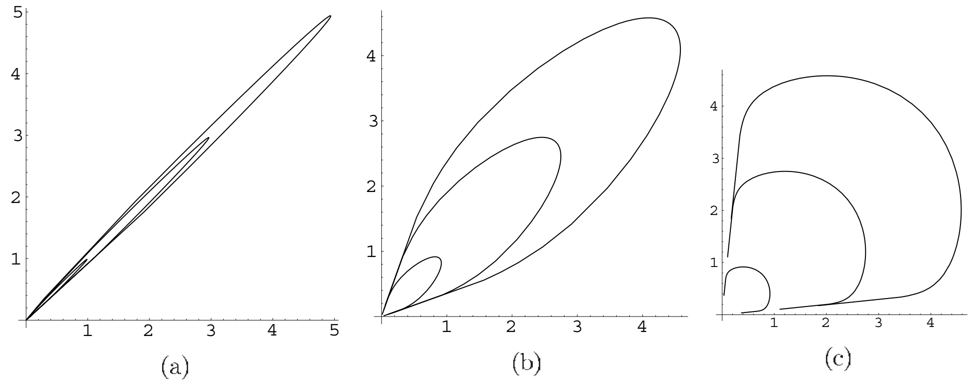

FIGURE 5. Finger dynamics: (a) $A = 1$, $B = 1$, $\beta = 0.16$; (b) $A = 1$, $B = 1$, $\beta = 0.1$; (c) $A = 1$, $B = 1$, $\beta = 0.05$.

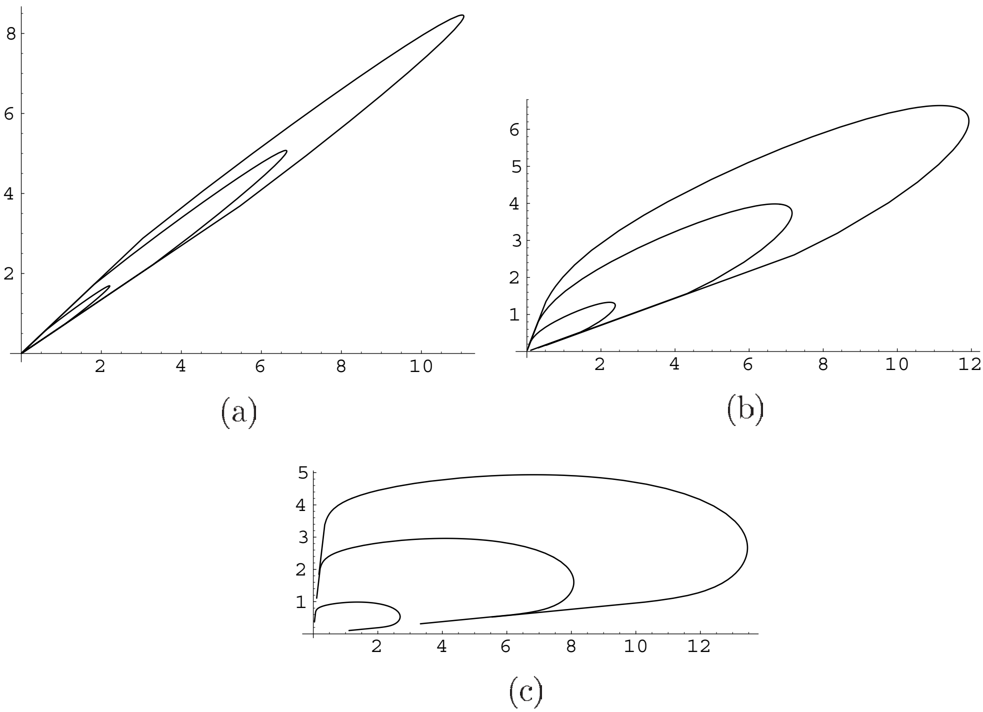

FIGURE 6. Finger dynamics: (a) $A = 1$, $B = 3$, $\beta = 0.16$; (b) $A = 1$, $B = 3$, $\beta = 0.1$; (c) $A = 1$, $B = 3$, $\beta = 0.05$.

For angles greater than π the procedure is the same. A corner of angle π implies other linearly independent solutions of the (3.1):

$$h_1(\zeta) = \frac{1}{\zeta^2}\mathbf{F}\left(\beta, \beta + 1, 2; \frac{1}{\zeta^2}\right),$$

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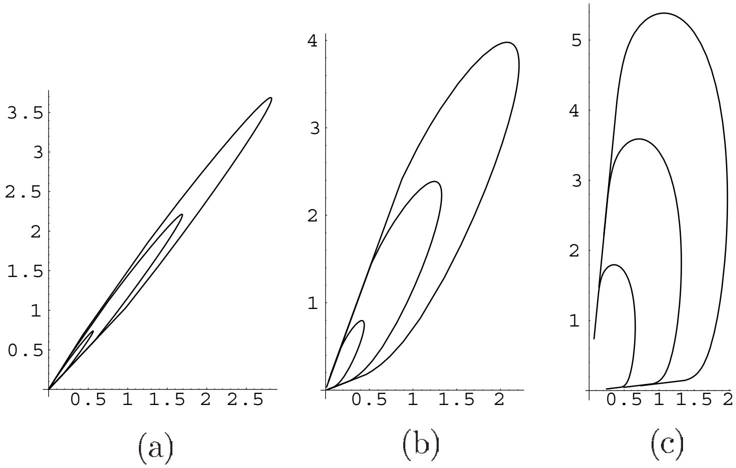

FIGURE 7. Finger dynamics: (a) $A = 1$, $B = 1/3$, $\beta = 0.16$; (b) $A = 1$, $B = 3$, $\beta = 0.1$; (c) $A = 1$, $B = 1$, $\beta = 0.05$.

$$h_2(\zeta) = \frac{-2\log\zeta}{\zeta^2}\mathbf{F}\left(\beta, \beta+1, 2; \frac{1}{\zeta^2}\right) + \sum_{k=1}^{\infty} \frac{\prod_{j=0}^{k-2}(\beta+j)^2(\beta-1)(\beta+k-1)}{\zeta^{2k+2}(k!)^2(k+1)} \left(2\left(\sum_{j=1}^{k-1}\frac{1}{\beta+j} - \sum_{j=2}^{k}\frac{1}{j}\right) + \frac{1}{\beta} + \frac{1}{\beta+k} - 1 - \frac{1}{k+1}\right) - \frac{1}{\beta(\beta+1)},$$

that can be treated similarly.

4 Conclusion

We have constructed several simple explicit time-dependent solutions for the corner flows in Hele–Shaw cells driven by the homogeneous pressure field. In the corner of an arbitrary angle α we established the existence of a two-parameter continuum of self-similar drop-shaped bubbles which, in general, are not symmetric even though the bubble-wall angles are equal. The moving interfaces are parameterized by conformal maps given in terms of hypergeometric functions.

Acknowledgements

This paper was completed while the authors were visiting OCIAM, Mathematical Institute, University of Oxford. They thank the staff for their hospitality. The authors are grateful to J. R. Ockendon, S. D. Howison for discussions and referees for helpful comments. Special thanks go to J. R. King and L. J. Cummings who drew our attention to the papers by

M. Ben Amar *et al.*. This work is partially supported by Projects Fondecyt (Chile) # 1030373, 1020067, and UTFSM #12.03.23.

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