J. Fluid Mech. (2006), *vol.* 559, *pp.* 299–306.
doi:10.1017/S0022112006000619 Printed in the United Kingdom

The effect of vibrations on sideways double-diffusive instabilities: strong stratification asymptotics

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(Received 13 February 2006 and in revised form 25 April 2006)

A body of salt-stratified fluid in a vertical slot can undergo double-diffusive instabilities when laterally heated. A previous study has indicated the possibility that vibrations could induce instabilities in the cases of a strong salinity gradient in regimes that are not only linearly stable, but also nonlinearly stable. We investigate this limit using the method of averaging and confirm that any level of high-frequency vibrations will lead to a reduction in the heating required for instability for a sufficiently strong salinity gradient, but that this is probably not of great importance to terrestrial experimenters.

1. Introduction

When a fluid has gradients of both temperature and composition instabilities can arise even though the fluid is gravitationally stable. The origin of the instabilities lies in the difference between the diffusivities of these two components and so the effect is often called double-diffusive convection (for an early review see Turner 1974). Such temperature and compositional gradients can occur in many situations such as in the oceans, magma chambers, and many industrial applications such as the growing of crystals from a solution (Schmitt 1994; Brandt & Fernando 1996).

The stability of fluid in a laterally heated vertical slot with a mean compositional gradient parallel to the walls was first examined by Thorpe, Hutt & Soulsby (1969) both experimentally and, in the limit of a strong vertical compositional gradient, theoretically. The full theoretical stability boundary was first examined by Thangam, Zebib & Chen (1981), and later by Young & Rosner (1998) and Kerr & Tang (1999). The last of these also identified further asymptotic regimes on the stability boundary in addition to the strong compositional gradient limit found previously.

A desire to improve the understanding of the flow in the fluid surrounding growing crystals has arisen in the context of experiments being conducted on the International Space Station. In this environment there is a much reduced gravity or mean acceleration; however there is often a significant degree of vibration from a variety of sources (Ostrach 1982) which can have a significant impact on convection (Wheeler *et al.* 1991; Gershuni *et al.* 1997). Motivated by this, Zebib (2001) repeated the previous analysis of instabilities in a vertical slot but including the effect of vibrations, or g-jitter, showing that it could significantly modify the stability boundary of the system for parameters appropriate to space.

One region of the stability boundary of interest was the behaviour of the critical thermal Rayleigh number, Ra_T, (a measure of the lateral temperature difference across the slot) as a function of the salt Rayleigh number, Ra_S, (a measure of the compositional stratification along the slot, both defined in the next section) as Ra_S becomes large. The boundaries on the log–log plot in figure 3 of Zebib (2001) seemed to be asymptoting to a straight line of slope 2/3 indicating $Ra_T \propto Ra_S^{2/3}$ for large Ra_S. This gradient is less than the gradient of 5/6 found in the absence of vibration (Thorpe *et al.* 1969; Kerr & Tang 1999). This 5/6 power law is also found for the linear stability boundary in the case of heating a salinity gradient from a single sidewall for strong stratification (Kerr 1989). In addition, for the single sidewall case, a boundary for nonlinear disturbances has been found (Kerr 1990) and this too has a slope of 5/6. A similar analysis for the case of a vertical slot with the same assumption that the vertical scale of the instabilities is limited by the Chen scale (Chen, Briggs & Wirtz 1971) yields the boundary for nonlinear stability of

$$Ra_T = (128\pi^4 (1-\tau)^3 \tau) Ra_S^{5/6}. \tag{1.1}$$

Here τ is the ratio of the salt diffusivity to the thermal diffusivity, usually a small number. Thus all the known stability boundaries for both a slot and a single sidewall in the limit of strong stratification are of similar form to (1.1) but with different factors. This leads to the possibility that if the apparent asymptotes of Zebib (2001) are real and extend all the way to infinity then, for any level of vibration, linear theory would predict instabilities in a region that in the absence of vibration would be stable to both linear and nonlinear disturbances. It is this possibility that is the focus of this paper.

Here we adapt the large-gradient asymptotics of double-diffusive instabilities of a vertical slot to include the effect of vibration or g-jitter. We also investigate the incursion of the stability boundary into the region in the Ra_T–Ra_S-plane where in the absence of vibration the system is stable to all disturbances, and show that the boundary does indeed cross into the region of linear and nonlinear stability for any level of vibration for sufficiently strong salinity gradients. However, we conclude that this is unlikely to be a major concern for terrestrial experimenters.

2. Problem formulation and results

In this analysis we will follow Zebib (2001) in considering the case where the slot walls are parallel to the mean acceleration (i.e. vertical on Earth) of strength g, and the vibration consists of oscillations perpendicular to the walls with acceleration $g_1 \cos \omega_1 t$. The governing equations for linear perturbations to an incompressible fluid experiencing such accelerations are

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla(\overline{W}(x)\hat{\boldsymbol{z}}) + \overline{W}(x)\frac{\partial \boldsymbol{u}}{\partial z} = -\frac{1}{\rho_0}\nabla p + (g\hat{\boldsymbol{z}} + g_1 \cos \omega_1 t\, \hat{\boldsymbol{x}})(\alpha T - \beta S) + \nu \nabla^2 \boldsymbol{u}, \tag{2.1a}$$

$$\frac{\partial T}{\partial t} + u\overline{T}_x + \overline{W}(x)\frac{\partial T}{\partial z} = \kappa_T \nabla^2 T, \tag{2.1b}$$

$$\frac{\partial S}{\partial t} + u\overline{S}_x + w\overline{S}_z + \overline{W}(x)\frac{\partial S}{\partial z} = \kappa_S \nabla^2 S, \tag{2.1c}$$

$$\nabla \cdot \boldsymbol{u} = 0. \tag{2.1d}$$

Here $\boldsymbol{u} = (u, v, w)$ is the velocity of the fluid, p, T and S the pressure, temperature and salinity perturbations, ν the kinematic viscosity, κ_T and κ_S the heat and salt diffusivities, α the coefficient of thermal expansion and β the coefficient of density increase with respect to the addition of salt. We assume that the density of the fluid is given by a linear relation of the form

$$\rho = \rho_0(1 - \alpha(T - T_0) + \beta(S - S_0)), \tag{2.2}$$

where T_0 and S_0 are some reference temperature and salinity, and ρ_0 the corresponding density of the fluid. We assume that the departures from this reference density are small and so the Boussinesq approximation can be applied. The vertical background salinity gradient, $\overline{S}_z$, is taken to be constant. The horizontal temperature gradient, $\overline{T}_x$, will be $\Delta T/d$ where ΔT is the temperature difference of the vertical walls, and d is the slot width. The background horizontal salinity gradient, $\overline{S}_x$, and vertical velocity, $\overline{W}$, are both functions of the horizontal coordinate, x. Because of the choice of vibration direction, these last two are identical to those found in the vibration-free slot (Thangam *et al.* 1981; Kerr & Tang 1999). This isolates the effect of the vibration on the instabilities from possible modification of the background flow.

The analysis follows that of Gershuni *et al.* (1997) by assuming that the vibrations are fast in comparison to the growth rate of the instabilities. This gives rise to a two-time-scale approximation, where the various components can be split into a part that oscillates with the vibration frequency, but has zero mean on the longer time scale, and a part that evolves on the longer time scale but is steady on the fast time scale. The vibrating components of the temperature and salinity do not, at leading order, play a role in the subsequent evolution of the instabilities. The oscillating component of the velocity is important, and we will require a disturbance streamfunction of the form

$$\psi(x)\exp\{\sigma t + \mathrm{i}\alpha z\} + f(x)\exp\{\sigma t + \mathrm{i}\omega_1 t + \mathrm{i}\alpha z\}, \tag{2.3}$$

where σ is the growth rate of the disturbances. The temperature and salinity perturbations take the form

$$T(x)\exp\{\sigma t + \mathrm{i}\alpha z\} \quad \text{and} \quad S(x)\exp\{\sigma t + \mathrm{i}\alpha z\}. \tag{2.4}$$

The details of the derivation of the non-dimensional governing equations for these instabilities to a salt-stratified fluid undergoing lateral heating in a slot with horizontal vibration are omitted here. They can be found in some detail in Zebib (2001). His four non-dimensional differential equations (his (11*a*)–(11*d*)), after a small amount of simplification, are

$$\left\{(\mathrm{D}^2-\alpha^2)^2 - \frac{\mathrm{i}\alpha}{Pr}[\overline{W}(\mathrm{D}^2-\alpha^2) - \overline{W}'']\right\}\psi + \mathrm{D}(Ra_T T - Ra_S S)$$
$$+ R_j[\mathrm{i}\alpha(Ra_T + Ra_S F') + Ra_S \mathrm{D}]f = \frac{\sigma}{Pr}(\mathrm{D}^2-\alpha^2)\psi, \tag{2.5a}$$

$$(\mathrm{D}^2-\alpha^2)T - \mathrm{i}\alpha\psi - \mathrm{i}\alpha\overline{W}T = \sigma T, \tag{2.5b}$$

$$\tau(\mathrm{D}^2-\alpha^2)S + \mathrm{i}\alpha F'\psi - \mathrm{i}\alpha\overline{W}S + \psi' = \sigma S, \tag{2.5c}$$

$$(\mathrm{D}^2-\alpha^2)f + \alpha^2(Ra_T T - Ra_S S) = 0, \tag{2.5d}$$

where all the variables are now non-dimensional, and D indicates a derivative with respect to x. The appropriate boundary conditions are

$$f = \psi = \psi' = T = S' = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \tag{2.6}$$

The fluid properties are given by the Prandtl number, $Pr = \nu/\kappa_T$, and the salt/heat diffusivity ratio, $\tau = \kappa_S/\kappa_T$. The parameters describing the vertical salinity gradient and the lateral heating are the salt and thermal Rayleigh numbers

$$Ra_S = \frac{g\phi_0 d^4}{\nu\kappa_T}, \qquad Ra_T = \frac{g\beta\Delta T d^3}{\nu\kappa_T}. \tag{2.7}$$

Here $\phi_0 = -\beta\overline{S}_z/\rho_0$ is the magnitude of the ratio of the density gradient due to the salt stratification to the reference density. The terms that are not present in the absence of g-jitter are those involving R_j and f. The former is the non-dimensional jitter parameter defined to be

$$R_j = \frac{g_1^2\kappa_T^2 Pr}{2g^2 d^4\omega_1^2}. \tag{2.8}$$

In these equations the background vertical velocity and horizontal salinity gradient are given by $\overline{W}$ and F' respectively. This change of notation for the salinity gradient is for comparability with Zebib (2001). In the limit of large Ra_S the background velocity becomes exponentially small everywhere except in thin boundary layers of width $O(Ra_S^{-1/4})$. The horizontal salinity gradient is also approximately constant, deviating from the constant gradient $-Ra_T/Ra_S$ by exponentially small amounts except in these thin boundary layers. These thin boundary layers do not play any part in the leading-order asymptotics and are ignored.

The instabilities were observed to grow monotonically by Zebib (2001), and so the appropriate growth rate, σ, for marginal stability is $\sigma = 0$.

Examination of possible balances between terms shows that in the large-Ra_S limit a possible set of scalings for the various variables is

$$\psi \propto Ra_S^0, \quad T \propto Ra_S^{-1/3}, \quad S \propto Ra_S^{-2/3}, \quad f \propto Ra_S^{1/3}, \quad \alpha \propto Ra_S^{1/3}, \quad Ra_T \propto Ra_S^{2/3}. \tag{2.9}$$

In this limit the vertical diffusion terms dominate the horizontal diffusion terms in the bulk of the fluid. This will lead to the loss of all the boundary conditions except $\psi = 0$ at $x = \pm 1/2$ for the solution in the core of the fluid. The flow will adjust to the other boundary conditions in thin boundary layers at the wall in a similar way to that found by Hart (1971) for a strong stratification in a vertical slot without vibrations. The effect of these thin boundary layers on the flow in the core of the slot is negligible at leading order, and can be ignored.

Setting

$$\left.\begin{aligned} \psi = \psi^*, \quad T = T^* Ra_S^{-1/3}, \quad S = S^* Ra_S^{-2/3}, \\ f = f^* Ra_S^{1/3}, \quad \alpha = \alpha^* Ra_S^{1/3}, \quad Ra_T = Ra_T^* Ra_S^{2/3}, \end{aligned}\right\} \tag{2.10}$$

we derive the leading-order equations for the rescaled equations in the bulk of the fluid for instabilities with zero growth rate:

$$\alpha^{*4}\psi^* + R_j f^{*\prime} = 0, \tag{2.11a}$$

$$-\alpha^{*2}T^* - \mathrm{i}\alpha^*\psi^* = 0, \tag{2.11b}$$

$$-\tau\alpha^{*2}S^* - \mathrm{i}\alpha^* Ra_T^*\psi^* + \psi^{*\prime} = 0, \tag{2.11c}$$

$$-\alpha^{*2}f^* + \alpha^{*2}(Ra_T^* T^* - S^*) = 0, \tag{2.11d}$$

with boundary conditions

$$\psi^* = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \tag{2.12}$$

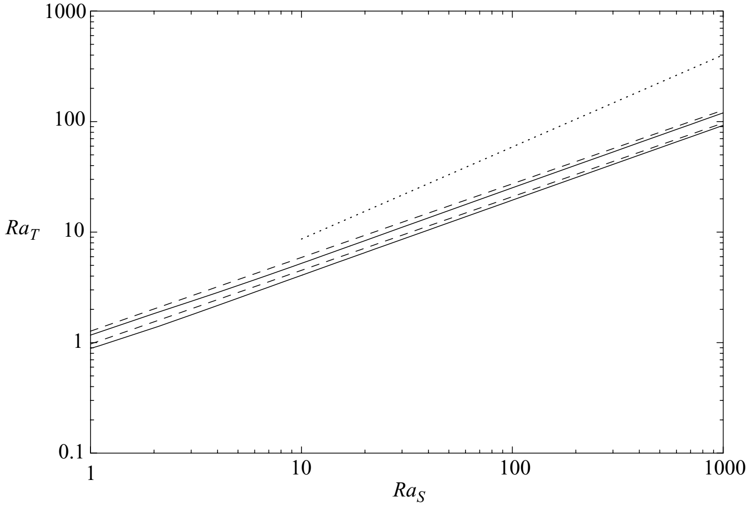

FIGURE 1. Comparison of the numerical results of Zebib (2001) (solid line) and the asymptotic predictions (dashed line) of the thermal Rayleigh number, Ra_T, for marginal stability as a function of the salt Rayleigh number, Ra_S. The upper pair of lines are for $R_j = 1$ and the lower pair for $R_j = 5$. In both cases $Pr = 1$ and $\tau = 0.0001$. The dotted line shows the large-Ra_S asymptotic behaviour for this case when vibration is absent.

These can be manipulated to give a second-order differential equation for ψ^*

$$\psi^{*\prime\prime} - \mathrm{i}(1-\tau)\alpha^* Ra_T^* \psi^{*\prime} - \frac{\alpha^{*6}\tau}{R_j}\psi^* = 0. \tag{2.13}$$

This equation allows non-trivial solutions if the following condition is met (see Kerr 1989):

$$2n\pi = \left((1-\tau)^2 \alpha^{*2} Ra_T^{*\,2} - \frac{4\alpha^{*6}\tau}{R_j} \right)^{1/2}, \tag{2.14}$$

for $n = 1, 2, \ldots,$ which is equivalent to

$$Ra_T^* = (1-\tau)^{-1} \left(\frac{4n^2\pi^2 + 4\alpha^{*6}\tau/R_j}{\alpha^{*2}} \right)^{1/2}, \tag{2.15}$$

and is minimized when $n = 1$ with

$$\alpha^* = \left(\frac{\pi^2 R_j}{2\tau} \right)^{1/6}. \tag{2.16}$$

This gives the leading-order asymptotics for the stability boundary for the large-Ra_S g-jitter instability of

$$Ra_T = \frac{6^{1/2}}{(1-\tau)} \left(\frac{2\pi^4\tau}{R_j} \right)^{1/6} Ra_S^{2/3}. \tag{2.17}$$

The convergence of the stability boundary calculated from the numerical solutions of Zebib (2001) to the asymptotic predictions for the cases of $R_j = 1$ and $R_j = 5$ are shown in figure 1. This shows close agreement over the whole range which improves as Ra_S increases. The asymptotic predictions follow the same variation as the numerical solutions as R_j varies. Also shown is the large-Ra_S asymptotic behaviour for the case

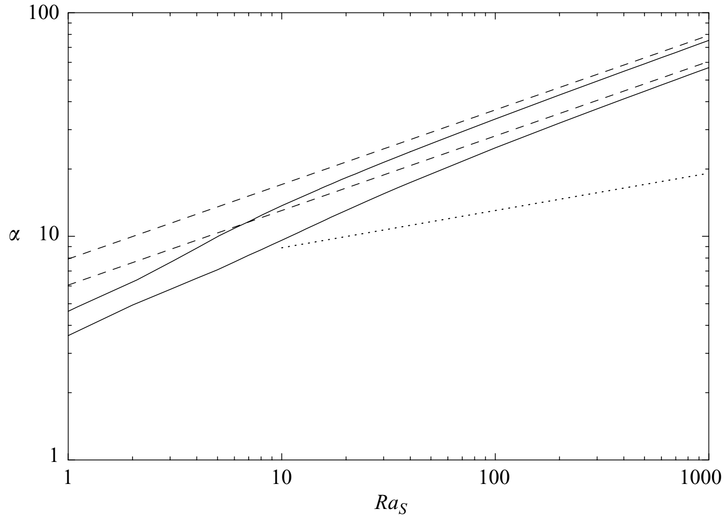

FIGURE 2. Comparison of the numerical results of Zebib (2001) (solid line) and the asymptotic predictions (dashed line) of the vertical wavenumber, α, for marginal stability as a function of the salt Rayleigh number, Ra_S. The lower pair of lines are for $R_j = 1$ and the upper pair for $R_j = 5$. In both cases $Pr = 1$ and $\tau = 0.0001$. The dotted line shows the large-Ra_S asymptotic behaviour for this case when vibration is absent.

with no vibration, showing the clear divergence of behaviour. The agreement between the calculated and predicted vertical wavenumbers for the same two cases is shown in figure 2. Again there is convergence of the two for larger values of Ra_S, but not as good for smaller values. Since the lower end of this range of salt Rayleigh numbers is close to the region where the $\alpha \to 0$ asymptotics dominate (Zebib 2001) this is not surprising. Indeed it probably indicates that the agreement between the calculated and asymptotic predictions for the Rayleigh number was fortuitous here. Again the trends as R_j is varied are well reproduced.

3. Discussion

For many terrestrial applications the typical value of Ra_S is likely to be large and the primary mode of instability in the absence of vibration will be that identified by Thorpe *et al.* (1969). In this limit instability occurs when

$$Ra_T(\tau^{-1} - 1) = 3^{1/2}(2\pi)^{2/3}(Ra_S/\tau)^{5/6}. \tag{3.1}$$

One way of estimating when the g-jitter mode could become important is to look for the value of the salt Rayleigh number at which the two asymptotic predictions (2.17) and (3.1) cross as a function of the jitter parameter, R_j. This crossing occurs when

$$8Ra_S = R_j^{-1}. \tag{3.2}$$

When considering the typical magnitude of Ra_S on Earth we can look at experiments in slots and in tanks. The former can be easily evaluated, while the latter need to be examined more closely. The value of Ra_S can be made ever larger for a given salinity gradient by using a wider tank. Instabilities in laterally heated wide tanks with salinity gradients are observed, but for times well short of that required for a linear temperature gradient to establish itself across the tank, or even for the effects of the heating to have propagated to the far wall. In such cases the salinity gradients have been heated from what is effectively an isolated wall (Thorpe *et al.* 1969; Chen *et al.* 1971; Narusawa & Suzukawa 1981; Tanny & Tsinober 1988). Tanny & Tsinober noted the instantaneous values of the salt and thermal Rayleigh numbers based on the length scale of the instantaneous thermal boundary layer. This is closely related to the thermal Rayleigh number used here, and instabilities occur at comparable values in the two situations (Kerr 1989). The largest salt Rayleigh number that they observed when one of their experiments became unstable was just under 10^8, which is the largest of any of these experiments.

A vertical slot on Earth of width of order 1 cm containing, say, water, and experiencing horizontal vibrations with a typical maximum velocity of $v\,\mathrm{m\,s^{-1}}$, will have a jitter parameter of approximately $R_j \sim v^2/10^7\,\mathrm{m^2\,s^{-2}}$. The levels of ambient vibration can vary greatly with location. According to Steffens (1966) people's perceptions of vibrations is related to the maximum velocity over a broad range of frequencies. Ambient vibrations are perceived by people to be 'annoying' when these velocities are of the order $2.2 \times 10^{-3}\,\mathrm{m\,s^{-1}}$ to $6.3 \times 10^{-3}\,\mathrm{m\,s^{-1}}$, and much above this to be 'painful'. Thus we can take these to be an upper limit for vibrations experienced in typical experiments, unless there is some local source of vibration. Hence we can see that on Earth the jitter parameter is typically very small – less than 4×10^{-12}. From (3.2) the g-jitter mode would only become important if the salt Rayleigh number were greater than around 3×10^{11}. This condition has not been met by past experiments. Even though the theory predicts that for any level of vibration the instability caused by the lateral heating of a slot will be further destabilized for sufficiently large salt Rayleigh numbers, on Earth-bound experiments this is unlikely to be important. However, in reduced gravity, or in cases where there is a local source of vibration, the instabilities identified in this asymptotic analysis could be encountered.

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