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DECIDABILITY OF MODULES OVER A BÉZOUT DOMAIN $D + XQ[X]$ WITH D A PRINCIPAL IDEAL DOMAIN AND Q ITS FIELD OF FRACTIONS

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Abstract. We describe the Ziegler spectrum of a Bézout domain $B = D+XQ[X]$ where D is a principal ideal domain and Q is its field of fractions; in particular we compute the Cantor–Bendixson rank of this space. Using this, we prove the decidability of the theory of B-modules when D is "sufficiently" recursive.

§1. Introduction. The model theory of modules over Bézout domains has been recently developed in [9]. This note is a further contribution to this theory, in which we analyze a particular class of Bézout domains obtained from principal ideal domains using the so-called D+M-construction (see [1, p. 7]).

Recall that a commutative domain B (with identity) is said to be *Bézout* if every 2-generated (and therefore every finitely generated) ideal of B is principal. Thus, for every pair of elements $a, b \in B$ one can introduce a *greatest common divisor* $\gcd(a, b)$ as a generator c of the ideal $aB + bB$ (this element is unique up to a multiplicative unit of B). Furthermore, the intersection $aB \cap bB$ is again a principal ideal dR (therefore B is coherent), and we call d a *least common multiple* of a and b (again defined up to a multiplicative unit). Under a suitable choice of lcm and gcd we have an equality $\operatorname{lcm}(a, b) \cdot \gcd(a, b) = ab$.

The D+M-construction produces from any principal ideal domain D, which is not a field, a Bézout domain which is not noetherian. In detail let

- $Q = Q(D)$ denote the quotient field of D,
- $B = B(D)$ be the subring of $Q[X]$ consisting of polynomials whose constant term is in D, that is $B = D + XQ[X]$.

Note that in the particular case when D is the ring of integers, $B = \mathbb{Z} + X\mathbb{Q}[X]$.

For basic properties of this construction see [1, pp. 7–8]. For instance (see [1, Example III.1.5]) B is a Bézout domain which is not noetherian. Namely for every prime (= irreducible) $p \in D$, we have a strictly ascending chain $XB \subset p^{-1}XB \subset p^{-2}XB \subset \dots$ of ideals of B. It follows that B is not a unique factorization domain (since being a UFD and being noetherian are equivalent for Bézout domains).

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Our aim in this note is to examine the decidability of the theory of modules over a Bézout domain $B = B(D)$ for a sufficiently recursive principal ideal domain D. With this purpose in mind, we will study in Section 2 for any such B (sufficiently recursive or not) the Ziegler spectrum of B, $\mathrm{Zg}(B)$, both the points and the topology, and we will compute the Krull–Gabriel dimension of B, equivalently, the Cantor–Bendixson rank of the spectrum. After that we will describe in Section 3 the right setting for analyzing decidability of modules over Bézout domains and we will single out the "effectively" given B for which our decision problem makes sense. Finally in Section 4 we tackle the decidability question for B-modules and we answer it positively when D is effectively given. For instance this is the case for our capital example $D = \mathbb{Z}$ and $B = \mathbb{Z} + X\mathbb{Q}[X]$.

We assume some familiarity with the basic model theory of modules, in particular with pp-formulae, pp-types, (indecomposable) pure injective modules and Ziegler topology. We refer the reader to [3,10] or also [6, Chapter 10]. We adopt the following notation: if φ, ψ are pp-formulae in one free variable over a given ring R and M is an R-module, then $\mathrm{Inv}(M, \varphi, \psi)$ denote the index of the subgroup $\varphi(M) \cap \psi(M)$ in $\varphi(M)$, which is either a positive integer k or ∞. Thus $\mathrm{Inv}(\varphi, \psi) = k$ and $\mathrm{Inv}(\varphi, \psi) \geq k$ are first order sentences in the language of R-modules saying that the index (in a given module) is exactly k, or at least k. Such statements are called *invariant sentences*.

For basic facts on model theory over Bézout domains we refer to [9]. Chapter 17 of [3] discusses the topic of decidability of modules. Modules are always assumed to be right.

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§**2. The Ziegler spectrum.** In this section we consider the Ziegler spectrum, $\mathrm{Zg}(B)$, of a Bézout domain $B = B(D) = D + XQ[X]$, where D is a principal ideal domain, not a field, and $Q = Q(D)$ denotes its quotient field. Recall that $\mathrm{Zg}(B)$ is a topological space whose points are (isomorphism classes) of indecomposable pure injective B-modules, and a basis of the topology is given by the compact open sets $(\varphi/\psi) := \{M \in \mathrm{Zg}(B) \mid \varphi(M) \cap \psi(M) \subset \varphi(M)\}$ (a strict inclusion), where φ and ψ range over pp-formulae over B in (at most) one free variable.

We want to calculate the Cantor–Bendixson rank of the Ziegler spectrum of B. Because the lattice L of pp-formulae over any Bézout domain is distributive, by [5, Corollary 5.3.29] this ordinal equals the Krull–Gabriel dimension of B, that is the m-dimension of $L(B)$. The latter invariant is determined by iterative factoring L (and what is obtained from it) by congruence relations collapsing intervals of finite length (see [5, Chapter 7]). For instance the m-dimension is undefined exactly when L contains a subchain isomorphic to the ordering of the rationals $(\mathbb{Q}, \leq)$.

As a preliminary step in our analysis let us describe the prime ideals of B.

LEMMA 2.1. *Any nonzero prime ideal P of B is one of the following*:

(1) pB, *where p is a prime element of D*;
(2) *for some irreducible polynomial $f(X) \in Q[X]$ whose constant term is* 1, *the ideal $P_f = f(x)B$*;
(3) $P_X = XQ[X]$.

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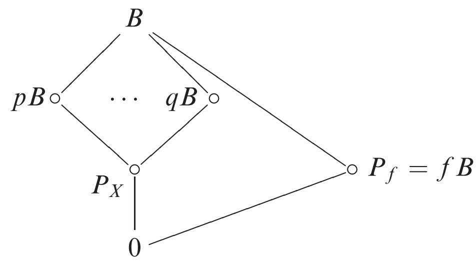
Furthermore, these prime ideals satisfy the following inclusion schema:



In particular, P_X *is not principal and* $P_X = \cap_p pB$.

Proof. The following arguments are standard (for instance, see [1, p. 8]). If A is a multiplicative subset of B (or any commutative domain) then there exists a natural 1-1 correspondence between prime ideals P of B such that $P \cap A = \emptyset$ and prime ideals of the localization B_A. This correspondence is defined as $P \mapsto P_A$ and $I \mapsto I \cap B$ (where I is a prime ideal of B_A).

We take $A = D \setminus 0 \subset B$. Then A is multiplicative and $B_A = Q[X]$. Let P be a prime ideal of B.

If $P \cap A \neq \emptyset$ then, since P is prime, it contains some prime p. Furthermore, $X = p \cdot (p^{-1}X) \in pB \subseteq P$. It follows that $B/pB \cong D/pD$ is a field, therefore pB is a maximal ideal of B. But then $P = pB$.

Suppose now that $P \cap A = \emptyset$, therefore P is obtained by restriction from a prime ideal $fQ[X]$ of $Q[X]$, where $f(X)$ is an irreducible polynomial. If the constant term of f is zero, we may assume that $f = X$, therefore $P = XQ[X] \cap B = XQ[X]$. Otherwise we may suppose that the constant term of f equals 1 (therefore $f \in B$) and $P = fQ[X] \cap B$. But clearly this intersection equals fB.

The remaining claims are straightforward. ⊣

Note that the Krull dimension of B (defined as a maximal length of a chain of prime ideals) is 2, and B is not catenary ($0 \subset P_f$ is another saturated chain of prime ideals of length 1).

Now we describe, for each prime P, the corresponding localization B_P. Since B is a Bézout domain, B_P must be a valuation domain. If $P = fB$, then B_f is a noetherian valuation domain whose ideals are of the form $B_f \supset fB_f \supset f^2B_f \supset \cdots$; and a similar description takes place for B_X.

Now consider the case $P = pB$ for a prime $p \in B$. Clearly $B_P = B_p = D_p + XQ[X]$, where D_p stands for localization of D with respect to pD. The principal ideals of B_p form the following chain:

$$B_p \supset pB_p \supset p^2B_p \supset \cdots \supset p^{-1}XB_p \supset XB_p \supset pXB_p \supset \cdots \supset X^2B_p \supset \cdots .$$

In particular, the Krull dimension of B_p equals 2.

Lemma 2.2. *The Krull–Gabriel dimension of* B *is at least* 4.

Proof. It suffices to prove that $\mathrm{KG}(B_p) = 4$ for some (in fact for any) prime p. Since B_p is a valuation domain, this is a standard procedure (see [6, Chapter 5]). Each indecomposable pure injective B_p-module M is uniquely determined by a pair of ideals (I, J) of B_p, therefore we will write $M = \mathrm{PE}(I, J)$, where I stands for the annihilator ideal of some element of M and J is its nondivisibility ideal. Furthermore, the Cantor–Bendixson rank of M equals $\mathrm{mdim}(I) \oplus \mathrm{mdim}(J)$, where $\mathrm{mdim}(I)$ is the m-dimension of the cut defined by I on the chain of principal ideals of B_p.

Note that the only cut on this chain of maximal m-dimension 2 corresponds to the zero ideal, and the cut defined by a principal ideal has m-dimension 0.

Thus, the unique point of maximal CB-rank in $\mathrm{Zg}(B_p)$ corresponds to the pair $(0, 0)$, hence isomorphic to $Q(X)$ (the generic point). Its CB-rank equals $2 + 2 = 4$. As we have already noticed this value coincides with the Krull–Gabriel dimension of B_p. ⊣

To simplify ongoing considerations let us make some general remarks. If P is a prime ideal of a commutative ring R, then by Zg_P we will denote the closed subspace of $\mathrm{Zg}(R)$ consisting of modules on which each $r \in R \setminus P$ acts as an automorphism. Clearly this set can be identified with the Ziegler spectrum of the localization R_P, that is $\mathrm{Zg}_P = \mathrm{Zg}(R_P)$.

Define a map $P \mapsto \mathrm{Zg}_P$ from the set of prime ideals of R ordered by inclusion to the collection of closed subsets of $\mathrm{Zg}(R)$.

Remark 2.3. *The map $P \mapsto \mathrm{Zg}_P$ preserves the ordering. Furthermore, if the intersection of prime ideals $\cap_{i \in I} P_i$ is a prime ideal (say, if the P_i form a chain), then this map preserves this intersection.*

Proof. Suppose that $P \subseteq Q$ are prime ideals and $M \in \mathrm{Zg}_P$. For any $r \notin Q$ we have $r \notin P$, therefore r acts as an isomorphism on M. But this means that $M \in \mathrm{Zg}_Q$. ⊣

Using this (though we do not need this) the above map can be extended to semiprime ideals, therefore (taking radicals) to all ideals of R.

If M is an indecomposable pure injective R-module, then consider the set $P = P(M)$ consisting of $r \in R$ which act as nonisomorphisms on M. It follows from [10, Theorem 5.4] that P is a prime ideal and M has a natural structure of an (indecomposable pure injective) R_P-module. Therefore, the whole Ziegler spectrum $\mathrm{Zg}(R)$ is covered by the union of closed subsets Zg_P.

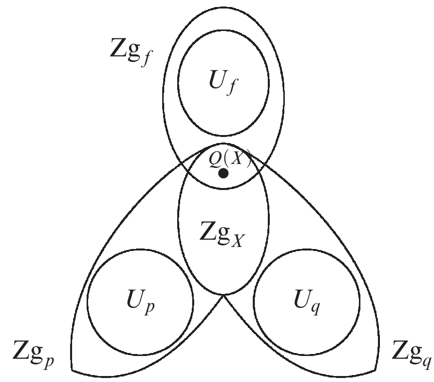
Now we are in a position to show that the above estimate of the Krull–Gabriel dimension of our B is sharp.

Theorem 2.4. *The Krull–Gabriel dimension of B equals 4 with $Q(X)$ being a unique point of maximal CB-rank. Moreover the Ziegler spectrum $\mathrm{Zg}(B)$ of B is the union of the closed subspaces*

- *$\mathrm{Zg}(B_{f(X)})$ where $f(X)$ ranges over the irreducible polynomials of $Q[X]$ with constant term 1,*
- *$\mathrm{Zg}(B_p)$ where p ranges over the prime elements of D.*

The latter subspaces include $\mathrm{Zg}(B_X)$, which is their intersection. The intersection of two different $\mathrm{Zg}(B_{f(X)})$, or of a $\mathrm{Zg}(B_{f(X)})$ and a $\mathrm{Zg}(B_p)$, reduces to the only point $Q(X)$.

Proof. The following is a schematic diagram of $\mathrm{Zg}(B)$ as described in the statement of the theorem: we imagine it as a bouquet of closed subspaces anchored in the generic point $Q(X)$. Recall that Zg_f, Zg_p and Zg_X abbreviate $\mathrm{Zg}(B_f)$, $\mathrm{Zg}(B_p)$, $\mathrm{Zg}(B_X)$ respectively.



We know the Ziegler spectrum of any valuation domain B_P with P a prime ideal of B, and know the relative CB-ranks of points measured in $\mathrm{Zg}(B_P)$. But $\mathrm{Zg}(B_P)$ is a closed subset of $\mathrm{Zg}(B)$ which is not open. Thus, if $M \in \mathrm{Zg}(B_P)$, the 'global' CB-rank of M could be larger than the CB-rank of M calculated in the relative topology. Measuring this jump is the main problem to take care of.

Let M be an indecomposable pure injective B-module and $P = P(M)$. Then M has a natural structure of a B_P-module.

First assume that $P = fB$ for an irreducible polynomial $f(X) \in Q[X]$ with 1 as a constant term. We have already mentioned that $B_f = B_{fB}$ is a noetherian valuation domain and described its ideals. It follows that either $M = B_f/f^n B_f$ is a finitely generated B_f-module, or M is Prüfer or adic, or $M = Q(X)$, the unique generic module.

Note that the basic open set $V_f = (xf = 0/x = 0)$ consists of points on which $f(X)$ acts with a nontrivial kernel. We claim that V_f is contained in $\mathrm{Zg}(B_f)$. Indeed, let M be a point in V_f and $P = P(M)$. Then $f \in P(M)$, therefore $P(M) = fB$ and hence every $r \notin fB$ acts as an isomorphism on M.

The same is true for the open set $W_f = (x = x/f \mid x)$. In the union $U_f = V_f \cup W_f$ of these open sets $f(X)$ acts as a nonisomorphism. Observe that U_f contains all points of $\mathrm{Zg}(B_f)$, but the generic, and $U_f \cap \mathrm{Zg}_p = \emptyset$ for any prime p.

Let n be a positive integer. Since $V = (xf^n = 0/f \mid x + xf^{n-1} = 0)$ isolates $B_f/f^n B_f$ in $\mathrm{Zg}(B_f)$, it follows that $V \cap U_f$ isolates this point in $\mathrm{Zg}(B)$. Since the Prüfer module $\mathrm{Pr}(B_f)$ has CB-rank 1 in $\mathrm{Zg}(B_f)$ and f has a nonzero kernel acting on it, it follows that $\mathrm{Pr}(B_f)$ has CB-rank 1 in $\mathrm{Zg}(B)$. Similarly, the adic module $\mathrm{PE}(B_f)$ has CB-rank 1 in $\mathrm{Zg}(B)$, as f is not onto when acting on this module.

As we will see later the only remaining point in Zg_f, that is, the generic point $Q(X)$ (whose CB-rank in $\mathrm{Zg}(B_f)$ equals 2) jumps to maximal CB-rank 4 in the whole space.

Now let us consider the points $M \in \mathrm{Zg}(B_p)$ for some prime $p \in D$. We have already mentioned the description of ideals of B_p and the fact that every point of $\mathrm{Zg}(B_p)$ is determined by a pair of ideals (I, J) of B_p; let $\mathrm{PE}(I, J)$ denote this point. Look at the open set U_p consisting of points on which p acts as a nonisomorphism. It is obvious that $U_p \subseteq \mathrm{Zg}(B_p)$ and its complement in Zg_P is Zg_X:

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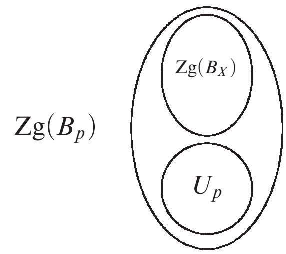

It is easily seen that a point $M = \mathrm{PE}(I, J)$ belongs to U_p if and only if either I or J is a principal nonzero ideal of B_p. For instance, if $I = J = pB_p$, then M is a simple B_p-module B_p/pB_p. The m-dimension of the cut defined by a principal nonzero ideal is 0 while the maximal m-dimension (i.e., the one of the cut defined by the zero ideal) is 2. Therefore, the pairs of m-dimensions of cuts defined by the ideals I, J of U_p are $(0, 0)$, $(0, 1)$, $(1, 0)$, $(2, 0)$ and $(0, 2)$. Their relative CB-ranks are the corresponding sums 0, 1 and 2.

Thus, intersecting U_p with an open set which isolates such a point $M = \mathrm{PE}(I, J)$ in $\mathrm{Zg}(B_p)$ at the corresponding level we see that its CB-rank does not change when passing to the ambient space $\mathrm{Zg}(B)$.

All the remaining points of $\mathrm{Zg}(B)$ are not included in U_p for any p. As each p acts as an isomorphism on these points, they belong to $\mathrm{Zg}(B_X)$. For instance if $I = J = \cup_n p^n XB_p$ and $M = \mathrm{PE}(I, J)$, then $M \notin U_p$ and its relative CB-rank equals $1 + 1 = 2$.

Thus, if M is one of the remaining points (and is not generic), then either $M = B_X/X^n B_X$ for some positive integer n or M is Prüfer or adic over B_X.

First we will prove that $M = B_X/X^n B_X$ has CB-rank 2 (by looking at the relative CB-rank, global rank is at least 2). For this we will use the basic open set $(xX^n = 0/X \mid x + xX^{n-1} = 0)$ and intersect it with U_X to avoid the various $\mathrm{Zg}(B_f)$ (clearly $U_X \cap \mathrm{Zg}(B_f) = \emptyset$). It suffices to show that this open set separates M from the points in $\mathrm{Zg}(B_p)$ (p a prime) of CB-rank 2 corresponding to the following pairs of m-dimensions: $(2, 0)$ and $(0, 2)$.

Those are the points $\mathrm{PE}(I, J)$, where one of the ideals is principal and nonzero and the other is zero. Suppose that the above pair opens on an element $m \in \mathrm{PE}(I, J)$, where we may assume that I is the annihilator of m and J is a 'nondivisibility' ideal of m. Since $X^n \in I$, it follows that I is nonzero. Similarly, as $X \in J$, one deduces that J is nonzero. But this contradicts the choice of I and J.

What remains in $\mathrm{Zg}(B_X)$ is the Prüfer point $\mathrm{Pr}(B_X)$, the adic point $\mathrm{PE}(B_X)$, and the generic point $Q(X)$. Clearly $(xX = 0/x = 0)$ separates $\mathrm{Pr}(B_X)$ from $\mathrm{PE}(B_X)$ and $Q(X)$, therefore $\mathrm{CB}(\mathrm{Pr}(B_X)) = 3$. The same is true for $\mathrm{PE}(B_X)$.

The only remaining point $Q(X)$ has CB-rank 4. ⊣

Note that the map in Remark 2.3 does not reflect intersections. Indeed, as follows from this remark, $\mathrm{Zg}_X = \cap_p \mathrm{Zg}_p$. But it is easily seen that for any primes $p \neq q$ we also have $\mathrm{Zg}_X = \mathrm{Zg}_p \cap \mathrm{Zg}_q$, but XB is a proper subset of $pB \cap qB$.

§3. Effectively given Bézout domains. We are going to consider decidability of B-modules. It is well known that some natural conditions are to be assumed on an

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arbitrary ring R (in particular on our B) to ensure that the decision problem of R-modules make sense (see [3, Section 17.1]). Let us briefly discuss this matter. For simplicity we refer to integral domains R with identity. The following definition is somewhat informal, but can be easily stated in a rigorous way via Turing Machines and Church Thesis.

DEFINITION 3.1. *A countable integral domain R is said to be effectively given if its elements can be recursively listed (possibly with repetitions) as*

$$r_0 = 0, r_1 = 1, r_2, \dots, r_k, \dots \qquad k \in \mathbb{N}$$

so that the following holds:

1) *there are algorithms which, given* $n, m \in \mathbb{N}$, *produce* $r_n + r_m$, $-r_n$ *and* $r_n \cdot r_m$ *(more precisely indices for these elements in the list);*
2) *there is an algorithm which, given* $n, m \in \mathbb{N}$, *decides whether* $r_n = r_m$ *or not;*
3) *there is an algorithm which, given* $n, m \in \mathbb{N}$, *establishes whether* $r_m \mid r_n$ *or not.*

Notice that, if R is effectively given, then the theory $T(R)$ of R-modules is recursively enumerable. Here are some further straightforward consequences of the same hypothesis.

REMARK 3.2. *Let R be an effectively given integral domain.*

4) *There is an algorithm which, given* $n, m \in \mathbb{N}$ *with* $r_m \mid r_n$, *provides* $r \in R$ *such that* $r_m \cdot r = r_m$ *(i.e., an index for this quotient in the list).*
5) *There is an algorithm which, given* $m \in \mathbb{N}$, *decides whether* r_m *is a unit of R or not and, if so, calculates its inverse.*
6) *Suppose that R is a Bézout domain. Then there is an algorithm which, given* $n, m \in \mathbb{N}$ *with* $r_n, r_m \neq 0$, *calculates a greatest common divisor of* r_n, r_m *(or rather an index of it).*

PROOF. 4) Just explore the list r_k, $k \in \mathbb{N}$, for every k calculate $r_m \cdot r_k$ and check whether this product equals r_n. As r_m divides r_n, one eventually finds such an index.

5) Apply 3) and 4) to $n = 1$.

6) Explore the list of all possible 4-types $(a, b, u, v) \in R^4$ (which can be obtained in a standard way from the list of R) looking at the solution of

$$r_n = (r_n \cdot u + r_m \cdot v) \cdot a, \qquad r_m = (r_n \cdot u + r_m \cdot v) \cdot b\,.$$

As R is Bézout, one eventually finds, after finitely many steps, a successful tuple (a, b, u, v). Then put $\gcd(r_n, r_m) = r_n \cdot u + r_m \cdot v$. ⊣

Note that in a Bézout domain R an algorithm for 6) yields procedures for 3) and consequently for 4) as well. Indeed to check whether r_m divides r_n calculate first $\gcd(r_m, r_n)$, divide r_m by it and look whether the quotient is invertible.

The following result shows that when analyzing pp-formulae over Bézout domains it suffices to consider only divisibility and annihilator conditions. Recall that, up to logical equivalence, if $\chi(x)$ and $\chi'(x)$ are pp-formulas in a single variable x, then also their conjunction $\chi(x) \wedge \chi'(x)$ and their sum $\chi(x) + \chi'(x)$, introduced as $\exists\, u\, \exists\, u'\, (\chi(u) \wedge \chi'(u') \wedge x = u + u')$, are pp-formulas. Moreover, the equivalence classes of pp-formulas are a lattice with respect to the corresponding operations.

LEMMA 3.3. *Every pp-formula* $\chi(x)$ *in one variable over a Bézout domain R is equivalent to a finite conjunction of formulae* $\varphi_{a,b} := (a \mid x + xb = 0)$, $a, b \in R$, *and also to a finite sum of formulae* $\psi_{c,d} := (c \mid x \wedge xd = 0)$.

Furthermore, if R is effectively given, then these formulae can be found effectively.

Proof. The existence of such formulas over an arbitrary Bézout domain R follows from [9, Lemma 2.3]. However, we have to find them effectively when R is effectively given. To do that, begin producing all the possible implications of $\chi(x)$ and the (recursively enumerable) theory $T(R)$ of R-modules via formal proofs. When this procedure provides a formula $\chi'(x)$ of the desired form—a suitable combination of divisibility and annihilator conditions—start producing implications from $T(R)$ and $\chi'(x)$, looking for $\chi(x)$. The existence result ensures that the procedure will eventually halt in a successful way, producing a formula $\chi'(x)$ equivalent to $\chi(x)$. ⊣

This lemma gives a good basis for the Ziegler topology.

Corollary 3.4. *Let R be an effectively given Bézout domain. Then the open sets $(\psi_{c,d}/\varphi_{a,b})$, $a, b, c, d \in R$ form a basis of the topology of $\mathrm{Zg}(R)$ which can be effectively enumerated.*

§4. **Decidability.** The aim of this section is to prove decidability of modules over effectively given (hence countable) Bézout domains B, and we have a range of methods at disposal. Using the fact that the Ziegler spectrum of B is countable (by Theorem 2.4) and its precise description, we can make an effective list of points M_k, $k \in \mathbb{N}$, of $\mathrm{Zg}(B)$. By Corollary 3.4 we also know an effective basis for this space. According to a general recipe of Ziegler [10, Theorem 9.4] (see also Prest's unpublished preprint [4]) it suffices to provide an algorithm which, given a point M_k, a basic open set (φ_i/ψ_i) and a positive integer l, decides whether $\mathrm{Inv}(\varphi_i, \psi_i) = l$ holds true in M_k.

It is possible to obtain the proof of decidability pursuing this approach, however (being partly logicians) we will produce another proof based on a recent result by Lorna Gregory on the decidability of modules over a valuation domain [2]. To do that, let us introduce some further notation: if V is a valuation domain, then $\mathrm{Jac}(V)$ will denote its Jacobson radical (= the set of nonunits) and $F = V/\mathrm{Jac}(V)$ is the residue field of V.

If V is effectively given, then (see [8, p. 273]) the decidability of V-modules yields the knowledge of the size of F (i.e., whether F is finite or infinite and, if finite, the number of elements in F). By [2] the converse is almost true.

Fact 4.1. [2] *Suppose that V is an effectively given valuation domain with known size of the residue field and with an algorithm checking for given $a, b \in V$ whether $a \in b^n V$ holds for some n. Then the theory of V-modules is decidable.*

Note that, if a principal ideal domain D is effectively given, then it is easily seen that the rings $Q = Q(D)$, $Q[X]$ and $B = D + XQ[X]$ are effectively given. However to reduce decidability to valuation domains, we will require of D some extra effectiveness. We say that (an effectively given) principal ideal domain D is *strongly effectively given* if it satisfies the following extra conditions:

1)′ there is an algorithm that lists all the prime elements of D;

2)′ there is an algorithm that lists all the irreducible polynomials of $Q[X]$;

3)′ for every prime p the size of the field D/pD is known.

It is well known that $\mathbb{Z}$ is strongly effectively given (for instance, by an algorithm going back to Kronecker for checking irreducibility of rational polynomials).

We do not know if these extra effectiveness conditions follow from the decidability of B-modules. The problem is that even when a localization B_p is given effectively, the theory of B_p-modules is defined in the theory of B-modules using an infinite set of axioms (so it is not clear in advance that the theory of B_p-modules must be decidable).

However, the previous restrictions are natural and satisfied by many examples. In fact the condition 2)′ rephrases, in the terminology of [3, p. 344], the property that Q has a splitting algorithm, and $\mathbb{Q}$ does admit it. On the basis of 1)′ and 2)′, one also gets algorithms to decompose a noninvertible element of D into a product of primes, in particular to decide whether it is irreducible or not; and we can do the same for polynomials in $Q[X]$.

We can even specify this for B, with an obvious proof.

LEMMA 4.2. *Let D be strongly effectively given. Every nonzero polynomial $F[X] \in B$ can be effectively decomposed as $r\, s^{-1} X^n F'(X)$, where r, s are coprime elements of D, n is a nonnegative integer and $F'(X)$ has constant term 1 and is (effectively) written as a product of irreducible polynomials with the same property.*

If a principal ideal domain D is effectively given, the same is clearly true for each localization B_p and B_f. Thus, in the remainder of this section we will refer to these localizations with a fixed effective enumeration.

LEMMA 4.3. *Let D be a strongly effectively given principal ideal domain. Then each localization B_p, p a prime, and B_f, f an irreducible polynomial of $Q[X]$ of constant term 1, has a decidable theory of modules.*

PROOF. Each such localization is an effectively given valuation domain. Furthermore, because $B_p/\operatorname{Jac}(B_p) \cong D/pD$ (the size of which is known), and $B_f/\operatorname{Jac}(B_f)$ is infinite, we know the sizes of residue fields. Using Gregory's result, it suffices to decide, for given elements a, b of any of these localizations V, whether $a \in b^n V$ holds true for some n.

This can be easily checked, because we can reduce a, b to polynomials in B and and then use their presentations from Lemma 4.2. ⊣

Now we are in a position to prove the following.

THEOREM 4.4. *Let D be a strongly effectively given principal ideal domain and let $B = D + XQ[X]$ be the corresponding Bézout domain. Then the theory $T(B)$ of B-modules is decidable.*

PROOF. Since B is effectively given, from axioms for B-modules we can generate a list of sentences true in any B-module (i.e., $T(B)$ is recursively enumerable). To prove decidability we have to enumerate the complement of $T(B)$, which is equivalent to listing in an effective way sentences true in some B-module.

Every indecomposable pure injective B-module localizes, therefore has a natural structure of either a B_p-module for some prime p or a B_f-module for some irreducible polynomial f with 1 as a constant term. Make an effective list of such modules with marks from which localization they stem.

In view of [4], in order to complete our proof, it suffices to restrict to modules M that are finite direct sums of indecomposable pure injective summands, $M = M_0 \oplus \cdots \oplus M_k$, and to produce a set of axioms for the theory of any such M, $Th(M)$.

By Baur–Monk theorem and because $Th(M)$ is complete, this theory is axiomatized by invariant sentences $\mathrm{Inv}(\varphi, \psi) \geq n$. We will list all such sentences σ and decide whether they are true in M. By additivity $M \models \sigma$ if and only if each $M_i \models \mathrm{Inv}(\varphi, \psi) \geq n_i$ and $n_1 \cdot \ldots \cdot n_k \geq n$, where we may assume that $n_i \leq n$.

Since the theory of each localization of B is decidable, each query $M_i \models \mathrm{Inv}(\varphi, \psi) \geq n_i$ can be answered effectively (using the localization at the marked prime ideal), hence so can σ. ⊣

Thus we obtain the result of our original interest.

COROLLARY 4.5. *The theory of* $\mathbb{Z} + X\mathbb{Q}[X]$*-modules is decidable.*

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