UNIT GROUPS OF CYCLIC EXTENSIONS

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Let Ω be an algebraic number field of finite degree, which we fix once for all, and let K be a cyclic extension over Ω such that the degree of K/Ω is a power l^ν of a prime number l. It is obvious that the norm group $N_{K/\Omega}\mathfrak{e}_K$ of the unit group $\mathfrak{e}_K$ of K, being a subgroup of the unit group $\mathfrak{e}$ of Ω, contains the group $\mathfrak{e}^{l^\nu}$ consisting of all l^ν-th powers ε^{l^ν} of $\varepsilon \in \mathfrak{e}$. The main aim of the present work is to prove the converse assertion of this fact in certain special case. Namely, it is verified that, if l is an odd prime number prime to the absolute discriminant $D(\Omega)$ of Ω, then, for any subgroup H of $\mathfrak{e}$ containing $\mathfrak{e}^{l^\nu}$, there is an infinite set $\mathfrak{K}$ of cyclic extensions of degree l^ν over Ω such that we have $N_{K/\Omega}\mathfrak{e}_K = H$ for every $K \in \mathfrak{K}$. More precisely, the infinite set $\mathfrak{K}$ is so chosen that, for every $K \in \mathfrak{K}$, the first cohomology group of $\mathfrak{e}_K$ is isomorphic to the direct product of the 0-th cohomology group of $\mathfrak{e}_K$ by a cyclic group $\mathfrak{Z}$ of degree l^ν, where the cohomology groups are defined by considering $\mathfrak{e}_K$ as an operator module of the Galois group of K/Ω. Thus we can also conclude that, if r_Ω is the dimension of $\mathfrak{e}$ and if A_0 is a subgroup of the direct product of r_Ω groups all isomorphic to $\mathfrak{Z}$, then there is an infinite set $\mathfrak{K}$ of cyclic extensions of degree l^ν over Ω such that the 0-th cohomology group of $\mathfrak{e}_K$ is isomorphic to A_0 and the first cohomology group of $\mathfrak{e}_K$ is isomorphic to $A_1 = A_0 \times \mathfrak{Z}$, where $K \in \mathfrak{K}$ and l is, still as before, an odd prime number prime to $D(\Omega)$.

In §1, we introduce the convenient notion of *fixed extensions*,[1] and, after preparations in §2, we deduce all the results in §3. As for the case of extensions with prime degree l, the results of this paper are already obtained in the previous paper of the author [4].

§1. Preliminaries

1. For a normal field K/Ω, we denote its Galois group by $\mathfrak{G}(K/\Omega)$. In

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[1] This was first introduced and studied in works of Hasse. See, e.g., Hasse [2].

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particular, if Ω and Ω_A are respectively the algebraic closure and the maximal abelian extension over Ω in the complex number field, then we put $\mathfrak{g}(\bar{\Omega}/\Omega) = G$ and $\mathfrak{g}(\Omega_A/\Omega) = G'$. Groups G, G' are always considered as compact topological groups in usual manner.

Let $\mathfrak{G}$ be a (discrete) finite group. We call a continuous homomorphism κ of G into $\mathfrak{G}$ a *fixed $\mathfrak{G}$-extension* over Ω. A fixed $\mathfrak{G}$-extension κ uniquely determines an overfield K_κ of Ω, i.e., the invariant field of the kernel of κ. It also determines a natural isomorphism between the Galois group of $\mathfrak{g}(K_\kappa/\Omega)$ and the subgroup $\kappa(G)$ of $\mathfrak{G}$. We call K_κ the *corresponding field* of κ. Some of the properties or invariants of the corresponding field K_κ of a fixed extension κ are expressed in the following as those of κ itself, e.g., we say κ is ramified at a place $\mathfrak{p}$ of Ω if K_κ/Ω is so, and the degree of κ means the degree $(K_\kappa : \Omega)$. If $\mathfrak{G}$ is of order n, then a fixed $\mathfrak{G}$-extension over Ω of degree n is said to be *proper*.

A fixed $\mathfrak{G}$-extension κ is naturally considered as a homomorphism of $\mathfrak{g}(K_\kappa/\Omega)$, and, if $\mathfrak{G}$ is an abelian group $\mathfrak{A}$, then κ is also considered as a homomorphism of G'. Furthermore, by the reciprocity law of class field theory, a fixed $\mathfrak{A}$-extension κ is considered as a homomorphism of the idèle group $\mathbf{I}$ of Ω or of the idèle class group C_Ω of Ω. These various interpretations of fixed extensions are occasionally applied as far as no confusion is possible.

The set of all fixed $\mathfrak{A}$-extensions over Ω forms an abelian group if we define the product $\kappa\kappa'$ of two fixed $\mathfrak{A}$-extensions κ, κ' by setting $\kappa\kappa'(\sigma) = \kappa(\sigma)\kappa'(\sigma)$ for any $\sigma \in G$.

Let κ be a fixed $\mathfrak{A}$-extension over Ω and $\mathfrak{p}$ be a finite or infinite place of Ω, then, using as usual the $\mathfrak{p}$-component of an idèle of Ω, we can attach to κ a continuous homomorphism $\kappa_\mathfrak{p}$ into $\mathfrak{A}$ of the multiplicative group $\Omega_\mathfrak{p}^\times$[2] of the $\mathfrak{p}$-completion $\Omega_\mathfrak{p}$ of Ω. By local class field theory, $\kappa_\mathfrak{p}$ is regarded as a homomorphism of the Galois group of a maximal abelian extension over $\Omega_\mathfrak{p}$ and therefore as a fixed $\mathfrak{A}$-extension over $\Omega_\mathfrak{p}$. We call $\kappa_\mathfrak{p}$ the $\mathfrak{p}$-component of κ.

2. Let $\mathbf{I}$, $\mathbf{U}$ be the idèle group and the unit idèle group[3] of Ω, respectively, and denote by $\Omega^\times$ the principal idèle group of Ω. Let $\mathfrak{S}$ be a finite set of places

[2] We always use the mark $\times$ to stand for the multiplicative group of non-zero elements of a field.

[3] In this paper, we settle no sign condition for the infinite components of a unit idèle, somewhat differently from the definition of Weil [6].

of Ω and κ_{U} be a homomorphism of $\mathbf{U}$ into $\mathfrak{Z}$ such that the $\mathfrak{q}$-component[4] of κ_{U} is trivial for every place $\mathfrak{q}$ of Ω outside $\mathfrak{S}$, where $\mathfrak{Z}$ is a cyclic group whose order l^{ν} is a power of a prime number l. Then κ_{U} is, in a natural way, regarded as a homomorphism of the group $\mathbf{U}_{\mathfrak{S},\nu} = \prod_{\mathfrak{p}\in\mathfrak{S}} U_{\mathfrak{p}}/U_{\mathfrak{p}}^{l^{\nu}}$, where $\mathbf{U}_{\mathfrak{p}}$ is the unit group of the $\mathfrak{p}$-completion $\Omega_{\mathfrak{p}}$ of Ω. On the other hand, set $B^{(\nu)} = \Omega^{\times} \cap \mathbf{I}^{l^{\nu}}\mathbf{U}$; then $B^{(\nu)}$ consists of numbers β of $\Omega^{\times}$ such that the principal ideal (β) is the l^{ν}-th power of an ideal of Ω, and, writing $\beta = \mathbf{a}^{l^{\nu}}\mathbf{u}$ $(\mathbf{a} \in \mathbf{I},\ \mathbf{u} \in \mathbf{U})$, the mapping $\beta \to \mathbf{u}$ followed by the natural mapping of $\mathbf{u}$ into $U_{\mathfrak{S},\nu}$ gives rise to a homomorphism $\iota_{\mathfrak{S},\nu}$ of $B^{(\nu)}$ into $U_{\mathfrak{S},\nu}$.

Now we state the following three Lemmas.[5]

LEMMA 1. *Let l^{ν} be a power of a prime number l, $\mathfrak{Z}$ be a cyclic group of order l^{ν} and let $\mathfrak{S}$ be a finite set of places of Ω. Then the restriction to* $\mathbf{U}$ *of a fixed $\mathfrak{Z}$-extension κ over Ω which is unramified at every place of Ω outside $\mathfrak{S}$ is characterized as a homomorphism κ_{U} of* $\mathbf{U}$ *into $\mathfrak{Z}$ which has trivial $\mathfrak{q}$-component for every place $\mathfrak{q}$ of Ω outside $\mathfrak{S}$ and which satisfies* $\kappa_{\mathrm{U}}(\iota_{\mathfrak{S},\nu}(B^{(\nu)})) = 1$.

LEMMA 2. *Let h_{ν} be the index* $(\mathbf{I} : \Omega^{\times}\mathbf{I}^{l^{\nu}}\mathbf{U})$. *Then the number of all fixed $\mathfrak{Z}$-extensions κ over Ω unramified at every place $\mathfrak{q}$ of Ω outside $\mathfrak{S}$ is equal to* $h_{\nu} \cdot (U_{\mathfrak{S},\nu} : \iota_{\mathfrak{S},\nu}(B^{(\nu)}))$.

LEMMA 3. *The kernel of $\iota_{\mathfrak{S},\nu}$ consists of the numbers $\beta \in B^{(\nu)}$ such that β is, for every $\mathfrak{p} \in \mathfrak{S}$, an l^{ν}-th power in the $\mathfrak{p}$-completion $\Omega_{\mathfrak{p}}$ of Ω.*

§ 2. Covering of an unramified field

3. Denote by $\mathbf{I}$, $\mathbf{U}$ the idèle group and the unit idèle group of Ω, respectively, and let $\mathfrak{Z}$ be a cyclic group whose order is a power l^{ν} of a prime number l. Set, as in § 1, **2**, $B^{(\nu)} = \Omega^{\times} \cap \mathbf{I}^{l^{\nu}}\mathbf{U}$ and consider the mapping $\beta \to \mathbf{a}$ defined for an element $\beta \in B^{(\nu)}$ with $\beta = \mathbf{a}^{l^{\nu}}\mathbf{u}$ $(\mathbf{a} \in \mathbf{I},\ \mathbf{u} \in \mathbf{U})$. If $\mathbf{e}$ is the unit group of Ω, then the above mapping gives rise to an isomorphism of $B^{(\nu)}/\mathbf{e}\Omega^{\times l^{\nu}}$ onto the group $C^{(\nu)}$ consisting of all elements of $\mathbf{I}/\Omega^{\times}\mathbf{U}$ whose orders divide l^{ν}. Any homomorphism χ of $C^{(\nu)}$ into $\mathfrak{Z}$ is therefore regarded as a homomorphism of $B^{(\nu)}/\mathbf{e}\Omega^{\times l^{\nu}}$ into $\mathfrak{Z}$, and *vice versa*. Whenever no confusion is possible, χ may also be considered as a homomorphism of $B^{(\nu)}$ or of a subgroup of $\mathbf{I}$. Take

4) This can be defined quite similarly to that of a fixed abelian extension.
5) As for the proofs of these lemmas, see Kubota [5], § 1.

such a homomorphism χ and denote by $B_\chi^{(\nu)}$ the subgroup of $B^{(\nu)}$ which is the kernel of χ. Suppose furthermore that $l \neq 2$. Then we have $(B^{(\nu)} : B_\chi^{(\nu)}) = (\Omega(\zeta_{l^\nu}, \sqrt[l^\nu]{B^{(\nu)}}) : \Omega(\zeta_{l^\nu}, \sqrt[l^\nu]{B_\chi^{(\nu)}}))$, where ζ_{l^ν} is a primitive l^ν-th root of unity.[6] Therefore, by Lemma 3 and by the theory of Kummer extensions, there are infinitely many prime ideals $\mathfrak{p}$ of Ω prime to l such that $N\mathfrak{p} - 1 \equiv 0 \pmod{l^\nu}$ and that, if we denote by $\iota_{\mathfrak{p},\nu}$ the homomorphism of §1, **2** with the set $\mathfrak{S} = \{\mathfrak{p}\}$ of a single place $\mathfrak{p}$, then the kernel of $\iota_{\mathfrak{p},\nu}$ coincides with $B_\chi^{(\nu)}$. We call such a $\mathfrak{p}$ a prime ideal of Ω which *belongs* to the homomorphism χ.

4. Let K' be an unramified cyclic extension over Ω such that the degree $(K' : \Omega)$ divides a power l^ν of a prime number l, and let K be an overfield of K' such that K/Ω is cyclic of degree l^ν and that there is at most one prime ideal of Ω which is ramified in K/Ω. Then we say that K is a *covering* of degree l^ν of K'. We propose to show that, for any K' and l^ν, we can always find a covering of degree l^ν of K', provided that $l \neq 2$. It suffices to prove that, if $\mathfrak{Z}$ is a cyclic group of order l^ν, then, for any unramified fixed $\mathfrak{Z}$-extension κ' over Ω, there is a covering κ of degree l^ν of κ', i.e., a proper fixed $\mathfrak{Z}$-extension κ over Ω such that κ is ramified at most at one prime ideal of Ω and that κ' is a power of κ.

Using the notations in **3**, let χ be the homomorphism of $C^{(\nu)}$ into $\mathfrak{Z}$ which is naturally induced by κ' and let $\mathfrak{p}$ be a prime ideal belonging to χ. If $U_\mathfrak{p}$ is the unit group of the $\mathfrak{p}$-completion $\Omega_\mathfrak{p}$ of Ω, then we can find an isomorphic mapping $\chi_\mathfrak{p}$ of $U_\mathfrak{p}/U_\mathfrak{p}^{l^\nu}$ onto $\mathfrak{Z}$ such that we have $\chi_\mathfrak{p}(\iota_{\mathfrak{p},\nu}(\beta)) = \chi(\beta)$ for every $\beta \in B^{(\nu)}$, where χ is considered as a homomorphism of $B^{(\nu)}$ as in **3**. Let $l^{\nu-r}$ be the degree of κ' and denote by κ_U the homomorphism of U into $\mathfrak{Z}$ whose $\mathfrak{p}$-component coincides with $\chi_\mathfrak{p}^{\nu-r}$ and whose $\mathfrak{q}$-component is trivial for every place $\mathfrak{q} \neq \mathfrak{p}$ of Ω. Then, since we have $\chi_\mathfrak{p}^{l^{\nu-r}}(\iota_{\mathfrak{p},\nu}(\beta)) = \chi^{l^{\nu-r}}(\beta) = 1$, there is, by Lemma 1, a fixed $\mathfrak{Z}$-extension κ_1 over Ω such that the restriction to U of κ_1 coincides with κ_U. If now $\mathbf{a}$ is an idèle of Ω which represents an element of $C^{(r)}$, then we have $\mathbf{a}^{l^r}\mathbf{u} = \alpha$ $(\mathbf{u} \in \mathrm{U},\ \alpha \in \Omega^\times)$ and consequently $\mathbf{a}^{l^\nu}\mathbf{u}^{l^{\nu-r}} = \alpha^{l^{\nu-r}} \in B^{(\nu)}$. Therefore we have $\kappa(\mathbf{a}) = \chi(\alpha^{l^{\nu-r}}) = \chi_\mathfrak{p}(\iota_{\mathfrak{p},\nu}(\alpha^{l^{\nu-r}})) = \chi_\mathfrak{p}(\mathbf{u}^{l^{\nu-r}}) = \kappa_1(\mathbf{u}) = \kappa_1^{-l^r}(\mathbf{a})$, where χ is considered as a homomorphism of $B^{(\nu)}$. This shows that $\kappa'\kappa_1^{l^r}$ induces a trivial mapping on $C^{(r)}$ and therefore we have $\kappa'\kappa_1^{l^r} = \kappa_2^{l^r}$ with an unramified fixed $\mathfrak{Z}$-extension κ_2 over Ω. Setting

[6] See Hasse [3], §1, Satz 1, 2.

$\kappa = \kappa_1^{-1}\kappa_2$, we have $\kappa' = \kappa^{l^r}$. Thus we see that, for every prime ideal $\mathfrak{p}$ of Ω belonging to χ, there is a fixed $\mathfrak{Z}$-extension κ over Ω with $\kappa' = \kappa^{l^r}$ and with at most one ramification place $\mathfrak{p}$, which proves our assertion.

5. Still using the same notations, we make another observation. We denote by $\mathfrak{S} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ a set of prime ideals, prime to l, of Ω and by $\mathbf{U}_{\mathfrak{S}}$ the group of unit idèles $\mathbf{u}$ of Ω such that, for every i, the $\mathfrak{p}_i$-component u_i of $\mathbf{u}$ satisfies the condition $u_i \equiv 1 \pmod{\mathfrak{p}_i}$. If $\mathfrak{p}_i$ completely decomposes in the field $\Omega(\zeta_{l^\nu}, \sqrt[l^\nu]{\mathbf{e}})$ and if the factor group $\mathbf{I}/\Omega^\times \mathbf{I}^{l^\nu} \mathbf{U}_{\mathfrak{S}}$ is isomorphic to the direct product of t cyclic groups of order l^ν, where t is the rank of the group $\mathbf{I}/\Omega^\times \mathbf{I}^l \mathbf{U}$, then we call $\mathfrak{S}$ a *parametric set* of degree l^ν of Ω and the class field $\tilde{Z}_\nu$ over $\Omega^\times \mathbf{I}^{l^\nu} \mathbf{U}_{\mathfrak{S}}$ the *complete covering* attached to $\mathfrak{S}$. It follows from Lemma 2 that, for any parametric set of degree l^ν of Ω, the order of $\iota_{\mathfrak{S},\nu}(B^{(\nu)})$ is equal to that of $\mathbf{I}/\Omega^\times \mathbf{I}^{l^\nu} \mathbf{U}$ and therefore the kernel of $\iota_{\mathfrak{S},\nu}$ is $\mathbf{e}\Omega^{\times l^\nu}$.

Now, we propose to prove the existence of parametric sets of arbitrary degree l^ν, provided that $l \neq 2$. Let $\tilde{c}_1, \ldots, \tilde{c}_t$ be a base of the group $\tilde{C}$ consisting of all elements of $\mathbf{I}/\Omega^\times \mathbf{U}$ whose orders are powers of l, and let $\chi_1, \ldots, \chi_t$ be a set of homomorphisms of $C^{(\nu)}$ into $\mathfrak{Z}$ such that the restriction of χ_i to the group $\{\tilde{c}_i\} \cap C^{(\nu)}$ is an isomorphism into $\mathfrak{Z}$ and that χ_i is trivial on $\{\tilde{c}_j\} \cap C^{(\nu)}$ $(i \neq j)$, where $\{\tilde{c}_i\}$ is the group generated by $\tilde{c}_i$. Then χ_i's form a base of the group consisting of all homomorphisms of $C^{(\nu)}$ into $\mathfrak{Z}$. Choose for every i a prime ideal $\mathfrak{p}_i$ of Ω belonging to χ_i and set $\mathfrak{S} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$. Then it follows from the results of **4** that, for every unramified fixed $\mathfrak{Z}$-extension κ_i' over Ω which is trivial on every $\{\tilde{c}_j\}$ with $i \neq j$, there is a covering κ_i of κ_i' which is unramified at every place of Ω except $\mathfrak{p}_i$. This means that the factor group $\mathbf{I}/\Omega^\times \mathbf{I}^{l^\nu} \mathbf{U}_{\mathfrak{S}}$ contains the direct product of t cyclic groups of order l^ν. On the other hand, $\chi_1, \ldots, \chi_t$ regarded as homomorphisms of $B^{(\nu)}$ form a base of the group consisting of all homorphisms of $B^{(\nu)}/\mathbf{e}\Omega^{\times l^\nu}$ into $\mathfrak{Z}$. Therefore it follows from the definition of $\mathfrak{S}$ that the kernel of $\iota_{\mathfrak{S},\nu}$ is $\mathbf{e}\Omega^{\times l^\nu}$ and consequently the order of $\iota_{\mathfrak{S},\nu}(B^{(\nu)})$ is equal to the order of the group $\mathbf{I}/\Omega^\times \mathbf{I}^{l^\nu} \mathbf{U}$. Hence, by Lemma 2, the number of all $\mathfrak{Z}$-extensions κ over Ω unramified at every place of Ω outside $\mathfrak{S}$ is equal to $l^{\nu t}$. Thus we see that the group $\mathbf{I}/\Omega^\times \mathbf{I}^{l^\nu} \mathbf{U}_{\mathfrak{S}}$ is just the direct product of t cyclic groups of order l^ν, whence $\mathfrak{S}$ is a parametric set of degree l^ν of Ω.

§ 3. Unit groups and their norms

6. The main purpose of this section is to prove the following

THEOREM 1. *Let* $\mathfrak{Z}$ *be a cyclic group whose order* l^ν *is a power of an odd prime number* l *prime to the absolute discriminant* $D(\Omega)$ *of* Ω. *Denote by* $\mathfrak{e}$ *the unit group of* Ω *and let* H *be a subgroup of* $\mathfrak{e}$ *containing* $\mathfrak{e}^{l^\nu}$. *Then there are infinitely many proper fixed* $\mathfrak{Z}$*-extensions* κ *over* Ω *such that we have* $N_{K_\kappa/\Omega}\mathfrak{e}_\kappa = H$, *where* $\mathfrak{e}_\kappa$ *is the unit group of the corresponding field* K_κ *of* κ.

7. Let $\mathbf{I}$, $\mathbf{U}$ be the idèle group and the unit idèle group of Ω, respectively, and let Z_1 be the class field over $\Omega^\times \mathbf{I}^l \mathbf{U}$. Then, under the assumptions in Theorem 1, we have $Z_1 \cap \tilde{\Omega}_\nu = \Omega$, where $\tilde{\Omega}_\nu = \Omega(\zeta_{l^\nu}, \sqrt[l^\nu]{B^{(\nu)}})$, $B^{(\nu)} = \Omega^\times \cap \mathbf{I}^{l^\nu}\mathbf{U}$ and ζ_{l^ν} is a primitive l^ν-th root of unity. For, since the assumptions imply that $\Omega(\zeta_{l^\nu})/\Omega$ is an extension of degree $l^{\nu-1}(l-1)$ containing no unramified subfield except Ω itself, $\tilde{\Omega}_\nu/\Omega$ has $\Omega(\zeta_{l^\nu})$ as the largest abelian subfield and has Ω itself as the largest unramified abelian subfield. From this follows that there is a parametric set $\mathfrak{S} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$ of degree l^ν of Ω such that the substitutions $\left(\frac{Z_1/\Omega}{\mathfrak{p}_i}\right)$ form a base of the Galois group $\mathfrak{g}(Z_1/\Omega)$, because the signifying condition in § 2 of $\mathfrak{p}_i$ concerns only the decomposition of $\mathfrak{p}_i$ in $\tilde{\Omega}_\nu$. We take such a parametric set $\mathfrak{S}$ and fix homomorphisms $\chi_{\mathfrak{p}_i}$ of $U_{\mathfrak{p}_i}$ onto $\mathfrak{Z}$, where $U_{\mathfrak{p}_i}$ is the unit group of $\mathfrak{p}_i$-completion $\Omega_{\mathfrak{p}_i}$ of Ω.

Now, we can find subgroups $H_1, \dots, H_s$ of $\mathfrak{e}$ containing H such that $\mathfrak{e}/H_i$ is cyclic and that we have $\bigcap_i H_i = H$. Let c_i be the index $(\mathfrak{e} : H_i)$ and ζ_{c_i} be a primitive c_i-th root of unity. Then, since we have $(\Omega(\zeta_{c_i}, \sqrt[c_i]{\mathfrak{e}}) : \Omega(\zeta_{c_i}, \sqrt[c_i]{H_i})) = c_i$,[7] there is a prime ideal $\mathfrak{q}_i$ of Ω prime to l such that $N\mathfrak{q}_i - 1 \equiv 0 \pmod{c_i}$ and that we have $H_i = \mathfrak{e} \cap U_{\mathfrak{q}_i}^{c_i}$, where $U_{\mathfrak{q}_i}$ is the unit group of the $\mathfrak{q}_i$-completion $\Omega_{\mathfrak{q}_i}$ of Ω and $\mathfrak{e}$ is regarded as a subgroup of $U_{\mathfrak{q}_i}$. We take such a prime ideal $\mathfrak{q}_i$ for every i, and fix homomorphisms $\chi_{\mathfrak{q}_i}$ of $U_{\mathfrak{q}_i}$ into $\mathfrak{Z}$ with the kernel $U_{\mathfrak{q}_i}^{c_i}$. Let π_i be a generator of the prime ideal of $\Omega_{\mathfrak{p}_i}$ and σ be a generator of $\mathfrak{Z}$. Then, setting $\chi_{\mathfrak{p}_i}(\pi_i) = 1$, we can extend $\chi_{\mathfrak{p}_i}$ to a homomorphism of the whole multiplicative group $\Omega_{\mathfrak{p}_i}^\times$. We also extend $\chi_{\mathfrak{q}_i}$ to a homomorphism into $\mathfrak{Z}$ of $\Omega_{\mathfrak{q}_i}^\times$ in an arbitrary way.

By the existence theorem of Grunwald,[8] there are infinitely many proper

[7] See footnote 6.
[8] See Hasse [3].

fixed $\mathfrak{Z}$-extensions κ over Ω such that we have $\chi_{\mathfrak{p}_i}=\bar{\chi}_{\mathfrak{p}_i}$, $\kappa_{\mathfrak{q}_i}=\bar{\chi}_{\mathfrak{q}_i}$ for the local components $\kappa_{\mathfrak{p}_i}$, $\kappa_{\mathfrak{q}_i}$ of κ and that there is only one ramification prime ideal $\mathfrak{x}$ outside the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_t, \mathfrak{q}_1, \ldots, \mathfrak{q}_s\}$.

8. We propose to show that the proper fixed $\mathfrak{Z}$-extensions κ in 7 have the required properties of Theorem 1. Since we have $\kappa\left(\frac{\alpha,\ K_\kappa/\Omega}{\mathfrak{p}_i}\right)$[9] $=\bar{\chi}_{\mathfrak{p}_i}(\alpha)$, $\kappa\left(\frac{\alpha,\ K_\kappa/\Omega}{\mathfrak{q}_i}\right)=\bar{\chi}_{\mathfrak{q}_i}(\alpha)$ for $\alpha\in\Omega^\times$, it follows from the definition of $\bar{\chi}_{\mathfrak{q}_i}$ that we have $N_{K_\kappa/\Omega}\mathrm{e}_\kappa\subset H$ and, on the other hand, it follows from the definition of $\chi_{\mathfrak{p}_i}$ and from a property in 5 of parametric sets that no element of $B^{(\nu)}$ outside $\mathrm{e}\Omega^{\times l^\nu}$ is a norm of K_κ/Ω. The latter result implies that, if an ideal $\mathfrak{a}$ of Ω is principal in K_κ, then it is principal in Ω, because from $\mathfrak{a}=(\alpha^\kappa)(\alpha^\kappa\in K_\kappa)$ necessarily follows $\mathfrak{a}^{l^\nu}=(N_{K_\kappa/\Omega}\alpha^\kappa)$ and $N_{K_\kappa/\Omega}\alpha^\kappa\in B^{(\nu)}$. Hence, denoting by (α) a principal ideal of Ω, by (α_0^κ) an "ambig" principal ideal of K_κ/Ω and by $\mathfrak{a}$ an ideal of Ω, we heve $((\alpha_0^\kappa)\cap\mathfrak{a} : (\alpha))=1$, where a general element of a group stands for the group itself. Therefore, if $\mathfrak{a}_0^\kappa$ is an "ambig" ideal of K_κ/Ω, we have $((\alpha_0^\kappa) : (\alpha))=((\alpha_0^\kappa)\mathfrak{a} : \mathfrak{a})=(\mathfrak{a}_0^\kappa : \mathfrak{a})/(\mathfrak{a}_0^\kappa : (\alpha_0^\kappa)\mathfrak{a})$. Since the group $(\alpha_0^\kappa)/(\alpha)$ is isomorphic to the first cohomology group of e_κ as a $\mathfrak{g}(K_\kappa/\Omega)$-group, we have, by Herbrand's relation,[10] $((\alpha_0^\kappa) : (\alpha))=l^\nu\cdot(\mathrm{e} : N_{K_\kappa/\Omega}\mathrm{e}_\kappa)$. Thus we obtain $(\mathrm{e} : N_{K_\kappa/\Omega}\mathrm{e}_\kappa)=l^{-\nu}\cdot(\mathfrak{a}_0^\kappa : \mathfrak{a})/(\mathfrak{a}_0^\kappa : (\alpha_0^\kappa)\mathfrak{a})$. The factor $l^{-\nu}\cdot(\mathfrak{a}_0^\kappa : \mathfrak{a})$ of this formula is estimated as follows: $l^{-\nu}\cdot(\mathfrak{a}_0^\kappa : \mathfrak{a})=l^{-\nu}\cdot\prod_i e(\mathfrak{p}_i)\cdot\prod_i e(\mathfrak{q}_i)\cdot e(\mathfrak{x}) \leqq l^{\nu t}\cdot(\mathrm{e} : H)$, where we denote by $e(\ )$ the ramification order with respect to K_κ/Ω. As for $(\mathfrak{a}_0^\kappa : (\alpha_0^\kappa)\mathfrak{a})$, we make the following investigation. Suppose that $\mathfrak{p}_i=\mathfrak{P}_i^{l^\nu}$ in K_κ and let $K_{\kappa,\mathfrak{P}_i}$ be the $\mathfrak{P}_i$-completion of K_κ. Then, since we have $\bar{\chi}_{\mathfrak{p}_i}(\pi_i)=\kappa\left(\frac{\pi_i,\ K_\kappa/\Omega}{\mathfrak{p}_i}\right)=1$, there is a generator Π_i of the prime ideal of $K_{\kappa,\mathfrak{P}_i}$ with the norm π_i to $\Omega_{\mathfrak{p}_i}$. If there is a relation $\mathfrak{P}_1^{m_1}\ldots\mathfrak{P}_t^{m_t}=(\alpha_0^\kappa)\mathfrak{a}$, then we have $\Pi_1^{m_1}\ldots\Pi_t^{m_t}\in K_\kappa^\times\mathbf{I}\mathbf{U}_\kappa$, where $K_\kappa^\times$ is the principal idèle group of K_κ, $\mathbf{U}_\kappa$ is the unit idèle group of K_κ and Π_i is regarded as an idèle of K_κ with $\mathfrak{P}_i$-component Π_i and with other components 1. Denoting by $\mathfrak{S}$ the parametric set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ and by $\mathbf{U}_\mathfrak{S}$ the group defined in 5, we have $K_{K_\kappa/\Omega}\mathbf{U}_\kappa\subset\mathbf{U}^{l^\nu}\mathbf{U}_\mathfrak{S}$, whence $\pi_1^{m_1}\ldots\pi_t^{m_t}\in\Omega^\times\mathbf{I}^{l^\nu}\mathbf{U}_\mathfrak{S}$. Since, however, the set of $\left(\frac{Z_1/\Omega}{\mathfrak{p}_i}\right)$ is a base of $\mathfrak{g}(Z_1/\Omega)$, an elementary property of finite abelian groups

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[9] This notation expresses the image by κ of the automorphism determined by the norm residue symbol.

[10] See Chevalley [1], §10.

of type $(l^\nu, \ldots, l^\nu)$ shows that, if $\tilde{Z}_\nu$ is the complete covering attached to $\mathfrak{Z}$ of Ω, then the set of reciprocal images $(\pi_i, \tilde{Z}_\nu/\Omega)$ form also a base of $\mathfrak{g}(\tilde{Z}_\nu/\Omega)$. This means that the relation $\pi_1^{m_1} \ldots \pi_t^{m_t} \in \Omega^\times \mathbf{I}^{l^\nu} \mathrm{U}_{\mathfrak{Z}}$ is impossible unless we have $m_1 \equiv \ldots \equiv m_t \equiv 0 \pmod{l^\nu}$. Thus we have $(\mathfrak{a}_0^\kappa : (\alpha_0^\kappa)\mathfrak{a}) \geqq l^{\nu t}$ and therefore $(\mathbf{e} : N_{K_\kappa/\Omega}\mathbf{e}_\kappa) \leqq (\mathbf{e} : H)$. This, together with $N_{K_\kappa/\Omega}\mathbf{e}_\kappa \subset H$ obtained above, proves our assertion.

9. We incidentally observe here the structure of the group $(\alpha_0^\kappa)/(\alpha)$ of **8.** Since $((\alpha_0^\kappa) \cap \mathfrak{a} : (\alpha)) = 1$, we have $(\alpha_0^\kappa)/(\alpha) \cong (\alpha_0^\kappa)\mathfrak{a}/\mathfrak{a}$. It is eventually shown in 8 that we have $((\alpha_0^\kappa) : (\alpha)) = l^\nu \cdot (\mathbf{e} : H)$ and that $\mathfrak{a}_0^\kappa/(\alpha_0^\kappa)\mathfrak{a}$ is the direct product of t cyclic groups of order l^ν. Therefore the character group of $\mathfrak{a}_0^\kappa/(\alpha_0^\kappa)\mathfrak{a}$ is a direct factor of the character group of $\mathfrak{a}_0^\kappa/\mathfrak{a}$. Since $\mathfrak{a}_0^\kappa/\mathfrak{a}$ is isomorphic to the direct product of $t+1$ cyclic groups of order l^ν by the group $\mathbf{e}/H$, $(\alpha_0^\kappa)\mathfrak{a}/\mathfrak{a} \cong (\alpha_0^\kappa)/(\alpha)$ must be isomorphic to the direct product of $\mathbf{e}/H$ by $\mathfrak{Z}$.

10. The unit group $\mathbf{e}_\kappa$ of the corresponding field K_κ of a proper fixed $\mathfrak{Z}$-extension κ over Ω is considered as a $\mathfrak{Z}$-group because the Galois group $\mathfrak{g}(K_\kappa/\Omega)$ is canonically isomorphic to $\mathfrak{Z}$, and the results which we hitherto obtained allow us to know a little about the cohomology groups of the $\mathfrak{Z}$-group $\mathbf{e}_\kappa$. Since $\mathfrak{Z}$ is cyclic, we may consider only the 0-th and the first cohomology groups. Namely, Theorem 1, together with **9**, immediately yields

THEOREM 2. *Let $\mathfrak{Z}$ be a cyclic group whose order l^ν is a power of an odd prime number l prime to the absolute discriminant $D(\Omega)$ of Ω. Denote by $\mathbf{e}_\kappa$ the unit group of the corresponding field K_κ of a proper fixed $\mathfrak{Z}$-extension κ over Ω and by $H^0(\mathfrak{Z}, \mathbf{e}_\kappa)$ resp. $H^1(\mathfrak{Z}, \mathbf{e}_\kappa)$ the 0-th resp. the first cohomology group of the $\mathfrak{Z}$-module $\mathbf{e}_\kappa$. Furthermore, let A_0 be any subgroup of the direct product of r_Ω groups all isomorphic to $\mathfrak{Z}$, where r_Ω is the dimension of the unit group $\mathbf{e}$ of Ω, and set $A_1 = A_0 \times \mathfrak{Z}$. Then there are infinitely many fixed $\mathfrak{Z}$-extensions κ over Ω such that we have $H^0(\mathfrak{Z}, \mathbf{e}_\kappa) \cong A_0$, $H^1(\mathfrak{Z}, \mathbf{e}_\kappa) \cong A_1$.*

It is easily seen that Theorem 1 and Theorem 2 hold even in the case where the order of the cyclic group $\mathfrak{Z}$ is not a power of a single prime number l but an odd natural number prime to the absolute discriminant $D(\Omega)$ of Ω.

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