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CARTAN SUBGROUPS OF GROUPS DEFINABLE IN O-MINIMAL STRUCTURES

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Abstract We prove that groups definable in o-minimal structures have Cartan subgroups, and only finitely many conjugacy classes of such subgroups. We also delineate with precision how these subgroups cover the ambient group.

Keywords: Lie groups; semialgebraic groups; groups definable in o-minimal structures; Cartan subgroups

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1. Introduction

If G is an arbitrary group, a subgroup Q of G is called a *Cartan subgroup* (*in the sense of Chevalley*) if it satisfies the following two conditions.

(1) Q is nilpotent and maximal with this property among subgroups of G.

(2) For any subgroup $X \leqslant Q$ which is normal in Q and of finite index in Q, the normalizer $N_G(X)$ of X in G contains X as a finite index subgroup.

The purely group-theoretic definition of a Cartan subgroup as above was designed by Chevalley in order to capture critical properties of very specific subgroups of Lie groups.

In connected affine algebraic groups over algebraically closed fields, Cartan subgroups are centralizers of maximal algebraic tori, they are connected, there is a unique conjugacy class of them, and their union is dense in the group (see [3, §§ 11 and 12]). In connected compact real Lie groups, Cartan subgroups are connected, there is a unique conjugacy class of them, and their union covers the group (see [8, Chapitre VI, § 5]). It is however worth emphasizing at the outset that in real Lie groups Cartan subgroups need not be connected in general, a point also noticed by Chevalley in the introduction of [8, Chapitre VI]: '*Il convient de noter que les groupes de Cartan de* G *ne sont en général pas connexes.*' The diagonal subgroup of $SL_2(\mathbb{R})$ is maybe the first example of a nonconnected Cartan subgroup that one should bear in mind. Most of the difficulties for the study of these subgroups in the past, notably in the early work of Cartan, have been this failure of connectedness. This is something that will eventually need considerable attention in the present paper as well.

We are going to study Cartan subgroups from the model-theoretic point of view of groups definable in an o-minimal structure, that is, a first-order structure $\mathcal{M} = \langle M, \leqslant, \cdots \rangle$ equipped with a total, dense, and without end-points definable order $\leqslant$, and such that every definable subset of M is a boolean combination of intervals with end-points in $M \cup \{\pm\infty\}$. The most typical example of an o-minimal structure is of course the ordered field $\mathbb{R}$ of the reals, but there are richer o-minimal structures, such as the field of the reals equipped in addition with the exponential function [35].

In order to deal with the nonconnectedness of Cartan subgroups in general, we will use the following notion. If G is a group definable in an arbitrary structure $\mathcal{M}$, then we say that it is *definably connected* if and only if it has no proper subgroup of finite index definable in the sense of $\mathcal{M}$. A subgroup of a group G definable in $\mathcal{M}$ is called a *Carter* subgroup of G if it is definable and definably connected (in the sense of $\mathcal{M}$ as usual), and nilpotent and of finite index in its normalizer in G. All the notions of definability depend on a ground structure $\mathcal{M}$, which in the present paper will typically be an o-minimal structure. The notion of a Carter subgroup first appeared in the case of finite groups as *nilpotent and selfnormalizing* subgroups. A key feature is that, in the case of finite solvable groups, they exist and are conjugate [7]. For infinite groups, the notion we are adopting here, incorporating definability and definable connectedness, comes from the theory of groups of finite Morley rank. That theory is another classical branch of group theory in model theory, particularly designed at generalizing algebraic groups over algebraically closed fields. Carter subgroups of groups of finite Morley rank exist [20], and it is conjectured that they are conjugate [15]. However, it is not known if Cartan subgroups of such groups are definably connected; if so, they would also be Carter subgroups of the corresponding group (see Lemma 5(a′) below). We note that the selfnormalization from the finite case becomes an almost selfnormalization property, and indeed the finite group $N_G(Q)/Q$ associated to a Carter subgroup Q typically generalizes the notion of the *Weyl group* relative to Q. This is something that will also make perfect sense here in the case of groups definable in o-minimal structures.

We will see shortly in § 2 that, for groups definable in o-minimal structures, and actually for groups with the mere descending chain condition on definable subgroups, there is an optimal correspondence between Cartan subgroups and Carter subgroups: the latter are exactly the definably connected components of the former. In particular, Cartan subgroups are automatically definable subgroups, a point not following from the definition of Chevalley in general, but which is always going to be true here.

In §§ 3–6, we will relate Cartan and Carter subgroups to a well-behaved notion of dimension for sets definable in an o-minimal structure, notably to *slight genericity* (having maximal dimension) or to *largeness* (having smaller codimension). We will

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mainly develop their *generous* analogs: *slightly generous* and *largely generous*, where one actually considers the slight (respectively, large) genericity of the union of conjugates of a given set. The techniques and results here will be substantial adaptations and generalizations from [20, 21] in the finite Morley rank case, and our arguments for Cartan and Carter subgroups of groups definable in o-minimal structure will very much depend on dimensional computations and generosity arguments. We will make such dimensional computations in a rather axiomatic framework, essentially with the mere existence of a definable and additive dimension, since they apply as such in many other contexts (for example, groups of finite Morley rank, groups in supersimple theories of finite rank, and groups definable over the p-adics).

Our main result can be summarized as follows.

Theorem 1. *Let G be a group definable in an o-minimal structure. Then Cartan subgroups of G exist, are definable, and fall into finitely many conjugacy classes.*

Our proof of Theorem 1 will also strongly depend on the main structural theorem about groups definable in o-minimal structures. It says in essence that any definably connected group G definable in an o-minimal structure is, modulo a largest normal solvable (and definable) subgroup $R(G)$, a direct product of finitely many definably simple groups which are essentially 'known' as groups of Lie type. Hence our proof will consist in an analysis of the interplay between these definably simple factors and the relevant definably connected solvable subgroups of G. Results specific about groups definable in an o-minimal structure which are used here will be reviewed in § 7.

A large part of the work will thus be concerned with the case of definably connected solvable groups. In this case we will make a strong use of the previously mentioned largeness and generosity arguments. Mixing them with more algebraic inductive arguments inspired by [13] in the finite Morley rank case, we will obtain the following result in § 8.

Theorem 40. *Let G be a definably connected solvable group definable in an o-minimal structure. Then Cartan subgroups of G exist and are conjugate, and they are definable, definably connected, selfnormalizing, and largely generous. Moreover, for any Cartan subgroup Q, the (definable) set of elements of Q contained in a unique conjugate of Q is large in Q and largely generous in G.*

A definably connected group is *semisimple* if it has a finite center and modulo that center abelian normal subgroups are trivial. One of the main theorems about groups definable in o-minimal structures actually says that any such semisimple group with a trivial center is a direct product of definably simple groups, with each factor a 'known' group of Lie type modulo certain elementary equivalences. We will review certain facts that are more or less classical about Cartan subgroups of Lie groups in § 9. In § 10, we will transfer the theory of Cartan subgroups of Lie groups to definably simple groups and get a quite complete description of Cartan subgroups of definably simple groups definable in o-minimal structures.

In § 11, we will elaborate further on the definably simple case to get a similarly quite complete description of Cartan subgroups of semisimple groups definable in o-minimal structures, obtaining the following general theorem.

Theorem 63 (light version). *Let G be a definably connected semisimple group definable in an o-minimal structure. Then G has (definable) Cartan subgroups, and the following hold.*

(1) *G has only finitely many conjugacy classes of Cartan subgroups.*

(2) *If Q_1 and Q_2 are Cartan subgroups and $Q_1^\circ = Q_2^\circ$, then $Q_1 = Q_2$.*

(3) *If Q is a Cartan subgroup, then $Z(G) \leqslant Q$, $Q' \leqslant Z(G)$, and $Q^\circ \leqslant Z(Q)$.*

(4) *If Q is a Cartan subgroup and $a \in Q$, then aQ° is slightly generous in G.*

(5) *The union of all Cartan subgroups, which is definable by* (1)*, is large in G.*

The general case of a definably connected group G definable in an o-minimal structure will be considered in § 12. In this case, we have both G not solvable and not semisimple, or, in other words,

$$G/R^\circ(G) \neq 1 \quad \text{and} \quad R^\circ(G) \neq 1.$$

In that case, Theorem 1 follows rapidly from Theorems 40 and 63, but some natural questions will remain without an answer here. The most important one is maybe the following: if Q is a Cartan subgroup of G, is it the case that $QR^\circ(G)/R^\circ(G)$ is a Cartan subgroup of the semisimple quotient $G/R^\circ(G)$? This question is indeed equivalent (see below Remark 77) to the fact that Cartan subgroups of $G/R^\circ(G)$ are exactly of the form $QR^\circ(G)/R^\circ(G)$ for some Cartan subgroup Q of G. We will only manage to prove that, for a Cartan subgroup Q of G, the group $QR^\circ(G)/R^\circ(G)$ is a finite index subgroup of a Cartan subgroup of $G/R^\circ(G)$, obtaining in particular the expected lifting for the corresponding Carter subgroups. Getting the exact lifting of Cartan subgroups seems to be related to interesting new problems of representation theory in a definable context. In any case, we will mention all that we managed to prove on the correlations between Cartan subgroups of G and of $G/R^\circ(G)$, trying also to work with a not necessarily definably connected ambient group G when possible. We will conclude in § 13 with further comments on certain specialized topics, including algebraic or compact factors and Weyl groups relative to the various Cartan subgroups.

In this paper, definability always means definability with parameters. We refer to [24] for a complete introduction to groups definable in o-minimal structures. By recent results in [12], every group interpretable in an o-minimal structure is also definable in it; hence our results on definable groups also apply to the interpretable ones. Actually, we will also work with quotient sets of definable groups. Even though an arbitrary o-minimal structure does not eliminate imaginaries in general, any group definable in an arbitrary o-minimal structure eliminates imaginaries, and actually has definable choice functions in a very strong sense [10, Theorem 7.2]. In particular, imaginaries coming from a group definable in an o-minimal structure will always be considered as definable in what follows, and can be equipped with a finite dimension as any definable set.

We refer to [32, Chapter 4] or [30] for the dimension of sets definable in o-minimal structures.

Before we enter into the core of the paper, maybe is worth mentioning that the results below are stated with rather weak assumptions. We have done so with the aim of both putting the results in their most natural context and being able to apply them to different contexts (as we have already mentioned). For that reason now, in the present paper, we do not use the machinery of Lie algebras available by [26] in o-minimal expansions of real closed field.

2. Cartan subgroups and Carter subgroups

We first consider the relations between Cartan and Carter subgroups of groups definable in o-minimal structures. Actually, by [30, Remark 2.13], such groups satisfy the *descending chain condition* on definable subgroups (*dcc* for short), and we will analyze these relations in the more natural context of groups with the *dcc*. Throughout the present section, G is a group definable in a structure $\mathcal{M}$, and definability may refer to $\mathcal{M}^{\mathrm{eq}}$, and we say that G satisfies the *dcc* if any strictly descending chain of definable subgroups is stationary after finitely many steps. Notice that the *dcc* always passes to quotients by definable normal subgroups.

We first list some general facts needed in what follows.

Fact 2 ([1, Fact 3.1]). *Let G be a definably connected group defined in an arbitrary structure.*

(a) *Any definable action of G on a finite set is trivial.*

(b) *If $Z(G)$ is finite, then $G/Z(G)$ is centerless.*

In a group with the *dcc*, any subset X is contained in a smallest definable subgroup $H(X)$ called the *definable hull* of X: take $H(X)$ to be the intersection of all definable subgroups of G containing X.

Fact 3 ([1, 3.3 and 3.4]). *Let G be a group with the dcc and X a subset of G.*

(a) *If X is K-invariant for some subset K of G, then $H(X)$ is K-invariant as well.*

(b) *If X is a nilpotent subgroup of G, then $H(X)$ is nilpotent of the same nilpotency class.*

We now mention an infinite version of the classical *normalizer condition* in finite nilpotent groups.

Lemma 4. *Let G be a nilpotent group with the dcc on definable subgroups, or merely such that each definable subgroup has a definably connected definable subgroup of finite index. If H is a definable subgroup of infinite index in G, then $N_G(H)/H$ is infinite.*

Proof. For instance, one may argue formally as in [31, Proposition 1.12]. □

We will use the notation [] when we apply right operators (e.g., the connected component in Lemma 5(b) below) to expressions involving several symbols.

Lemma 5. *Let G be a group with the dcc.*

(a) *If Q is a maximal nilpotent subgroup of G, then Q is definable.*

(a′) *If Q is a Cartan subgroup of G, then Q is definable and Q° is a Carter subgroup of G.*

(b) *If Q is a Carter subgroup of G, then Q is contained in a maximal nilpotent subgroup $\tilde{Q}$ of G, and any such subgroup $\tilde{Q}$ is a Cartan subgroup of G with $[\tilde{Q}]^\circ = Q$.*

Proof. (a) By Fact 3(b).

(a′) Q is definable by item (a). Since Q° is a normal subgroup of Q of finite index in Q, Q° is a finite index subgroup of $N_G(Q^\circ)$, and Q° is a Carter subgroup of G.

(b) A definable nilpotent subgroup H containing Q must satisfy $H^\circ = Q$, by Lemma 4, and thus $H \leqslant N_G(H^\circ) = N_G(Q)$. Now Fact 3(b) implies that any nilpotent subgroup H containing Q satisfies $Q \leqslant H \leqslant N_G(Q)$. Since $N_G(Q)/Q$ is finite, there are maximal such subgroups, proving our first claim.

Now fix any such maximal nilpotent subgroup $\tilde{Q}$. It is definable by item (a), and we have already seen that $Q = [\tilde{Q}]^\circ$, and $\tilde{Q} \leqslant N_G([\tilde{Q}]^\circ) = N_G(Q)$. We now check that $\tilde{Q}$ is a Cartan subgroup. Let X be any normal subgroup of finite index of $\tilde{Q}$. We first observe that $H^\circ(X) = Q$: since $\tilde{Q}$ is definable we get $H^\circ(X) \leqslant [\tilde{Q}]^\circ = Q$, and since $H^\circ(X)$ must have finite index in $\tilde{Q}$ we get the desired equality. Now, by Fact 3(a), $N_G(X)$ normalizes $H^\circ(X) = Q$, so $X \leqslant N_G(X) \leqslant N_G(Q)$. Since X has finite index in $\tilde{Q}$ and $\tilde{Q}$ has finite index in $N_G(Q)$, X has finite index in $N_G(Q)$, and in particular X has finite index in $N_G(X)$. □

Applying Lemma 5, we have thus that in groups definable in o-minimal structures Carter subgroups are *exactly* the definably connected components of Cartan subgroups, with the latter always definable. We also note that Lemma 5(a) gives the automatic definability of unipotent subgroups in those contexts where it implies maximal nilpotence (see [14] for different concepts of unipotence). However, such unipotent subgroups are in general not almost selfnormalizing, e.g., the subgroup of $\mathrm{SL}_2(\mathbb{R})$ formed by the upper unitriangular matrices. This subgroup also serves as an example that condition (a) in the definition of a Cartan subgroup is not enough (the Cartan subgroups of $\mathrm{SL}_2(\mathbb{R})$ are described below in Remark 57).

We also note that, if Q is a maximal nilpotent subgroup of a group G with the *dcc*, then Q is a Cartan subgroup of G if and only if Q° is a Carter subgroup (by Lemma 5) and the latter is also equivalent to $N_G(Q^\circ)/Q^\circ$ being finite. Finally, a selfnormalizing Carter subgroup must be a Cartan subgroup, by Lemma 5(b), a definably connected Cartan subgroup must be a Carter subgroup, and Carter subgroups are maximal among definably connected nilpotent subgroups.

Definably connected nilpotent groups definable in o-minimal structures are divisible, by [10, Theorem 6.10], so it is worth bearing in mind that the following always applies in groups definable in o-minimal structures.

Fact 6 ([1, Lemma 3.10]). *Let G be a nilpotent group with the dcc and such that G° is divisible. Then $G = B * G^\circ$ (central product, i.e., $G = BG^\circ$ and $[B, G^\circ] = 1$) for some finite subgroup B of G.*

When Fact 6 applies, one can strengthen Lemma 5(b) as follows. Again the following statement is valid in groups definable in an o-minimal structure, because they cannot contain an infinite increasing chain of definably connected subgroups (by the existence of a well-behaved notion of dimension [24, Corollary 2.4]).

Henceforth, the notation $K < G$ denotes that K is a *proper* subgroup of G.

Lemma 7. *Let G be a group with the dcc. Assume that definably connected definable nilpotent subgroups of G are divisible, and that G contains no infinite increasing chain of such subgroups. Then any definably connected definable nilpotent subgroup of G is contained in a maximal nilpotent subgroup of G.*

Proof. Let N be a definably connected nilpotent subgroup of G. By assumption, N is contained in a definably connected definable nilpotent subgroup N_1 which is maximal for inclusion. It suffices to show that N_1 is then contained in a maximal nilpotent subgroup of G, and by Fact 3(b) we may consider only definable nilpotent subgroups containing N_1. It suffices then to show that any strictly increasing chain of definable nilpotent subgroups $N_1 < N_2 < \cdots$ is stationary after finitely many steps.

Assume towards a contradiction that $N_1 < N_2 < \cdots$ is such an infinite increasing chain of definable nilpotent subgroups. Recall that $N_1 = N_1^\circ$, and notice also that $N_i^\circ = N_1$ for each i, since N_1 is maximal subject to being definably connected and containing N. By Fact 6, each N_i has the form $B_i * N_1$ for some finite subgroup $B_i \leqslant N_i$, and in particular $N_i \leqslant C_G(N_1) \cdot N_1$. We may thus replace G by the definable subgroup $C_G(N_1) \cdot N_1$.

Let X be the union of the groups N_i. Working modulo the normal subgroup N_1, we have an increasing chain of finite nilpotent groups. Now X/N_1 is a periodic locally nilpotent group with the *dcc* on centralizers, and by [6, Theorem A] it is nilpotent-by-finite. Replacing X by a finite index subgroup of X if necessary, we may thus assume that X/N_1 is nilpotent and infinite. Since $G = C_G(N_1) \cdot N_1$, the nilpotency of X/N_1 and of N_1 forces X to be nilpotent (of nilpotency class bounded by the sum of that of X/N_1 and N_1). Replacing X by $H(X)$, we may now assume with Fact 3(b) that X is a definable nilpotent subgroup containing N_1 as a subgroup of infinite index. Then $N_1 < X^\circ$, a contradiction to the maximality of N_1. □

Before moving ahead, it is worth mentioning concrete examples of Cartan subgroups of real Lie groups to be kept in mind in the present paper. In $\mathrm{SL}_2(\mathbb{R})$ there are up to conjugacy two Cartan subgroups, the subgroup of diagonal matrices $Q_1 \simeq \mathbb{R}^\times$, noncompact and not connected, with corresponding Carter subgroup $Q_1^\circ \simeq \mathbb{R}^{>0}$, and $Q_2 = \mathrm{SO}_2(\mathbb{R})$ isomorphic to the circle group, compact and connected and hence also a Carter subgroup (see [22, p. 141–142] for more details). More generally, and again referring to [22], the group $\mathrm{SL}_n(\mathbb{R})$ has up to conjugacy $\lfloor \frac{n}{2} \rfloor + 1$ Cartan subgroups:

$$Q_j \simeq [\mathbb{C}^\times]^{j-1} \times [\mathbb{R}^\times]^{n-2j+1} \quad \text{where } 1 \leqslant j \leqslant \left\lfloor \frac{n}{2} \right\rfloor + 1,$$

except in the case $n = 2(j-1)$, where we also have $Q_{\frac{n}{2}+1} \simeq [\mathbb{C}^\times]^{\frac{n}{2}-1} \times \mathrm{SO}_2(\mathbb{R})$.

We will need the following lemma relating the center to Cartan and Carter subgroups. For any group G we define the iterated centers $Z_n(G)$ as follows: $Z_0(G) = \{1\}$; and by induction $Z_{n+1}(G)$ is the preimage in G of the center $Z(G/Z_n(G))$ of $G/Z_n(G)$.

Lemma 8. *Let G be a group, and for $n \geqslant 0$ let $Z_n := Z_n(G)$.*

(a) *If Q is a Cartan subgroup of G, then $Z_n \leqslant Q$ and Q/Z_n is a Cartan subgroup of G/Z_n, and conversely every Cartan subgroup of G/Z_n has this form.*

(b) *If G satisfies the dcc, then Carter subgroups of G/Z_n are exactly subgroups of the form $Q^\circ Z_n/Z_n$, for Q a Cartan subgroup of G.*

Proof. We may freely use the fact that the preimage in G of a nilpotent subgroup of G/Z_n is nilpotent.

(a) Clearly $Z_n \leqslant Q$, by maximal nilpotence of Q. Clearly also, Q/Z_n is nilpotent maximal in the quotient $\overline{G} = G/Z_n$. Let $\overline{X}$ be a normal subgroup of finite index of $\overline{Q} = Q/Z_n$, for some subgroup X of G containing Z_n. The preimage in G of $N_{\overline{G}}(\overline{X})$ normalizes X, which clearly is normal and has finite index in Q. Since Q is a Cartan subgroup of G, we easily get that $\overline{X}$ has finite index in $N_{\overline{G}}(\overline{X})$.

Conversely, let Q be a subgroup of G containing Z_n such that Q/Z_n is a Cartan subgroup of $\overline{G} = G/Z_n$. Clearly Q has to be maximal nilpotent in G. Let X be a normal finite index subgroup of Q. $N_G(X)$ normalizes $\overline{X}$ modulo Z_n, so it must contain $\overline{X}$ as a finite index subgroup, and then X is also a finite index subgroup of $N_G(X)$.

(b) By item (a), Cartan subgroups of G/Z_n are exactly of the form Q/Z_n for a Cartan subgroup Q of G containing Z_n. So Carter subgroups of G/Z_n are by Lemma 5 exactly of the form $[Q/Z_n]^\circ = Q^\circ Z_n/Z_n$, for Q a Cartan subgroup of G. □

Finally, we will also use the following lemma describing Cartan subgroups of central products.

Lemma 9. *Let $G = G_1 * \cdots * G_n$ be a central product of finitely many and pairwise commuting groups G_i. Then Cartan subgroups of G are exactly of the form $Q_1 * \cdots * Q_n$, where each Q_i is a Cartan subgroup of G_i.*

Proof. It suffices to prove our claim for $n = 2$. For $i = 1$ and 2, and X an arbitrary subset of G, let $\pi_i(X) = \{g \in G_i \mid \exists h \in G_{i+1}\ gh \in X\}$, where the indices i are of course considered modulo 2. It is clear that, when X is a subgroup of G, $\pi_i(X)$ is a subgroup G_i. If X is nilpotent (of nilpotency class k), then $\pi_i(X)$ is nilpotent (of nilpotency class at most $k+1$): it suffices to consider G/G_{i+1} and to use the fact that $G_1 \cap G_2 \leqslant Z(G_i)$.

Let Q be a Cartan subgroup of $G_1 * G_2$. Since $Q \leqslant \pi_1(Q) * \pi_2(Q)$, the maximal nilpotence of Q forces equality. Now it is clear that each $\pi_i(Q)$ is maximal nilpotent in G_i, by maximal nilpotence of Q again. Let now X be a normal subgroup of $\pi_1(Q)$ of finite index. Then $N_{G_1}(X) * \pi_2(Q)$ normalizes $X * \pi_2(Q)$ and, as the latter is a normal subgroup of finite index in Q, one concludes that X has finite index in $N_{G_1}(X)$. Hence $\pi_1(Q)$ is a Cartan subgroup of G_1. Similarly, $\pi_2(Q)$ is a Cartan subgroup of G_2.

Conversely, let Q be a subgroup of G of the form $Q_1 * Q_2$ for some Cartan subgroups Q_i of G_i. Since each Q_i is maximal nilpotent in G_i, it follows, considering projections as above, that Q is maximal nilpotent in G. Let now X be a normal subgroup of Q of finite index. Then $\pi_i(N_G(X))$ normalizes the normal subgroup of finite index $\pi_i(X)$ of Q_i. Since Q_i is a Cartan subgroup of G_i, it follows that $\pi_i(X)$ has finite index in $\pi_i(N_G(X))$. Finally, since $X \leqslant \pi_1(X) * \pi_2(X) \leqslant Q$, we get that X has finite index in $N_G(X)$. □

The special case of a direct product in Lemma 9 has also been observed in [8, Chapter VI, §4, Proposition 3].

Corollary 10. *Let $G = G_1 \times \cdots \times G_n$ be a direct product of finitely many groups G_i. Then Cartan subgroups of G are exactly of the form $Q_1 \times \cdots \times Q_n$, where each Q_i is a Cartan subgroup of G_i.*

3. Dimension and unions

In this section, we work with a structure such that each nonempty definable set is equipped with a dimension in $\mathbb{N}$ satisfying the following axioms for any nonempty definable sets A and B.

(A1) **(Definability)** If f is a definable function from A to B, then the set $\{b \in B \mid \dim(f^{-1}(b)) = m\}$ is definable for every m in $\mathbb{N}$.

(A2) **(Additivity)** If f is a definable function from A to B, whose fibers have constant dimension m in $\mathbb{N}$, then $\dim(A) = \dim(\mathrm{Im}(f)) + m$.

(A3) **(Finite sets)** A is finite if and only if $\dim(A) = 0$.

(A4) **(Monotonicity)** $\dim(A \cup B) = \max(\dim(A), \dim(B))$.

In an o-minimal structure, definable sets are equipped with a finite dimension satisfying all these four axioms, by [32, Chapter 4] or [30]. Hence our reader only interested in groups definable in o-minimal structures may read all the following dimensional computations in the restricted context of such groups. But, as mentioned in the introduction, such computations are relevant in other contexts as well (for example, groups of finite Morley rank, groups in supersimple theories of finite rank, and groups definable over the p-adics), and thus we will proceed with the mere axioms A1–A4.

Axioms A2 and A3 guarantee that, if f is a definable bijection between two definable sets A and B, then $\dim(A) = \dim(B)$. Axiom A4 is a strong form of monotonicity in the sense that $\dim(A) \leqslant \dim(B)$ whenever $A \subseteq B$.

Definition 11. Let $\mathcal{M}$ be a first-order structure equipped with a dimension dim on definable sets, and let $X \subseteq Y$ be two definable sets. We say that X is:

(a) *slightly generic* in Y whenever $\dim(X) = \dim(Y)$;

(b) *generic* in Y whenever Y is a definable group covered by finitely many translates of X;

(c) *large* in Y whenever $\dim(Y \setminus X) < \dim(Y)$.

Clearly, genericity and largeness both imply slight genericity when the dimension satisfies axioms A1–A4. If G is a group definable in an o-minimal structure and X

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is a large definable subset of G, then X is generic: see [30, Lemma 2.4] for a proof by compactness, and [25, § 5] for a proof with precise bounds on the number of translates needed for genericity. In what follows we are only going to use dimensional computations: hence the notions of slight genericity and of largeness. We are not going to use the notion of genericity (which is imported from the theory of stable groups in model theory), but we will make some apparently quite new remarks on genericity and Cartan subgroups in real Lie groups (Remark 57 below).

Our arguments for Cartan subgroups in groups definable in o-minimal structures will very much depend on computations of the dimension of their unions in the style of [20], and to compute the dimension of a union of definable sets we adopt the following geometric argument essentially due to Cherlin.

Assume from now on that X_a is a uniformly definable family of definable sets, with a varying in a definable set A and such that $X_a = X_{a'}$ if and only if $a = a'$. We have now a combinatorial geometry, where the set of points is $U := \bigcup_{a \in A} X_a$, the set of lines is the set $\{X_a \mid a \in A\}$ in definable bijection with A, and the incidence relation is the natural one. The set of *flags* is then defined to be the subset of couples (x, a) of $U \times A$ such that $x \in X_a$. By projecting the set of flags on the set of points, one sees with axiom A1 that, for any r such that $0 \leqslant r \leqslant \dim(A)$, the set

$$U_r := \{x \in U \mid \dim(\{a \in A \mid x \in X_a\}) = r\}$$

is definable. In particular, each subset of the form $[X_a]_r := X_a \cap U_r$, i.e., the set of points x of X_a such the set of lines passing through x has dimension r, is definable as well.

Proposition 12. *In a structure equipped with a dimension satisfying axioms* A1– 2*, let X_a be a uniformly definable family of sets, with a varying in a definable set A and such that $X_a = X_{a'}$ if and only if $a = a'$. Suppose, for some r such that $0 \leqslant r \leqslant \dim(A)$, that $[X_a]_r$ is nonempty and that $\dim([X_a]_r)$ is constant as a varies in A. Then*

$$\dim(U_r) + r = \dim(A) + \dim([X_a]_r).$$

Proof. One can consider the definable subflag associated to $U_r = [\bigcup_{a \in A} X_a]_r$ in the point/line incidence geometry described above. By projecting this definable set on the set of points and on the set of lines respectively, one finds using axiom A2 of the dimension the desired equality as in [20, § 2.3]. □

Given a permutation group (G, Ω) and a subset X of Ω, we denote by $N(X)$ and by $C(X)$ the *setwise* and the *pointwise* stabilizer of X, respectively; that is, $G_{\{X\}}$ and $G_{(X)}$ in the usual permutation group theory notation. We also denote by X^G the set $\{x^g \mid (x, g) \in X \times G\}$, where x^g denotes the image of x under the action of g, as in the case of an action by conjugation. Subsets of the form X^g for some g in G are also called G-*conjugates* of X. Notice that the set X^G can be seen, alternatively, as the union of G-orbits of elements of X, or also as the union of G-conjugates of X. When considering the action of a group on itself by conjugation, as we will do below, all these terminologies and notations are the usual ones, with $N(X)$ and $C(X)$ the *normalizer* and the *centralizer* of X respectively.

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We shall now apply Proposition 12 in the context of permutation groups in a way much reminiscent of [21, Fact 4]. For that purpose we will need that the dimension is well defined on certain imaginaries, and for that purpose we will make the simplifying assumption that the theory considered eliminates such specific imaginaries. We recall that groups definable in o-minimal structures eliminate all imaginaries, by [10, Theorem 7.2], so these technical assumptions will always be verified in this context. (And our arguments are also valid in any context where the dimension is well defined and compatible in the relevant imaginaries.) For any quotient $X/\sim$ associated to an equivalence relation $\sim$ on a set X, we call any subset of X intersecting each equivalence class in exactly one point *transversal*. Throughout the paper we are going to consider quotients with definable transversals. We do so to simplify notation, even though it would be enough to consider the weaker property that each relevant quotient has a bijective definable set.

Corollary 13. *Let (G, Ω) be a definable permutation group in a structure equipped with a dimension satisfying axioms* A1–A3*, and let X be a definable subset of Ω such that $G/N(X)$ (right cosets) has a definable transversal A. Suppose that, for some r between 0 and $\dim(A)$, the definable subset $X_r := \{x \in X \mid \dim(\{a \in A \mid x \in X^a\}) = r\}$ is nonempty. Then*

$$\dim(X_r{}^G) = \dim(G) + \dim(X_r) - \dim(N(X)) - r.$$

Proof. We can apply Proposition 12 with the uniformly definable family of G-conjugates of X, which is parameterized as $\{X^a \mid a \in A\}$ since A is a definable transversal of $G/N(X)$. Notice that the sets $[X^a]_r$ are in definable bijection, as pairwise G-conjugates, and hence all have the same dimension. Notice also that $\dim(A) = \dim(G) - \dim(N(X))$, by the additivity of the dimension and its invariance under definable bijections. □

The following corollary, which is crucial in what follows, can be compared to [21, Corollary 5].

Corollary 14. *Assume furthermore in Corollary 13 that the dimension satisfies axiom A4, and that $\dim(G) = \dim(\Omega)$ and $\dim(X) \leqslant \dim(N(X))$. Then*

$$\dim(X^G) = \dim(\Omega) \text{ if and only if } \dim(X_0) = \dim(N(X))(= \dim(X)).$$

In this case, $X_0{}^G$ is large in X^G.

Proof. If $\dim(X^G) = \dim(\Omega)$, then one has for some r as in Corollary 13 that $\dim(X_r{}^G) = \dim(\Omega)$ by axiom A4, and then

$$0 \leqslant r = \dim(X_r) - \dim(N(X)) \leqslant \dim(X) - \dim(N(X)) \leqslant 0$$

by monotonicity of the dimension, showing that all these quantities are equal to 0. In particular, $r = 0$, and $\dim(X_0) = \dim(N(X))$. Conversely, if $\dim(X_0) = \dim(N(X))$, then $\dim(X_0{}^G) = \dim(G) = \dim(\Omega)$, by Corollary 13.

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Assume now that the equivalent conditions above are satisfied. The first part of the proof above shows that $\dim({X_r}^G) = \dim(X^G)$ $(= \dim(\Omega))$ can occur only for $r = 0$. Hence ${X_0}^G$ is large in X^G by axiom A4 again. □

Remark 15. In general it seems that one cannot conclude also that X_0 is large in X in Corollary 14. Hence, in principle, one could have $\dim(X_r) = \dim(X)$ for some $r > 0$ and $\dim({X_r}^G) = \dim(\Omega) - r$.

In the remainder of the section we will always consider the action of a group G on itself by conjugation, so the condition $\dim(G) = \dim(\Omega)$ will always be met in Corollary 14. Then we can apply Corollary 14 with X any normalizing coset of a definable subgroup H of G, as commented in [21, page 1064]. More generally, we now see that we can apply it simultaneously to finitely many such cosets. We first elaborate on the notion of *generosity* defined in [20, 21] in the finite Morley rank case.

Definition 16. Let X be a definable subset of a group G definable in a structure equipped with a dimension satisfying axioms A1–A4. We say that X is:

(a) *slightly generous* in G whenever X^G is slightly generic in G;

(b) *generous* in G whenever X^G is generic in G;

(c) *largely generous* in G whenever X^G is large in G.

Corollary 17. *Suppose that H is a definable subgroup of a group G definable in a structure equipped with a dimension satisfying axioms* A1*–A4, and suppose that W is a finite subset of $N(H)$ such that $G/N(WH)$ has a definable transversal. Then WH is slightly generous in G if and only if*

$$\dim([WH]_0) = \dim(N(WH)).$$

In this case, $[WH]_0^G$ is large in $[WH]^G$, and $\dim([WH]_0) = \dim(WH) = \dim(H) = \dim(N(WH))$.

Proof. Let $X = WH$. Since W is finite, X is definable. In order to apply Corollary 14, one needs to check that $\dim(X) \leqslant \dim(N(X))$. Of course, the subgroup H normalizes each coset wH, for each $w \in W \subseteq N(H)$, and in particular $H \leqslant N(WH)$. We get thus that $\dim(X) = \dim(WH) = \dim(H) \leqslant \dim(N(WH)) = \dim(N(X))$.

Now Corollary 14 gives our necessary and sufficient condition, and the largeness of $[WH]_0^G$ in $[WH]^G$. It also gives $\dim(X_0) = \dim(X) = \dim(N(X))$. We have seen already that $\dim(X) = \dim(H)$. □

The following lemma is a fundamental trick that is used below.

Lemma 18. *Let G be a group definable in a structure equipped with a dimension satisfying axioms* A1*–A4 and with the dcc. Let X be a definable subset of G, let X_0 be the subset of elements of X contained in only finitely many G-conjugates of X, and let U be a definable subset of X such that $U \cap X_0 \neq \emptyset$. Then $N^\circ(U) \leqslant N(X)$.*

Proof. As in [20, Lemma 3.3]. First, we note that $N(U)$ normalizes $U \cap X_0$. For, if $g \in N(U)$ and $h \in U \cap X_0$, then $h^g \in U \subseteq X$; hence $h^g \in U \cap X_0$. Therefore, $N^\circ(U)$ acts on the finite set $\{X^g \mid X^g \supseteq U \cap X_0, g \in N^\circ(U)\}$, by Fact 2(a) the action is trivial; so $N^\circ(U)$ fixes $X(\supseteq U \cap X_0)$, which means that $N^\circ(U) \leqslant N(X)$. □

4. Cosets arguments

Corollary 17 will be used at the end of this paper in certain arguments reminiscent of a theory of Weyl groups from [21]. Since such specific arguments follow essentially from Corollary 17, we insert here, as a warm up, a short section devoted to them.

Theorem 19. *Let G be a group definable in a structure equipped with a dimension satisfying axioms* A1–*A4 and with the dcc, let H be a slightly generous definable subgroup of G, and let w be an element normalizing H and such that $G/N(H)$ has a definable transversal. Then one of the following must occur.*

(a) *The coset wH is slightly generous in G.*
(b) *The definable set $\{h^{w^{n-1}} h^{w^{n-2}} \cdots h \mid h \in H\}$, where h^{w^i} is $(w^i)^{-1}hw^i$, is not large in H for any multiple n of the (necessarily finite) order of w modulo H. If w centralizes H, then $\{h^n \mid h \in H\}$ is not large in H.*

Proof. We proceed essentially as in [21, Lemmas 11–12]. Assume that wH is not slightly generous. In particular, $w \in N(H) \setminus H$, since H is slightly generous by assumption. By Corollary 17, H_0 is slightly generic in $N(H)$; in particular, H has finite index in $N(H)$. Of course, $N(wH) \leqslant N(H)$, since $H = \{ab^{-1} : a, b \in wH\}$, and one sees then that $N(wH)$ is exactly the preimage in $N(H)$ of the centralizer of w modulo H. To summarize, $H \leqslant N(wH) \leqslant N(H)$, with $N(H)/H$ finite. In particular, w has finite order modulo H. Notice also at this stage that $G/N(wH)$ has a definable transversal (of the form AX, where X is a definable transversal of $G/N(H)$ and A is a definable transversal of the finite quotient $N(H)/N(wH)$). Since we assume that wH is not slightly generous, Corollary 17 implies that $[wH]_0$ is not slightly generic in wH. In other words, the (definable) set of elements of the coset wH contained in infinitely many G-conjugates of wH is large in wH.

Assume towards a contradiction that $\{h^{w^{n-1}} h^{w^{n-2}} \cdots h \mid h \in H\}$ is large in H for n a multiple of the finite order of w modulo H. Let $\phi : wh \mapsto (wh)^n$ denote the definable map, from wH to H, consisting of taking n-powers. As

$$\phi(wH) = w^n \cdot \{h^{w^{n-1}} h^{w^{n-2}} \cdots h \mid h \in H\},$$

our contradictory assumption forces that $\phi(wH)$ must be large in H.

Then $H_0 \cap \phi(wH)$ must be slightly generic in H. Since the dimension can only decrease when taking images by definable functions, $\phi^{-1}(H_0 \cap \phi(wH))$ necessarily has to be slightly generic in the coset wH. Therefore one finds an element x in the intersection of this preimage with the large subset $[wH] \setminus [wH]_0$ of elements of wH contained in infinitely many G-conjugates of wH. Now, since $w^n \in H$ and $N(wH)$ has finite index in $N(H)$, it follows that $\phi(x) = x^n$ belongs to infinitely many G-conjugates of H, a contradiction, since $\phi(x)$ belongs to H_0. This proves our main statement in case (b).

For our last remark in case (b), notice that when w centralizes H one has $\{h^{w^{n-1}}h^{w^{n-2}}\cdots h \mid h \in H\} = \{h^n \mid h \in H\}$. □

Corollary 20. *Suppose additionally in Theorem 19 that w has order n modulo H and that H is n-divisible ($n \geqslant 1$). Then one of the following must occur.*

(a) *The coset wH is slightly generous in G.*

(b) *$C_H(w)$ is a proper subgroup of H.*

Proof. Suppose that both alternatives fail. Then $\{h^n \mid h \in H\}$ is not large in H, by Theorem 19, a contradiction, since this set is H by n-divisibility. □

The following corollary of Theorem 19 will be particularly adapted in what follows to Cartan subgroups of groups definable in o-minimal structures.

Corollary 21. *Suppose additionally in Theorem 19 that H is definably connected and divisible, and that $\langle w\rangle H$ is nilpotent. Then the coset wH is slightly generous in G.*

Proof. This is clear if w is in H, so we may assume that $w \in N(H) \setminus H$. As above, w has finite order modulo $H = H^\circ$. By the *dcc* of the ambient group and [1, Lemma 3.10], the coset wH contains a torsion element which commutes with $H = H^\circ$, and thus we may assume that $C_H(w) = H$. By divisibility of $H = H^\circ$, $\{h^n \mid h \in H\} = H$ is large in H, and by Theorem 19 the coset wH must be slightly generous in G. □

We will also use the following more specialized results in the same spirit, which apply as usual to nilpotent groups definable in o-minimal structures, by [10, Theorem 6.10].

Lemma 22. *Let H be a nilpotent divisible group definable in a structure equipped with a dimension satisfying axioms* A1–*A4, with the dcc, and with no infinite elementary abelian p-subgroups for any prime p. Let ϕ be the map consisting of taking n*th *powers for some $n \geqslant 1$. If X is a slightly generic definable subset of H, then $\phi(X)$ is slightly generic as well.*

Proof. Considering the dimension, it suffices to show that ϕ has finite fibers. Suppose that $a^n = b^n$ for some elements a and b in H. If $aZ(H) = bZ(H)$, then our assumption forces, with a fixed, that b can only vary in a finite set, as desired. Hence, working in $H/Z(H)$, it suffices to show that $a^n = b^n$ implies that $a = b$. But, by [1, Lemma 3.10(a′)], all definable sections of $H/Z(H)$ are torsion-free, and our claim follows easily by induction on the nilpotency class of $H/Z(H)$. □

Corollary 23. *Let Q be a nilpotent group definable in a structure equipped with a dimension satisfying axioms* A1–*A4, with the dcc, and with no infinite elementary abelian p-subgroups for any prime p. Suppose that Q° is divisible, and let $a \in Q$, let n be a multiple of the order of a modulo Q°, and let ϕ be the map consisting of taking n*th *powers. If X is a slightly generic definable subset of aQ°, then $\phi(X)$ is a slightly generic subset of Q°.*

Proof. By [1, Lemma 3.9], we may assume that a centralizes Q°. Now for any $x \in Q^\circ$ we have $\phi(ax) = a^n x^n$. Hence, if x varies in a slightly generic definable subset of Q°, then $\phi(ax)$ also varies in a slightly generic definable subset of Q° by Lemma 22 with $H = Q^\circ$. □

5. Generosity and lifting

In the present section, we study the behavior of slight or large generosity when passing to quotients by definable normal subgroups. We continue with the mere axioms A1–A4 of § 3 for the dimension. *We will assume that all the relevant quotients have definable transversals.* Everything in this section applies in particular to groups definable in o-minimal structures.

Proposition 24. *Let G be a group definable in a structure equipped with a dimension satisfying axioms* A1*–A4, let N be a definable normal subgroup of G, let H be a definable subgroup of G containing N, and let Y be a definable subset of H large in H.*

(a) *If H/N is slightly generous in G/N, then Y is slightly generous in G.*

(b) *If H/N is largely generous in G/N, then Y is largely generous in G.*

Proof. First note that H^G is a union of cosets of N, since $N \leqslant H$ and $N \trianglelefteq G$. Hence the slight (respectively, large) generosity of H/N in G/N forces the slight (respectively, large) generosity of H in G. In any case, $\dim(H^G) = \dim(G)$.

Replacing Y by Y^H if necessary, we may assume that $H \leqslant N(Y)$ and Y are large in H.

Claim 25. *Let $Z = H \setminus Y$. Then Z^G cannot be slightly generic in H^G.*

Proof. Suppose that Z^G is slightly generic in H^G. Then $\dim(Z^G) = \dim(H^G) = \dim(G)$. Since $Z \subseteq H \subseteq N_G(Z)$, Corollary 14 yields $\dim(Z) = \dim(N_G(Z))$. In particular, $\dim(Z) = \dim(H)$, a contradiction to the largeness of Y in H. □

(a) Since $\dim(H^G) = \dim(G)$ and $H^G = Y^G \cup Z^G$, claim 25 yields $\dim(Y^G) = \dim(G)$.

(b) In this case, H^G is large in G. Since $G = (G \setminus H^G) \sqcup (H^G \setminus Y^G) \sqcup Y^G$, claim 25 now forces Y^G to be large in G. □

Corollary 26. *Assume that G, N, H, and Y are as in Proposition 24, and that $Y = Q^H$ for some largely generous definable subgroup Q of H.*

(a) *If H/N is slightly generous in G/N, then so is Q in G*

(b) *If H/N is largely generous in G/N, then so is Q in G.*

Proof. It suffices to apply Proposition 24 with $Y = Q^H$, noticing that $Y^G = Q^G$. □

Corollary 27. *Assume furthermore that Q is a Carter subgroup of H in Corollary 26. Then, in both cases (a) and (b), Q is a Carter subgroup of G.*

Proof. By definition, Q is definable, definably connected, and nilpotent. So it suffices to check that Q is a finite index subgroup of $N_G(Q)$. But in any case, it follows from the slight generosity of Q in G given in Corollary 26 and from Corollary 17 that $\dim(Q) = \dim(N_G(Q))$. Now axiom A3 applies. □

6. Slightly generous nilpotent subgroups

In the present section, we shall rework arguments from [20] concerning slightly generous Carter subgroups. Throughout the section, G is a group definable in a structure with a dimension satisfying axioms A1–A4, and with the *dcc*. As in § 5, we will assume that all relevant quotients have definable transversals. Again, everything applies to groups definable in an o-minimal structure.

Lemma 28. *Let G be a group definable in a structure with a dimension satisfying axioms A1–A4, and with the dcc. Let H be a definable subgroup of G such that $N^\circ(H) = H^\circ$, let H_0 be the set of elements of H contained in only finitely many conjugates of H, and let N be a definable nilpotent subgroup of G such that $N \cap H_0$ is nonempty. Then $N^\circ \leqslant H^\circ$.*

Proof. Let $U = N \cap H$. By assumption, $U \cap H_0$ is nonempty, so, by Lemma 18, $N^\circ(U) \leqslant N^\circ(H) = H^\circ$. In particular, $N_N^\circ(U) \leqslant (N \cap H)^\circ = U^\circ$, which shows that U has finite index in $N_N(U)$. Now Lemma 4 shows that U must have finite index in N, and in particular $U^\circ = N^\circ$. Hence, $N^\circ = (N \cap H)^\circ \leqslant H^\circ$. □

Corollary 29. *Let G be a group definable in a structure with a dimension satisfying axioms A1–A4, and with the dcc. Let Q be a definable nilpotent slightly generous subgroup of G, and let Q_0 denote the set of elements of Q contained in only finitely many conjugates of Q. Then, the following hold.*

(a) *For any definable nilpotent subgroup N such that $N \cap Q_0 \neq \emptyset$, we have $N^\circ \leqslant Q^\circ$.*

(b) *For any g in G such that $Q_0 \cap Q^g \neq \emptyset$, we have that $Q^\circ = [Q^\circ]^g$.*

Proof. (a) As Q is slightly generous, we have $N^\circ(Q) = Q^\circ$, by Corollary 17. Hence Lemma 28 gives $N^\circ \leqslant Q^\circ$. (b) Item (a) applied with $N = Q^g$ yields $[Q^\circ]^g = [Q^g]^\circ \leqslant Q^\circ$. Now applying Lemma 4 shows that $[Q^\circ]^g$ cannot be of infinite index in Q° (as otherwise we would contradict that $N^\circ(Q) = Q^\circ$), and thus $[Q^\circ]^g = Q^\circ$. □

Corollary 30. *Suppose in addition in Corollary 29 that Q is a Carter subgroup of G. Then, for any $g \in Q_0$ and any definably connected definable nilpotent subgroup N containing g, we have $N \leqslant Q$. In particular, Q is the unique maximal definably connected definable nilpotent subgroup containing g, and the distinct conjugates of Q_0 are indeed disjoint, forming thus a partition of a slightly generic subset of G.*

Proof. First, let g and N be as in the statement, so in particular $N \cap Q_0 \neq \emptyset$, and by Corollary 29(a) we have $N \leqslant Q$. Suppose now that $Q_0^g \cap Q_0^h \neq \emptyset$ for some $g, h \in G$. Then $Q_0 \cap Q_0^{hg^{-1}} \neq \emptyset$, and hence by Corollary 29(b) we have $Q = Q^{hg^{-1}}$, so $Q_0^g = [Q^g]_0 = [Q^h]_0 = Q_0^h$. □

As a result, one also obtains the following general theorem, which can be compared to the main result of [20].

Theorem 31. *Let G be a group definable in a structure with a dimension satisfying axioms A1–A4, and with the dcc. Then G has at most one conjugacy class of largely*

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generous Carter subgroups. If such a Carter subgroup exists, then the set of elements contained in a unique conjugate of that Carter subgroup is large in G.

Proof. Let P and Q be two largely generous Carter subgroups of G. We want to show that P and Q are conjugate. We have ${P_0}^G$ and ${Q_0}^G$ large in G, by Corollary 17. Since the intersection of two large sets is nontrivial (and in fact large as well), we get that ${P_0}^G \cap {Q_0}^G$ is nonempty, so after conjugation we may thus assume that $P_0 \cap Q_0$ is nonempty. But then Corollary 30 gives $P = Q$.

Our last claim follows also from Corollary 30. □

7. On groups definable in o-minimal structures

We shall now collect results specific to groups definable in o-minimal structures which are needed in what follows. We recall that groups definable in o-minimal structures satisfy the *dcc* on definable subgroups [30, Remark 2.13], and o-minimal structures are equipped with a dimension satisfying axioms A1–A4 considered in the previous sections [32, Chapter 4]. As commented before, we can freely apply all the results of the preceding sections to the specific case of groups definable in an o-minimal structure. We also recall that all the technical assumptions on the existence of transversals in §§ 3–6 are satisfied, since groups definable in o-minimal structures eliminate all imaginaries, by [10, Theorem 7.2]. As mentioned in the introduction, we consider only *definable* groups, but by the results in [12] this class also includes groups *interpretable* in o-minimal structures.

Fact 32 ([1, § 6]). *Let G be a group definable in an o-minimal structure, with G° solvable, and let A and B be two definable subgroups of G normalizing each other. Then $[A, B]$ is definable, and definably connected whenever A and B are. In particular, if G is moreover definably connected and solvable, then G' is definable and definably connected.*

Any group G definable in an o-minimal structure has a largest normal nilpotent subgroup $F(G)$, which is also definable [1, Fact 3.5], and a largest normal solvable subgroup $R(G)$, which is also definable [1, Lemma 4.5].

Fact 33. *Let G be a definably connected solvable group definable in an o-minimal structure.*

(a) [10, *Theorem 6.9*]*G' is nilpotent.*

(b) [1, *Proposition 5.5*]*$G' \leqslant F^\circ(G)$. In particular, $G/F^\circ(G)$ and $G/F(G)$ are divisible abelian groups.*

(c) [1, *Corollary 5.6*] *If G is nontrivial, then $F^\circ(G)$ is nontrivial. In particular, G has an infinite abelian characteristic definable subgroup.*

(d) [1, *Lemma 3.6*] *If G is nilpotent and H is an infinite normal subgroup of G, then $H \cap Z(G)$ is infinite.*

In general, for any structure and any H and G definable subgroups of a given definable group, with G normalizing H, a *G-minimal* subgroup of H is an infinite G-invariant definable subgroup of H, which is minimal with respect to these properties. If moreover H satisfies the *dcc* on definable subgroups, then G-minimal subgroups of H always exist. As the definably connected component of a definable subgroup is a definably characteristic subgroup, we get also in this case that any G-minimal subgroup of H must be definably connected.

Lemma 34. *Let G be a definably connected solvable group definable in an o-minimal structure, and let A be a G-minimal subgroup of G. Then $A \leqslant Z^{\circ}(F(G))$, and $C_G(a) = C_G(A)$ for every nontrivial element a in A.*

Proof. By Fact 33(c), A has an infinite characteristic abelian definable subgroup. Therefore the G-minimality of A forces A to be abelian. In particular, $A \leqslant F(G)$. Since A is normal in $F(G)$, Fact 33(d) and the G-minimality of A now force that $A \leqslant Z(F(G))$. Since A is definably connected, we have indeed $A \leqslant Z^{\circ}(F(G))$.

Now $F(G) \leqslant C_G(A)$, and $G/C_G(A)$ is definably isomorphic to a quotient of $G/F(G)$. In particular, $G/C_G(A)$ is abelian, by Fact 33(b). If $A \leqslant Z(G)$, then clearly $C_G(a) = C_G(A)$ $(=G)$ for every a in A, and thus we may assume that $G/C_G(A)$ is infinite. Consider the semidirect product $A \rtimes (G/C_G(A))$. Since A is G-minimal, A is also $G/C_G(A)$-minimal. Now an o-minimal version of Zilber's Field Interpretation Theorem for groups of finite Morley rank [27, Theorem 2.6] applies directly to $A \rtimes (G/C_G(A))$. It says that there is an infinite interpretable field K, with $A \simeq K_+$ and $G/C_G(A)$ an infinite subgroup of $K^{\times}$, and such that the action of $G/C_G(A)$ on A corresponds to scalar multiplication. In particular, $G/C_G(A)$ acts *freely* (or *semiregularly* in another commonly used terminology) on $A \setminus \{1\}$. This means exactly that, for any nontrivial element a in A, $C_G(a) \leqslant C_G(A)$; i.e., $C_G(a) = C_G(A)$. □

For definably connected groups definable in an o-minimal structure which are not solvable, our study of Cartan subgroups will make heavy use of the main theorem about groups definable in o-minimal structures. It can be summarized as follows, compiling several papers to which we will refer immediately after the statement. Recall that a group is *definably simple* if the only definable normal subgroups are the trivial and the full subgroup.

Fact 35. *Let G be a definably connected group definable in an o-minimal structure $\mathcal{M}$. Then*

$$G/R(G) = G_1 \times \cdots \times G_n,$$

where each G_i is a definably simple infinite definable group. Furthermore, for each i, there is an $\mathcal{M}$-definable real closed field R_i such that G_i is $\mathcal{M}$-definably isomorphic to a semialgebraically connected semialgebraically simple linear semialgebraic group, definable in R_i over the subfield of real algebraic numbers of R_i.

Besides, for each i, either

(a) *$\langle G_i, \cdot\rangle$ and $\langle R_i(\sqrt{-1}), +, \cdot\rangle$ are bi-interpretable; in this case, G_i is definably isomorphic in $\langle G_i, \cdot\rangle$ to the $R_i(\sqrt{-1})$-rational points of a linear algebraic group; or*

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(b) $\langle G_i, \cdot \rangle$ *and* $\langle R_i, +, \cdot \rangle$ *are bi-interpretable; in this case,* G_i *is definably isomorphic in* $\langle G_i, \cdot \rangle$ *to the connected component of the* R_i*-rational points of an algebraic group without nontrivial normal algebraic subgroups defined over* R_i*.*

The description of $G/R(G)$ as direct product of definably simple definable groups can be found in [26, 4.1]. The second statement about definably simple groups is in [26, 4.1 and 4.4], with the remark concerning the parameters in the proof of [28, 5.1]. The final alternative for each factor, essentially between the complex case and the real case, is in [27, 1.1].

When applying Fact 35 in what follows, we will also use the following.

Remark 36. Let $\mathcal{M}$ be an o-minimal structure, let R be a real closed field definable in $\mathcal{M}$, and let X be an R-definable subset of some R^n. Then $\dim_{\mathcal{M}}(X) = \dim_R(X)$.

Proof. By o-minimality, $\mathcal{M}$ is a geometric structure [27, Definition 3.2]. Moreover, since $\dim_{\mathcal{M}}(R) = 1$, by [30, Proposition 3.11], we deduce that R itself is R-minimal in the sense of [27, Definition 3.3]. Hence, by [27, Lemma 3.5], we have that $\dim_{\mathcal{M}}(X) = \dim_{\mathcal{M}}(R) \dim_R(X) = \dim_R(X)$. □

We finish the present section with specific results about definably compact groups which might be used when such specific groups are involved in what follows.

Fact 37. *Let* G *be a definably compact definably connected group definable in an o-minimal structure.*

(a) [29, *Corollary 5.4*] *Either* G *is abelian or* $G/Z(G)$ *is semisimple. In particular, if* G *is solvable, then it is abelian.*

(b) [11, *Proposition 1.2*]G *is covered by a single conjugacy class of a definably connected definable abelian subgroup* T *such that* $\dim(T)$ *is maximal among dimensions of abelian definable subgroups of* G*.*

With Fact 37 we can entirely clarify properties of Cartan subgroups in the specific case of definably compact groups definable in o-minimal structures, with a picture entirely similar to that in compact real Lie groups.

Corollary 38. *Let* G *be a definably compact definably connected group definable in an o-minimal structure. Then Cartan subgroups* T *of* G *exist and are abelian, definable, definably connected, and conjugate, and* $G = T^G$*.*

Proof. Let T be a definably connected abelian subgroup as in Fact 37(b). We show that T is a Cartan subgroup of G. Indeed, since $G = T^G$, T is in particular slightly generous, and thus of finite index in its normalizer, by Corollary 17. Hence T is a Carter subgroup of G. Since $G = T^G$ again, and $t \in T \leqslant C^\circ(t)$ for every $t \in T$, we have the property that $g \in C^\circ(g)$ for every g in G.

We now prove our statement by induction on $\dim(G)$. By Lemma 5, $T \leqslant Q$ for some Cartan subgroup such that $Q^\circ = T$. This takes care of the existence of Cartan subgroups of G, and their definability follows from Lemma 5(a). We also have $G = T^G$.

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We now claim that $T = Q$. Otherwise, $T = Q^\circ < Q$, and we find by Fact 6 an element a in $Q \setminus T$ centralizing T. Since $a \in T^g$ for some $g \in G$, we have T and T^g in $C^\circ(a)$. Now the Carter subgroups T and T^g of $C^\circ(a)$ are conjugate by an element of $C^\circ(a)$, obviously if $C^\circ(a) = G$ and by induction otherwise. Since $a \in T^g \leqslant C^\circ(a)$, we get $a \in T$, a contradiction. Hence $T = Q$ is a Cartan subgroup of G.

It remains just to show that Cartan subgroups of G are conjugate. Let Q_1 be an arbitrary Cartan subgroup of G, and let z be a nontrivial element of $Z(Q_1)$ (Fact 6 and Fact 33(d)). We also have $z \in T^g$ for some $g \in G$, and thus $Q_1, T^g \leqslant C(z)$. If $C^\circ(z) < G$, the induction hypothesis applied in $C^\circ(z)$ yields the conjugacy of Q_1° and of T, giving also $Q_1 = Q_1^\circ$ by maximal nilpotence of T. So we may assume that $z \in Z(G)$. If $Z(G)$ is finite, then $G/Z(G)$ has a trivial center, by Fact 2(b), and the previous argument applied in $G/Z(G)$, together with Lemma 8(a), yields the conjugacy of Q_1 and T. There remains the case of $Z(G)$ infinite: then, applying the induction hypothesis in $G/Z(G)$, and using Lemma 8(a), also gives the conjugacy of Q_1 and T. This completes our proof. □

We have seen in the proof of Corollary 38 that the T of Fact 37(b) must be Cartan subgroups; then both types of subgroup coincide (by the conjugacy of Cartan subgroups) and so such T are conjugate. Note that the latter conjugacy result was also proved under an additional unnecessary hypothesis in [11, Proposition 1.2]. On the other hand, in [2], working over expansions of real closed fields, the author considers maximal tori of a definably compact group as those maximal abelian definably connected subgroups proving that they are all conjugate (so they coincide with the T of Fact 37(b)) and that their union is the whole group. Making use of all these results and over an expansion of a real closed field, the proof of Corollary 38 can be simplified.

Besides, we note that the maximal nilpotence of a Cartan subgroup T of a group G always implies that $C_G(T) = Z(T)$. In particular, in Corollary 38, $C_G(T) = T$, and the 'Weyl group' $W(G, T) := N_G(T)/C_G(T)$ acts faithfully on T.

Finally, we take this opportunity to mention, parenthetically, a refinement of Fact 37(a).

Corollary 39. *Let G be a definably compact definably connected group definable in an o-minimal structure. Then $R(G) = Z(G)$.*

Proof. By Fact 37(a) and [1, Lemma 3.13]. □

8. The definably connected solvable case

In the present section, we are going to prove the following theorem.

Theorem 40. *Let G be a definably connected solvable group definable in an o-minimal structure. Then Cartan subgroups of G exist and are conjugate, and they are definably connected, selfnormalizing, and largely generous in G. Moreover, for any Cartan subgroup Q, the (definable) set of elements of Q contained in a unique conjugate of Q is large in Q and largely generous in G.*

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To prove Theorem 40, we first look at the minimal configuration for our analysis, which can be thought of as an abstract analysis of Borel subgroups of SL_2 (over $\mathbb{C}$ or $\mathbb{R}$), first studied by Nesin in the case of groups of finite Morley rank [4, Lemma 9.14]. We recall that, given a group G and a normal subgroup N of G, a *complement* of N in G is any subgroup H of G such that $G = HN$ and $H \cap N = 1$.

Lemma 41. *Let G be a definably connected solvable group definable in an o-minimal structure, with G' a G-minimal subgroup and $Z(G)$ finite. Then $G = G' \rtimes Q$ for some (abelian) selfnormalizing definably connected definable largely generous complement Q. Moreover, we have the following.*

(a) $F(G) = Z(G) \times G' = C_G(G')$.

(b) *For any x in $G \setminus F(G)$, $xG' = x^{G'}$, $G = G' \rtimes C_G(x)$, and $C_G(x)$ is the unique conjugate of $C_G(x)$ containing x. And any two complements of G' are G'-conjugate.*

Proof. We elaborate on the proof given in [15, Theorem 3.14] in the finite Morley rank case. Since $Z(G)$ is finite, the definably connected group G is not nilpotent, by Fact 33(d), and in particular $C_G(G') < G$. By G-minimality of G' and Lemma 34, $G' \leqslant Z^\circ(F(G))$ and $C_G(a) = C_G(G')$ for every nontrivial element a of G'.

For any element x in $G \setminus C_G(G')$, we now show that $Q := C_G(x)$ is a required complement of G'. Since $x \notin C_G(G')$, $C_{G'}(x) = 1$ and in particular $\dim(x^G) \geqslant \dim(G')$. On the other hand, $x^G \subseteq xG'$ as G/G' is abelian, and it follows that $\dim(x^G) = \dim(G')$, or in other words that $\dim(G/Q) = \dim(G')$. Since $Q \cap G' = 1$, the definable subgroup $G' \rtimes Q$ has maximal dimension in G, and since G is definably connected we get that $G = G' \rtimes Q$. Of course $Q \simeq G/G'$ is abelian, and definably connected as G is. We also see that $N_{G'}(Q) = C_{G'}(Q) = 1$, since $C_{G'}(x) = 1$, and thus the definable subgroup $Q = C_G(x)$ is selfnormalizing.

(a) The finite center $Z(G)$ is necessarily in $Q = C_G(x)$ in the previous paragraph, and in particular $Z(G) \cap G' = 1$. Since $G = G' \rtimes Q$ and Q is abelian, $C_Q(G') \leqslant Z(G)$, and since $G' \leqslant Z(F(G))$ one gets $Z(G) \times G' \leqslant F(G) \leqslant C_G(G') \leqslant Z(G) \times G'$, proving item (a).

(b) Let again x be any element in $G \setminus F(G)$. The map $G' \to G' : u \mapsto [x, u]$ is a definable group homomorphism, since G' is abelian, with trivial kernel as $C_{G'}(x) = 1$, and an isomorphism onto G', since the latter is definably connected. It follows that any element of the form xu', for $u' \in G'$, has the form $xu' = x[x, u] = x^u$ for some $u \in G'$; i.e., $xG' = x^{G'}$.

Next, notice that any complement Q_1 of G' is of the form $Q_1 = C_G(x_1)$ for any $x_1 \in Q_1 \setminus Z(G)$. Indeed, $x_1 \notin Z(G)$ and Q_1 abelian imply that $x_1 \notin C_G(G')$, and as above $C_G(x_1)$ is a definably connected complement of G' containing Q_1, and comparing the dimensions we get $Q_1 = C_G(x_1)$.

Moreover, if $Q_1 = C_G(x_1)$ and $Q_2 = C_G(x_2)$ are two complements of G', we can always choose x_1 and x_2 in the same G'-coset; then they are G'-conjugate, as well as Q_1 and Q_2. It is also now clear that, for any $x \in G \setminus F(G)$, $C_G(x)$ is the unique complement of G' containing x, proving item (b).

It is clear from item (b) that two complements of G' are G'-conjugate, and that such complements are largely generous in G. □

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Corollary 42. *Let G be a group as in Lemma 41. Then the following hold.*

(a) *If X is an infinite subgroup of a complement Q of G', then $N_G(X) = Q$, and so $N_G(X) \cap G' = 1$.*

(b) *If X is a nilpotent subgroup of G not contained in $F(G)$, then X is in an abelian complement of G'.*

(c) *Complements of G' in G are both Carter and Cartan subgroups of G, and all are of this form.*

Proof. (a) Since Q is abelian and X is contained in Q, we have $Q \leqslant N_G(X)$. And thus, $G = G' \rtimes Q$ implies that $N_G(X) = N_{G'}(X) \rtimes Q$. On the other hand, $[N_{G'}(X), X] \leqslant G' \cap X = 1$; hence $N_{G'}(X) \subseteq C_G(X)$. Now, let $x \in X \setminus Z(G)$ (which exists since X is infinite and $Z(G)$ finite); then $x \notin C_G(G')$, since $x \in Q$, which is abelian, and $G = G' \rtimes Q$. Then $N_{G'}(X) \subseteq C_G(X) \subseteq C_G(x)$. Then, by Lemma 41(b), $C_G(x) = Q$, and so $N_{G'}(X) = 1$. Hence $N_G(X) = Q$, and so $N_G(X) \cap G' = 1$.

(b) X contains an element x outside of $F(G) = C_G(G')$. Replacing X by its definable hull $H(X)$, and using Fact 3(b), we may assume without loss of generality that X is definable. As in the proof of Lemma 41, $X \cap G' = \{[x, u] \mid u \in X \cap G'\}$, and the nilpotency of X forces that $X \cap G' = 1$. Hence X is abelian, and in the complement $C_G(x)$ of G'.

(c) Complements of G' are selfnormalizing Carter subgroups, by Lemma 41, and thus also Cartan subgroups, by Lemma 5. Conversely, one sees easily that a Carter or a Cartan subgroup of G cannot be contained in $F(G)$, and then must be a complement of G', by item (b). □

Crucial in our proof of Theorem 40, the next point shows that any definably connected nonnilpotent solvable group has a quotient as in Lemma 41.

Fact 43 (*Cf.* [13, Proposition 3.5]). *Let G be a definably connected nonnilpotent solvable group definable in an o-minimal structure. Then G has a definably connected definable normal subgroup N such that $(G/N)'$ is G/N-minimal and $Z(G/N)$ is finite.*

Proof. The proof works formally exactly as in [13, Proposition 3.5] in the finite Morley rank case. All facts used there about groups of finite Morley rank have their formal analogs in Fact 33(a) and Lemma 34 in the o-minimal case. We also use the fact that lower central series and derived series of definably connected solvable groups definable in o-minimal structures are definable and definably connected, which follows from Fact 32 here. □

In the proof of Theorem 40, and further on, we will make use of the following classical group-theoretic argument.

Fact 44 (Frattini argument). *Let G be a group, and let L be a normal subgroup of G. Let $\mathcal{S}$ be a collection of subgroups of L closed under G-conjugation and all of them L-conjugate. Then, for any $Q \in \mathcal{S}$, $G = LN_G(Q)$.*

Proof. Let $Q \in \mathcal{S}$ and $g \in G$; then $Q^g \in \mathcal{S}$. Hence there is $h \in L$ such that $Q^g = Q^h$; thus $h^{-1}g \in N_G(Q)$. And so $g \in LN_G(Q)$. □

We now pass to the proof of the general theorem, 40. At this stage we could follow the analysis by *abnormal* subgroups of [7] in finite solvable groups, as developed in the case of infinite solvable groups of finite Morley rank in [13]. However, we provide a more conceptual proof of Theorem 40, mixing the use of Fact 43 with our general genericity arguments, in particular those of § 6. We note that the proof of Theorem 40 we give here would work equally in the finite Morley rank case (in that case there is no elimination of imaginaries, but the dimension is well defined on imaginaries), providing a somewhat more conceptual proof of the analogous theorem in [13] in that case.

Proof of Theorem 40. We proceed by induction on $\dim(G)$. Clearly a minimal counterexample G has to be nonnilpotent, and then has a definably connected definable normal subgroup N, as in Fact 43. In what follows, we use the notation '—' to denote quotients by N. Notice that $\overline{G}$ is necessarily infinite in Fact 43, and that N is a subgroup of infinite index in G.

Claim 45. *G contains a definably connected and selfnormalizing Cartan subgroup Q which is largely generous in the following sense: the (definable) set of elements of Q contained in a unique conjugate of Q is large in Q and largely generous in G.*

Proof. Let H be a definable subgroup of G containing N such that $\overline{H}$ is a selfnormalizing largely generous Carter subgroup of $\overline{G}$, as in Lemma 41. Notice that H is definably connected, since $\overline{H}$ and N are. As $\overline{G}'$ is infinite, $\dim(\overline{H}) < \dim(\overline{G})$, and $\dim(H) < \dim(G)$. We can thus apply the induction hypothesis in H, and assume that H contains a definably connected and selfnormalizing Cartan subgroup Q with the strong large generosity property: the set of elements of Q contained in a unique H-conjugate of Q is large in Q and largely generous in H. We will show that Q is the required subgroup.

First note that Q, being definably connected, is a largely generous Carter subgroup of H. By Corollaries 26 and 27, Q must be a largely generous Carter subgroup of G. We now show that Q is selfnormalizing in G. Notice that Q has an infinite image in $\overline{H}$, since it is largely generous in H and N is normal and proper in H. If $x \in N_G(Q)$, then $\overline{x} \in N_{\overline{G}}(\overline{Q}) = \overline{H}$, by Corollary 42(a), and since Q is selfnormalizing in H it follows that $x \in N_H(Q) = Q$. Thus Q is selfnormalizing in G. By Lemma 5, Q is also a Cartan subgroup of G.

It remains just to show the largeness issue. Let Q_0 denote the set of elements of Q contained in a unique H-conjugate of Q. We know that Q_0 is large in Q and that $[Q_0]^H$ is large in H, so $[Q_0]^G$ $(=[[Q_0]^H]^G)$ is large in G, by Proposition 24. This shows that Q is largely generous in G, and thus it remains only to show it is in the strong sense of our claim. For that purpose, one easily sees that it is enough to show that the subset X of elements of Q_0 contained in a unique *G-conjugate* of Q is still large in Q_0, given the large partition of G, as in Corollary 30 and Theorem 31 (see also Proposition 12). Since Q is largely generous in H and the preimage L in H of $F(\overline{G})$ is normal and proper in H, we get that $Q \not\leq L$, and thus it suffices to show that $Q_0 \setminus X$ is in L. Suppose towards a contradiction that an element x in Q_0 and not in L is in Q^g for some g not in $N_G(Q)$. Looking at images in $\overline{G}$, and since $\overline{x} \in \overline{H} \setminus Z(\overline{G})$, we then see with Lemma 41 that $\overline{g} \in N_{\overline{G}}(\overline{H}) = \overline{H}$, and thus $g \in H$. Then $x \in Q \cap Q^g$ for some $g \in H \setminus N_H(Q)$,

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a contradiction, since x is in a unique H-conjugate of Q. This completes our proof of claim 45. □

Claim 46. *Carter subgroups of G are conjugate.*

Proof. There are indeed at this stage two quick ways to argue for the conjugacy of Carter subgroups, either by quotienting by a G-minimal subgroup of G, as in [15, Proof of Theorem 3.11], or still looking at the quotient $\overline{G}$. Since we have already used $\overline{G}$ for the existence of a largely generous Carter subgroup, we keep on this second line of arguments.

Let Q_1 be an arbitrary Carter subgroup of G. By Theorem 31, it suffices to prove that Q_1 is a largely generous Carter subgroup of G. Let L be the preimage of $[\overline{G}]'$ in G; notice that L is definably connected, as $[\overline{G}]'$ and N are. If $Q_1 \leqslant L$, then a Frattini argument (see Fact 44) applied in L, using the induction hypothesis in L, gives $G = L \cdot N_G(Q_1)$, and since Q_1 is a Carter subgroup this gives that L has finite index in G, a contradiction. Therefore $Q_1 \not\leqslant L$, and since $\overline{Q_1}$ is definably connected we also get $\overline{Q_1} \not\leqslant F(\overline{G})$, by Lemma 41(a). In particular, by Corollary 42(b), Q_1 is contained in a definably connected definable subgroup H, as in the proof of claim 45. Since $H < G$, the induction hypothesis applies in H, and thus Q_1 must be conjugate in H to a largely generous Carter subgroup Q of H. In particular, by the proof of claim 45, Q_1 is a largely generous Carter subgroup of G, as required. □

The Cartan subgroup Q provided by claim 45 is also a Carter subgroup, by definable connectedness and Lemma 5(a′). If Q_1 is an arbitrary Cartan subgroup, then Q_1° is a Carter subgroup, by Lemma 5(a′), and hence a conjugate of Q, by claim 46, and the maximal nilpotence of Q forces $Q_1^\circ = Q_1$. Hence Cartan subgroups are definably connected and conjugate. This completes the proof of Theorem 40. □

Corollary 47. *In a definably connected solvable group definable in an o-minimal structure, Cartan subgroups and Carter subgroups coincide.*

Proof. If Q is a Cartan subgroup, then it is definably connected, by Theorem 40, and thus a Carter subgroup, by Lemma 5(a′). If Q is a Carter subgroup, then Q is the definably connected component of a Cartan subgroup $\tilde{Q}$, by Lemma 5, and thus $Q = \tilde{Q}$, by Theorem 40. □

There are other aspects refining further the structure of definably connected solvable groups that we will not follow here, but which could be followed. They include the already mentioned approach of Cartan/Carter subgroups as *minimal* abnormal subgroups [7, 13], as well as covering properties of nilpotent quotients by Cartan/Carter subgroups (see also [15, §§ 4–5]), and also the peculiar theory of 'generalized centralizers' of [13, § 5.3]. We merely mention the most basic covering property, but before that we mention a Frattini argument following Theorem 40.

Corollary 48. *Let G be a group definable in an o-minimal structure, let N be a definably connected definable normal solvable subgroup, and let Q be a Cartan/Carter subgroup of N. Then $G = N_G(Q)N$.*

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Proof. By a standard Frattini argument (see Fact 44), following the conjugacy in Theorem 40. □

Lemma 49. *Let G be a definably connected solvable group definable in an o-minimal structure, let N be a definable normal subgroup such that G/N is nilpotent, and let Q be a Cartan/Carter subgroup of G. Then $G = QN$.*

Proof. Suppose that $QN < G$. Then QN/N is a definable subgroup of infinite index in the definably connected nilpotent group G/N. By Lemma 4, and since $N_G(QN)$ is the preimage in G of $N_{G/N}(QN/N)$, we have thus QN of infinite index in $K := N_G(QN)$. But Q is a Cartan/Carter subgroup of the definably connected solvable group $[QN]^\circ$, normal in K, and thus $K = N_K(Q)[QN]^\circ = N_K(Q)N^\circ$, by Corollary 48. Since Q is a Carter subgroup, we get that QN must have finite index in K, a contradiction. □

We note that Lemma 49 always applies with $N = F^\circ(G)$, in view of Fact 33(b), giving thus in particular $G = QF^\circ(G)$ for any definably connected solvable group G and any Cartan/Carter subgroup Q of G.

9. On Lie groups

In this section, we collect properties needed in what follows concerning Cartan subgroups (in the sense of Chevalley as usual) of Lie groups.

By a *Lie algebra*, we mean a finite-dimensional real Lie algebra. We are going to make use of the following concepts about Lie algebras: subalgebras, commutative, nilpotent, and semisimple Lie algebras [5, I.1.1, I.1.3, I.4.1 and I.6.1]. If $\mathfrak{g}$ is a Lie algebra and $x \in \mathfrak{g}$, the linear map $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g} : y \mapsto [x, y]$ is called the *adjoint map* of x. If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, the *normalizer* of $\mathfrak{h}$ in $\mathfrak{g}$ is $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) := \{x \in \mathfrak{g} \colon \mathrm{ad}_x(\mathfrak{h}) \subseteq \mathfrak{h}\}$, and the *centralizer* of $\mathfrak{h}$ in $\mathfrak{g}$ is $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}) := \{x \in \mathfrak{g} \colon [\mathrm{ad}_x]_{|\mathfrak{h}} = id_{|\mathfrak{h}}\}$.

Definition 50. Let $\mathfrak{g}$ be a Lie algebra, and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. We say that $\mathfrak{h}$ is a *Cartan subalgebra* of $\mathfrak{g}$ if $\mathfrak{h}$ is nilpotent and selfnormalizing in $\mathfrak{g}$.

The two following facts can be found in [33, Theorem 4.1.2] and [33, Theorem 4.1.5], respectively.

Fact 51. *Every Lie algebra has a Cartan subalgebra.*

Fact 52. *Let $\mathfrak{g}$ be a semisimple Lie algebra, and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ if and only if the following hold.*

(a) *$\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$.*

(b) *For every $x \in \mathfrak{h}$, ad_x is a semisimple endomorphism of $\mathfrak{g}$; i.e., ad_x is diagonalizable over $\mathbb{C}$.*

By a *Lie group*, we mean a finite-dimensional real Lie group G. The *connected component of the identity* is denoted by G°. The Lie algebra of G is denoted by $\mathfrak{L}(G)$. A connected Lie group G is called a *semisimple Lie group* if $\mathfrak{L}(G)$ is a semisimple

Lie algebra (equivalently, every normal commutative connected immerse subgroup of G is trivial [5, Proposition III.9.8.26]). If g is an element of a Lie group G, then $\mathrm{Ad}(g) : \mathfrak{L}(G) \to \mathfrak{L}(G)$ denotes the differential at the identity of G of the map from G to G mapping h to ghg^{-1}, for each $h \in G$. If $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, the *centralizer* of $\mathfrak{h}$ in G is $Z_G(\mathfrak{h}) := \{g \in G : \mathrm{Ad}(g)(x) = x \text{ for every } x \in \mathfrak{h}\}$.

In most of the current literature, Cartan subgroups of a Lie group G are defined as centralizers in G of a Cartan subalgebra of $\mathfrak{g}$. This definition coincides with the intrinsic one given by Chevalley if the Lie group is reductive (e.g., semisimple), as is shown in Fact 53 below. In general, if G is a connected Lie group, H is a Cartan subgroup of G, and $\mathfrak{g}$ and $\mathfrak{h}$ are their corresponding Lie algebras, then $Z_G(\mathfrak{h})$ can be strictly contained in H [23].

Fact 53. *Let G be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$, and let H be a subgroup of G. Then H is a Cartan subgroup of G if and only if $H = Z_G(\mathfrak{h})$ for some Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Moreover, in this case, $\mathfrak{h}$ is $\mathfrak{L}(H)$.*

Proof. As G is connected, [23, Theorem A.4] implies that H is a Cartan subgroup of G if and only if

(C0) H is a closed subgroup of G;

(C1) $\mathfrak{h}(=\mathfrak{L}(H))$ is a Cartan subalgebra of $\mathfrak{g}$; and

(C2) $H = C(\mathfrak{h})$.

Here, $C(\mathfrak{h})$ is defined by a centralizer-like condition. To avoid introducing more notation, instead of properly defining $C(\mathfrak{h})$, we make use of [23, Lemma I.5], which states that $C(\mathfrak{h}) = Z_G(\mathfrak{h})$, provided that $\mathfrak{h}$ is reductive in $\mathfrak{g}$, which is our case, since $\mathfrak{g}$ is semisimple.

For the converse, we observe that, if $H = Z_G(\mathfrak{h})$ for some Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, then H is closed in G and $\mathfrak{L}(H) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$, by [5, Proposition III.9.3.7]. Since $\mathfrak{h}$ is abelian, by Fact 52, we have $\mathfrak{h} \subseteq \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$. Therefore, $\mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$, by maximality of $\mathfrak{h}$. We then conclude as above, first applying Lemma I.5 and then Theorem A.4 from [23]. □

Fact 54. *Let G be a connected semisimple centerless Lie group, and let H be a subgroup of G. If H is a Cartan subgroup of G, then H is abelian.*

Proof. By Fact 53, $H = Z_G(\mathfrak{h})$, with $\mathfrak{h} = \mathfrak{L}(H)$ a Cartan subalgebra of $\mathfrak{g}$. By [17, Lemma 8, p. 556], we have that $H/Z(G)$ is abelian (see also [34, Theorem 1.4.1.5], noting that since G is semisimple the general assumption (1.1.5) holds). Hence H is abelian. □

We note that the assumption that $Z(G) = 1$ is essential to get the Cartan subgroup abelian in Fact 54. For example, $\mathrm{SL}_3(\mathbb{R})$ has a simply connected double covering with non-abelian Cartan subgroups [22, p. 141], an example which can also occur in the context of our Theorem 63 below.

Let us also note, even if we do not make use of it, that, if G is a connected Lie group and H a Cartan subgroup of it, then $\mathfrak{h}$, the Lie algebra of H, is a Cartan subalgebra of $\mathfrak{g}$ (see the proof of Fact 53).

Fact 55. *Let G be a connected semisimple Lie group. Then the following hold.*

(a) *There are only finitely many conjugacy classes of Cartan subgroups of G. All Cartan subgroups of G have the same dimension, and their connected components are abelian.*

(b) *If H_1 and H_2 are two Cartan subgroups of G with ${H_1}^\circ = {H_2}^\circ$, then $H_1 = H_2$. In particular, if ${H_1}^\circ$ and ${H_2}^\circ$ are conjugate, then H_1 and H_2 are conjugate as well.*

(c) *For any Cartan subgroup H of G, the set of elements of H contained in a unique conjugate of H is dense in H.*

Proof. (a) Let $\mathfrak{g} = \mathfrak{L}(G)$. Then $\mathfrak{g}$ is semisimple, and it has finitely many Cartan subalgebras, say $\mathfrak{h}_1, \ldots, \mathfrak{h}_s$, such that any Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is conjugate to one of them by an element of $\mathrm{Ad}(G)$; i.e., $\mathrm{Ad}(g)(\mathfrak{h}) = \mathfrak{h}_i$ for some $i \in \{1, \ldots, s\}$ and some $g \in G$ (see [16, Corollary to Lemma 2] or [34, Corollary 1.3.1.11]).

Next, note that, for every g in G and every (Cartan) subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we have $Z_G(\mathrm{Ad}(g)(\mathfrak{h})) = gZ_G(\mathfrak{h})g^{-1}$. This is because $h \in Z_G(\mathrm{Ad}(g)(\mathfrak{h}))$ if and only if $\mathrm{Ad}(h)\mathrm{Ad}(g)x = \mathrm{Ad}(g)x$ for every $x \in \mathfrak{h}$, and the latter is equivalent to $g^{-1}hg \in Z_G(\mathfrak{h})$. Therefore, conjugate Cartan subalgebras correspond to conjugate centralizers, and by Fact 53 to conjugate Cartan subgroups.

We prove the second part. By Fact 53, the Lie algebra of a Cartan subgroup is a Cartan subalgebra. By [33, Corollary 4.1.4], all Cartan subalgebras have the same dimension. If H is Cartan, then H° is abelian, because $\mathfrak{L}(H^\circ) = \mathfrak{L}(H)$ is abelian (see Fact 52).

(b) It is clear, since $\mathfrak{L}(H_i) = \mathfrak{L}({H_i}^\circ)$, for $i = 1, 2$, and $H_i = Z_G(\mathfrak{L}(H_i))$. (Actually, to prove (b) we do not need G to be semisimple: just consider the $C(\mathfrak{L}(H_i))$ of the proof of Fact 53, instead of the centralizers.)

(c) We essentially refer to [18]. Recall that, by Fact 53 and its proof, in the semisimple case, our notion of a Cartan subgroup is the same as that used in that paper, and $C(\mathfrak{h}) = Z_G(\mathfrak{h})$ for any Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g} := \mathfrak{L}(G)$. Let $\mathrm{Reg}(G)$ be the set of regular elements of G, as defined after Lemma 1.3 in [18]. We first show that each element g of $\mathrm{Reg}(G)$ lies in a unique Cartan subgroup of G. Fix $g \in \mathrm{Reg}(G)$. By the proof of [18, Proposition 1.5], we have that $\mathfrak{g}^1(\mathrm{Ad}(g)) := \{x \in \mathfrak{g} : (\exists n \in \mathbb{N})(\mathrm{Ad}(g) - 1)^n x = 0\}$ is a Cartan subalgebra of $\mathfrak{g}$, and g belongs to the Cartan subgroup $Z_G(\mathfrak{g}^1(\mathrm{Ad}(g)))$. To show the uniqueness, let H be a Cartan subgroup of G containing g. By Fact 53, $H = Z_G(\mathfrak{h})$, with $\mathfrak{h} = \mathfrak{L}(H)$ a Cartan subalgebra of $\mathfrak{g}$. Since $g \in Z_G(\mathfrak{h})$, we have that $\mathfrak{h} \subseteq \mathfrak{g}^1(\mathrm{Ad}(g))$, and hence $\mathfrak{h} = \mathfrak{g}^1(\mathrm{Ad}(g))$, by maximality of Cartan subalgebras. Therefore $H = Z_G(\mathfrak{g}^1(\mathrm{Ad}(g)))$.

Finally, by [18, Proposition 1.6], the subset $\mathrm{Reg}(G) \cap H$ is dense in H for all Cartan subgroups H of G. □

For the following, we refer directly to [36, Proposition 5] and (the proof of) [36, Lemma 11] respectively.

Fact 56. *Let G be a connected Lie group. Then the following hold.*

(a) *The union of all Cartan subgroups of G is dense in G.*

(b) *For any Cartan subgroup H of G, $[H^\circ]^G$ contains an open subset.*

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We finish this section with a remark which, as far as we know, does not seem to have been made before. We will show later that all Cartan and Carter subgroups of a group definable in an o-minimal structure are, as indicated by Fact 56(b), slightly generous in the sense of Definition 16(a). Our remark is essentially that the stronger notion of generosity of Definition 16(b) may be satisfied or not, depending of the Carter subgroups considered, and this phenomenon occurs even inside $\mathrm{SL}_2(\mathbb{R})$. Recall that the Cartan subgroups of $\mathrm{SL}_2(\mathbb{R})$ are, up to conjugacy, the subgroup Q_1 of diagonal matrices and $Q_2 = \mathrm{SO}_2(\mathbb{R})$. Considering the characteristic polynomial, the two following equalities are easily checked:

$$Q_1^{\mathrm{SL}_2(\mathbb{R})} = \{A \in \mathrm{SL}_2(\mathbb{R}) : |\mathrm{tr}(A)| > 2\} \cup \{I, -I\}$$
$$Q_2^{\mathrm{SL}_2(\mathbb{R})} = \{A \in \mathrm{SL}_2(\mathbb{R}) : |\mathrm{tr}(A)| < 2\} \cup \{I, -I\}$$

Remark 57. Let $G = \mathrm{SL}_2(\mathbb{R})$. Then, according to Definition 16(b), the following hold.

(a) The Cartan subgroup Q_1 of diagonal matrices is generous in G.

(b) The Cartan subgroup $Q_2 = \mathrm{SO}_2(\mathbb{R})$ is not generous in G.

Proof. (a) Fix $a, b \in (0, \frac{1}{13})$ and consider the matrices $A_1 = I$,

$$A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}, \quad \text{and} \quad A_4 = \begin{pmatrix} 0 & -b^{-1} \\ b & 0 \end{pmatrix}.$$

We show that $G = \cup_{i=1}^4 A_i Q_1^G$. Suppose there exists

$$M = \begin{pmatrix} x & y \\ u & v \end{pmatrix} \in G,$$

with $M \notin \cup_{i=1}^4 A_i Q_1^G$. Since $M \notin A_1 Q_1^G \cup A_2 Q_1^G$, we have $x = \epsilon - v$ and $y = u + \delta$ for some $\epsilon, \delta \in [-2, 2]$. Since $M \notin A_3 Q_1^G$, we have that $|ax + a^{-1}v| = |a(\epsilon - v) + a^{-1}v| \leqslant 2$, so $v \in [\frac{-2a - a^2\epsilon}{1-a^2}, \frac{2a - a^2\epsilon}{1-a^2}]$. Since $\epsilon \in [-2, 2]$, we deduce that $v \in [\frac{-2a}{1-a}, \frac{2a}{1-a}]$. Similarly, it follows from $M \notin A_4 Q_1^G$ that $u \in [\frac{-2b}{1-b}, \frac{2b}{1-b}]$.

Finally, since $a, b < \frac{1}{13}$, we have that $|v|, |u| < \frac{1}{6}$ and $|x|, |y| < 2 + \frac{1}{6} < 3$. In particular, $\det(M) = |xv - uy| \leqslant |x||v| + |u||y| < 1$, a contradiction.

(b) We show that the family of matrices

$$M_x = \begin{pmatrix} x^2 & x - 1 \\ 1 & x^{-1} \end{pmatrix}$$

with $x > 0$ cannot be covered by finitely many translates of Q_2^G. It suffices to prove that, for a fixed matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G,$$

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we have that $\{x \in \mathbb{R}^{>0} : |\mathrm{tr}(A^{-1}M_x)| > 2\} \subseteq \{x \in \mathbb{R}^{>0} : M_x \notin AQ_2^G\}$ is not bounded. Since $\mathrm{tr}(A^{-1}M_x) = x^2 d - b - c(x-1) + ax^{-1}$ and x is positive, it follows that $|\mathrm{tr}(A^{-1}M_x)| > 2$ if and only if one of the following two conditions holds:

$$dx^3 - cx^2 - (b - c + 2)x + a > 0 \tag{1}$$

$$dx^3 - cx^2 - (b - c - 2)x + a < 0. \tag{2}$$

It is easy to check that, if $d \neq 0$, then either (1) or (2) is satisfied for large enough x. If $d = 0$, then $c \neq 0$ (otherwise $\det(A) = 0$), and again the same holds. □

In Remark 57, the generous Cartan subgroup is noncompact and the nongenerous one is compact. One can then wonder about the various possibilities for generosity depending on compactness. But considering $Q_1 \times Q_2$ in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, one gets from Remark 57 a nongenerous and noncompact Cartan subgroup. Besides, any compact group is typically covered by a single conjugacy class of compact Cartan subgroups, by Corollary 38, and these compact Cartan subgroups are in particular generous. As far as we know, it is not known if every simple Lie group has a generous Cartan subgroup.

10. From Lie groups to definably simple groups

We now return to the context of groups definable in o-minimal structures. In the present section, we prove the following theorem, essentially transferring via Fact 35 the results of § 9 on Lie groups to definably simple groups definable in an o-minimal structure.

Theorem 58. *Let G be a definably simple group definable in an o-minimal structure. Then G has definable Cartan subgroups, and the following hold.*

(1) *G has only finitely many conjugacy classes of Cartan subgroups.*

(2) *If Q_1 and Q_2 are Cartan subgroups of G and ${Q_1}^\circ = Q_2^\circ$, then $Q_1 = Q_2$.*

(3) *Cartan subgroups of G are abelian, and have the same dimension.*

(4) *If Q is a Cartan subgroup of G, then the set of elements of Q contained in a unique conjugate of Q is large in Q. In particular, if $a \in Q$, then the set of elements of aQ° contained in a unique conjugate of aQ° is large in aQ°, and aQ° is slightly generous in G.*

(5) *The union of all Cartan subgroups of G, which is definable by* (1), *is large in G.*

Before passing to the proof of Theorem 58, we explain the 'In particular' part of item (4). So let Q be a Cartan subgroup such that the set Q_0 of elements of Q contained in a unique conjugate of Q is large in Q. Let $[aQ^\circ]_0$ be the set of elements of aQ° contained in a unique conjugate of aQ°, for some $a \in Q$. We see easily that $Q_0 \cap aQ^\circ \subseteq [aQ^\circ]_0$, and since Q_0 is large in Q we get that $Q_0 \cap aQ^\circ$ is large in aQ°, and hence also $[aQ^\circ]_0$ is large in aQ°. Now, since $Q^\circ \leqslant N(aQ^\circ) \leqslant N(Q^\circ)$ and $\dim(Q^\circ) = \dim(N(Q^\circ))$, we get that $\dim([aQ^\circ]_0) = \dim(N(aQ^\circ))$, and Corollary 17 gives the slight generosity of aQ°.

We now embark on the proof of Theorem 58, bearing in mind that for item (4) we only need to prove the first statement. We first prove some results that will allow us to transfer statements between different structures. For our purposes it will be enough to

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work over o-minimal expansions of ordered groups where we have an absolute control of the relevant parameters (see the comments at the end of this section).

Lemma 59. *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group with a distinguished element different from* 0*, and let $\{G_t : t \in T\}$ be a definable family of definable groups over $\emptyset$. Then the subfamily of nilpotent subgroups is definable over $\emptyset$. Moreover, in all cases the obtained definable families give the desired families in any model of Th($\mathcal{M}$).*

Proof. For the 'Moreover' part, just note that the proof is actually independent of the model. We can assume that all G_t have the same dimension (because of the definability of dimension). If $\dim(G_t) = 0$, then all are finite, and there is a uniform bound on the cardinality of the groups. In particular, we have a definable partition of the definable family into a finite number of isomorphic types, and the result holds. If $\dim(G_t) > 0$, then by Fact 33(d) we can assume that each $Z(G_t)$ is infinite. Since a group G_t is nilpotent if and only if $G_t/Z(G_t)$ is nilpotent, the subfamily is definable by induction. □

Theorem 60. *Let G be a definable group over a set of parameters A in an o-minimal expansion of an ordered group $\mathcal{M}$ with a distinguished element different from* 0*. Then the family of Cartan subgroups of G is a definable family $\{Q_s : s \in S\}$ over A. Moreover, for every model $\mathcal{N}$ of Th($\mathcal{M}_A$), the realization $\{Q_s(N) : s \in S(N)\}$ of the family in $\mathcal{N}$ is the family of Cartan subgroups of $G(N)$.*

Proof. We first prove by induction that, if $\{G_t : t \in T\}$ is a definable family over $\emptyset$ of groups, then there is a definable family over $\emptyset$ of groups

$$\{H_{t,s} : t \in T, s \in S_t\}$$

such that for each $t \in T$ the groups $\{H_{t,s} : s \in S_t\}$ are exactly the maximal definably connected nilpotent subgroups of G_t. We point out that in the following process each of the definable families obtained gives actually the maximal definably connected nilpotent subgroups in any model of Th($\mathcal{M}$).

First note that we can assume that all G_t are definably connected, and with the same dimension. If $\dim(G_t) = 0$, then we know that there is a uniform bound on the cardinal of the groups G_t; thus we have a definable partition of the definable family into a finite number of isomorphic types, and the result is obvious. If $\dim(G_t) > 0$, then we can assume that one of the following three cases holds.

(1) All G_t are centerless. We first observe that any definably connected nilpotent subgroup of G_t is contained in $C_{G_t}(x)$ for some $x \in G_t \setminus \{1\}$. Indeed, if the subgroup is trivial, then we are done. If it is not, then it is infinite, and therefore it is enough to pick a point $x \neq 1$ in its infinite center. Thus, the family of maximal definably connected nilpotent subgroups of the definable family $\{G_t : t \in T\}$ coincides with the one of the definable family $\{C_{G_t}(x) : t \in T, x \in G_t \setminus \{1\}\}$. Then the result follows by induction, since $\dim(C_{G_t}(x)) < \dim(G_t)$.

(2) All G_t have infinite center or all have finite center. Then, either by induction or by (1), the family of maximal definably connected nilpotent subgroups of $\{G_t/Z(G_t) : t \in T\}$

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is a definable family $\{H_{t,s} : t \in T, s \in S_t\}$ over $\emptyset$. Then $\{\pi_t^{-1}(H_{t,s})^\circ : t \in T, s \in S_t\}$ is the family of maximal definably connected nilpotent subgroups of $\{G_t : t \in T\}$, as required.

All in all, we deduce that the family of maximal definably connected nilpotent subgroups of a definable group G over A is a definable family $\{H_s : s \in S\}$ over A. In fact, the subfamily of $\{H_s : s \in S\}$ of those subgroups which have finite index in its normalizer is again definable over A, because we have a uniform bound on the finite groups of $\{N_G(H_s)/H_s : s \in S\}$. In other words, the family of Carter subgroups of G is a definable family $\{Q_s^\circ : s \in S\}$ over A. Again, note that this definable family gives the Carter subgroups in any model of $\mathrm{Th}(\mathcal{M}_A)$.

Finally, since there is a uniform bound on the cardinal of the finite groups $\{N_G(Q_s^\circ)/Q_s^\circ : s \in S\}$, the family of all subgroups of $N_G(Q_s^\circ)$ containing Q_s°, $s \in S$, is again definable over A. In particular, the subfamily of the nilpotent subgroups is definable by Lemma 59. From that family of nilpotent subgroups we can extract definably over A the maximal ones, i.e., the Cartan subgroups of G. □

Corollary 61. *Let G be a definable group over a set of parameters A in an o-minimal expansion of an ordered group $\mathcal{M}$ with a distinguished element different from* 0*. Then G satisfies properties* (1)–(5) *of Theorem 58 if and only if the realization of G in any model of $Th(\mathcal{M}_A)$ satisfies properties* (1)–(5)*.*

Proof. Properties (1)–(5) are definable, and therefore the result follows directly from Theorem 60. □

Corollary 62. *Let G be a definable group in an o-minimal expansion of an ordered group $\mathcal{M}$ over a set of parameters A containing an element different from* 0*. Assume that G has a finite number of Cartan subgroups up to conjugacy. Then, in each conjugacy class there exists a Cartan subgroup definable over A.*

Proof. The definable closure $\mathrm{dcl}_{\mathcal{M}}(A)$ is a model of $\mathrm{Th}(\mathcal{M}_A)$, and therefore the result follows again from Theorem 60. □

Proof of Theorem 58. Let $\mathcal{M}$ denote the ground o-minimal structure. By Fact 35, there is an $\mathcal{M}$-definable real closed field R (with no extra structure) such that G is $\mathcal{M}$-definably isomorphic to a semialgebraically connected semialgebraically simple semialgebraic group, definable in R over the real algebraic numbers $\mathbb{R}_{alg}$. By Remark 36, the dimensions of sets definable in R, computed in $\mathcal{M}$ or R, are the same. Since $\mathcal{M}$-definable bijections preserve dimensions, all the conclusions of Theorem 58 would then be true if we prove them in this semialgebraic group definable in R. Therefore, replacing $\mathcal{M}$ by R, we may suppose that $\mathcal{M}$ is a pure real closed field, and that $G = G(M)$ is a semialgebraically connected semialgebraically simple group defined over $\mathbb{R}_{alg}$. Then, by Corollary 61, it suffices to show our statements for $G(\mathbb{R})$.

Now, we observe that $G(\mathbb{R})$ is a finite-dimensional semisimple centerless connected real Lie group. By Facts 51 and 53, it has Cartan subgroups, necessarily definable as usual by Lemma 5(b). It remains just to notice that all items (1)–(5) are true in the connected real Lie group $G(\mathbb{R})$, by Facts 54, 55 and 56(a). For item (4), we recall that it

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suffices to prove the first claim, as explained just after the statement of Theorem 58. It follows from Fact 55(c), noticing that a definable subset has maximal dimension if and only it has interior [30, Proposition 2.14], and thus is dense if and only if it is large. □

We note that the second claim in Theorem 58(4) could also have been shown using Fact 56(b).

As commented by the referee, in the hypothesis of Theorem 60 we could just have required that G is A-definable in an o-minimal structure (without the expansion of a group assumption). In this case, we would conclude that the family of Cartan subgroups is $(A \cup c)$-definable for some finite set of parameters c. To do this, it suffices to consider in Lemma 59 and in the proof of Theorem 60 families of (quotients of) subgroups of a given definable group. Then, as we have mentioned, we can apply [10, Theorem 7.2] to consider these quotients as definable objects, taking into account that we need extra parameters (see [10, proof of Theorem 2.5]).

11. The semisimple case

We now prove a version of Theorem 58 for definably connected semisimple groups definable in an o-minimal structure. Recall that a definably connected group G is semisimple if $R(G) = Z(G)$ is finite; modulo that finite center, G is a direct product of finitely many definably simple groups, by Fact 35.

Theorem 63. *Let G be a definably connected semisimple group definable in an o-minimal structure. Then G has definable Cartan subgroups and the following hold.*

(1) *G has only finitely many conjugacy classes of Cartan subgroups.*
(2) *If Q_1 and Q_2 are Cartan subgroups of G and $Q_1^\circ = Q_2^\circ$, then $Q_1 = Q_2$.*
(3) *If Q is a Cartan subgroup, then $Z(G) \leqslant Q$, $Q' \leqslant Z(G)$, and $Q^\circ \leqslant Z(Q)$. Furthermore, all Cartan subgroups have the same dimension.*
(4) *If Q is a Cartan subgroup of G and $a \in Q$, then the set $[aQ^\circ]_0$ of elements of aQ° contained in a unique conjugate of aQ° is large in aQ°, and aQ° is slightly generous in G. In addition, if a_1 belongs to another Cartan subgroup Q_1, then either $[aQ^\circ]_0 \cap a_1Q_1^\circ = \emptyset$ or $aQ^\circ = a_1Q_1^\circ$.*
(5) *The union of all Cartan subgroups of G, which is definable by (1), is large in G.*
(6) *There are finitely many pairwise disjoint definable sets of the form $[aQ^\circ]_0^G$ with Q a Cartan subgroup of G and $a \in Q$, each slightly generic and consisting of pairwise disjoint conjugates of $[aQ^\circ]_0$, whose union is large in G.*

Proof. Assume first that $R(G) = Z(G) = 1$. By Fact 35, $G = G_1 \times \cdots \times G_n$, where each G_i is an infinite definably simple definable factor. Now, by Corollary 10, Cartan subgroups Q of G are exactly of the form

$$Q = \tilde{Q}_1 \times \cdots \times \tilde{Q}_n,$$

with $\tilde{Q}_i$ a Cartan subgroup of G_i for each i. In particular, G has definable Cartan subgroups, by Theorem 58. Since $Q^\circ = \tilde{Q}_1{}^\circ \times \cdots \times \tilde{Q}_n{}^\circ$ and the dimension is additive,

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items (1)–(3) follow easily from Theorem 58(1)–(3). By additivity of the dimension, the first claim in item (4) also transfers readily from Theorem 58(4). If some element α belongs to $[aQ^\circ]_0 \cap a_1 Q_1^\circ$, for some Cartan subgroups Q and Q_1 and some $a \in Q$ and $a_1 \in Q_1$, then $Q_1^\circ \leqslant C_G{}^\circ(\alpha) = Q^\circ$, by the commutativity of Q and Q_1 and Lemma 18, and $Q^\circ = Q_1^\circ$. In particular, $aQ^\circ = \alpha Q^\circ = \alpha Q_1^\circ = a_1 Q_1^\circ$, proving item (4). For items (5) and (6), notice that, if some $[aQ^\circ]_0^G \cap [a_1 Q_1^\circ]_0^G$ is nonempty in item (4), then $aQ^\circ = [a_1 Q_1^\circ]^g$ for some g (conjugating in particular Q° to Q_1°), so the finitely many slightly generic definable sets of the form $[aQ^\circ]_0^G$ are pairwise disjoint and consist of a disjoint union of G-conjugates of $[aQ^\circ]_0$. By the largeness of $[aQ^\circ]_0^G$ in $[aQ^\circ]^G$ provided by Corollary 17, and the largeness of the union of all Cartan subgroups provided by Theorem 58(5), the union of all these sets $[aQ^\circ]_0^G$ is large in G, proving items (5) and (6).

Assume now just $R(G) = Z(G)$ finite, and let the notation '—' denote the quotients by $Z(G)$. By the centerless case, all the conclusions of Theorem 63 hold in $\overline{G}$. By Lemma 8, Cartan subgroups of G contain $Z(G)$, and are exactly the preimages in G of Cartan subgroups of $G/Z(G)$. In particular, G has definable Cartan subgroups, and we now check that they still satisfy (1)–(6).

(1) Since $Z(G)$ is contained in each Cartan subgroup, item (1) transfers from the centerless case. (2) If Q_1 and Q_2 are two Cartan subgroups of G with $Q_1^\circ = Q_2^\circ$, then $\overline{Q_i{}^\circ} = [\overline{Q_i}]^\circ$ and $\overline{Q_1} = \overline{Q_2}$ by (2) in $\overline{G}$, giving $Q_1 = Q_2$. (3) By the centerless case, $\overline{Q}$ is abelian, and thus $Q' \leqslant Z(G)$. In particular, $[Q, Q^\circ]$ is in the finite center $Z(G)$, but, since $[Q, Q^\circ]$ is definable and definably connected by [1, Corollary 6.5], we get $[Q, Q^\circ] = 1$, proving the first claim of (3). Since the natural (and definable) projection from G onto $\overline{G}$ has finite fibers, one gets by axioms A2–A3 of the dimension that $\dim(\overline{Q}) = \dim(Q)$, transferring also from $\overline{G}$ to G the second claim of (3). (4) Let Q and Q_1 be two Cartan subgroups, $a \in Q$, and $a_1 \in Q_1$. If some element α belongs to $[aQ^\circ]_0 \cap a_1 Q_1^\circ$, one sees, as in the centerless case, still using Lemma 18, but now the fact that $Q^\circ \leqslant Z(Q)$ and $Q_1^\circ \leqslant Z(Q_1)$, that $aQ^\circ = a_1 Q_1^\circ$. We now show that $[aQ^\circ]_0$ is large in aQ°. For that purpose, first notice that $[aQ^\circ]_0$ is exactly the set of elements of aQ° contained in finitely many conjugates of aQ°: this is because, if α is in aQ° and in only finitely many of its conjugates, say $(aQ^\circ)^{g_1}, \ldots, (aQ^\circ)^{g_k}$, then, as above, Lemma 18 yields $Q^\circ = C^\circ(\alpha)$, and $aQ^\circ = (aQ^\circ)^{g_1} = \cdots = (aQ^\circ)^{g_k}$. For the largeness of $[aQ^\circ]_0$ in aQ°, it suffices as in item (3) to show that $[aQ^\circ]_0$ contains the preimage of the set of elements $\overline{\alpha}$ of $\overline{aQ^\circ}$ contained in a unique $\overline{G}$-conjugate of $\overline{aQ^\circ}$. So assume towards a contradiction that there exists an element α in aQ°, in infinitely many G-conjugates of aQ°, but such that $\overline{\alpha}$ is in a unique conjugate of $\overline{aQ^\circ}$. Now, for g varying in infinitely many cosets of $N(aQ^\circ)$, and in particular in infinitely many cosets of $N^\circ(aQ^\circ) = N^\circ(Q^\circ) = Q^\circ$, we have $aZ(G)Q^\circ = [aZ(G)Q^\circ]^g$. But such elements g must normalize the subgroup $Z(G)Q^\circ$, and in particular $[Z(G)Q^\circ]^\circ = Q^\circ$, and hence cannot vary in infinitely many cosets of Q°. This contradiction proves that $[aQ^\circ]_0$ is large in aQ°, and the slight generosity of aQ° in G follows as usual with Corollary 17. (5) Using the projection from G to $\overline{G}$, the non-slight genericity of the complement of the union of all Cartan subgroups passes from $\overline{G}$ to G, and thus the union of all Cartan subgroups of G is large in G. Item (6) also follows, as in the case $Z(G) = 1$. □

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In Theorem 63(3), Cartan subgroups need not be abelian outside of the centerless case, since the simply connected double covering of $\mathrm{SL}_3(\mathbb{R})$ with non-abelian Cartan subgroups mentioned after Fact 54 is definable in $\mathbb{R}$. The following question then arises naturally.

Question 64. *Let G be a definably connected semisimple group definable in an o-minimal structure, and let Q be a Cartan subgroup of G. When it is the case that $Q = Q^\circ Z(G)$, or more generally, Q is abelian?*

For Carter subgroups, one gets the following corollary of Theorem 63.

Corollary 65. *Let G be a definably connected semisimple group definable in an o-minimal structure. Then G has finitely many conjugacy classes of Carter subgroups. Each Carter subgroup Q° is abelian and slightly generous in the following strong sense: the set of elements of Q° contained in a unique conjugate of Q° is large in Q° and slightly generous in G.*

Proof. We know by Lemma 5 that Carter subgroups are exactly the definably connected components Q° of Cartan subgroups Q of G. In particular, item (3) of Theorem 63 shows that $Q^\circ \leqslant Z(Q)$, and hence Q° is abelian. The other conclusions follow immediately from items (1) and (4) in Theorem 63. □

Before moving to more general situations, we make a few additional remarks about the semisimple case. We first mention a general result on control of fusion, reminiscent from [9, Corollary 2.12] in the finite Morley rank case.

Lemma 66 (Control of fusion). *Let G be a group definable in an o-minimal structure, let Q be a Cartan subgroup of G, and let X and Y be two G-conjugate subsets of $C(Q^\circ)$ such that $C^\circ(Y)$ has a single conjugacy class of Carter subgroups. Then $Y = X^g$ for some g in $N(Q^\circ)$.*

Proof. Let g in G be such that $Y = X^g$. Then $C^\circ(Y) = C^\circ(X)^g$ contains both Q° and $Q^{\circ g}$, so our assumption forces that $[Q^\circ]^{g\gamma} = Q^\circ$ for some γ in $C^\circ(Y)$. Now $g\gamma$ normalizes Q°, and $X^{g\gamma} = Y^\gamma = Y$. □

Lemma 67. *Let G be a definably connected semisimple group G definable in an o-minimal structure, and let Q be a Cartan subgroup of G. Then $Q = F(N_G(Q^\circ))$.*

Proof. Any definable nilpotent subgroup containing the Carter subgroup Q° is a finite extension of it, by Lemma 4, and hence is in $N_G(Q^\circ)$. By Theorem 63(2), there is a unique maximal nilpotent subgroup containing Q°. This proves that $Q \trianglelefteq N_G(Q^\circ)$. Hence $Q \leqslant F(N_G(Q^\circ))$, and in fact there is equality by maximal nilpotence of Q. □

With Lemma 66, we can rephrase the last part of Theorem 63(4).

Corollary 68. *Let G be a definably connected semisimple group definable in an o-minimal structure, and let Q be a Cartan subgroup of G. If a_1 and a_2 are two G-conjugate elements of Q such that $a_i \in [a_iQ^\circ]_0$ as in Theorem 63(4) for $i = 1$ and 2, then a_1Q° and a_2Q° are $N(Q)$-conjugate.*

Proof. By Theorem 63(3), $a_i \in C(Q^\circ)$ for each i, and, by Lemma 18, $Q^\circ = C^\circ(a_1) = C^\circ(a_2)$. Lemma 66 implies then that $a_2 = {a_1}^g$ for some g in $N(Q^\circ)$. But since $Q \trianglelefteq N_G(Q^\circ)$, by Lemma 67, $g \in N_G(Q)$. □

As just seen in Corollary 68, if Q is a Cartan subgroup of a definably connected semisimple group G definable in an o-minimal structure, then

$$N_G(Q) = N_G(Q^\circ).$$

Now the finite group $W(G,Q) := N_G(Q)/Q = N_G(Q^\circ)/Q$ can naturally be called the *Weyl group relative to* Q, or, equivalently, *relative to* Q°. If G is definably simple, then one has the two alternatives at the end of Fact 35. In the first case, G is essentially a simple algebraic group over an algebraically closed field (of characteristic 0). It is well known in this case that there is only one conjugacy of Cartan subgroups, the maximal algebraic (and split) tori, which are also Carter subgroups by divisibility (see [3, 12.6 and 13.17]). Then there is only one relative Weyl group for such G, and their classification for G varying among these definable groups is provided by the classification of the simple algebraic groups. In the second alternative at the end of Fact 35, the group is essentially a simple real Lie group, and again the Weyl groups relative to the various Cartan subgroups, corresponding to the various split or non-split tori, are classified in this case. For a general definably connected semisimple ambient group G, the structure of the Weyl groups is inherited from that of the definably simple factors of $G/R(G)$, as we will see in § 13.

Theorem 63(6) equips any definably connected semisimple group with some kind of partition into finitely many canonical 'generic types'. We finish this section by counting them precisely.

Remark 69. The number $n(G)$ of slightly generic definable sets of the form $[aQ^\circ]_0^G$ as in Theorem 63(6) is clearly bounded by the sum $\Sigma_{Q\in\mathcal{Q}}|Q/Q^\circ|$, where $\mathcal{Q}$ is a system of representatives of the set of Cartan subgroups of G. But it might happen in Theorem 63(4) that two distinct sets of the form aQ° and $a'Q^\circ$, for a and a' in a common Cartan subgroup Q, are conjugate by the action of the Weyl group $W(G,Q) = N_G(Q)/Q$. If one denotes by $\sim_Q$ the equivalence relation on Q/Q° by the action of $W(G,Q)$ naturally induced by conjugation on Q/Q°, then one sees indeed with Corollary 68 that

$$n(G) = \Sigma_{Q\in\mathcal{Q}}|[Q/Q^\circ]_{\sim_Q}|.$$

12. The general case

We now analyze the general case of a group definable in an o-minimal structure. As far as possible, we will restrict ourselves to definably connected groups only when necessary. We start by lifting Carter subgroups.

Lemma 70. *Let G be a group definable in an o-minimal structure, and let N be a definable normal subgroup of G such that N° is solvable. Then, H is a Carter subgroup of G/N if and only if there is a Carter subgroup Q of G such that $H = QN/N$.*

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Proof. We use the notation '—' to denote the quotients by N and $\pi : G \to G/N$ to denote the quotient map. Suppose first that Q is a Carter subgroup of G. Then $\overline{Q} = QN/N$ is nilpotent and definably connected, so it suffices to prove that it is almost selfnormalizing to conclude that is a Carter subgroup of G/N. Let $S = \pi^{-1}(N_{\overline{G}}(\overline{Q}))(=N_G(QN))$. Since S normalizes $[QN]^\circ = QN^\circ$ and Q is also a Carter subgroup of the definable subgroup QN°, by Corollary 48 we have that $S = N_S(Q)QN^\circ \leqslant N_G(Q)N$. Hence $\overline{Q}$ must have finite index in its normalizer in $\overline{G}$, and is thus a Carter subgroup of $\overline{G}$. Conversely, let X/N be a Carter subgroup of $\overline{G}$ for some subgroup X of G containing N. Since X/N is definable, X must be definable. By Theorem 40, X° has a Carter subgroup Q, and of course Q must also be a Carter subgroup of X. We now prove that $X = QN$. Since $X = X^\circ N$ and $X^\circ = Q(X^\circ \cap N)$ by Lemma 49, we get that $X = QN$. It suffices to prove that Q is almost selfnormalizing in G, to conclude that Q is the required Carter subgroup of G. Since Q is a Carter subgroup of QN, the index of Q in $N_{QN}(Q)$ must be finite. On the other hand, $\overline{QN}$ is a Carter subgroup of $\overline{G}$, and so QN has finite index in $N_G(QN)$. Since $QN \leqslant N_G(Q)N \leqslant N_G(QN)$, we get that Q has finite index in $N_G(Q)$. □

The following special case of Lemma 70 with $N = R^\circ(G)$ is of major interest, and for the rest of the paper one should bear in mind that

$$R^\circ(G) = R^\circ(G^\circ).$$

Corollary 71. *Let G be a group definable in an o-minimal structure. Then Carter subgroups of $G/R^\circ(G)$ are exactly of the form $QR^\circ(G)/R^\circ(G)$ for Q a Carter subgroup of G.*

At this stage, we can prove our general theorem, 1, giving the existence, the definability, and the finiteness of the set of conjugacy classes of Cartan subgroups in an arbitrary group definable in an o-minimal structure.

Proof of Theorem 1. Let G be our arbitrary group definable in an arbitrary o-minimal structure. The quotient $G^\circ/R^\circ(G)$ is semisimple, by Fact 2, and has Carter subgroups, by Theorem 63. Hence G° has Carter subgroups, by Corollary 71. This takes care of the existence of Carter subgroups of G°, and of course of G as well. Now G has Cartan subgroups, by Lemma 5. Their definability is automatic as usual, in view of Lemma 5(a′). To prove that Cartan subgroups fall into only finitely many conjugacy classes, it suffices by Lemma 5(a′) to prove it for Carter subgroups. We may then assume G to be definably connected. Now groups of the form $QR^\circ(G)/R^\circ(G)$, for Q a Carter subgroup of G, are Carter subgroups of the semisimple quotient $G/R^\circ(G)$. By Theorem 63(1), there are only finitely many $G/R^\circ(G)$-conjugacy classes of groups of the form $QR^\circ(G)/R^\circ(G)$, and thus only finitely many G-conjugacy classes of groups of the form $QR^\circ(G)$. Replacing G by such a $QR^\circ(G)$, we may thus assume G to be definably connected and solvable. But now in G there is only one conjugacy class of Carter subgroups, by Theorem 40. This completes our proof of Theorem 1. □

We mention some consequences of Theorem 1.

Corollary 72. *Let G be a definable group in an o-minimal expansion of an ordered group $\mathcal{M}$ over a set of parameters A containing an element different from 0. Then, in each conjugacy class there exists a Cartan subgroup definable over A.*

Proof. By Theorem 1 and Corollary 62. □

Corollary 73. *Let G be a definably connected group definable in an o-minimal structure, and let N be a definable normal subgroup of G. Then $G = N_G{}^{\circ}(Q)N^{\circ}$ for any Cartan subgroup Q of N.*

Proof. Clearly, for any element g of G, Q^g is a Cartan subgroup of N. On the other hand, the set $\mathcal{Q}$ of conjugacy classes of Cartan subgroups of N is finite, by Theorem 1, and the action of G on N by conjugation naturally induces a definable action on the finite set $\mathcal{Q}$. Since G is definably connected, Fact 2(a) shows that this action must be trivial. Hence, for any g in G, Q^g is indeed in the same N-conjugacy class as Q, i.e., $Q^g = Q^h$ for some $h \in N$; in particular, $g = gh^{-1}h \in N_G(Q)N$. Hence $G = N_G(Q)N$, and in fact $G = N_G{}^{\circ}(Q)N^{\circ}$ by definable connectedness. □

We shall now inspect case by case what survives of Theorem 63(2)–(5) in the general case. We first consider Theorem 63(2).

Theorem 74. *Let G be a definably connected group definable in an o-minimal structure, and let Q be a Cartan subgroup of G. Then there is a unique (definable) subgroup K_Q of G containing $R^{\circ}(G)$ such that $K_Q/R^{\circ}(G)$ is the unique Cartan subgroup of $G/R^{\circ}(G)$ containing $Q^{\circ}R^{\circ}(G)/R^{\circ}(G)$. Moreover, $QR(G) \leqslant K_Q$ and*

$$Q = F(N_{K_Q}(Q^{\circ})) = C_{K_Q}(Q^{\circ})Q^{\circ} = C_G(Q^{\circ})Q^{\circ}.$$

Proof. By Corollary 71, the group $Q^{\circ}R^{\circ}(G)/R^{\circ}(G)$ is a Carter subgroup of the semisimple quotient $G/R^{\circ}(G)$. By Theorem 63(2), it is contained in a unique Cartan subgroup, of the form $K/R^{\circ}(G)$ for some subgroup K containing $R^{\circ}(G)$ and necessarily definable by Lemma 5(a′). We will show that $K_Q = K$ satisfies all our claims. Since $QR^{\circ}(G)/R^{\circ}(G)$ is nilpotent and contains the Carter subgroup $Q^{\circ}R^{\circ}(G)/R^{\circ}(G)$, we have $QR^{\circ}(G) \leqslant K$. Since $R(G)/R^{\circ}(G)$ is the center of $G/R^{\circ}(G)$, it is contained in $K/R^{\circ}(G)$, by Lemma 8(a), and thus $R(G) \leqslant K$. Hence, $QR(G) \leqslant K$.

To prove our last equalities, we first show that $F(N_K(Q^{\circ})) = C_K(Q^{\circ})Q^{\circ}$. Since $Q^{\circ} = F^{\circ}(N_K(Q^{\circ}))$ by Lemma 4, the inclusion from left to right follows from Fact 6. For the reverse inclusion, notice that $C_K(Q^{\circ})Q^{\circ}$ is normal in $N_K(Q^{\circ})$. Since Cartan subgroups of $G/R^{\circ}(G)$ are nilpotent in two steps by Theorem 63(3), the second term of the descending central series of $C_K(Q^{\circ})Q^{\circ}$ is in $R^{\circ}(G)$, and thus in Q°, because Q° is selfnormalizing in $Q^{\circ}R^{\circ}(G)$, by Theorem 40. By keeping taking descending central series and using the nilpotency of Q°, we then see that $C_K(Q^{\circ})Q^{\circ}$ is nilpotent, and thus in $F(N_K(Q^{\circ}))$, by normality in $N_K(Q^{\circ})$.

Since $C_G(Q^{\circ}) \leqslant K$, clearly by considering its image modulo $R^{\circ}(G)$, our last equality is true. Finally, $Q = C_Q(Q^{\circ})Q^{\circ}$, by Fact 6, and thus $Q \leqslant C_K(Q^{\circ})Q^{\circ} = F(N_K(Q^{\circ}))$. Now the maximal nilpotence of Q forces $Q = F(N_K(Q^{\circ}))$, and our proof is complete. □

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With Theorem 74 one readily gets the analog of Theorem 63(2). Of course definable connectedness is a necessary assumption here, since a finite group may have several Cartan subgroups.

Corollary 75. *Let G be a definably connected group definable in an o-minimal structure, and let Q_1 and Q_2 be two Cartan subgroups. If $Q_1^\circ = Q_2^\circ$, then $Q_1 = Q_2$.*

The main question we are facing with at this stage is the following.

Question 76. *Is it the case, in Theorem 74, that $QR^\circ(G)/R^\circ(G)$ is a Cartan subgroup of G/N? Or equivalently, is it the case in Theorem 74 that $K_Q = QR^\circ(G)$?*

Remark 77. Let G be a definably connected group definable in an o-minimal structure. Let N be a normal definable definably connected and solvable subgroup of G. Suppose that, for any Cartan subgroup of G, its image in G/N is a Cartan subgroup of G/N. Then the Cartan subgroups of G/N are exactly of the form QN/N for some Q Cartan subgroup of G.

Proof. Let $X = H/N$ be a Cartan subgroup of G/N. So $X^\circ = H^\circ/N$ is a Carter subgroup of G/N. By Lemma 70, $X^\circ = Q_1N/N$ for some Q_1 Carter subgroup of G. By Lemma 5, there is a Cartan subgroup Q of G with $Q^\circ = Q_1$. Hence $X^\circ = Q_1N/N = [QN/N]^\circ$, and both X and QN/N are Cartan subgroups of G/N. By Corollary 75, we have that $X = QN/N$, and thus X is of the required form. □

In Theorem 74, we also get that $QR^\circ(G)$ is normal in K_Q, and actually has a quite stronger uniqueness property in K_Q.

Corollary 78. *We use the same assumptions and notation as in Theorem 74. Then $[K_Q]^\circ = Q^\circ R^\circ(G)$, and $QR^\circ(G)$ is invariant under any automorphism of K_Q leaving $[K_Q]^\circ$ invariant.*

Proof. The first equality comes from Lemma 70.

Let σ be an arbitrary automorphism of K_Q leaving $[K_Q]^\circ$ invariant. Since Q° is a Cartan subgroup of $[K_Q]^\circ$, by Corollary 47, its image by σ is also a Cartan subgroup of $[K_Q]^\circ$, and with Theorem 40 one gets $[Q^\circ]^\sigma = [Q^\circ]^k$ for some k in $[K_Q]^\circ$. Since $QR^\circ(G)$ is normalized by k, we can thus assume that σ leaves Q° invariant. But now σ leaves $F(N_{K_Q}(Q^\circ))$ invariant. Hence, by Theorem 74, Q is left invariant by σ, and thus σ leaves $Q[K_Q]^\circ = QR^\circ(G)$ invariant. □

Question 76 has *a priori* stronger forms, which are indeed equivalent, as the following lemma shows.

Lemma 79. *Under the assumptions and notation of Theorem 74, the following are equivalent.*

(a) *$K_Q = PR^\circ(G)$ for some Cartan subgroup P of G.*

(b) *$K_Q = PR^\circ(G)$ for any Cartan subgroup P of K_Q.*

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Proof. Assume that $K_Q = P_1 R^\circ(G)$ for some Cartan subgroup P_1 of G, and suppose that P_2 is a Cartan subgroup of K_Q. Then P_1° and P_2° are Carter subgroups of $[K_Q]^\circ$, by Lemma 5(a′). Since they are $[K_Q]^\circ$-conjugate by Theorem 40, we may assume that $P_1^\circ = P_2^\circ$ up to conjugacy. Now applying Theorem 74 with the Cartan subgroup P_1, or just Corollary 75, we see that $P_1 = P_2$ up to conjugacy, and thus $K_Q = P_2 R^\circ(G)$.

Conversely, suppose that $K_Q = PR^\circ(G)$ for any Cartan subgroup P of K_Q. This applies in particular to the Cartan subgroup Q of G. □

By the usual Frattini argument following the conjugacy of Cartan/Carter subgroups in $[K_Q]^\circ$, we have that $K_Q = \hat{Q} R^\circ(G)$, where

$$\hat{Q} = N_{K_Q}(Q^\circ).$$

The subgroup $\hat{Q}$ is solvable and nilpotent-by-finite, and with the selfnormalization property of Q° in the definably connected solvable group $Q^\circ R^\circ(G)$ one sees easily that $\hat{Q}/Q \simeq K_Q/(QR^\circ(G))$. Hence Question 76 is equivalent to proving that the finite quotient $\hat{Q}/Q$ is trivial.

Retaining all the notation introduced so far, Theorem 63(3) takes the following form for a general definably connected group.

Theorem 80. *We use the same assumptions and notation as in Theorem 74. Then* $[K_Q]' \leqslant R(G)$, *and* $[\hat{Q}, [\hat{Q}]'] \leqslant Q^\circ \cap R^\circ(G)$, *where* $\hat{Q} = N_{K_Q}(Q^\circ)$.

Proof. By Theorem 63(3), $[K_Q]' \leqslant R(G)$ and $[K_Q, [K_Q]'] \leqslant R^\circ(G)$. The second inclusion shows in particular that $[\hat{Q}, [\hat{Q}]'] \leqslant R^\circ(G)$, and, since Q° is selfnormalizing in $Q^\circ R^\circ(G)$ by Theorem 40, we get inclusion in Q° as well. □

With respect to Theorem 63(3), we also have the following.

Question 81. *Do all the Cartan subgroups of a given group definable in an o-minimal structure have equal dimension?*

We now consider Theorem 63(4) and give its most general form in the general case (working in particular without any assumption of definable connectedness of the ambient group).

Theorem 82. *Let G be a group definable in an o-minimal structure, let Q be a Cartan subgroup of G, and let $a \in Q$. Then aQ° is slightly generous in G. In fact, the set of elements of aQ° contained in a unique conjugate of aQ° is large in aQ°. Furthermore, if G is definably connected, then the set of elements of Q contained in a unique conjugate of Q is large in Q.*

Proof. We first prove that the set of elements of Q° contained in a unique G-conjugate of Q° is large in Q°. For that purpose, it suffices by Corollary 30 to show that the set of elements of Q° contained in only finitely many G-conjugates of Q° is large in Q°. Assume towards a contradiction that the set Q_∞ of elements of Q° contained in infinitely many G-conjugates of Q° is slightly generic in Q°. By Theorem 40, we may restrict Q_∞ to

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the subset of elements contained in a unique $Q^\circ R^\circ(G)$-conjugate of Q°, and still have a slightly generic subset of Q°. Now Q_∞ must have a slightly generic image in Q° modulo $R^\circ(G)$. By Theorem 63(4), we must then find an element $x \in Q_\infty$ which, modulo $R^\circ(G)$, is in a unique conjugate of Q° (i.e., if $\overline{x}$ is in $\overline{Q^{\circ}}^{\overline{g}}$ then $\overline{g} \in N_{\overline{G}}(\overline{Q^\circ})$, where '—' means quotient by $R^\circ(G)$). Then we have infinitely many Carter subgroups of $Q^\circ R^\circ(G)$ passing through x, a contradiction, since they are all $Q^\circ R^\circ(G)$-conjugate, by Theorem 40.

We now consider the full Cartan subgroup Q, and an arbitrary element a in Q. For the slight generosity of aQ° in G, it suffices to use our general Corollary 21. Indeed, by Corollary 17, it suffices to show the stronger property that the set of elements of aQ° in a unique conjugate of aQ° is large in aQ°. Assume towards a contradiction that the set X of elements of aQ° in at least two distinct conjugates of aQ° is slightly generic in aQ°. If n is the order of a modulo Q°, then the set of nth powers of elements of X would be slightly generic in Q°, by Corollary 23. Hence by the preceding paragraph one would find an element x in X such that x^n is in a unique conjugate of Q°. This is a contradiction as usual, since xQ° must then be the unique conjugate of aQ° containing x.

We now prove our last claim about Q when G is definably connected. Assume towards a contradiction that the set X of elements in Q and in at least two distinct conjugates of Q is slightly generic in Q. Then it should meet one of the cosets aQ° of Q° in Q in a slightly generic subset, say X'. By Corollary 23 again, one finds an element x in X' such that $x^{|Q/Q^\circ|}$ is in a unique conjugate of Q°. Now all the conjugates of Q passing through x should have the same definably connected component, and thus are $N_G(Q^\circ)$-conjugate. Then they are all equal by Corollary 75, a contradiction. □

When Question 76 fails, we unfortunately found no way of proving Theorem 82 for a in $\hat{Q} \setminus Q$. Besides, our method for proving the slight generosity of aQ° in G does not seem to be appropriate for attacking the following more refined question.

Question 83. *Let G, Q, and a be as in Theorem 82, with G definably connected and such that, modulo $R^\circ(G)$, a is in a unique conjugate of aQ°.*

(a) *Is it the case that $[aQ^\circ]^{R^\circ(G)}$ is large in $aQ^\circ R^\circ(G)$?*

(b) *Suppose now a is in $\hat{Q}$, and the rest of the conditions remain the same. Is it the case that $[aQ^\circ]^{R^\circ(G)}$ is large in $aQ^\circ R^\circ(G)$?*

By Theorem 82, the union of Cartan subgroups of a group definable in an o-minimal structure must be slightly generic, but the much stronger statement of Theorem 63(5) now becomes a definite question.

Question 84. *Let G be a definably connected group definable in an o-minimal structure. Is it the case that the union of its Cartan subgroups forms a large subset?*

We now prove that Question 84 can be seen on top of both Questions 76 and 83.

Proposition 85. *Let G be a definably connected group definable in an o-minimal structure whose Cartan subgroups form a large subset. Then the following hold.*

(a) *Cartan subgroups of $G/R^\circ(G)$ are exactly of the form $QR^\circ(G)/R^\circ(G)$ with Q a Cartan subgroup of G.*

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(b) *For every Cartan subgroup Q and a in Q such that, modulo $R^\circ(G)$, a is in a unique conjugate of aQ°, $[aQ^\circ]^{R^\circ(G)}$ is large in $aQ^\circ R^\circ(G)$.*

Proof. (a) Assume towards a contradiction that, for some Cartan subgroup Q, and with the previously used notation, we have $QR^\circ(G) < K_Q$. Let $\overline{B}$ be the large subset of $(K_Q/R^\circ(G)) \setminus (QR^\circ(G)/R^\circ(G))$ then provided by Theorem 63(4), and B its pull back in G. By additivity of the dimension, B^G must be slightly generic in G. Now the largeness of the set of Cartan subgroups forces the existence of an element g in $B \cap P$ for some Cartan subgroup P of G. Let $\overline{g}$ denote the image of g in $G/R^\circ(G)$. We have $g \in K_Q \setminus QR^\circ(G)$, and $C^\circ(\overline{g}) = Q^\circ R^\circ(G)/R^\circ(G)$ by considering the structure of Cartan subgroups in the semisimple quotient $G/R^\circ(G)$ and the uniqueness property of $\overline{g}$. By Lemma 70, the group P°, modulo $R^\circ(G)$, is a Carter subgroup of $G/R^\circ(G)$. Now P, modulo $R^\circ(G)$, is included in a Cartan subgroup of $G/R^\circ(G)$, and its definably connected component centralizes $\overline{g}$, by Theorem 63(3). We then get $P^\circ R^\circ(G)/R^\circ(G) \leqslant C^\circ(\overline{g}) = Q^\circ R^\circ(G)/R^\circ(G)$, and actually equality since the first group is a Carter subgroup. Hence $P^\circ R^\circ(G) = Q^\circ R^\circ(G)$, and Theorem 74 yields $P \leqslant K_Q$. Since Q° and P° are conjugate in $Q^\circ R^\circ(G)$, by Theorem 40, we may also assume without loss of generality that $P^\circ = Q^\circ$. But then $P = Q$ by Corollary 75, a contradiction, since $g \notin QR^\circ(G)$.

(b) Let A be the pull back in G of the large set of $G/R^\circ(G)$ provided in Theorem 63(5), and let

$$A = A_1 \sqcup \cdots \sqcup A_{n(G)}$$

be the pull back in G of the corresponding partition of that large set equally provided in Theorem 63(6). Here $n(G)$ is the number of 'generic types' of $G/R^\circ(G)$ computed with precision in Remark 69. By additivity of the dimension, A is large in G, and each A_i is slightly generic. Our claim is that, for Q a Cartan subgroup of G and $a \in Q \cap A_i$ for some i, the set $[aQ^\circ]^{R^\circ(G)}$ is large in $aQ^\circ R^\circ(G)$. Since Q° normalizes the coset aQ°, this is equivalent to showing that $[aQ^\circ]^{Q^\circ R^\circ(G)}$ is large in $aQ^\circ R^\circ(G)$. But, by Theorem 63(4)–(5) applied in $G/R^\circ(G)$, one can see that the largeness of the set of Cartan subgroups of G and the additivity of the dimension forces $[aQ^\circ]^{Q^\circ R^\circ(G)}$ to be large in $aQ^\circ R^\circ(G)$. □

For instance, if G is a definably connected real Lie group definable in an o-minimal expansion of $\mathbb{R}$, then its Cartan subgroups form a large subset by Fact 56(a) and the fact that density implies largeness for definable sets (as seen in the proof of Theorem 58). Hence, by Proposition 85, such a G cannot produce a counterexample to either Question 76 or Question 83. Attacking Question 84 in general would seem to rely on an abstract version of Fact 56(a), but with *a priori* no known abstract analog of regular elements (as in the proof of Fact 55(c)) it seems difficult to find any spark plug.

13. Final remarks

We begin this final section with additional comments on Question 76 in special cases. If G is a definably connected group definable in an o-minimal structure, then by Fact 35 we have

$$G/R(G) = G_1/R(G) \times \cdots \times G_n/R(G)$$

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for some definable subgroups G_i containing $R(G)$ and such that $G_i/R(G)$ is definably simple. For each i, $G_i/R(G)$ is definably connected, and thus $G_i = {G_i}^\circ R(G)$. From the decomposition $G = G_1 \cdots G_n$, we get $G = {G_1}^\circ \cdots {G_n}^\circ R(G)$. By definable connectedness of G, we also get a decomposition

$$G = {G_1}^\circ \cdots {G_n}^\circ, \qquad (*)$$

where each ${G_i}^\circ$ is definably connected, contains $R^\circ(G)$, and ${G_i}^\circ/R^\circ(G)$ is finite-by-(definably simple), as $R({G_i}^\circ) = {G_i}^\circ \cap R(G)$ and ${G_i}^\circ/R({G_i}^\circ)$ is definably isomorphic to $G_i/R(G)$. We may analyze certain factors ${G_i}^\circ$ individually with the following result.

Remark 86. Let $\mathcal{M}$ be an o-minimal structure, and let G be a definably connected group definable in $\mathcal{M}$ with $R(G) = Z(G)$ finite and $G/Z(G)$ definably simple. If G is stable as a pure group, then it has a single conjugacy class of Cartan subgroups, which are divisible, definably connected, and largely generous.

Proof. Since $G/Z(G)$ is stable and definably simple, G is definably isomorphic to an algebraic group over an algebraic closed field (see [19, 4.4(ii), 6.3]). Since G is moreover reductive, the maximal algebraic tori coincide with Cartan subgroups (see [3, 12.6 and 13.17]). Hence they are isomorphic to a direct product of finitely many copies of the multiplicative group of the ground field (where the number of copies is the Lie rank of the group seen as a pure algebraic group). In particular, they are divisible, and thus with no proper subgroup of finite index. Moreover, by Theorem 63(5), they are largely generous in G. □

Consider the decomposition (*) of a definably connected group G as above, and let $I = \{1, \ldots, n\}$. Let I_1 be the subset of elements $i \in I$ such that ${G_i}^\circ/R^\circ({G_i}^\circ)$ is stable as a pure group or definably compact (notice that it suffices to require the definably simple group ${G_i}^\circ/R({G_i}^\circ)$ to be stable or definably compact). Let I_2 be the subset of elements $i \in I$ such that Cartan subgroups of ${G_i}^\circ/R^\circ({G_i}^\circ)$ are definably connected. Finally, let I_3 be the subset of elements $i \in I$ such that in ${G_i}^\circ$ Question 76 has a positive answer for any Cartan subgroup. Corollary 38 and Remark 86 show that $I_1 \subseteq I_2$ and Lemma 70 shows that $I_2 \subseteq I_3$. Hence

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq I,$$

and the inclusion $I_1 \subseteq I_3$ reads informally as the fact that the definably simple factors of $G/R(G)$ which are algebraic or compact cannot produce any counterexample to the lifting problem of Question 76. More precisely, we have the following statement.

Remark 87. If $I_2 = I$, then, for every Q Cartan subgroup of G, $K_Q = QR^\circ(G)$ in Question 76.

Proof. First, one can check that, modulo $R^\circ(G)$, the decomposition (*) of G becomes a central product:

$$G/R^\circ(G) = {G_1}^\circ/R^\circ(G) * \cdots * {G_1}^\circ/R^\circ(G).$$

Indeed, if $i \neq j$, then $[G_i{}^\circ, G_j{}^\circ] \leqslant R(G)$, and $R(G)$ is finite modulo $R^\circ(G)$. Hence any element in $G_i{}^\circ/R^\circ(G)$ has a centralizer of finite index in the other factor $G_j{}^\circ/R^\circ(G)$, which must then be the full factor $G_j{}^\circ/R^\circ(G)$ by definable connectedness. Therefore the factors $G_i{}^\circ/R^\circ(G)$ pairwise commute, as claimed. Now Lemma 9 gives that Cartan subgroups of $G/R^\circ(G)$ are exactly of the form $Q_1/R^\circ(G) * \cdots * Q_n/R^\circ(G)$ with, for each i, $Q_i/R^\circ(G)$ a Cartan subgroup of $G_i{}^\circ/R^\circ(G)$.

Assuming now that $I_2 = I$ we get that, for each i, each Cartan subgroup $Q_i/R^\circ(G)$ of $G_i{}^\circ/R^\circ(G)$ is definably connected. We then see that Cartan subgroups of $G/R^\circ(G)$ must be definably connected as well. Now Lemma 70 implies that Question 76 is positively satisfied for every Cartan subgroup of G (and that such Cartan subgroups of G are all definably connected and Carter subgroups by Corollary 47). □

The decomposition (*) of a definably connected group G as above is also convenient for describing the various relative Weyl groups. If Q is a Cartan subgroup of G, then we still have that $N_G(Q^\circ) = N_G(Q)$ by Corollary 75. If Question 76 is positively satisfied for Q, then, retaining the notation of § 12 and using the notation '—' for quotients modulo $R^\circ(G)$, we get, as after Lemma 79, that

$$W(\overline{G}, \overline{K_Q}) \simeq N_G(Q)/Q.$$

We also see, with Theorem 74 or just Lemma 8(a), that $\overline{R(G)}$ does not contribute to the Weyl group $W(\overline{G}, \overline{K_Q})$. Hence the latter is isomorphic to the direct product of the Weyl groups in $G_i/R(G)$ relative to the factors of $QR(G)/R(G)$ in its decomposition along the decomposition $G_1/R(G) \times \cdots \times G_n/R(G)$ of $G/R(G)$ (Corollary 10). Since the group $N_G(Q)/Q$ is isomorphic to $W(\overline{G}, \overline{K_Q})$, it has the same isomorphism type, and may be called the *Weyl group relative to* Q.

Without assuming the exact lifting of Question 76 for the Cartan subgroup Q we only get, with Corollary 78, and as after Lemma 79, that

$$W(\overline{G}, \overline{K_Q}) \simeq (N_G(Q)/Q)/(\hat{Q}/Q).$$

In this case, the Weyl group $W(\overline{G}, \overline{K_Q})$ has the same description as above, but $N_G(Q)/Q$ just has a quotient isomorphic to $W(\overline{G}, \overline{K_Q})$.

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