Proceedings of the Royal Society of Edinburgh, **142A**, 81–114, *2012*

Multiple existence of solutions for a singularly perturbed nonlinear elliptic problem on a Riemannian manifold

Norimichi Hirano
Department of Mathematics,
Graduate School of Environment and Information Sciences,
Yokohama National University, Tokiwadai, Hodogayaku, Yokohama,
Japan (hirano@math.sci.ynu.ac.jp)

(MS received 27 May 2010; accepted 25 March 2011)

Let (M, g) be a compact smooth N-dimensional Riemannian manifold without boundary. We consider the multiple existence of positive solutions of the problem

$$-\varepsilon^2 \Delta_g u + u = f(u) \quad \text{in } M,$$

where Δ_g stands for the Laplacian in M and $f \in C^2(M)$.

1. Introduction

Let $2 \leqslant N \leqslant 4$ and let (M, g) be a compact smooth N-dimensional Riemannian manifold without boundary. We consider the multiple existence of positive solutions of problem

$$-\varepsilon^2 \Delta_g u + u = f(u) \quad \text{in } M, \tag{P}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous mapping and Δ_g denotes the Laplacian in M.

Let $\Omega \subset \mathbb{R}^N$ be a domain. The existence and multiplicity of positive solutions of the problem

$$\left.\begin{aligned} -\varepsilon^2 \Delta u + u &= u^{p-1} && \text{in } \Omega, \\ Bu &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

has been investigated by many authors in the case that p is subcritical, i.e. $1 < p < 2^*$, where $2^* = 2N/(N-2)$ and B is the Dirichlet or Neumann boundary operator. It is known the multiplicity of the solution depends on the geometry and topology of Ω (cf. [1, 2, 4–6] and references therein). For the problem (P) with $f(t) = t^{p-1}$, Benci *et al.* [3] established multiple existence results for (P). They showed that if M has a rich topology, problem (P) has multiple solutions using Lyusternik–Schnirelman category. As far as we know, their results are the first multiple existence results for (P) on a Riemannian manifold. In [7], Hirano showed that the number of solutions of (P) is affected by the topology of suitable subset of M. More recently, Micheletti and Pistoia [8] studied the role of the scalar curvature for the multiple existence of positive solutions of (P) with $f(t) = t^{p-1}$. They showed

https://doi.org/10.1017/S0308210510000892 Published online by Cambridge University Press

that, for each C^1-suitable critical set K, there exists a solution u_ε of (P) under the assumption that the limiting problem

$$-\Delta U + U = f(U), \quad U \in H^1(\mathbb{R}^N) \tag{1.2}$$

has a unique least energy solution U up to translations and which is non-degenerate, i.e. the solutions of the linearized problem

$$-\Delta\varphi + \varphi = f'(U)\varphi, \quad \varphi \in H^1(\mathbb{R}^N) \tag{1.3}$$

consist only of translations of U. Their argument is based on Lyapunov–Schmidt reduction procedure. That is, they reduce the problem to finding the critical point of a functional on the finite-dimensional manifold M. Our purpose in this paper is to obtain an analogous result to the result obtained in [8] for a more general class of functions f. In case that $f(t) = t^{p-1}$, the uniqueness of the least energy solution of problem (1.2) up to translations plays a crucial role. In general, we cannot expect such uniqueness. Moreover, the Lyapunov–Schmidt reduction procedure is involved and it is not clear if the procedure works for general cases such that the set of the positive solutions of (1.2) is degenerate [8]. In this paper, we will show the multiple existence of positive solutions of (P) under some assumptions on f using homology group arguments.

To state our main result, we need some notation. We denote by $\tau\colon M \to \mathbb{R}$ the scalar curvature of M. Let $K(\tau) = \{x \in M\colon \nabla\tau(x) = 0\}$ and

$$\mathcal{K}(\tau) = \{L \text{is a connected subset of } K(\tau)\}.$$

For each $L \in \mathcal{K}(\tau)$, we set $c_L = -\tau(x)$, where $x \in L$. We denote by $\mathcal{K}_C(\tau)$ the subset of $\mathcal{K}(\tau)$ such that each $L \in \mathcal{K}_C(\tau)$ satisfies

$$H_*((-\tau)^{c_L} \cap U, ((-\tau)^{c_L} \setminus L) \cap U) \neq \{0\}$$

for an open neighbourhood U of L such that $U \cap K(\tau) = L$. We denote by $\#\mathcal{K}_C(\tau)$ the number of elements of $\mathcal{K}_C(\tau)$. We assume that the function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the following conditions.

(f1) There exists $\mu > 1$ such that $f'(t)t > \mu f(t) > 0$ for all $t \in \mathbb{R}^+$.

(f2) There exists $M_0 > 0$ such that $|f(t)| \leqslant M_0(|t|^{q-1} + |t|^{p-1})$ for $t \in \mathbb{R}$, where $1 < q \leqslant 2 < p < 3$.

(f3) $f''' \in C(\mathbb{R})$ is bounded and $f'''(t) < 0$ for $t \in \mathbb{R}^+$.

We can now state our main results.

Theorem 1.1. *There exists $\varepsilon_1 > 0$ such that problem (P) has at least $\#K_C(\tau)$ positive solutions for each $\varepsilon \in (0, \varepsilon_1)$.*

Remark 1.2. Through the argument below, we can see that, for each $L \in \mathcal{K}_C(\tau)$, there exists solutions u_ε of (P) such that u_ε concentrate at a point of L as $\varepsilon \to 0$.

Remark 1.3. We do not need condition (f3) in the case that the non-degeneracy of problem (1.3) is guaranteed. That is, theorem 1.1 covers the case that $f(t) = t^{p-1}$,

$t > 0$. Conditions (f1)–(f3) are satisfied by a broad class of functions, e.g. $f(t) = t\log(t+1)$ and $f(t) = (1+t)t^2/(1+2t)$, $t > 0$. We conjecture whether the condition $f'''(t) < 0$ can be removed.

The text is organized as follows: the next section is devoted to preliminaries. In § 3, we give a few lemmas that are crucial for our argument. Section 4 is devoted to the proof of theorem 1.1. In Appendix A, we give a few estimates which are needed for our computation.

2. Preliminaries

Let (X, Y) be a pair of topological space with $Y \subset X$. We denote by $H_*(X, Y)$ the singular relative homology group. For each $q \in \mathbb{N} \cup \{0\}$ and a q-singular chain α, $[\alpha]$ stands for the element of $H_q(X, Y)$ induced from α [13]. Let E be a real Hilbert manifold and $F\colon E \to \mathbb{R}$ be a functional. Then, for each $c \in \mathbb{R}$, we denote by F^c the level set with respect to F, i.e. $F^c = \{v \in E\colon F(v) \leqslant c\}$. For each C^1-Hilbert manifold E, we denote by T_vE the tangent space of E at $v \in H$ and TE the tangent space, respectively. Let $\varepsilon \in \mathbb{R}^+$. We denote by $\|\cdot\|_{\mathbb{R}^N,\varepsilon}$ the norm of $H^1(\mathbb{R}^N)$ defined by

$$\|v\|^2_{\mathbb{R}^N,\varepsilon} = \frac{1}{\varepsilon^N}\int_{R^N}(\varepsilon^2|\nabla v|^2 + |v|^2)\,\mathrm{d}\mu \quad \text{for all } v \in H^1(\mathbb{R}^N),$$

where $\mathrm{d}\mu$ stands for the Euclidian measure. The inner product of $H^1(\mathbb{R}^N)$ corresponding to the norm $\|\cdot\|_{\mathbb{R}^N,\varepsilon}$ is denoted by $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathbb{R}^N,\varepsilon}$. In addition, we denote by $|\cdot|_{\mathbb{R}^N,p,\varepsilon}$ the norm of $L^p(\mathbb{R}^N)$ defined by

$$|v|^p_{\mathbb{R}^N,p,\varepsilon} = \frac{1}{\varepsilon^N}\int_{R^N}|v|^p\,\mathrm{d}\mu \quad \text{for } v \in L^p(\mathbb{R}^N).$$

We set

$$\langle u, v\rangle_{\mathbb{R}^N,\varepsilon} = \frac{1}{\varepsilon^N}\int_{R^N} uv\,\mathrm{d}\mu \quad \text{for } u, v \in H^1(\mathbb{R}^N).$$

We denote by exp the exponential mapping $\exp\colon TM \to M$ (cf. [10]). We denote by $B_r(x)$ the open ball in $\mathbb{R}^N$ centred at x with radius r. Then, for $\varepsilon > 0$ sufficiently small, the Riemannian manifold M has the special set of charts $\{\exp_x\colon B_\varepsilon(0) \to M\colon x \in M\}$. Corresponding to this chart, by choosing a orthogonal coordinate system $(x^1, \dots, x^N)$ of $\mathbb{R}^N$ and identifying T_xM with $\mathbb{R}^N$ for $x \in M$, we can define a coordinates called Riemannian normal coordinates. Let $\mathcal{C}$ be an atlas of M whose charts are given by the exponential map. For each $R > 0$ and $x \in M$, we denote by $B_g(x, R)$ the ball in M centred at x with radius R with respect to metric induced by g. Let $\varepsilon > 0$. We denote by $\|\cdot\|_\varepsilon$ the norm of $H = H^1(M)$ given by

$$\|u\|^2_\varepsilon = \frac{1}{\varepsilon^N}\int_M(\varepsilon^2|\nabla_g u|^2 + |u|^2)\,\mathrm{d}\mu_g \quad \text{for } u \in H,$$

where ∇_g denotes the gradient and μ_g is the volume form on M associated to the metric tensor g. We omit $\mathrm{d}\mu$ and $\mathrm{d}\mu_g$ when there is no fear of confusion. We denote by $\langle\!\langle\cdot,\cdot\rangle\!\rangle_\varepsilon$ the inner product of H corresponding to the norm $\|\cdot\|_\varepsilon$. We also define an inner product of $L^2(M)$ and a norm of $L^r(M)$, $1 < r < 2^*$, by

$$\langle u, v\rangle_\varepsilon = \frac{1}{\varepsilon^N}\int_M uv\,\mathrm{d}\mu_g \quad \text{for } u, v \in L^2(M)$$

and

$$|w|^r_{r,\varepsilon} = \frac{1}{\varepsilon^N}\int_M |w|^r\,\mathrm{d}\mu_g \quad \text{for } w \in L^r(M).$$

For each $w \in H$, we set

$$|w(z)|^2_\varepsilon = \varepsilon^2|\nabla_g w(z)|^2 + |w(z)|^2 \quad \text{for } z \in M.$$

By Sobolev's embedding theorem, we have that there exists $c_p > 0$, independent of ε, such that $|v|_{p,\varepsilon} \leqslant c_p\|v\|_\varepsilon$ for all $v \in H$ and

$$\left(\frac{1}{\varepsilon^N}\int_{B_g(x,a)} |w|^p_\varepsilon\,\mathrm{d}\mu_g\right)^{2/p} \leqslant c_p\left(\frac{1}{\varepsilon^N}\int_{B_g(x,a)} |w|^2_\varepsilon\,\mathrm{d}\mu_g\right) \quad \text{for } w \in H \text{ and } a \in (0,1]. \tag{2.1}$$

For each $x \in M$, $H^1(B_g(x,1))$ stands for the restriction of H on $B_g(x,1)$ and $H^1_0(B_g(x,1))$ denotes the closure of $C^\infty_0(B_g(x,1))$ in $H^1(B_g(x,1))$. For each function $u \in H$, we have

$$\int_M |\nabla_g u|^2\,\mathrm{d}\mu_g = \sum_{C\in\mathcal{C}}\int_C \varphi_C(x)|\nabla_g u|^2\,\mathrm{d}\mu_g.$$

If $C = B_g(x,1) \in \mathcal{C}$ for some $x \in M$ and $\operatorname{supp} u \in C$, then

$$\int_C |\nabla_g u|^2\,\mathrm{d}\mu_g = \int_{B(0,1)}\left(\sum_{i,j=1}^n g^{ij}_x \frac{\partial u(\exp_x(z))}{\partial z_i}\frac{\partial u(\exp_x(z))}{\partial z_j}\right)\sqrt{|g_x|}\,\mathrm{d}z_1\cdots\mathrm{d}z_N,$$

where $|g_x|$ denotes the determinant of the metric matrix $\{(g_x)_{ij}\}$ for each $x \in M$ and $\{g^{ij}_x\}$ is the inverse matrix of $\{(g_x)_{ij}\}$. The Laplacian Δ_g on M is given by

$$\Delta_g v = \sum_{i,j}\frac{1}{\sqrt{|g_x|}}\partial_i(g^{ij}_x\sqrt{|g_x|}\partial_j v) \quad \text{for } v \in H. \tag{2.2}$$

For each $x \in M$, there exists $\varepsilon > 0$ such that $\exp_x\colon B_\varepsilon(0) \to B_g(x,\varepsilon)$ is a diffeomorphism for all $x \in M$. We may assume, for simplicity, that $\varepsilon = 1$ for each $x \in M$ and each $C \in \mathcal{C}$ has the form $C = B_g(x,1)$ for some $x \in M$. For $x \in M$ and $(r,\zeta) \in [0,1) \times S^{N-1}$, we have

$$r^{N-1}\sqrt{|g_x(\exp(r\zeta))|} = r^{N-1} - \tfrac{1}{6}\rho_x(\zeta)r^{N+1} + O(r^{N+2}), \tag{2.3}$$

where ρ_x stands for the Ricci curvature [12]. It is known that

$$\int_{S^{N-1}} \rho_x(\zeta)\,\mathrm{d}\zeta = \varpi\tau(x), \tag{2.4}$$

where ϖ is the volume of unit ball $B_1(0)$ in $\mathbb{R}^N$ [12]. We set

$$\eta_x(r,\zeta) = \sqrt{|g_x(\exp(r\zeta))|}$$

https://doi.org/10.1017/S0308210510000892 Published online by Cambridge University Press

for $(r,\zeta)\in[0,1)\times S^{N-1}$. Then we have

$$\int_M u\,\mathrm{d}\mu_g=\int_{S^{N-1}}\mathrm{d}\zeta\int_{B_1(0)}u(r,\zeta)\eta_x r^{N-1}\,\mathrm{d}r\quad\text{for } u\in L^1(B_g(x,1)). \tag{2.5}$$

By (2.3), we also have that

$$\eta_x(r,\zeta)=1-\tfrac{1}{6}\rho_x(\zeta)r^2+O(r^3)\quad\text{for } (r,\zeta)\in\mathbb{R}^+\times S^{N-1}. \tag{2.6}$$

We simply write g and g_{ij} instead of g_x and $(g_x)_{ij}$, respectively, when there is no fear of confusion. For $x\in M$, each function $u\in H_0^1(B_g(x,1))$ is regarded as an element of H by putting $u=0$ on $M\setminus B_g(x,1)$, that is, we have $H_0^1(B_g(x,1))\subset H^1(M)$. Set

$$H_{x,\mathrm{rad}}=\{u\in H_0^1(B_g(x,1))\colon u \text{ is radial with respect to } x\}\quad\text{for } x\in M.$$

Then for each $u\in H_{x,\mathrm{rad}}$, we have that

$$\int_M|\nabla_g u|^2\,\mathrm{d}\mu_g=\int_{B(0,1)}\left|\frac{\partial u(\exp_x(r))}{\partial r}\right|^2\eta_x r^{N-1}\,\mathrm{d}r. \tag{2.7}$$

We also recall [11, ch. 2.4] that for $u\in H_{x,\mathrm{rad}}$,

$$\Delta_g u=\frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial u}{\partial r}\right)+\frac{1}{\eta_x}\frac{\partial\eta_x}{\partial r}\frac{\partial u}{\partial r}. \tag{2.8}$$

Since our purpose is to find positive solutions, we may assume without any loss of generality that $f(t)=0$ for $t\leqslant 0$. We set

$$F(t)=\int_0^t f(\tau)\,\mathrm{d}\tau\quad\text{for } t\in\mathbb{R}.$$

From (f1), it is easy to verify that

$$f(t)t>(\mu+1)F(t)\quad\text{for all } t\in\mathbb{R}. \tag{2.9}$$

Next we retrieve some known facts related to problem (P). It is known that problem (1.2) has a radial positive solution $U_0\in C^\infty(\mathbb{R}^N)\cap H^2(\mathbb{R}^N)$ such that

$$c_0=I_{\mathbb{R}^N,1}(U_0)=\min\{I_{\mathbb{R}^N,1}(v)\colon v\in\mathcal{N}_{\mathbb{R}^N,1}\}, \tag{2.10}$$

where

$$I_{\mathbb{R}^N,\varepsilon}(v)=\frac{1}{\varepsilon^N}\left(\frac{1}{2}\int_{\mathbb{R}^N}(\varepsilon^2|\nabla v|^2+|v|^2)\,\mathrm{d}\mu-\int_{R^N}F(v)\,\mathrm{d}\mu\right) \tag{2.11}$$

for $(\varepsilon,v)\in\mathbb{R}^+\times H^1(\mathbb{R}^N)$, and

$$\mathcal{N}_{\mathbb{R}^N,\varepsilon}=\left\{v\in H^1(\mathbb{R}^N)\colon\int_{R^N}(\varepsilon^2|\nabla v|^2+|v|^2)\,\mathrm{d}\mu=\int_{R^N}f(v)v\,\mathrm{d}\mu\right\}\quad\text{for }\varepsilon\in\mathbb{R}^+.$$

We also have by (f1) that

$$I_{\mathbb{R}^N,1}(v)\geqslant I_{\mathbb{R}^N,1}(tv)\quad\text{for all } v\in\mathcal{N}_{\mathbb{R}^N,1}\text{ and } t\in\mathbb{R}.$$

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Since U_0 is radial, $U_0(r)$ is a solution of the following problem:

$$-\frac{\partial^2 U}{\partial r^2} - \frac{(N-1)}{r}\frac{\partial U}{\partial r} + U = f(U) \quad \text{for } r \geqslant 0. \tag{2.12}$$

Since $\lim_{t\to 0} f(t)/t = 0$ by (f2), we have that U_0 satisfies the condition that, for $\mathrm{d}U_0/\mathrm{d}r < 0$ on $(0,\infty)$,

$$\limsup_{r\to\infty} U_0(r)\mathrm{e}^{(1-\delta)r} < \infty \quad \text{and} \quad \limsup_{r\to\infty}\left|\frac{\mathrm{d}U_0(r)}{\mathrm{d}r}\right|\mathrm{e}^{(1-\delta)r} < \infty \tag{2.13}$$

for each $\delta \in (0,1)$ [9]. Let $a \in (\frac{1}{2}, 1)$ be a constant. From the inequality above, we have that there exists $l_0 > 0$ and $\varepsilon_0 > 0$ such that

$$|U_0(r)| \leqslant l_0 \mathrm{e}^{-r/2} \quad \text{and} \quad \left|\frac{\mathrm{d}U_0(r)}{\mathrm{d}r}\right| \leqslant l_0\mathrm{e}^{-r/2} \quad \text{for all } r \geqslant \frac{1}{4\varepsilon_0^{1-\alpha}}. \tag{2.14}$$

Combining (2.12) and (2.13) with (f2), we find that

$$\limsup_{r\to\infty}\left|\frac{\partial^2 U_0}{\partial r^2}\right|\mathrm{e}^{(1-\delta)r} < \infty \quad \text{for each } \delta \in (0,1). \tag{2.15}$$

It is obvious that each element of the set $S = \{U_0(\cdot - x) \colon x \in \mathbb{R}^N\}$ is a solution of problem (1.2). Let $\varepsilon \in \mathbb{R}^+$. We put $U_\varepsilon(x) = U_0(x/\varepsilon)$ on $\mathbb{R}^N$. Then $U_\varepsilon \in H^1(\mathbb{R}^N)$ is a positive radial solution of problem

$$-\varepsilon^2 \Delta U + U = f(U) \quad \text{on } \mathbb{R}^N, \tag{2.16}$$

and each U_ε is a critical point of $I_{\mathbb{R}^N,\varepsilon}$ with $I_{\mathbb{R}^N,\varepsilon}(U_\varepsilon) = c_0$. Since U_ε is radial, i.e. $U_\varepsilon(x) = U_\varepsilon(|x|)$, it follows from (1.2) that

$$-\varepsilon^2 \frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial U_\varepsilon}{\partial r}\right) + U_\varepsilon = f(U_\varepsilon). \tag{2.17}$$

It is obvious that for each $x \in \mathbb{R}^N$, $U_\varepsilon(\cdot - x)$ is a solution of (2.16). We put

$$S_\varepsilon = \{U_\varepsilon(\cdot - x) \colon x \in \mathbb{R}^N\} \quad \text{and} \quad T_0 S_\varepsilon = T_{U_\varepsilon} S_\varepsilon = \operatorname{span}\left\{\frac{\partial U_\varepsilon}{\partial x_i} \colon 1 \leqslant i \leqslant N\right\}.$$

By multiplying (2.16) by U_ε we have that

$$\|U_\varepsilon\|^2_{\mathbb{R}^N,\varepsilon} = \frac{1}{\varepsilon^N}\int_{R^N} f(U_\varepsilon)U_\varepsilon \,\mathrm{d}\mu. \tag{2.18}$$

Then it follows that

$$c_0 = I_{\mathbb{R}^N,\varepsilon}(U_\varepsilon) = \tfrac{1}{2}\|U_\varepsilon\|^2_{\mathbb{R}^N,\varepsilon} - \frac{1}{\varepsilon^N}\int_{R^N} F(U_\varepsilon) = \frac{1}{\varepsilon^N}\int_{R^N} (\tfrac{1}{2} f(U_\varepsilon)U_\varepsilon - F(U_\varepsilon))\,\mathrm{d}\mu. \tag{2.19}$$

The functional $I_\varepsilon \in C^2(H, \mathbb{R})$ associated with the problem (P) is given by

$$I_\varepsilon(u) = \frac{1}{\varepsilon^N}\int_M (\tfrac{1}{2}\varepsilon^2|\nabla_g u|^2 + \tfrac{1}{2}|u|^2 - F(u))\,\mathrm{d}\mu_g \quad \text{for } u \in H.$$

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We denote by $\mathcal{N}_\varepsilon$ the Nehari manifold defined by

$$\mathcal{N}_\varepsilon = \left\{ v \in H^1(M)\colon \|v\|_\varepsilon^2 = \frac{1}{\varepsilon^N}\int_M f(v)v\,\mathrm{d}\mu_g \right\}.$$

From (f1), we can see that for each $v \in H \setminus \{0\}$, there exists a unique positive number t such that $tv \in \mathcal{N}_\varepsilon$ [14]. We set $\mathcal{N}_\varepsilon(v) = tv$. Then $\mathcal{N}_\varepsilon(\cdot)\colon H \setminus \{0\} \to \mathcal{N}_\varepsilon$ is a continuous mapping and $\mathcal{N}_\varepsilon$ is a submanifold of H with codimension 1. It is clear that $\mathcal{N}_\varepsilon(v)$ is transversal to $\mathcal{N}_\varepsilon$. It is easy to see that each critical point $u \in H^1(M)\setminus\{0\}$ of I_ε is contained in $\mathcal{N}_\varepsilon$. By the definition of $\mathcal{N}_\varepsilon$ and (f1), we have that, for each $v \in \mathcal{N}_\varepsilon$,

$$I_\varepsilon(v) = \frac{1}{\varepsilon^N}\int_M (\tfrac{1}{2}f(v)v - F(v))\,\mathrm{d}\mu_g > 0.$$

We next consider the eigenvalue problem of the linearized problem of (1.2)

$$-\Delta w + w - \mu f'(U_0)w = 0. \tag{2.20}$$

From the definition of U_0, we have that the first eigenvalue μ_1 of (2.20) is less than 1. We also have $\mu_2 = 1$ and the eigenspace $N(U_0)$ corresponding to μ_2 is a space whose dimension is at most $N+1$ and contains the tangent space $T_{U_0}S$. For simplicity, we write T_0S instead of $T_{U_0}S$, i.e. $T_0S = \mathrm{span}\{\partial U_0/\partial x_i\colon 1 \leqslant i \leqslant N\}$ of S.

If $\dim N(U_0) = N$, the reduction method employed in [8] is valid and condition (f3) is not needed. Then, in the following, we assume that $\dim N(U_0) = N+1$ and give the proof of theorem 1.1 under this assumption. Our argument is valid for the case that $\dim N(U_0) = N$. Let $v_1 \in H^1(\mathbb{R}^N)$ be a normalized eigenfunction corresponding to the first eigenvalue μ_1 and $v_0 \in H^1(\mathbb{R}^N) \cap (T_0S)^\perp$ be an normalized eigenfunction corresponding to $\mu_1 = 1$.

3. Lemmas

Henceforth, we fix $\alpha \in (0,1)$ such that $\frac{1}{2} < \alpha$ and $10(1-\alpha) > 4$. Let $\varphi \in C^\infty(\mathbb{R}, [0,1])$ be a monotone decreasing function such that $\varphi(x) = 1$ on $(-\infty, \frac{1}{2}]$, $\varphi(x) = 0$ on $[1,\infty)$ and $|\varphi|_{C^1} \leqslant 2$. For each $\varepsilon \in (0,1)$, we set

$$\varphi_\varepsilon(x) = \varphi\biggl(\frac{2|x|}{\varepsilon^\alpha}\biggr) \quad \text{for } x \in \mathbb{R}^N,$$

and define a function $W_\varepsilon \in H_0^1(B_1(0))$ by

$$W_\varepsilon(z) = \varphi_\varepsilon(z)U_\varepsilon(z) \quad \text{for } z \in \mathbb{R}^N.$$

From the definition of φ, we have that $W_\varepsilon(z) = 0$ for $z \in \mathbb{R}^N \setminus B_{\varepsilon^\alpha/2}(0)$ and $W_\varepsilon(z) = U_\varepsilon(z)$ for $z \in B_{\varepsilon^\alpha/4}(0)$. For each $(\varepsilon, x) \in \mathbb{R}^+ \times M$, we define a function $W_{\varepsilon,x} \in H$ by

$$W_{\varepsilon,x}(z) = \begin{cases} W_\varepsilon(\exp_x^{-1}(z)) & \text{for } z \in B_g(x,1), \\ 0 & \text{for } z \in M \setminus B_g(x,1). \end{cases}$$

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We put $\tilde{S}_\varepsilon = \{W_{\varepsilon,x} \colon x \in M\}$ and define a neighbourhood $D_\delta(\tilde{S}_\varepsilon) \subset H^1(M)$ of $\tilde{S}_\varepsilon$ by

$$D_\delta(\tilde{S}_\varepsilon) = \{W_{\varepsilon,x} + w \colon x \in M,\ w \in H^1(M) \text{ with } \|w\|_\varepsilon < \delta\} \quad \text{for each } \delta > 0.$$

LEMMA 3.1. *There exists $\delta_0 \in (0,1)$ such that, for each $\varepsilon \in (0,\varepsilon_0)$, there exists a metric projection P_ε from $D_{\delta_0}(\tilde{S}_\varepsilon)$ onto $\tilde{S}_\varepsilon$.*

Proof. We identify $B_g(x,1)$ with $B_1(0)$ for each $x \in M$. Then, from the definition of $W_{1,x}$, we have that there exists $C > 0$ such that

$$\left\|\frac{\partial W_{1,x}}{\partial x_i}\right\|_1 \leqslant C \quad \text{and} \quad \left\|\frac{\partial^2 W_{1,x}}{\partial x_i \partial x_j}\right\|_1 \leqslant C \quad \text{for all } x \in M \text{ and } 1 \leqslant i,j \leqslant N.$$

For each $\varepsilon \in (0,\varepsilon_0)$ and $(x,y) \in M \times B_g(x,1)$, we define a curve $\rho_{\varepsilon,x,y}(t) = W_{\varepsilon,x+ty}$, for $|t|$ sufficiently small. Let $\tau_\varepsilon(x,y)$ be the curvature of the curve at $t = 0$. Then, to prove the assertion, it is sufficient to show that $\{\tau_\varepsilon(x,y) \colon \varepsilon \in (0,\varepsilon_0), (x,y) \in M \times B_g(x,1)\}$ is bounded. From the definition,

$$\begin{aligned}\tau_\varepsilon(x,y) = \left\|\frac{\partial W_{\varepsilon,x}}{\partial y}\right\|_\varepsilon^{-1} \bigg\langle\!\!\bigg\langle \frac{\partial W_{\varepsilon,x}}{\partial y}, \frac{\partial^2 W_{\varepsilon,x}}{\partial y^2}\bigg\rangle\!\!\bigg\rangle_\varepsilon \frac{\partial W_{\varepsilon,x}}{\partial y} \\ - \left\|\frac{\partial W_{\varepsilon,x}}{\partial y}\right\|_\varepsilon \frac{\partial^2 W_{\varepsilon,x}}{\partial y^2}\left(\left\|\frac{\partial W_{\varepsilon,x}}{\partial y}\right\|_\varepsilon^3\right)^{-1}\end{aligned}$$

for $\varepsilon \in (0,\varepsilon_0)$ and $(x,y) \in M \times B_g(x,1)$. Then noting that

$$\left\|\frac{\partial W_{\varepsilon,x}}{\partial y}\right\|_\varepsilon = \varepsilon^{-1}\left\|\frac{\partial W_{1,x}}{\partial y}\right\|_1 \quad \text{and} \quad \left\|\frac{\partial^2 W_{\varepsilon,x}}{\partial y^2}\right\|_\varepsilon = \varepsilon^{-2}\left\|\frac{\partial^2 W_{1,x}}{\partial y^2}\right\|_1,$$

one can see that $\{\tau_\varepsilon(x,y) \colon \varepsilon \in (0,\varepsilon_0),\ x \in M,\ y \in B_g(x,1)\}$ is bounded. Then the assertion holds. □

In the following, we will find solutions of (P) in $D_{\delta_0}(\tilde{S}_\varepsilon)$. By the definition of $D_{\delta_0}(\tilde{S}_\varepsilon)$, we have that $M_1 = \sup\{\|v\|_\varepsilon \colon \varepsilon \in (0,\varepsilon_0) \text{ and } v \in D_{\delta_0}(\tilde{S}_\varepsilon)\} < \infty$. Then we have

$$m_{\mathcal{N}} = \inf\left\{\frac{1}{\varepsilon^N}\int_M f(v)v \,\mathrm{d}\mu_g \colon \varepsilon \in (0,\varepsilon_0), v \in D_{\delta_0}(\tilde{S}_\varepsilon)\right\} > 0. \tag{3.1}$$

From (f1) and (f3), we have that there exists $M_f > 0$ such that

$$0 \leqslant f'(t) \leqslant M_f(1+t) \quad \text{for } t \in \mathbb{R}^+ \quad \text{and} \quad \sup\{|f''(t)| \colon t \in \mathbb{R}\} < \infty. \tag{3.2}$$

LEMMA 3.2. *For each $\varepsilon \in (0,\varepsilon_0)$ and $x \in M$,*

$$I_\varepsilon(W_{\varepsilon,x}) = c_0 - A\varepsilon^2\tau(x) + O(\varepsilon^3),$$

where $A = \frac{1}{6}\varpi(A_0 + NA_1)$,

$$A_0 = \int_{S^{N-1}} \mathrm{d}\zeta \int_0^\infty (\tfrac{1}{2}f(U_0)U_0 - F(U_0))r^{N+1}\,\mathrm{d}r$$

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and

$$A_1 = \int_{S^{N-1}} \mathrm{d}\zeta \int_0^\infty U_0^2 r^{N-1}\,\mathrm{d}r.$$

Proof. Let $(\varepsilon, x) \in (0, \varepsilon_0) \times M$. For simplicity, we set $\eta(z) = \eta_x(z)$, $U(z) = U_\varepsilon(z)$, $\psi(z) = \varphi_\varepsilon(z)$ for $z \in B_1(0)$ and $\tilde{U} = U_\varepsilon(\exp_x^{-1}(z))$ and $\tilde{\psi}(z) = \psi(\exp_x^{-1}(z))$ for $z \in M$. Then, by definition, $W_\varepsilon = \psi U$ and $W_{\varepsilon,x} = \tilde{\psi}\tilde{U}$. Since ψ and U are radial with respect to the origin, we have by (2.8), (2.17) and (A 8) that, for each $w \in H$,

$$\begin{aligned}
&\langle -\varepsilon^2 \Delta_g W_{\varepsilon,x} + W_{\varepsilon,x}, w\rangle_\varepsilon \\
&\quad = \varepsilon^2 \langle -\tilde{\psi}\Delta_g \tilde{U} - 2\nabla_g \tilde{\psi}\nabla_g \tilde{U} - \tilde{U}\Delta_g \tilde{\psi}, w\rangle_\varepsilon + \langle W_{\varepsilon,x}, w\rangle_\varepsilon \\
&\quad = -\frac{\varepsilon^2}{\varepsilon^N}\int_{S^{N-1}} \mathrm{d}\zeta \int_0^1 \psi \frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial U}{\partial r}\right) w(\exp_x(r\zeta))\eta r^{N-1}\,\mathrm{d}r \\
&\qquad - \frac{\varepsilon^2}{\varepsilon^N}\int_{S^{N-1}} \mathrm{d}\zeta \int_0^1 \psi \frac{\partial U}{\partial r}\frac{\partial \eta}{\partial r} w(\exp_x(r\zeta)) r^{N-1}\,\mathrm{d}r + \langle W_{\varepsilon,x}, w\rangle_\varepsilon \\
&\qquad - \varepsilon^2 \langle 2\nabla_g \tilde{\varphi}\cdot \nabla_g \tilde{U} + \tilde{U}\Delta_g \tilde{\varphi}, w\rangle_\varepsilon \\
&\quad = \frac{1}{\varepsilon^N}\int_{S^{N-1}} \mathrm{d}\zeta \int_0^1 \psi f(U) w(\exp_x(r\zeta))\eta r^{N-1}\,\mathrm{d}r \\
&\qquad - \frac{\varepsilon^2}{\varepsilon^N}\int_{S^{N-1}} \mathrm{d}\zeta \int_0^1 \psi \frac{\partial U}{\partial r}\frac{\partial \eta}{\partial r} w(\exp_x(r\zeta)) r^{N-1}\,\mathrm{d}r \\
&\qquad + O\biggl(\exp\left(-\frac{1}{8\varepsilon^{1-\alpha}}\right)\|w\|_\varepsilon\biggr).
\end{aligned} \tag{3.3}$$

Then setting $w = W_{\varepsilon,x} = (\psi U)(\exp_x^{-1}(z))$ for $z \in M$, we have by (3.3) that

$$\begin{aligned}
&\int_M (\varepsilon^2 |\nabla_g W_{\varepsilon,x}|^2 + |W_{\varepsilon,x}|^2)\,\mathrm{d}\mu_g \\
&\quad = \int_{S^{N-1}} \mathrm{d}\zeta \int_0^1 \psi^2 f(U) U \eta r^{N-1}\,\mathrm{d}r \\
&\qquad - \varepsilon^2 \int_{S^{N-1}} \mathrm{d}\zeta \int_0^1 \psi^2 \left(U\frac{\partial U}{\partial r}\right)\frac{\partial \eta}{\partial r} r^{N-1}\,\mathrm{d}r + O\biggl(\exp\left(-\frac{1}{8\varepsilon^{1-\alpha}}\right)\biggr).
\end{aligned} \tag{3.4}$$

Therefore, we have that

$$\begin{aligned}
I_\varepsilon(W_{\varepsilon,x}) &= \frac{1}{\varepsilon^N}\left(\frac{1}{2}\int_M (\varepsilon^2 |\nabla_g W_{\varepsilon,x}|^2 + W_{\varepsilon,x}^2)\,\mathrm{d}\mu_g - \int_M F(W_{\varepsilon,x})\,\mathrm{d}\mu_g\right) \\
&= \frac{1}{\varepsilon^N}\int_{S^{N-1}} \mathrm{d}\zeta \int_0^1 (\tfrac{1}{2}\psi^2 f(U) U - F(\psi U))\eta r^{N-1}\,\mathrm{d}r \\
&\quad - \frac{\varepsilon^2}{\varepsilon^N}\int_{S^{N-1}} \mathrm{d}\zeta \int_0^1 \left(\psi^2\left(U\frac{\partial U}{\partial r}\right)\frac{\partial \eta}{\partial r}\right) r^{N-1}\,\mathrm{d}r + O\biggl(\exp\left(-\frac{1}{8\varepsilon^{1-\alpha}}\right)\biggr).
\end{aligned}$$

Then, by (A 7) and (A 11), the assertion follows. □

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LEMMA 3.3. *For each* $(\varepsilon, x) \in (0, \varepsilon_0) \times M$,

$$\|\mathcal{N}_\varepsilon(W_{\varepsilon,x}) - W_{\varepsilon,x}\|_\varepsilon = O(\varepsilon^{1+\alpha}) \quad \text{and} \quad |I_\varepsilon(\mathcal{N}_\varepsilon(W_{\varepsilon,x})) - I_\varepsilon(W_{\varepsilon,x})| = O(\varepsilon^{2(1+\alpha)}). \tag{3.5}$$

Proof. Let $(\varepsilon, x) \in (0, \varepsilon_0) \times M$. We set $W = W_{\varepsilon,x}$. Let $\tau \in \mathbb{R}$ such that $(1+\tau)W = \mathcal{N}_\varepsilon(W)$. That is

$$(1+\tau)^2\|W\|_\varepsilon^2 = \frac{1}{\varepsilon^N}\int_M f((1+\tau)W)(1+\tau)W \,\mathrm{d}\mu_g.$$

It then follows that

$$\|W\|_\varepsilon^2 + 2\tau\|W\|_\varepsilon^2 = \frac{1}{\varepsilon^N}\int_M (f(W) + \tau(f(W) + f'(W)W)W)\,\mathrm{d}\mu_g + O(\tau^2).$$

Then we have

$$\begin{aligned}\tau = -\|W\|_\varepsilon^2 - \frac{1}{\varepsilon^N}\int_M f(W)W\,\mathrm{d}\mu_g \\ \times\biggl(2\biggl(\|W\|_\varepsilon^2 - \frac{1}{\varepsilon^N}\int_M f(W)W\,\mathrm{d}\mu_g\biggr) + \frac{1}{\varepsilon^N}\int_M (f(W) - f'(W)W)W\,\mathrm{d}\mu_g\biggr)^{-1} \\ + O(\tau^2).\end{aligned} \tag{3.6}$$

We can see from (f1) that

$$L_0 = \int_{\mathbb{R}^N} (f'(U_0)U_0^2 - f(U_0)U_0)\,\mathrm{d}\mu \geqslant \int_M (\mu - 1)f(U_0)U_0\,\mathrm{d}\mu > 0. \tag{3.7}$$

Then since $\lim_{\varepsilon\to 0}(W_{\varepsilon,x} - U_\varepsilon(\exp_x^{-1}(\cdot))) = 0$ on $B_g(x,1)$, we have that

$$\frac{1}{\varepsilon^N}\int_M (f'(W)W^2 - f(W)W)\,\mathrm{d}\mu_g \geqslant \frac{L_0}{2}. \tag{3.8}$$

for $\varepsilon > 0$ sufficiently small. On the other hand, from lemma A.4 with $w = W_{\varepsilon,x}$, we have that

$$\|W\|_\varepsilon^2 - \frac{1}{\varepsilon^N}\int_M f(W)W\,\mathrm{d}\mu_g = O(\varepsilon^{1+\alpha})$$

and then $\tau = O(\varepsilon^{1+\alpha})$. Therefore, $\|\mathcal{N}_\varepsilon(W) - W\|_\varepsilon = O(\varepsilon^{1+\alpha})$. Since

$$\begin{aligned}I_\varepsilon(\mathcal{N}_\varepsilon(W)) &= \frac{(1+\tau)^2}{2}\|W\|_\varepsilon^2 - \frac{1}{\varepsilon^N}\int_M F((1+\tau)W)\,\mathrm{d}\mu_g \\ &= I_\varepsilon(W) + \tau\biggl(\|W\|_\varepsilon^2 - \frac{1}{\varepsilon^N}\int_M f(W)W\,\mathrm{d}\mu_g\biggr) + O(\tau^2),\end{aligned} \tag{3.9}$$

we obtain that

$$I_\varepsilon(\mathcal{N}_\varepsilon(W)) = I_\varepsilon(W) + O(\varepsilon^{2(1+\alpha)}).$$

□

LEMMA 3.4. *There exist* $\delta_1 \in (0, \delta_0)$ *and* $m_1 > 0$ *such that*

$$I_{\mathbb{R}^N,1}(U_0 + w) \geqslant I_{\mathbb{R}^N,1}(U_0) + m_1\|w\|_{\mathbb{R}^N,1}^4$$

for all $w \in H^1(\mathbb{R}^N)$ *with* $w \in (T_0S)^\perp \cap \mathcal{N}_{\mathbb{R}^N,1}$ *and* $\|w\|_{\mathbb{R}^N,1} < \delta_1$.

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Proof. For simplicity, we write $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ instead of $\|\cdot\|_{\mathbb{R}^N,1}$ and $\langle\cdot,\cdot\rangle_{\mathbb{R}^N,1}$, respectively. The Nehari manifold $\mathcal{N}=\mathcal{N}_{\mathbb{R}^N,1}$ is a C^1-submanifold of $H^1(\mathbb{R}^N)$ with codimension 1. That is, there exists $\tau_0\in C^1(H^1(\mathbb{R}^N)\setminus\{-U_0\},\mathbb{R})$ such that $(1+\tau_0(w))(U_0+w)\in\mathcal{N}$ for each $w\in H^1(\mathbb{R}^N)\setminus\{-U_0\}$ [14]. By lemma A.1, $|\tau_0(w)|\leqslant L_1\|w\|_{\mathbb{R}^N,1}$ for w with $\|w\|_{\mathbb{R}^N,1}$ sufficiently small. From the property of τ_0, it is easy to see that $(T_0S)^\perp\cap\mathcal{N}$ is a C^1-submanifold of $(T_0S)^\perp$ with codimension 1. Let E be a subspace of $(T_0S)^\perp$ such that codimension of E is 1 and $E\cap\{tU_0\colon t\in\mathbb{R}\}=\{0\}$. Then, for $\delta>0$ small,

$$\{(1+\tau_0(w))(U_0+w)\colon w\in E \text{ with } \|w\|_{\mathbb{R}^N,1}<\delta\}$$

form a neighbourhood of U_0 in $(T_0S)^\perp\cap\mathcal{N}$. To show the assertion, it is sufficient to show that there exists $\tilde m>0$ such that

$$I_{\mathbb{R}^N,1}((1+\tau_0(w))(U_0+w))\geqslant I_{\mathbb{R}^N,1}(U_0)+\tilde m\|w\|^4 \tag{3.10}$$

for each $w\in E$ with $\|w\|$ sufficiently small. In fact, setting

$$w'=\tau_0(w)U_0+(1+\tau_0(w))w,$$

we can rewrite the inequality above as

$$I_{\mathbb{R}^N,1}(U_0+w')\geqslant I_{\mathbb{R}^N,1}(U_0)+\tilde m\|w\|^4.$$

Since

$$\|w'\|=\|\tau_0(w)U_0+(1+\tau_0(w))w\|\leqslant(L_1\|U_0\|+(1+L_1\|w\|))\|w\|,$$

we have that there exists $m_1>0$, independent of w, such that

$$I_{\mathbb{R}^N,1}(U_0+w')\geqslant I_{\mathbb{R}^N,1}(U_0)+m_1\|w'\|^4$$

for $\|w'\|$ sufficiently small. Since $w'\in(T_0S)^\perp$, we have the assertion.

Next we shall show that (3.10) holds for some $\tilde m>0$. We set

$$E=\{v\in H^1(\mathbb{R}^N)\colon\langle-\Delta U_0+U_0-f'(U_0)U_0,v\rangle=0\}\cap(T_0S)^\perp.$$

Then E is a subspace of $(T_0S)^\perp$ with codimension 1, and $E\cap\{tU_0\colon t\in\mathbb{R}\}=\{0\}$. We also have that there exists $c_1>0$ such that

$$\langle-\Delta w+w-f'(U_0)w,w\rangle\geqslant c_1\|w\|^2\quad\text{for } w\in E\cap\{tv_0\colon t\in\mathbb{R}\}^\perp. \tag{3.11}$$

Let $w\in E$. In the following, we decompose w into the form $w=w_1+w_0$, where $w_0\in V_0=\{tv_0\colon t\in\mathbb{R}\}$ and $w_1\in V_0^\perp$. Noting that $I_{\mathbb{R}^N,1}((1+\tau_0(w))(U_0+w))\geqslant I_{\mathbb{R}^N,1}(U_0+w)$, we find from (A 5) that

$$\begin{aligned}I_{\mathbb{R}^N,1}(U_0+w)\geqslant{}& I_{\mathbb{R}^N,1}(U_0)+\tfrac12\langle-\Delta w_1+w_1-f'(U_0)w_1,w_1\rangle\\&-\int_{R^N}\left(\frac{1}{3!}f''(U_0)(w_0+w_1)^3+\frac{1}{4!}f'''(U_0+\theta w)(w_0+w_1)^4\right)\end{aligned} \tag{3.12}$$

for some $\theta\in(0,1)$. In case that $w_1=0$ in (3.12), one can see from lemma A.1(ii) and (f3) that if $\|w\|=\|w_0\|$ is sufficiently small, the assertion holds. We note that

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since $N=2,3,4$, condition (3.2) implies that there exists $L_2>0$ such that, for $w=w_0+w_1\in H^1(\mathbb{R}^N)$,

$$\left|\int_{R^N} f^{(n)}(U_0+\xi w)w_0^k w_1^{n+1-k}\right|\leqslant L_2\|w_0\|^k\|w_1\|^{n+1-k},\quad 2\leqslant n\leqslant 3,\ 0\leqslant k\leqslant n+1.$$

Then from (3.11), lemma A.1(ii) and (3.12), we can see that there exist $L_3>0$, $d_1>0$ and $l_1>0$ such that, for each $w\in E$ with $\|w\|<d_1$ and $\|w_1\|\geqslant L_3\|w_0\|^2$,

$$I_{\mathbb{R}^N,1}(U_0+w)\geqslant I_{\mathbb{R}^N,1}(U_0)+l_1\|w\|^4. \tag{3.13}$$

We next claim that

$$\langle -\Delta w+w-f'(U_0)w,w\rangle-\frac{1}{2c}\left(\frac{1}{3!}\int_{R^N} f''(U_0)wv_0^2\right)^2\geqslant 0 \tag{3.14}$$

for $w\in E$ with $\|w\|=1$, where

$$c=-\frac{1}{4!}\int_{R^N} f'''(U_0)v_0^4+L_0^{-1}\left(\int_{R^N} f''(U_0)U_0v_0^2\right).$$

Here L_0 is the constant defined in (3.7). Moreover, we will show that there exists at most one pair $\pm v\in E\cap V^{\perp}$ such that $\|v\|=1$ and the equality holds with $w=\pm v$ in (3.14).

We note that condition (f3) implies that $c>0$. We prove (3.14) by contradiction. Suppose that there exists $v\in E$ such that $\|v\|=1$ and

$$\langle -\Delta v+v-f'(U_0)v,v\rangle-\frac{1}{2c}\left(\frac{1}{3!}\int_{R^N} f''(U_0)vv_0^2\right)^2<0. \tag{3.15}$$

Let $w\in E$ with $\|w_1\|\leqslant L_3\|w_0\|^2$. Put $\tau=\tau_0(w)$. Since

$$\langle -\Delta U_0+U_0-f'(U_0)U_0,w\rangle=0,$$

we have by (A 2) that

$$\|U_0+w_0\|^2-\langle f(U_0+w),U_0+w\rangle=\langle -\Delta w-f(U_0)w,w\rangle-\langle \tfrac{1}{2}f''(U_0)w^2,U_0\rangle+O(\|w\|^3). \tag{3.16}$$

Since

$$|\tau|=L_0^{-1}|\|U_0+w\|^2-\int_{R^N} f(U_0+w)(U_0+w)|(1+O(\|w\|))$$

by (A 3), we have from the quality above that

$$\tau\left(\|U_0+w\|^2-\int_{R^N} f(U_0+w)(U_0+w)\right)=L_0^{-1}\left(\frac{1}{2}\int_{R^N} f''(U_0)U_0w_0^2\right)^2+O(\|w_0\|^5).$$

On the other hand, we have by (A 5) that

$$\begin{aligned}&I_{\mathbb{R}^N,1}((1+\tau)(U_0+w))\\&\qquad=I_{\mathbb{R}^N,1}(U_0+w)+\tau\left(\|U_0+w\|^2-\int_{R^N} f(U_0+w)(U_0+w)\right)\end{aligned}$$

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$$= I_{\mathbb{R}^N,1}(U_0) + \tfrac{1}{2}\langle -\Delta w + w - f'(U_0), w\rangle - \int_{R^N}\left(\frac{1}{3!}f''(U_0)w^3 + \frac{1}{4!}f'''(U_0+\theta w)w^4\right) + \tau\left(\|U_0+w\|^2 - \int_{R^N} f(U_0+w)(U_0+w)\right).$$

Then, combining the equalities above, we have

$$I_{\mathbb{R}^N,1}((1+\tau)(U_0+w)) - I_{\mathbb{R}^N,\varepsilon}(U_0) = \tfrac{1}{2}\langle -\Delta w_1 + w_1 - f'(U_0), w_1\rangle - \int_{R^N}\left(\frac{1}{3!}f''(U_0)w_1w_0^2 + \frac{1}{4!}f'''(U_0)w_0^4\right) + L_0^{-1}\left(\frac{1}{2}\int_{R^N} f''(U_0)U_0w_0^2\right)^2 + O(\|w_0\|^5). \tag{3.17}$$

Here we set $t = \|w_0\|$, $x_w = \|w_1\|/t$,

$$a_w = \frac{1}{2\|w_1\|^2}\langle -\Delta w_1 + w_1 - f'(U_0)w_1, w_1\rangle \quad \text{and} \quad b_w = \frac{1}{3!}\int_{R^N} f''(U_0)\left(\frac{w_1}{\|w_1\|}\right)v_0^2.$$

Then (3.17) is rewritten as

$$\begin{aligned} I_{\mathbb{R}^N,1}(1+\tau)(U_0+w) - I_{\mathbb{R}^N,1}(U_0) &= (a_w x_w^2)t^2 - (b_w x_w)t^3 + ct^4 + O(t^5) \\ &= t^2\left(c\left(t - \frac{b_w x_w}{2c}\right)^2 + \frac{x_w^2(4a_wc - b_w^2)}{4c}\right) + O(t^5), \end{aligned} \tag{3.18}$$

where c is the constant defined above. We note that $a_w = 0$ if and only if $w_1 = 0$, and then we have by of lemma A.1(ii) that if $w_1 = 0$,

$$I_{\mathbb{R}^N,1}(1+\tau)(U_0+w) \geqslant I_{\mathbb{R}^N,1}(U_0) + ct^4 = I_{\mathbb{R}^N,1}(U_0) + c\|w\|^4. \tag{3.19}$$

Now we suppose that $w_1 = sv$ for $s > 0$. Then. by (3.15), it follows that $(b_wx_w)^2 - a_wcx_w^2 = x_w^2(b_w^2 - 4a_wc) > 0$. Then setting $t = b_wx_w/2c$ with t sufficiently small, we have from (3.18) that $I_{\mathbb{R}^N,1}((1+\tau)(U_0+w)) < I_{\mathbb{R}^N,1}(U_0)$. This contradicts (2.10). Then we find (3.14) holds as claimed. Next we assume that there exists $v \in E \cap V_0^{\perp}$ such that

$$\langle -\Delta v + v - f'(U_0)v, v\rangle - \frac{1}{2c}\left(\frac{1}{3!}\int_{R^N} f''(U_0)vv_0^2\right)^2 = 0. \tag{3.20}$$

That is $v \in E$ is the minimizer of the minimization problem

$$\min\left\{\langle -\Delta w + w - f'(U_0)w, w\rangle - \frac{1}{2c}\left(\frac{1}{3!}\int_{R^N} f'(U_0)wv_0^2\right)^2 : w \in E \text{ with } \|w\| = 1\right\}.$$

Noting that $\langle -\Delta w + w - f'(U_0)w, U_0\rangle = 0$ for $w \in E$, we have that

$$-\Delta v + v - f'(U_0)v = \frac{1}{2c}\left(\frac{1}{3!}\int_{R^N} f'(U_0)vv_0^2\right)\left(f'(U_0)v_0^2 - \left(\frac{\int_{R^N} f'(U_0)v_0^2U_0}{\int_{R^N} U_0^2}\right)U_0\right).$$

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Then, from the definitions of E and V_0, we find $\pm v \in E \cap V_0^{\perp}$ is the unique pair satisfying (3.20), as claimed.

Thus we find that there exists $m > 0$ such that

$$\langle -\Delta w + w - f'(U_0)w, w\rangle - \frac{1}{2c}\biggl(\int_{R^N} f''(U_0)wv_0^2\biggr)^2 \geqslant m$$

for all $w \in E \cap \operatorname{span}\{v, v_0\}^{\perp}$ with $\|w\| = 1$. That is $(4a_w c - b_w^2)/4c \geqslant m$ for all $w \in E \cap \{tv\colon t \in \mathbb{R}\}^{\perp}$ with $\|w\| = 1$. Therefore, we find from (3.18) that there exist $d_2 > 0$ and $l_2 > 0$ such that

$$I_{\mathbb{R}^N,1}(U_0 + w) \geqslant I_{\mathbb{R}^N,1}(U_0) + l_2\|w\|^4 \tag{3.21}$$

for all $w \in E \cap \operatorname{span}\{v, v_0\}^{\perp}$ such that $\|w\| \leqslant d_2$ and $\|w_1\| \leqslant L_3\|w_0\|^2$.

It follows from (3.18) and the definitions of v and v_0 that there exist $d_3 > 0$ and $l_3 > 0$ such that

$$I_{\mathbb{R}^N,1}(U_0 + w) \geqslant I_{\mathbb{R}^N,1}(U_0) + l_3\|w\|^4 \tag{3.22}$$

for $w \in \operatorname{span}\{v, v_0\}$ such that $\|w\| \leqslant d_3$ and $\|w_1\| \leqslant L_3\|w_0\|$. From (3.21) and (3.22), it is easy to see, using (3.18) again, that there exists $d_4 > 0$ and $l_4 > 0$ such that

$$I_{\mathbb{R}^N,1}(U_0 + w) \geqslant I_{\mathbb{R}^N,1}(U_0) + l_4\|w\|^4 \tag{3.23}$$

for $w \in E$ such that $\|w\| \leqslant d_3$ and $\|w_1\| \leqslant L_3\|w_0\|^2$. Then, combining (3.13) and (3.23), we obtain the assertion. □

Here we put

$$E_{\varepsilon,x} = \operatorname{span}\biggl\{\frac{\partial W_{\varepsilon,x}}{\partial x_i}\colon 1 \leqslant i \leqslant N\biggr\} \quad \text{for each } x \in M.$$

Then we have the following.

Lemma 3.5. *There exists $\delta_2 \in (0, \delta_1)$ and $m_2 > 0$ such that for $(\varepsilon, x) \in (0, \varepsilon_0) \times M$ and $w \in E_{\varepsilon,x}^{\perp} \cap \mathcal{N}_\varepsilon$ with $\|w\|_\varepsilon < \delta_2$, there exists $w' \in (E_{\varepsilon,x}^{\perp} \cap \mathcal{N}_\varepsilon) \cap H_0^1(B_g(x, \varepsilon^\alpha))$ such that*

$$\|w'\|_\varepsilon^2 \leqslant m_2\|w\|_\varepsilon^2, \tag{3.24}$$

and satisfies one of the following conditions:

(i) $I_\varepsilon(W_{\varepsilon,x} + w') < I_\varepsilon(W_{\varepsilon,x} + w)$;

(ii) $\|w - w'\|_\varepsilon \leqslant m_2\varepsilon^2\|w\|_\varepsilon$ *and* $I_\varepsilon(W_{\varepsilon,x} + w') \leqslant I_\varepsilon(W_{\varepsilon,x} + w) + m_2\varepsilon^2\|w\|_\varepsilon$.

Proof. Let $(\varepsilon, x) \in (0, \varepsilon_0) \times M$. By (f2) and Sobolev's embedding theorem, we can choose $\delta_2 \in (0, \delta_1)$ sufficiently small that

$$\frac{1}{\varepsilon^N}\int_M f(w)w \,\mathrm{d}\mu_g \leqslant \frac{1}{4\varepsilon^N}\int_M |w|_\varepsilon^2 \,\mathrm{d}\mu_g \tag{3.25}$$

and

$$\frac{1}{\varepsilon^N}\int_{M\setminus B_g(x,a)} f(w)w \,\mathrm{d}\mu_g \leqslant \frac{1}{4\varepsilon^N}\int_{M\setminus B_g(x,a)} |w|_\varepsilon^2 \,\mathrm{d}\mu_g \tag{3.26}$$

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for all $a \in (\varepsilon, \frac{1}{2})$ and $w \in H$ with $\|w\|_\varepsilon < \delta_2$. Recall that $W_{\varepsilon,x} = 0$ on $M \backslash B_g(x, \frac{1}{2}\varepsilon^\alpha)$. Let $w \in E_{\varepsilon,x}^\perp \cap \mathcal{N}_\varepsilon$ with $\|w\|_\varepsilon \leqslant \delta_2$. We set

$$n = \left[\frac{1}{20\varepsilon^{1-\alpha}}\right].$$

Here $[t]$ denotes the minimal natural number greater than or equals to $t \in \mathbb{R}^+$. Then, noting that

$$\int_M f(w)w \leqslant \tfrac{1}{4}\|w\|_\varepsilon^2,$$

there exists $k_1 \in \mathbb{N}$ such that $10n \leqslant k_1 \leqslant 11n - 1$ and

$$\frac{1}{\varepsilon^N}\int_{B_g(x,\varepsilon(k_1+1))\backslash B_g(x,\varepsilon k_1)} (|w|_\varepsilon^2 + f(w)w)\,\mathrm{d}\mu_g \leqslant \tfrac{5}{4}\cdot 20\varepsilon^{1-\alpha}\|w\|_\varepsilon^2 = 25\varepsilon^{1-\alpha}\|w\|_\varepsilon^2. \tag{3.27}$$

Let φ be the function defined at the beginning of this section. We set

$$\varphi_1(z) = \varphi(\varepsilon^{-1}\cdot \mathrm{dist}(B_g(x, \varepsilon k_1), z)) \quad \text{for } z \in M.$$

where $\mathrm{dist}(A, x)$ stands for the distance of $x \in M$ from $A \subset M$ with respect to the Riemannian metric g. Then $\varphi_1(z) = 0$ on $M \setminus B_g(x, \varepsilon(k_1 + 1))$, $\varphi_1(z) = 1$ on $B_g(x, \varepsilon k_1)$ and $|\nabla_g \varphi_i| \leqslant C_0\varepsilon^{-1}$ on M for some $C_0 > 0$. We also set

$$w_1(z) = \varphi_1(z)w(z) \quad \text{on } M. \tag{3.28}$$

Then since $w_1 = w$ on $\operatorname{supp} \partial W_{\varepsilon,x}/\partial x_i$ for $1 \leqslant i \leqslant N$, we find that $w_1 \in E_{\varepsilon,x}^\perp$. On the other hand, we have

$$\begin{aligned}
\int_{B_g(x,\varepsilon(k_1+1))} &|\nabla_g(W_{\varepsilon,x} + w_1)|^2\,\mathrm{d}\mu_g \\
&= \int_{B_g(x,\varepsilon(k_1+1))} |\nabla_g(W_{\varepsilon,x} + w) + (\nabla_g\varphi_1)w + (\varphi_1 - 1)\nabla_g w|^2\,\mathrm{d}\mu_g \\
&= \int_{B_g(x,\varepsilon(k_1+1))} |\nabla_g(W_{\varepsilon,x} + w)|^2\,\mathrm{d}\mu_g \\
&\quad + \int_{B_g(x,\varepsilon(k_1+1))\backslash B_g(x,\varepsilon k_1)} |(\nabla_g\varphi_1)w + (\varphi_1 - 1)\nabla_g w|^2\,\mathrm{d}\mu_g \\
&\quad + 2\int_{B_g(x,\varepsilon(k_1+1))\backslash B_g(x,\varepsilon k_1)} \nabla_g w\cdot((\nabla_g\varphi_1)w + (\varphi_1 - 1)\nabla_g w)\,\mathrm{d}\mu_g.
\end{aligned}$$

Noting that

$$\varepsilon^2\int_{B_g(x,\varepsilon(k_1+1))\backslash B_g(x,\varepsilon k_1)} |(\nabla_g\varphi_1)w|^2\,\mathrm{d}\mu_g \leqslant 25\varepsilon^{1-\alpha}C_0^2\|w\|_\varepsilon^2$$

and $|\nabla_g\varphi_1| \leqslant C_0\varepsilon^{-1}$, we have that, for some $C > 0$,

$$\left|\frac{\varepsilon^2}{\varepsilon^N}\int_{B_g(x,\varepsilon(k_1+1))\backslash B_g(x,\varepsilon k_1)} \nabla_g w\cdot((\nabla_g\varphi_1)w + (\varphi_1 - 1)\nabla_g w)\right| \leqslant C\varepsilon^{1-\alpha}\|w\|_\varepsilon^2$$

https://doi.org/10.1017/S0308210510000892 Published online by Cambridge University Press

and

$$\frac{\varepsilon^2}{\varepsilon^N}\int_{B_g(x,\varepsilon(k_1+1))\setminus B_g(x,\varepsilon k_1)}|(\nabla_g\varphi_1)w+(\varphi_1-1)\nabla_g w|^2\leqslant C\varepsilon^{1-\alpha}\|w\|_\varepsilon^2.$$

Then we find that there exists $C_1>0$ such that

$$\left|\frac{1}{\varepsilon^N}\int_{B_g(x,\varepsilon(k_1+1))}|W_{\varepsilon,x}+w_1|_\varepsilon^2\,\mathrm{d}\mu_g-\frac{1}{\varepsilon^N}\int_{B_g(x,\varepsilon(k_1+1))}|W_{\varepsilon,x}+w|_\varepsilon^2\,\mathrm{d}\mu_g\right| \leqslant C_1\varepsilon^{1-\alpha}\|w\|_\varepsilon^2. \quad (3.29)$$

Similarly, we have from (3.27) and (f1) that, by choosing C_1 sufficiently large,

$$\frac{1}{\varepsilon^N}\int_{B_g(x,\varepsilon(k_1+1))}F(W_{\varepsilon,x}+w_1)\,\mathrm{d}\mu_g \geqslant\frac{1}{\varepsilon^N}\int_{B_g(x,\varepsilon(k_1+1))}F(W_{\varepsilon,x}+w)\,\mathrm{d}\mu_g-\tfrac{1}{2}C_1\varepsilon^{1-\alpha}\|w\|_\varepsilon^2.$$

Then we obtain

$$\begin{aligned}\frac{1}{\varepsilon^N}&\int_{B_g(x,\varepsilon(k_1+1))}(\tfrac12|W_{\varepsilon,x}+w_1|_\varepsilon^2-F(W_{\varepsilon,x}+w_1))\,\mathrm{d}\mu_g\\&\leqslant\frac{1}{\varepsilon^N}\int_{B_g(x,\varepsilon(k_1+1))}(\tfrac12|W_{\varepsilon,x}+w|_\varepsilon^2-F(W_{\varepsilon,x}+w))\,\mathrm{d}\mu_g+C_1\varepsilon^{1-\alpha}\|w\|_\varepsilon^2.\end{aligned} \quad (3.30)$$

Then, since $W_{\varepsilon,x}=0$ on $M\setminus B_g(x,\varepsilon k_1)$ and $F(t)\leqslant f(t)t$ for $t\in\mathbb{R}$, we obtain from (3.26) and (3.30) that

$$\begin{aligned}I_\varepsilon&(W_{\varepsilon,x}+w_1)\\&<I_\varepsilon(W_{\varepsilon,x}+w)-\left(\frac{1}{\varepsilon^N}\int_{M\setminus B_g(x,\varepsilon(k_1+1))}(\tfrac12|w|_\varepsilon^2-F(w))\,\mathrm{d}\mu_g-C_1\varepsilon^{1-\alpha}\|w\|_\varepsilon^2\right)\\&\leqslant I_\varepsilon(W_{\varepsilon,x}+w)-\left(\frac{1}{4\varepsilon^N}\int_{M\setminus B_g(x,\varepsilon(k_1+1))}|w|_\varepsilon^2\,\mathrm{d}\mu_g-C_1\varepsilon^{1-\alpha}\|w\|_\varepsilon^2\right).\end{aligned} \quad (3.31)$$

Here we consider two cases.

Case 1. Suppose that

$$\frac{1}{4\varepsilon^N}\int_{M\setminus B_g(x,\varepsilon(k_1+1))}|w|_\varepsilon^2\,\mathrm{d}\mu_g\geqslant 2C_1\varepsilon^{1-\alpha}\|w\|^2.$$

For simplicity of notation, we set $W=W_{\varepsilon,x}+w_1$. Then we have, from the inequality above, that

$$I_\varepsilon(W)=I_\varepsilon(W_{\varepsilon,x}+w_1)<I_\varepsilon(W_{\varepsilon,x}+w)-\frac{1}{8\varepsilon^N}\int_{M\setminus B_g(x,\varepsilon(k_1+1))}|w|_\varepsilon^2\,\mathrm{d}\mu_g. \quad (3.32)$$

On the other hand, noting that

$$\|W_{\varepsilon,x}+w\|_\varepsilon^2-\frac{1}{\varepsilon^N}\int_M f(W_{\varepsilon,x}+w)(W_{\varepsilon,x}+w)\,\mathrm{d}\mu_g=0,$$

we have by (3.29) that

$$\begin{aligned}\biggl|\|W\|_\varepsilon - \frac{1}{\varepsilon^N}\int_M f(W)W\,\mathrm{d}\mu_g\biggr| &= \biggl|\|W_{\varepsilon,x}+w\|^2_\varepsilon - \|W\|^2_\varepsilon - \frac{1}{\varepsilon^N}\int_M (f(W_{\varepsilon,x}+w)(W_{\varepsilon,x}+w) - f(W)W)\,\mathrm{d}\mu_g\biggr| \\ &\leqslant |\|W_{\varepsilon,x}+w\|^2_\varepsilon - \|W_{\varepsilon,x}+w_1\|^2_\varepsilon| + \frac{1}{\varepsilon^N}\int_{M\setminus B_g(x,\varepsilon k_1)} f(w)(w)\,\mathrm{d}\mu_g \\ &\leqslant C_1\varepsilon^{1-\alpha}\|w\|^2_\varepsilon + \frac{1}{\varepsilon^N}\int_{M\setminus B_g(x,\varepsilon k_1)} f(w)(w)\,\mathrm{d}\mu_g.\end{aligned}$$

Then, again by (3.26) and (3.27), we have that there exists $C>0$ such that

$$\begin{aligned}\biggl|\|W\|^2_\varepsilon - \frac{1}{\varepsilon^N}\int_M f(W)W\,\mathrm{d}\mu_g\biggr| &\leqslant \frac{1}{4\varepsilon^N}\int_{M\setminus B_g(x,\varepsilon(k_1+1))} |w|^2_\varepsilon\,\mathrm{d}\mu_g + C\varepsilon^{1-\alpha}\|w\|^2_\varepsilon \\ &\leqslant \biggl(\frac{2C_1+C}{8C_1}\biggr)\frac{1}{\varepsilon^N}\int_{M\setminus B_g(x,\varepsilon(k_1+1))} |w|^2_\varepsilon. \qquad (3.33)\end{aligned}$$

Now let $\tau\in\mathbb{R}$ such that $(1+\tau)(W_{\varepsilon,x}+w_1)\in\mathcal{N}_\varepsilon$. Note that $(1+\tau)(W_{\varepsilon,x}+w_1)\in E^\perp_{\varepsilon,x}$. We shall see that $I_\varepsilon((1+\tau)(W_{\varepsilon,x}+w_1)) < I_\varepsilon(W_{\varepsilon,x}+w)$. That is, we claim that (i) holds with $w' = \tau W_{\varepsilon,x} + (1+\tau)w_1$. From the definition of τ, we have, by (3.6) and (3.8) with $W = W_{\varepsilon,x}+w_1$, that

$$\begin{aligned}|\tau| &\leqslant \frac{2}{L_0}\biggl(\|W\|^2_\varepsilon - \frac{1}{\varepsilon^N}\int_M f(W)W\,\mathrm{d}\mu_g\biggr) \\ &\leqslant \biggl(\frac{2C_1+C}{8C_1}\biggr)\frac{2}{L_0\varepsilon^N}\int_{M\setminus B_g(x,\varepsilon(k_1+1))} |w|^2_\varepsilon\,\mathrm{d}\mu_g. \qquad (3.34)\end{aligned}$$

Then by (3.32) and (3.9) with $W = W_{\varepsilon,x}+w_1$, we have

$$\begin{aligned}&I_\varepsilon((1+\tau)(W_{\varepsilon,x}+w_1)) \\ &\quad= I_\varepsilon(W) + \tau\biggl(\|W\|^2_\varepsilon - \frac{1}{\varepsilon^N}\int_M f(W)W\,\mathrm{d}\mu_g\biggr) + O(\tau^2) \\ &\quad< I_\varepsilon(W_{\varepsilon,x}+w) - \frac{1}{8\varepsilon^N}\int_{M\setminus B_g(x,\varepsilon(k_1+1))} |w|^2_\varepsilon\,\mathrm{d}\mu_g \\ &\qquad + \frac{2}{L_0}\biggl[\biggl(\frac{2C_1+C}{8C_1}\biggr)\frac{1}{\varepsilon^N}\int_{M\setminus B_g(x,\varepsilon(k_1+1))} |w|^2_\varepsilon\,\mathrm{d}\mu_g\biggr]^2 + O(\tau^2).\end{aligned}$$

Since $\|w_1\|_\varepsilon \to 0$ as $\varepsilon\to 0$, we find that, for $\varepsilon>0$ sufficiently small,

$$I_\varepsilon((1+\tau)(W_{\varepsilon,x}+w_1)) < I_\varepsilon(W_{\varepsilon,x}+w)$$

holds, as claimed.

Case 2. We next assume that

$$\frac{1}{4\varepsilon^N}\int_{M\setminus B_g(x,\varepsilon(k_1+1))} |\nabla_g w|^2_\varepsilon\,\mathrm{d}\mu_g \leqslant 2C_1\varepsilon^{1-\alpha}\|w\|^2_\varepsilon.$$

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Then we can find $k_2 \in \mathbb{N}$ such that $11n \leqslant k \leqslant 12n-1$ and

$$\frac{1}{\varepsilon^N}\int_{B_g(x,\varepsilon(k_2+1))\setminus B_g(x,\varepsilon k_2)}(|\nabla_g w|_\varepsilon^2 + f(w)w)\,\mathrm{d}\mu_g \leqslant 200C_1\varepsilon^{2(1-\alpha)}\|w\|_\varepsilon^2.$$

We set

$$\varphi_2(z)=\varphi(\varepsilon^{-1}\cdot \operatorname{dist}(B_g(x,\varepsilon k_2),z)) \quad \text{and} \quad w_2(z)=\varphi_2(z)w(z) \quad \text{on } M.$$

Then, by the same argument as above, we have that there exists $C_2>0$ such that

$$I_\varepsilon(W_{\varepsilon,x}+w_2) < I_\varepsilon(W_{\varepsilon,x}+w) - \biggl(\frac{1}{4\varepsilon^N}\int_{M\setminus B_g(x,\varepsilon(k_2+1))}|w|_\varepsilon^2\,\mathrm{d}\mu_g - C_2\varepsilon^{2(1-\alpha)}\|w\|_\varepsilon^2\biggr).$$

If

$$\frac{1}{4}\int_{M\setminus B_g(x,\varepsilon(k_2+1))}|\nabla_g w|_\varepsilon^2\,\mathrm{d}\mu_g > 2C_2\varepsilon^{2(1-\alpha)}\|w\|_\varepsilon^2,$$

then we find, for $\tau\in\mathbb{R}$ such that $(1+\tau)(W_{\varepsilon,x}+w_2)\in\mathcal{N}_\varepsilon$, that

$$I_\varepsilon((1+\tau)(W_{\varepsilon,x}+w_2)) < I_\varepsilon(W_{\varepsilon,x}+w),$$

and then (i) holds with $w' = \tau W_{\varepsilon,x} + (1+\tau)w_2$.

Repeating this procedure, we can define positive integers k_i and functions w_i for $i\geqslant 3$. That is, if (i) does not hold, then there exist $C_{10}>0$ and $k_{10}\in\mathbb{N}$ with $19n\leqslant k_{10}\leqslant 20n-1$ such that

$$\frac{1}{4}\int_{M\setminus B_g(x,\varepsilon(k_{10}+1))}|\nabla_g w|_\varepsilon^2\,\mathrm{d}\mu_g \leqslant 2C_{10}\varepsilon^{10(1-\alpha)}\|w\|_\varepsilon^2.$$

We set

$$\varphi_{10}(z)=\varphi(\varepsilon^{-1}\cdot \operatorname{dist}(B_g(x,\varepsilon k_{10}),z)) \quad \text{and} \quad w_{10}(z)=\varphi_{10}(z)w(z) \quad \text{on } M.$$

Then since $10(1-\alpha)>4$ by the assumption, we find that

$$\|w-w_{10}\|_\varepsilon^2 = O(\varepsilon^{10(1-\alpha)}\|w\|_\varepsilon^2) = o(\varepsilon^4\|w\|_\varepsilon^2),$$

and then by a parallel argument as above we find that $|\tau| = O(\varepsilon^2\|w\|_\varepsilon^2)$. That is, by setting $w'=\tau W_{\varepsilon,x}+(1+\tau)w_{10}$, we have that $\|w-w'\|_\varepsilon = o(\varepsilon^2\|w\|_\varepsilon)$ and then, by the definition of I_ε, we can deduce that

$$I_\varepsilon(W_{\varepsilon,x}+w') \leqslant I_\varepsilon(W_{\varepsilon,x}+w) + O(\varepsilon^2\|w\|_\varepsilon).$$

This completes the proof. □

Lemma 3.6. *There exists $\delta_3\in(0,\delta_2)$ such that*

$$I_\varepsilon(W_{\varepsilon,x}+w) \geqslant I_\varepsilon(W_{\varepsilon,x}) + m_3\|w\|_\varepsilon^4 - m_4(\varepsilon^{1+\alpha}\|w\|_\varepsilon + \varepsilon^{2\alpha}\|w\|_\varepsilon^2) \tag{3.35}$$

for all $(\varepsilon,x)\in(0,\varepsilon_0)\times M$ and $w\in E_{\varepsilon,x}^\perp$ such that $W_{\varepsilon,x}+w\in\mathcal{N}_\varepsilon$ and $\|w\|_\varepsilon\leqslant\delta_3$.

Proof. We first note that we can choose $\delta_3 \in (0, \delta_2)$ sufficiently small that, for $w \in H$ with

$$\int_{B_g(x,\varepsilon^\alpha/2)} |w|_\varepsilon^2 \,\mathrm{d}\mu_g < \delta_3,$$

we have

$$\frac{1}{\varepsilon^N}\int_{B_g(x,\varepsilon^\alpha/2)} (w^2 + |w|^3)\,\mathrm{d}\mu_g < \frac{2}{\varepsilon^N}\int_{B_g(x,\varepsilon^\alpha/2)} |w|_\varepsilon^2 \,\mathrm{d}\mu_g$$

for all $(\varepsilon, x) \in (0, \varepsilon_0) \times M$. On the other hand, we have from (3.2) that, for some $m > 0$,

$$\frac{1}{\varepsilon^N}\int_{B_g(x,\varepsilon^\alpha/2)} f'(W_{\varepsilon,x} + \theta w)w^2 \,\mathrm{d}\mu_g \leqslant \frac{m}{\varepsilon^N}\int_{B_g(x,\varepsilon^\alpha/2)} (w^2 + |w|^3)\,\mathrm{d}\mu_g$$

for $\theta \in [0, 1]$ and $w \in H$. Therefore, combining the inequalities above, we have that

$$\frac{1}{\varepsilon^N}\int_{B_g(x,\varepsilon^\alpha/2)} f'(W_{\varepsilon,x} + \theta w)w^2 \,\mathrm{d}\mu_g \leqslant \frac{2m}{\varepsilon^N}\int_{B_g(x,\varepsilon^\alpha/2)} |w|_\varepsilon^2 \,\mathrm{d}\mu_g \tag{3.36}$$

for all $(\varepsilon, x) \in (0, \varepsilon_0) \times M$ and $w \in H$ with

$$\int_{B_g(x,\varepsilon^\alpha/2)} |w|_\varepsilon^2 \,\mathrm{d}\mu_g < \delta_3.$$

Now let $(\varepsilon, x) \in (0, \varepsilon_0) \times M$. From lemma A.4 and the inequality above, we have that

$$\begin{aligned}
\int_{B_g(x,\varepsilon^\alpha/2)} &[(\tfrac{1}{2}|W_{\varepsilon,x} + w|_\varepsilon^2 - F(W_{\varepsilon,x} + w)) - (\tfrac{1}{2}|W_{\varepsilon,x}|^2 - F(W_{\varepsilon,x}))]\,\mathrm{d}\mu_g \\
&\geqslant \int_{B_g(x,\varepsilon^\alpha/2)} (-\Delta_g W_{\varepsilon,x} + W_{\varepsilon,x} - f(W_{\varepsilon,x}))w \,\mathrm{d}\mu_g \\
&\quad + \frac{1}{2}\int_{B_g(x,\varepsilon^\alpha/2)} (|w|_\varepsilon^2 - f'(W_{\varepsilon,x} + \theta)w^2)\,\mathrm{d}\mu_g \\
&\geqslant -\bar{m}\varepsilon^{1-\alpha}\|w\|_\varepsilon - \frac{2m}{\varepsilon^N}\int_{B_g(x,\varepsilon^\alpha/2)} |w|_\varepsilon^2 \,\mathrm{d}\mu_g.
\end{aligned} \tag{3.37}$$

On the other hand, we have by (3.26) that

$$\begin{aligned}
\int_{M\setminus B_g(x,\varepsilon^\alpha/2)} &[(\tfrac{1}{2}|W_{\varepsilon,x} + w|_\varepsilon^2 - F(W_{\varepsilon,x} + w)) - (\tfrac{1}{2}|W_{\varepsilon,x}|^2 - F(W_{\varepsilon,x}))]\,\mathrm{d}\mu_g \\
&= \int_{M\setminus B_g(x,\varepsilon^\alpha/2)} (\tfrac{1}{2}|w|_\varepsilon^2 - F(w))\,\mathrm{d}\mu_g \\
&\geqslant \frac{1}{4}\int_{M\setminus B_g(x,\varepsilon^\alpha/2)} |w|_\varepsilon^2 \,\mathrm{d}\mu_g.
\end{aligned} \tag{3.38}$$

Now suppose that

$$2m\int_{B_g(x,\varepsilon^\alpha/2)} |w|_\varepsilon^2 \,\mathrm{d}\mu_g \leqslant \frac{1}{8}\int_{M\setminus B_g(x,\varepsilon^\alpha/2)} |w|_\varepsilon^2 \,\mathrm{d}\mu_g.$$

Then from (3.37) and (3.38), we have

$$\begin{aligned} I_\varepsilon(W_{\varepsilon,x}+w) &\geqslant I_\varepsilon(W_{\varepsilon,x}) + \frac{1}{8}\int_{M\setminus B_g(x,\varepsilon^\alpha/2)} |w|_\varepsilon^2 \,\mathrm{d}\mu_g - \bar{m}\varepsilon^{1+\alpha}\|w\|_\varepsilon \\ &\geqslant I_\varepsilon(W_{\varepsilon,x}) + m'\|w\|_\varepsilon^2 - \bar{m}\varepsilon^{1+\alpha}\|w\|_\varepsilon \\ &\geqslant I_\varepsilon(W_{\varepsilon,x}) + m'\|w\|_\varepsilon^4 - \bar{m}\varepsilon^{1+\alpha}\|w\|_\varepsilon, \end{aligned}$$

where $m' = 2m/(16m+1)$. That is (3.35) holds for some m_3 and m_4.

We next assume that

$$2m\int_{B_g(x,\varepsilon^\alpha/2)} |w|_\varepsilon^2 \,\mathrm{d}\mu_g \geqslant \frac{1}{8}\int_{M\setminus B_g(x,\varepsilon^\alpha/2)} |w|_\varepsilon^2 \,\mathrm{d}\mu_g. \tag{3.39}$$

Then, by lemma 3.5, there exists $w' \in (E_{\varepsilon,x}^\perp \cap \mathcal{N}_\varepsilon) \cap H_0^1(B_g(x,\varepsilon^\alpha))$ such that (i) or (ii) of lemma 3.5 is satisfied.

We first consider the case that (i) holds. From the proof of lemma 3.5, we have that w' has the form $w' = \tau W_{\varepsilon,x} + (1+\tau)w_i$, $1 \leqslant i \leqslant 10$, where $w_i = w$ on $B_g(x, \frac{1}{2}\varepsilon^\alpha)$ and $\tau \in \mathbb{R}$. From (3.34) with k_1 replaced by k_i, we have that $|\tau| = O(\|w\|_\varepsilon^2)$. Then from (3.39), by choosing δ_3 sufficiently small, we have that there exists $d_1 > 0$ such that $\|w'\|_\varepsilon \geqslant d_1\|w\|_\varepsilon$. Here we set

$$H(v) = \frac{1}{v}(F(W_{\varepsilon,x}+v) - F(W_{\varepsilon,x})) - f(W_{\varepsilon,x}) \quad \text{for } (\varepsilon, x) \in (0,\varepsilon_0)\times M \text{ and } v \in H.$$

We also set

$$\tilde{H}(v) = \frac{1}{v}(F(W_\varepsilon+v) - F(W_\varepsilon)) - f(W_\varepsilon) \quad \text{for } v \in H^1(\mathbb{R}^N).$$

Then we have $H(v) = \tilde{H}(\tilde{v})$ for $v \in H_0^1(B_g(x,1))$, where $\tilde{v}(z) = v(\exp_x^{-1}(z))$ for $z \in B_1(0)$. For each $v \in H$ with $\|v\|_\varepsilon$ sufficiently small, we have that there exists $\theta \in (0,1)$ such that

$$\langle H(w), v\rangle_\varepsilon = \langle f'(W_{\varepsilon,x}+\theta w)w, v\rangle_\varepsilon.$$

Then noting that (3.36) holds with $M \setminus B_g(x,\varepsilon^\alpha/2)$ replaced by M, we find that there exists $C_h > 0$ such that

$$\langle H(w), v\rangle_\varepsilon \leqslant C_h(|w|_{2,\varepsilon} + |w|_{3,\varepsilon}^2)|v|_{3,\varepsilon} \quad \text{for } v, w \in H. \tag{3.40}$$

That is $H(w) \in L^{3/2}(B_g(x,\varepsilon^\alpha/2))$ and $|H(w)|_{3/2} \leqslant C_h(|w|_{2,\varepsilon} + |w|_{3,\varepsilon}^2)$. Then noting that $\|w\|_\varepsilon$ is small, we have by lemma A.5 that

$$\langle H(w'), w'\rangle_\varepsilon - \langle \tilde{H}(w'), w'\rangle_{\mathbb{R}^N,\varepsilon} = O(\varepsilon^{2\alpha}(|w|_{2,\varepsilon} + |w|_{3,\varepsilon}^2)|w|_{3,\varepsilon}) = O(\varepsilon^{2\alpha}\|w\|_\varepsilon^2). \tag{3.41}$$

From the definition of H and I_ε, we have

$$\begin{aligned} I_\varepsilon(W_{\varepsilon,x}+w') - I_\varepsilon(W_{\varepsilon,x}) &= \langle -\Delta_g W_{\varepsilon,x} + W_{\varepsilon,x} - f(W_{\varepsilon,x}), w'\rangle_\varepsilon + \langle H(w'), w'\rangle_\varepsilon \\ &= O(\varepsilon^{1+\alpha}\|w\|_\varepsilon) + \langle H(w'), w'\rangle_\varepsilon. \end{aligned}$$

Then by (3.41), we have

$$I_\varepsilon(W_{\varepsilon,x}+w') - I_\varepsilon(W_{\varepsilon,x}) = O(\varepsilon^{1+\alpha}\|w\|_\varepsilon) + O(\varepsilon^{2\alpha}\|w'\|_\varepsilon^2) + \langle \tilde{H}(w'), w'\rangle_{\mathbb{R}^N,\varepsilon}, \tag{3.42}$$

where we identify w' with the function $w'(\exp_x^{-1}(\cdot))$ on $B_{\varepsilon^\alpha}(0)$. By lemma A.5, we have that there exists $v' \in H_0^1(B_{\varepsilon^\alpha}(0)) \cap \mathcal{N}_{\mathbb{R}^N,\varepsilon}$ and $\|v' - w'\|_{\mathbb{R}^N,\varepsilon} = O(\varepsilon^{2\alpha}\|w'\|_\varepsilon)$. Then we have

$$|\langle \tilde{H}(v'), v'\rangle_{\mathbb{R}^N,\varepsilon} - \langle \tilde{H}(w'), w'\rangle_{\mathbb{R}^N,\varepsilon}| = O(\varepsilon^{2\alpha}\|w'\|_\varepsilon^2), \tag{3.43}$$

On the other hand, we have, by lemma 3.4,

$$\begin{aligned} m_1\|v'\|_{\mathbb{R}^N,\varepsilon}^4 &\leqslant I_{\mathbb{R}^N,\varepsilon}(W_\varepsilon + v') - I_{\mathbb{R}^N,\varepsilon}(W_\varepsilon) \\ &= \tfrac{1}{2}\langle -\Delta W_\varepsilon + W_\varepsilon - f(W_\varepsilon), v'\rangle_{\mathbb{R}^N,\varepsilon} + \langle \tilde{H}(v'), v'\rangle_{\mathbb{R}^N,\varepsilon} \\ &= O\biggl(\exp\biggl(-\frac{1}{8\varepsilon^{1-\alpha}}\biggr)\|v'\|_{\mathbb{R}^N,\varepsilon}\biggr) + \langle H(v'), v'\rangle_{\mathbb{R}^N,\varepsilon}. \end{aligned} \tag{3.44}$$

Then combining (3.42)–(3.44), assuming that ε is sufficiently small, we obtain that

$$I_\varepsilon(W_{\varepsilon,x} + w') - I_\varepsilon(W_{\varepsilon,x}) \geqslant m_1\|v'\|_{\mathbb{R}^N,\varepsilon}^4 + O(\varepsilon^{1+\alpha}\|w\|_\varepsilon) + O(\varepsilon^{2\alpha}\|w'\|_\varepsilon^2).$$

Therefore,

$$I_\varepsilon(W_{\varepsilon,x}+w) \geqslant I_\varepsilon(W_{\varepsilon,x}+w') \geqslant I_\varepsilon(W_{\varepsilon,x})+m_1\|v'\|_{\mathbb{R}^N,\varepsilon}^4+O(\varepsilon^{1+\alpha}\|w\|_\varepsilon)+O(\varepsilon^{2\alpha}\|w'\|_\varepsilon^2).$$

Recalling that $\|w'\|_\varepsilon \geqslant d_1\|w\|_\varepsilon$ and $\|w' - v'\|_{\mathbb{R}^N,\varepsilon} = O(\varepsilon^{2\alpha}\|w'\|_\varepsilon)$, we have

$$I_\varepsilon(W_{\varepsilon,x} + w) \geqslant I_\varepsilon(W_{\varepsilon,x}) + m_1 d_1^4\|w\|_{\mathbb{R}^N,\varepsilon}^4 + O(\varepsilon^{1+\alpha}\|w\|_\varepsilon) + O(\varepsilon^{2\alpha}\|w\|_\varepsilon^2).$$

That is (3.35) holds for some m_3 and m_4. By a parallel argument as above, we find that there exist m_3 and m_4 such that (3.35) holds for w satisfying (3.39) and lemma 3.5(ii). This completes the proof. □

4. Proof of theorem 1.1

For each $L \in \mathcal{K}(\tau)$, we choose open neighbourhoods $U_L^{(1)}, U_L^{(2)}$ and V_L of L satisfying the following conditions:

$$\bar{V}_L \subset U_L^{(1)} \subset \overline{U_L^{(1)}} \subset U_L^{(2)} \quad \text{and} \quad \overline{U_L^{(2)}} \cap \overline{U_{L'}^{(2)}} = \emptyset \quad \text{if } L, L' \in \mathcal{K}(\tau) \quad \text{and} \quad L \neq L'.$$

Then for $L \in \mathcal{K}_C(\tau)$,

$$H_*((-\tau)^{c_L} \cap V_L, ((-\tau)^{c_L} \setminus L) \cap V_L) \cong H_*((-\tau)^{c_L} \cap U_L^{(2)}, ((-\tau)^{c_L} \setminus L) \cap U_L^{(2)}) \neq \{0\}.$$

Since M is compact, we have that

$$m_d = \min_{x \in M \setminus \mathcal{V}_K} \biggl\{ \max \biggl|\frac{\partial \tau(x)}{\partial x_i}\biggr| : 1 \leqslant i \leqslant N \biggr\} > 0, \tag{4.1}$$

where $\mathcal{V}_K = \bigcup\{V_L : L \in \mathcal{K}(\tau)\}$. Then we have the following lemma.

Lemma 4.1. *There exist $a_0 \in (0,1)$ and $\delta_4 \in (0, \delta_3)$ such that, for each $\varepsilon \in (0, \varepsilon_0)$, $x \in M$ and $w \in E_{\varepsilon,x}^\perp$ with $W_{\varepsilon,x} + w \in \mathcal{N}_\varepsilon$ and $\|w\|_\varepsilon \leqslant \delta_4$,*

$$\|Q_{W_{\varepsilon,x}+w} \nabla I_\varepsilon(W_{\varepsilon,x} + w)\|_\varepsilon \geqslant a_0\|\nabla I_\varepsilon(W_{\varepsilon,x} + w)\|_\varepsilon,$$

where Q_v is the projection from H to $T_v\mathcal{N}_\varepsilon$ for each $v \in \mathcal{N}_\varepsilon$.

Proof. Since U_0 is transversal to $\mathcal{N}_{\mathbb{R}^N,1}$, we have that there exists $a \in (0,1)$ such that

$$\langle\!\langle U_0, v\rangle\!\rangle_{\mathbb{R}^N,1} \leqslant a\|U_0\|_{\mathbb{R}^N,1}\|v\|_{\mathbb{R}^N,1} \quad \text{for all } v \in T_{U_0}\mathcal{N}_{\mathbb{R}^N,1}. \tag{4.2}$$

Then, for any $\varepsilon > 0$,

$$\langle\!\langle U_\varepsilon, v\rangle\!\rangle_{\mathbb{R}^N,\varepsilon} \leqslant a\|U_\varepsilon\|_{\mathbb{R}^N,\varepsilon}\|v\|_{\mathbb{R}^N,\varepsilon} \quad \text{for all } v \in T_{U_\varepsilon}\mathcal{N}_{\mathbb{R}^N,\varepsilon}.$$

Let $a_1, a_2 \in (0,1)$ such that $a < a_1 < a_2$. Since (2.14) holds, by the definition of W_ε, assuming ε_0 is sufficiently small, we have that

$$\langle\!\langle W_\varepsilon, v\rangle\!\rangle_{\mathbb{R}^N,\varepsilon} \leqslant a_1\|W_\varepsilon\|_{\mathbb{R}^N,\varepsilon}\|v\|_{\mathbb{R}^N,\varepsilon} \quad \text{for all } v \in T_{\mathcal{N}(W_\varepsilon)}\mathcal{N}_{\mathbb{R}^N,\varepsilon}.$$

Then recalling that $\operatorname{supp} W_{\varepsilon,x} \subset B_g(x, \varepsilon^\alpha/2)$ for $x \in M$ and $(g_x)_{ij}(\varepsilon^\alpha z) \to 1$, as $\varepsilon \to 0$ uniformly on $B_1(0)$, again assuming that ε_0 is sufficiently small, for each $\varepsilon \in (0, \varepsilon_0)$ and $x \in M$, we have that

$$\langle\!\langle W_\varepsilon, v\rangle\!\rangle_\varepsilon \leqslant a_2\|W_{\varepsilon,x}\|_\varepsilon\|v\|_\varepsilon \quad \text{for all } v \in T_{\mathcal{N}_\varepsilon(W_{\varepsilon,x})}\mathcal{N}_\varepsilon. \tag{4.3}$$

Then it is easy to verify that there exists $\delta_4 \in (0, \delta_3)$ and $a_3 \in (a_2, 1)$ such that for each $\varepsilon \in (0, \varepsilon_0)$ and $w \in H$ such that $\|w\|_\varepsilon \leqslant \delta_4$ and $W_{\varepsilon,x} + w \in \mathcal{N}_\varepsilon$,

$$\langle\!\langle W_{\varepsilon,x} + w, v\rangle\!\rangle_\varepsilon \leqslant a_3\|W_{\varepsilon,x} + w\|_\varepsilon\|v\|_\varepsilon \quad \text{for all } v \in T_{\mathcal{N}_\varepsilon(W_{\varepsilon,x}+w)}\mathcal{N}_\varepsilon. \tag{4.4}$$

By definition,

$$\langle\!\langle \nabla I_\varepsilon(\mathcal{N}_\varepsilon(W_{\varepsilon,x} + w)), W_{\varepsilon,x} + w\rangle\!\rangle_\varepsilon = 0$$

for each $w \in H$, that is $\nabla I_\varepsilon(\mathcal{N}_\varepsilon(W_{\varepsilon,x} + w))$ is orthogonal to $W_{\varepsilon,x} + w$ for $(\varepsilon, x) \in (0, \varepsilon_0) \times M$ and $w \in H$. Then the assertion follows from (4.4). □

Lemma 4.2. *There exists $m_5 > 0$ such that for $(\varepsilon, x) \in (0, \varepsilon_0) \times (M \setminus \mathcal{V}_K)$ and $w \in E^\perp_{\varepsilon,x}$ with $\|w\|_\varepsilon \leqslant \delta_4$,*

$$\|Q_{\varepsilon,x,S}\nabla I_\varepsilon(W_{\varepsilon,x})\|_\varepsilon \geqslant m_5\varepsilon^2,$$

where $Q_{\varepsilon,x,S}$ is the projection from H to $T_{W_{\varepsilon,x}}\tilde{S}_\varepsilon$ for each $x \in M$.

Proof. Let $(\varepsilon, x) \in (0, \varepsilon_0) \times (M \setminus \mathcal{V}_K)$. Then noting that

$$\frac{\partial W_{\varepsilon,x}}{\partial x_i} = \frac{\mathrm{d}}{\mathrm{d}t} W_{\varepsilon,x+tx_i}\bigg|_{t=0} \quad \text{for } x \in M \text{ and } 1 \leqslant i \leqslant N,$$

we have by (2.4), (2.6) and (2.14) that

$$\begin{aligned}
&\frac{\partial}{\partial x_i} I_\varepsilon(W_{\varepsilon,x}) \\
&\quad = \left\langle -\varepsilon^2\Delta W + W - f(W), \frac{\partial W_{\varepsilon,x}}{\partial x_i}\right\rangle_\varepsilon \\
&\quad = \frac{1}{\varepsilon^N}\int_{S^{N-1}} \mathrm{d}\zeta \int_0^1 (\tfrac{1}{2}(\varepsilon^2|\nabla W_\varepsilon|^2 + |W_\varepsilon|^2) - F(W_\varepsilon))\frac{\partial \eta_x}{\partial x_i} r^{N-1}\,\mathrm{d}r
\end{aligned}$$

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$$
\begin{aligned}
&= \varpi \int_0^{1/\varepsilon} (\tfrac{1}{2}(|\nabla(\varphi_1 U_0)|^2 + \varphi_1^2|U_0|^2) - F(\psi U_0))\biggl(\frac{\varepsilon^2}{3} y \frac{\partial \tau}{\partial x_i} + O(\varepsilon^2 y^2)\biggr) y^{N-1}\,\mathrm{d}y \\
&= \frac{\varepsilon^2\varpi}{3}\int_0^{\infty} (\tfrac{1}{2}(|\nabla U_0|^2 + |U_0|^2) - F(U_0))\frac{\partial \tau}{\partial x_i} y^N\,\mathrm{d}y \\
&\quad + \frac{\varepsilon^2\varpi}{3}\int_{1/4\varepsilon^{1-\alpha}}^{\infty} (\tfrac{1}{2}(|\nabla(\varphi_1 U_0)|^2 - |\nabla U_0|^2 \\
&\qquad\qquad + (\varphi_1^2 - 1)|U_0|^2) - F(\psi U_0) + F(U_0))\frac{\partial \tau}{\partial x_i} y^N\,\mathrm{d}y + O(\varepsilon^3) \\
&= \frac{\varepsilon^2\varpi}{3} A_2 \frac{\partial \tau(x)}{\partial x_i} + O(\varepsilon^3)
\end{aligned}
$$

where

$$
A_2 = \int_0^{\infty} (\tfrac{1}{2}(|\nabla U_0|^2 + |U_0|^2) - F(U_0)) r^N\,\mathrm{d}r.
$$

Then by (4.1), the assertion follows. □

Here we define a semiflow on $\mathcal{N}_\varepsilon$ associated with the functional I_ε. For each $\varepsilon \in (0, \varepsilon_0)$, we can define, by a standard argument, a pseudo-gradient vector field $V_\varepsilon \colon \mathcal{N}_\varepsilon \to T\mathcal{N}_\varepsilon$ satisfying the following conditions:

(i) for each $v \in \mathcal{N}_\varepsilon$, $V_\varepsilon(v) \in T_v\mathcal{N}_\varepsilon$;

(ii) $\|V_\varepsilon(v)\|_\varepsilon \leqslant 2\|Q_v \nabla I_\varepsilon(v)\|_\varepsilon$ for $v \in \mathcal{N}_\varepsilon$;

(iii) $\langle\!\langle V_\varepsilon(v), \nabla I_\varepsilon(v)\rangle\!\rangle_\varepsilon \geqslant \frac{1}{2}\|Q_v \nabla I_\varepsilon(v)\|_\varepsilon^2$ for $v \in \mathcal{N}_\varepsilon$.

We define a semiflow $\rho_\varepsilon \colon [0, \infty) \times \mathcal{N}_\varepsilon \to \mathcal{N}_\varepsilon$ by

$$
\begin{aligned}
\frac{\mathrm{d}\rho_\varepsilon(t, v)}{\mathrm{d}t} &= -V_\varepsilon(\rho_\varepsilon(t, v)) \quad \text{for } t > 0, \\
\rho_\varepsilon(0, v) &= v \in \mathcal{N}_\varepsilon.
\end{aligned}
$$

Similarly, we can construct a pseudo-gradient vector field $V_{\varepsilon,S} \colon \tilde{S}_\varepsilon \to T\tilde{S}_\varepsilon$ satisfying (i)–(iii) with Q_v replaced by $Q_{\varepsilon,x,S}$. Then we define a semiflow $\rho_{\varepsilon,S} \colon [0, \infty) \times \tilde{S}_\varepsilon \to \tilde{S}_\varepsilon$ in the same way as above, with V_ε replaced by $V_{\varepsilon,S}$.

Let $l_0 > 0$ such that

$$
m_3 l_0 > 4A \max\biggl\{\biggl|\frac{\partial \tau(x)}{\partial x_i}\biggr| : x \in M,\ 1 \leqslant i \leqslant N\biggr\},
$$

where A is the constant defined in lemma 3.2. By lemmas 3.2 and 3.3 we can choose $\varepsilon_1 \in (0, \varepsilon_0)$ so small that

$$
I_\varepsilon(\mathcal{N}_\varepsilon(W_{\varepsilon,x})) \leqslant c_0 + 2A\varepsilon^2|\tau(x)| \quad \text{for all } \varepsilon \in (0, \varepsilon_1) \quad \text{and} \quad x \in M. \tag{4.5}
$$

We may assume ε_1 is so small that

$$
m_3\|w\|_\varepsilon^4 - m_4(\varepsilon^{1+\alpha}\|w\|_\varepsilon + \varepsilon^{2\alpha}\|w\|_\varepsilon^2) \geqslant \tfrac{1}{2} m_3\|w\|_\varepsilon^4 \tag{4.6}
$$

for $\varepsilon \in (0, \varepsilon_1)$ and $w \in H$ with $\|w\|_\varepsilon = l_0\sqrt{\varepsilon}$, where m_3 and m_4 are constants given in lemma 3.6. Then, from lemma 3.6 and the inequalities above, we have that

$$
I_\varepsilon(\mathcal{N}_\varepsilon(W_{\varepsilon,x})) < c_0 + 2A\varepsilon^2|\tau(x)| < I_\varepsilon(W_{\varepsilon,x} + w) \tag{4.7}
$$

for $(\varepsilon, x) \in (0, \varepsilon_1) \times M$ and $w \in \partial D_{l_0\sqrt{\varepsilon}}(\tilde{S}_\varepsilon) \cap \mathcal{N}_\varepsilon$. By the definition of semiflows ρ_ε, we have $I_\varepsilon(\rho_\varepsilon(t, \mathcal{N}_\varepsilon(W_{\varepsilon,x}))) \leqslant I_\varepsilon(\mathcal{N}_\varepsilon(W_{\varepsilon,x}))$ for all $(t, x) \in [0, \infty) \times M$. Then we find that

$$\rho_\varepsilon(t, \tilde{S}_\varepsilon) \subset D_{d_0\sqrt{\varepsilon}}(\tilde{S}_\varepsilon) \quad \text{for all } (\varepsilon, t) \in (0, \varepsilon_1) \times [0, \infty). \tag{4.8}$$

We set

$$\Omega_\varepsilon(L) = \{v \in D_{l_0\sqrt{\varepsilon}}(\tilde{S}_\varepsilon) \colon P_\varepsilon v \in V_L\}$$

and

$$\Omega_{i,\varepsilon}(L) = \{v \in D_{l_0\sqrt{\varepsilon}}(\tilde{S}_\varepsilon) \colon P_\varepsilon v \in U_L^{(i)}\}$$

for $(\varepsilon, L) \in (0, \varepsilon_1) \times \mathcal{K}(\tau)$ and $i = 1, 2$. For each $L \in \mathcal{K}_C(\tau)$ and $\varepsilon \in (0, \varepsilon_1)$, we set

$$d_1 = \inf_{L \in \mathcal{K}(\tau), \varepsilon \in (0,\varepsilon_1)} \inf\{\|W_{\varepsilon,x} - W_{\varepsilon,y}\|_\varepsilon \colon x \in \partial U_L^{(1)} \text{ and } y \in \partial V_L \cup \partial U_L^{(2)}\}.$$

Then by the definition $\Omega_{i,\varepsilon}$, assuming that δ_0 is sufficiently small, we find that $d_1 > 0$.

LEMMA 4.3. *Let $\varepsilon \in (0, \varepsilon_1)$ and $L \in \mathcal{K}_C(\tau)$. Suppose that there exists no critical point of I_ε in $I_\varepsilon^d \cap \Omega_{2,\varepsilon}(L)$, where $d = \sup\{I_\varepsilon(\mathcal{N}(W_{\varepsilon,x})) \colon x \in U_L^{(2)}\}$. Then, for $x \in \Omega_{2,\varepsilon}(L)$, there exist $\tilde{t}_x \in [0, \infty)$ and a semiflow $\tilde{\rho}_\varepsilon \colon \{(t, x) \colon t \in [0, \tilde{t}_x),\ x \in V_L\} \to \Omega_{2,\varepsilon}(L)$ such that, for each $x \in V_L$:*

(a) *$P_\varepsilon\tilde{\rho}_\varepsilon(t, x) \in U_L^{(2)}$ for $t \in [0, \tilde{t}_x)$ and $P_\varepsilon\tilde{\rho}_\varepsilon(\tilde{t}_x, x) \in \partial U_L^{(2)}$;*

(b) *$I_\varepsilon(\tilde{\rho}_\varepsilon(\tilde{t}_x, x)) \leqslant I_\varepsilon(W_{\varepsilon,x}) - m_6\varepsilon^2$, where m_6 is a positive number independent of L and x.*

Proof. Let $\varepsilon \in (0, \varepsilon_1)$ and $L \in \mathcal{K}_C(\tau)$. Assume that there exists no critical point of I_ε in $I_\varepsilon^d \cap \Omega_{2,\varepsilon}(L)$. Let $x \in V_L$. If $\rho_\varepsilon(t, \mathcal{N}_\varepsilon(W_{\varepsilon,x})) \in \Omega_{1,\varepsilon}(L)$ for all $t \geqslant 0$, then we have

$$\lim_{t\to\infty} I_\varepsilon(\rho_\varepsilon(t, \mathcal{N}_\varepsilon(W_{\varepsilon,x}))) = -\infty$$

by assumption. Since $\inf\{I_\varepsilon(\mathcal{N}_\varepsilon(W_{\varepsilon,x})) \colon x \in M\} > 0$, we find that there exists $t_{1,x} \geqslant 0$ such that

$$\rho_\varepsilon(t, \mathcal{N}_\varepsilon(W_{\varepsilon,x})) \in \Omega_{1,\varepsilon}(L) \quad \text{for } t \in (0, t_{1,x})$$

and

$$P_\varepsilon\rho_\varepsilon(t_{1,x}, \mathcal{N}_\varepsilon(W_{\varepsilon,x})) \in \partial U_L^{(1)}.$$

Set $y = P_\varepsilon\rho_\varepsilon(t_{1,x}, \mathcal{N}_\varepsilon(W_{\varepsilon,x}))$. Noting that $\inf\{I_\varepsilon(\mathcal{N}_\varepsilon(W_{\varepsilon,x})) \colon x \in M\} > -\infty$, we have, by lemma 4.2 and the definition of $\rho_{\varepsilon,S}$, that there exists $t_{2,x} > 0$ such that

$$P_\varepsilon\rho_{\varepsilon,S}(t_{2,x}, W_{\varepsilon,y}) \in \partial U_L^{(2)} \cup \partial V_L.$$

From the definition of $\rho_{\varepsilon,S}$, we have

$$\|\rho_{\varepsilon,S}(t, W_{\varepsilon,y}) - W_{\varepsilon,y}\|_\varepsilon \leqslant \int_0^t \left\|\frac{\mathrm{d}\rho_{\varepsilon,S}(\tau, W_{\varepsilon,y})}{\mathrm{d}\tau}\right\|_\varepsilon \mathrm{d}\tau \leqslant \int_0^t \|V_{\varepsilon,S}(\rho_\varepsilon(\tau, W_{\varepsilon,y}))\|_\varepsilon \,\mathrm{d}t.$$

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Since $P_\varepsilon \rho_{\varepsilon,S}(t_{2,x}, W_{\varepsilon,x}) \in \partial U_L^{(2)} \cup \partial V_L$, we have, by the definition of d_1, that

$$\|\rho_{\varepsilon,S}(t_{2,x}, W_{\varepsilon,y}) - W_{\varepsilon,y}\|_\varepsilon \geqslant d_1.$$

It then follows from lemma 4.2 and the definition of $\rho_{\varepsilon,S}$ that

$$\begin{aligned} I_\varepsilon(\rho_{\varepsilon,S}(t_{2,x}, W_{\varepsilon,y})) &\leqslant I_\varepsilon(W_{\varepsilon,y}) - \int_0^{t_{2,x}} \left\langle\!\!\left\langle \nabla I_\varepsilon(\rho_{\varepsilon,S}(\tau, W_{\varepsilon,y})), \frac{\mathrm{d}\rho_{\varepsilon,S}(\tau, W_{\varepsilon,y})}{\mathrm{d}t} \right\rangle\!\!\right\rangle_\varepsilon \mathrm{d}\tau \\ &\leqslant I_\varepsilon(W_{\varepsilon,y}) - \frac{1}{2}\int_0^{t_{2,x}} \|Q_{\varepsilon, P_\varepsilon \rho_{\varepsilon,S}(\tau, W_{\varepsilon,y}), S} \nabla I_\varepsilon(\rho_{\varepsilon,S}(\tau, W_{\varepsilon,y}))\|_\varepsilon^2 \, \mathrm{d}t \\ &\leqslant I_\varepsilon(W_{\varepsilon,y}) - \tfrac{1}{4} m_5 \varepsilon^2 \int_0^{t_{2,x}} \|V_{\varepsilon,S}(\rho_{\varepsilon,S}(\tau, W_{\varepsilon,y}))\|_\varepsilon \, \mathrm{d}t \\ &\leqslant I_\varepsilon(W_{\varepsilon,y}) - \tfrac{1}{4} m_5 \varepsilon^2 d_1. \end{aligned}$$

Since

$$z = P_\varepsilon \rho_{\varepsilon,S}(t_{2,x}, W_{\varepsilon,x}) \in \partial V_L \cup \partial U_L^{(2)},$$

we have that there exists $t_{3,x} \geqslant 0$ such that $P_\varepsilon \rho_\varepsilon(t_{3,x}, \mathcal{N}_\varepsilon(W_{\varepsilon,z})) \in \partial U_L^{(2)}$. Clearly, $t_{3,x} = 0$ if $z \in \partial U_L^{(2)}$. We now define a semiflow as follows: let $x \in V_L$. We set $\tilde{t}_{1,x} = t_{1,x} + 1$, $\tilde{t}_{2,x} = t_{1,x} + t_{2,x} + 1$ and $\tilde{t}_x = t_{1,x} + t_{2,x} + t_{3,x} + 2$. Then we set

$$\tilde{\rho}_\varepsilon(t, x) = \begin{cases} \rho_\varepsilon(t, \mathcal{N}_\varepsilon(W_{\varepsilon,x})) & \text{for } 0 \leqslant t \leqslant t_{1,x}, \\ (t_{1,x} + 1 - t) p_\varepsilon(t_{1,x}, W_{\varepsilon,x}) + (t - t_{1,x}) W_{\varepsilon,y} & \text{for } t_{1,x} < t \leqslant \tilde{t}_{1,x}, \\ \rho_{\varepsilon,S}(t - \tilde{t}_{1,x}, W_{\varepsilon,y}) & \text{for } \tilde{t}_{1,x} < t \leqslant \tilde{t}_{2,x}, \\ (\tilde{t}_{2,x} + 1 - t) W_{\varepsilon,z} + (t - \tilde{t}_{2,x}) \mathcal{N}_\varepsilon(W_{\varepsilon,z}) & \text{for } \tilde{t}_{2,x} < t \leqslant \tilde{t}_{2,x} + 1, \\ \rho_\varepsilon(t - \tilde{t}_{2,x} - 1, \mathcal{N}_\varepsilon(W_{\varepsilon,z})) & \text{for } \tilde{t}_{2,x} + 1 < t \leqslant \tilde{t}_x. \end{cases}$$

From the definition of $\tilde{\rho}_\varepsilon$, we have

$$I_\varepsilon(\tilde{\rho}_\varepsilon(t_{1,x}, x)) \leqslant I_\varepsilon(\mathcal{N}_\varepsilon(W_{\varepsilon,x})). \tag{4.9}$$

Then, by lemma 3.3, it follows that

$$I_\varepsilon(\tilde{\rho}_\varepsilon(\tilde{t}_{1,x}, x)) \leqslant I_\varepsilon(\mathcal{N}_\varepsilon(W_{\varepsilon,x})) + O(\varepsilon^{2(1+\alpha)}). \tag{4.10}$$

Similarly, we have

$$\begin{aligned} I_\varepsilon(\tilde{\rho}_\varepsilon(\tilde{t}_{2,x} + 1, x)) &\leqslant I_\varepsilon(\tilde{\rho}_\varepsilon(\tilde{t}_{2,x}, x)) + O(\varepsilon^{2(1+\alpha)}) \\ &\leqslant I_\varepsilon(\tilde{\rho}_\varepsilon(\tilde{t}_{1,x}, x)) - \tfrac{1}{4} m_5 \varepsilon^2 d_1 + O(\varepsilon^{2(1+\alpha)}). \end{aligned}$$

Finally, combining the above inequality with (4.10), we find that

$$I_\varepsilon(\tilde{\rho}_\varepsilon(\tilde{t}_x, x)) \leqslant I_\varepsilon(\tilde{\rho}_\varepsilon(\tilde{t}_{2,x} + 1, x)) \leqslant I_\varepsilon(\mathcal{N}_\varepsilon(W_{\varepsilon,x})) - \tfrac{1}{4} m_5 \varepsilon^2 d_1 + O(\varepsilon^{2(1+\alpha)}).$$

Then by assuming ε_1 is sufficiently small, we obtain assertion (b). □

We can now complete the proof of theorem 1.1.

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Proof of theorem 1.1. Since $(U_L \setminus L) \cap K(\tau) = \emptyset$ for each $L \in \mathcal{K}(\tau)$, we can choose $d > 0$ so small that $d < m_6/4A$ and

$$\begin{aligned} H_*((-\tau)^{c_L} \cap V_L, ((-\tau)^{c_L} \setminus L) \cap V_L) \\ &\cong H_*((-\tau)^{c_L} \cap V_L, (-\tau)^{c_L - 2d} \cap V_L) \\ &\cong H_*((-\tau)^{c_L} \cap U_L, (-\tau)^{c_L - d} \cap U_L) \quad \text{for each } L \in \mathcal{K}_C(\tau). \end{aligned}$$

Now fix $\varepsilon \in (0, \varepsilon_1)$ and $L \in \mathcal{K}_C(\tau)$ and set

$$d = \sup\{I_\varepsilon(\mathcal{N}_\varepsilon(W_{\varepsilon,x}))\colon x \in U_L^{(2)}\}.$$

We prove the assertion by contradiction. That is, we assume that there exists no critical point of I_ε in $\Omega_{2,\varepsilon}(L) \cap I_\varepsilon^d$. We define a homotopy of mappings $\sigma\colon [0,1] \times V_L \to U_L$ by

$$\sigma(t,x) = \begin{cases} P_\varepsilon(2t\mathcal{N}_\varepsilon(W_{\varepsilon,x}) + (1-2t)W_{\varepsilon,x}) & \text{if } t \in [0, \tfrac{1}{2}], \\ P_\varepsilon\tilde{\rho}_\varepsilon(\tilde{t}_x(2t-1), \mathcal{N}_\varepsilon(W_{\varepsilon,x})) & \text{if } t \in (\tfrac{1}{2}, 1], \end{cases}$$

for $x \in V_L$, where $\tilde{t}_x \in [0,\infty)$ defined in lemma 4.3. Then by lemma 4.3, we have

$$I_\varepsilon(\tilde{\rho}_\varepsilon(\tilde{t}_x, \mathcal{N}_\varepsilon(W_{\varepsilon,x}))) \leqslant I_\varepsilon(W_{\varepsilon,x}) - m_6\varepsilon^2.$$

Let $z = P_\varepsilon\tilde{\rho}_\varepsilon(\tilde{t}_x, \mathcal{N}_\varepsilon(W_{\varepsilon,x}))$. Then $\tilde{\rho}_\varepsilon(\tilde{t}_x, \mathcal{N}_\varepsilon(W_{\varepsilon,x})) = W_{\varepsilon,z} + w$ and $w \in E_{\varepsilon,z}^{\perp}$ with $\|w\|_\varepsilon \leqslant l_0\sqrt{\varepsilon}$. Then, by lemma 3.6,

$$\begin{aligned} I_\varepsilon(W_{\varepsilon,z}) &\leqslant I_\varepsilon(W_{\varepsilon,z} + w) + m_4(\varepsilon^{1+\alpha}\|w\|_\varepsilon + \varepsilon^{2\alpha}\|w\|_\varepsilon^2) \\ &\leqslant I_\varepsilon(W_{\varepsilon,x}) + (l_0\varepsilon^{3/2+\alpha} + l_0^2\varepsilon^{1+2\alpha}) - m_6\varepsilon^2. \end{aligned}$$

From the inequality above, by choosing ε_1 sufficiently small, we have that

$$I_\varepsilon(W_{\varepsilon,z}) \leqslant I_\varepsilon(W_{\varepsilon,x}) - \tfrac{1}{2}m_6\varepsilon^2.$$

Then we have by lemma 3.2 that

$$-\tau(\sigma(1,x)) = -\tau(z) \leqslant -\tau(x) - \frac{m_6}{2A} + O(\varepsilon).$$

Then again assuming that ε_1 is sufficiently small, we have

$$-\tau(\sigma(1,x)) \leqslant -\tau(x) - \frac{m_6}{4A}.$$

On the other hand, it also follows from lemma 3.3 and the definition of $\tilde{\rho}_\varepsilon$ that there exists $m_7 > 0$ such that

$$I_\varepsilon(W_{\varepsilon,x(t)}) \leqslant I_\varepsilon(W_{\varepsilon,x}) + m_7\varepsilon^{2(1+\alpha)} \quad \text{for all } t \in [0, \tilde{t}_x], \tag{4.11}$$

where $x(t) = P_\varepsilon\tilde{\rho}_\varepsilon(t,x)$ for $t \in [0, \tilde{t}_x]$. Then, assuming ε_1 is sufficiently small, we have

$$-\tau(\sigma(t,x)) \leqslant -\tau(x) + \tfrac{1}{2}d \quad \text{for } t \in [0,1] \text{ and } x \in V_L. \tag{4.12}$$

From the assumption, there exist $q \in \mathbb{N}$ and a q-singular chain α such that $[\alpha]$ is a non-trivial element of $H_q((-\tau)^{c_L} \cap V_L, (-\tau)^{c_L - 2d} \cap V_L)$. For simplicity, we assume that α is a q-singular simplex, i.e. $\alpha\colon \Delta_q \to (-\tau)^{c_L} \cap V_L$ is a continuous mapping. Then $[\alpha]$ is considered as a non-trivial element of

$$H_q((-\tau)^{c_L} \cap U_L, (-\tau)^{c_L-2d} \cap U_L).$$

We set $m = \sup\{-\tau(\sigma(t,x))\colon (t,x) \in [0,1] \times \alpha\}$. Since there is no critical point of $-\tau$ in $\{x \in U_L\colon -\tau(x) > c_L\}$, we have

$$H_q((-\tau)^{c_L+m} \cap U_L, (-\tau)^{c_L-d} \cap U_L) = H_q((-\tau)^{c_L} \cap U_L, (-\tau)^{c_L-d} \cap U_L)$$

and $[\alpha]$ is a non-trivial element of $H_q((-\tau)^{c_L+m} \cap U_L, (-\tau)^{c_L-d} \cap U_L)$. The homotopy of mappings $\sigma(\cdot, \alpha(\cdot))\colon [0,1] \times \Delta_q \to M$ satisfies

$$\sigma(t, \alpha(x)) \in (-\tau)^{c_L+m} \cap U_L \quad \text{for each } (t,x) \in [0,1] \times \Delta_q,$$

and

$$\sigma(t, \alpha(x)) \in (-\tau)^{c_L-d} \cap U_L \quad \text{for } (t,x) \in [0,1] \times \partial\Delta_q$$

by (4.12). That is $[\alpha] = [\sigma(t,\alpha)]$ in $H_q((-\tau)^{c_L+m} \cap U_L, (-\tau)^{c_L-d} \cap U_L)$ for each $t \in [0,1]$. On the other hand, we have

$$\sigma(1, \alpha(x)) \in (-\tau)^{c_L-d} \cap U_L \quad \text{for all } x \in \Delta_q.$$

That is $[\sigma(1,\alpha)] = \{0\}$ in $H_q((-\tau)^{c_L+m} \cap U_L, (-\tau)^{c_L-d} \cap U_L)$. This is a contradiction. Therefore, we have that there exists a critical point of I_ε in $\Omega_{2,\varepsilon}(L) \cap I_\varepsilon^d$. Since $\Omega_{2,\varepsilon}(L) \cap \Omega_{2,\varepsilon}(L') = \emptyset$ for $L \neq L'$, the assertion of theorem 1.1 holds. $\square$

Acknowledgements

The author express his hearty thanks to the referee for reading this paper very carefully and giving valuable suggestions.

Appendix A.

Let $\tau_0\colon H^1(\mathbb{R}^N) \setminus \{-U_0\} \to \mathbb{R}$ such that $(1 + \tau_0(w))(U_0 + w) \in \mathcal{N}_{\mathbb{R}^N,1}$ for $w \in H^1(\mathbb{R}^N) \setminus \{-U_0\}$. Then we have the following.

Lemma A.1.

(i) *There exists $L_1 > 0$ and $d_1 > 0$ such that*

$$|\tau_0(w)| \leqslant L_1 \|w\|_{\mathbb{R}^N,1} \quad \textit{for } w \in H^1(\mathbb{R}^N) \textit{ with } \|w\|_{\mathbb{R}^N,1} < d_1.$$

(ii) $\displaystyle\int_{R^N} f''(U_0) v_0^3 \, \mathrm{d}\mu = 0.$

Proof. For simplicity, we write $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ instead of $\|\cdot\|_{\mathbb{R}^N,1}$ and $\langle\cdot,\cdot\rangle_{\mathbb{R}^N,1}$, respectively. Let $w \in H^1(\mathbb{R}^N)$. From the definition of $\tau = \tau_0(w)$, $(1+\tau)(U_0+w) \in \mathcal{N}_{\mathbb{R}^N,1}$, i.e.

$$(1+\tau)^2 \|U_0 + w\|^2 = \int_{R^N} f((1+\tau)(U_0+w))((1+\tau)(U_0+w))$$

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holds. Then, by the same computation as in the proof of lemma 3.3, we find

$$\tau = -\|U_0 + w\|^2 - \int_{R^N} f(U_0+w)(U_0+w) \times \biggl(2\biggl(\|U_0+w\|^2 - \int_{R^N} f(U_0+w)(U_0+w)\biggr) + \int_{R^N} (f(U_0+w)(U_0+w) - f'(U_0+w)(U_0+w)^2)\biggr)^{-1}. \quad (\text{A }1)$$

On the other hand, we have by the Taylor expansion that, for some $\theta \in (0,1)$,

$$\begin{aligned}(\|U_0+w\|^2 - \langle f(U_0+w), U_0+w\rangle) &= \|U_0\|^2 + 2\langle\langle U_0, w\rangle\rangle_{\mathbb{R}^N,1} + \|w\|^2 \\ &\quad - \biggl\langle f(U_0) + f'(U_0)w + \tfrac{1}{2}f''(U_0)w^2 + \frac{1}{3!}f'''(U_0+\theta w)w^3, U_0 + w\biggr\rangle \\ &= \langle -\Delta U_0 + U_0 - f'(U_0)U_0, w\rangle + \langle -\Delta w + w - f(U_0)w, w\rangle \\ &\quad - \langle \tfrac{1}{2} f''(U_0)w^2, U_0\rangle + O(\|w\|^3). \end{aligned} \quad (\text{A }2)$$

We note that condition (3.2) ensure that $f^{(n)}(U_0+\theta w)w^{n+1} \in L^3(\mathbb{R}^N)$ for $n \geqslant 1$. Then we can see that

$$\|U_0+w\|^2 - \int_{R^N} f(U_0+w)(U_0+w) = O(\|w\|).$$

Then by (3.7) and (A 1), we find

$$|\tau| = \frac{1}{L_0}\biggl|\|U_0+w\|^2 - \int_{R^N} f(U_0+w)(U_0+w)\biggr|(1 + O(\|w\|)) = O(\|w\|). \quad (\text{A }3)$$

for $\|w\|$ sufficiently small. This completes the proof of (i).

(ii) Let $w \in H^1(\mathbb{R}^N)$ with $\|w\|$ small. Then recalling that $-\Delta v_0 + v_0 - f'(U_0)v_0 = 0$, we have from A 2 with $w = \alpha v_0$, $\alpha \in \mathbb{R}$ that

$$\|U_0+w\|^2 - \int_{R^N} f(U_0+w)(U_0+w) = -\frac{1}{2}\int_{R^N} f''(U_0)U_0 w^2 + O(\|w\|^3). \quad (\text{A }4)$$

Again by the Taylor expansion, we have that

$$\begin{aligned} I_{\mathbb{R}^N,1}(U_0+w) &= \tfrac{1}{2}(\|U_0\|^2 + 2\langle\langle U_0, w\rangle\rangle_{\mathbb{R}^N,1} + \|w\|^2) - \int_{R^N} F(U_0+w) \\ &= I_{\mathbb{R}^N,1}(U_0) + \langle\langle U_0, w\rangle\rangle_{\mathbb{R}^N,1} + \tfrac{1}{2}\|w\|^2 \\ &\quad - \int_{R^N}\biggl(f(U_0)w + \tfrac{1}{2}f'(U_0)w^2 + \frac{1}{3!}f''(U_0)w^3 + \frac{1}{4!}f'''(U_0+\theta w)w^4\biggr) \\ &= I_{\mathbb{R}^N,1}(U_0) + \tfrac{1}{2}\langle -\Delta w + w - f'(U_0)w, w\rangle \\ &\quad - \int_{R^N}\biggl(\frac{1}{3!}f''(U_0)w^3 + \frac{1}{4!}f'''(U_0+\theta w)w^4\biggr). \end{aligned} \quad (\text{A }5)$$

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Here we assume that

$$d = \int_{R^N} f''(U_0)v_0^3 \neq 0.$$

Then by setting $w = \alpha v_0$ in (A 5) with $\alpha d < 0$, we have

$$I_{\mathbb{R}^N,1}(U_0 + \alpha v_0) = I_{\mathbb{R}^N,1}(U_0) + \tfrac{1}{3}d\alpha^3 + O(\alpha^4). \tag{A 6}$$

By (A 1) and (A 4), we have $\tau = O(\alpha^2)$ and

$$\|U_0 + w\|^2 - \int_{R^N} f(U_0 + w)(U_0 + w) = O(\alpha^2).$$

Then

$$\begin{aligned} I_{\mathbb{R}^N,1}&((1+\tau)(U_0 + \alpha v_0)) \\ &= \tfrac{1}{2}(1+\tau)^2\|U_0 + \alpha v_0\|^2 - \int_{R^N} F((1+\tau)(U_0+\alpha v_0)) \\ &= \tfrac{1}{2}(1+2\tau)\|U_0 + \alpha v_0\|^2 \\ &\quad - \int_{R^N} (F(U_0+\alpha v_0) + \tau f(U_0+\alpha v_0)(U_0+\alpha v_0)) + O(\tau^2) \\ &= I_{\mathbb{R}^N,1}(U_0+\alpha v_0) - \tau\biggl(\|U_0+\alpha v_0\|^2 - \int_{R^N} f(U_0+\alpha v_0)(U_0+\alpha v_0)\biggr) + O(\alpha^4) \\ &= I_{\mathbb{R}^N,1}(U_0) + \tfrac{1}{3}d\alpha^3 + O(\alpha^4). \end{aligned}$$

Therefore, we have that $I_{\mathbb{R}^N,1}((1+\tau)(U_0+\alpha v_0)) < I_{\mathbb{R}^N,1}(U_0)$ for α sufficiently small with $\alpha d < 0$. This contradicts (2.10). □

LEMMA A.2. *For each* $(\varepsilon, x) \in (0, \varepsilon_0) \times M$,

$$\frac{1}{\varepsilon^N}\int_{S^{N-1}} \mathrm{d}\zeta \int_0^1 \varphi_\varepsilon^2\biggl(U_\varepsilon \frac{\partial U_\varepsilon}{\partial r}\biggr)\frac{\partial \eta_x}{\partial r} r^{N-1}\,\mathrm{d}r = \tfrac{1}{6}N\varpi\tau(x)A_1 + O(\varepsilon), \tag{A 7}$$

and

$$\left.\begin{aligned} \frac{1}{\varepsilon^N}\int_M |\nabla_g\tilde{\varphi}\cdot\nabla_g\tilde{U}|^2\,\mathrm{d}\mu_g &= O\biggl(\exp\biggl(-\frac{1}{4\varepsilon^{1-\alpha}}\biggr)\biggr), \\ \frac{1}{\varepsilon^N}\int_M |\tilde{U}\Delta_g\tilde{\varphi}|^2\,\mathrm{d}\mu_g &= O\biggl(\exp\biggl(-\frac{1}{4\varepsilon^{1-\alpha}}\biggr)\biggr), \end{aligned}\right\} \tag{A 8}$$

where $\tilde{U}(z) = U_\varepsilon(\exp_x^{-1}(z))$ *and* $\tilde{\varphi}(z) = \varphi_\varepsilon(\exp_x^{-1}(z))$ *for* $z \in B_g(x,1)$,

Proof. Let $(\varepsilon, x) \in \mathbb{R}^+ \times M$. We put $U = U_\varepsilon$, $\psi = \varphi_1$ and $\eta = \eta_x$. Then ψ and U are radial, i.e. $\psi(x) = \psi(|x|)$ and $U(x) = U(|x|)$ for $x \in \mathbb{R}^N$. Integrating the left-hand side of (A 7) by parts, we have

$$\begin{aligned} \int_{S^{N-1}} \mathrm{d}\zeta &\int_0^1 \varphi_\varepsilon^2\biggl(U\frac{\mathrm{d}U}{\mathrm{d}r}\biggr)\frac{\partial\eta}{\partial r} r^{N-1}\,\mathrm{d}r \\ &= -\frac{1}{2}\int_{S^{N-1}} \mathrm{d}\zeta \\ &\quad \times \int_0^1 U^2\biggl(\varphi_\varepsilon^2\frac{\partial^2\eta}{\partial r^2}r^{N-1} + 2\varphi_\varepsilon\frac{\partial\varphi_\varepsilon}{\partial r}\frac{\partial\eta}{\partial r}r^{N-1} + (N-1)r^{N-2}\varphi_\varepsilon^2\frac{\partial\eta}{\partial r}\biggr)\,\mathrm{d}r. \end{aligned}$$

Noting that

$$\frac{\partial^2\eta}{\partial r^2} = -\frac{\rho_x(\zeta)}{3} + O(r) \quad \text{for } [r,\zeta] \in (0,1)\times S^{N-1},$$

we find by setting $y = \varepsilon^{-1} r$ that

$$\begin{aligned}
\frac{1}{\varepsilon^N}\int_{S^{N-1}} \mathrm{d}\zeta \int_0^1 \varphi_\varepsilon^2 U^2 \frac{\partial^2\eta}{\partial r^2} r^{N-1}\,\mathrm{d}r
&= \int_{S^{N-1}} \mathrm{d}\zeta \int_0^{1/\varepsilon} \psi^2 U_0^2(y)(-\tfrac{1}{3}\rho_x(\zeta) + O(\varepsilon y)) y^{N-1}\,\mathrm{d}y \\
&= \int_{S^{N-1}} \mathrm{d}\zeta \int_0^{\infty} U_0^2(y)(-\tfrac{1}{3}\rho_x(\zeta) + O(\varepsilon y)) y^{N-1}\,\mathrm{d}y \\
&\quad + \int_{S^{N-1}} \mathrm{d}\zeta \int_{1/4\varepsilon^{1-\alpha}}^{\infty} (\psi^2 - 1) U_0^2(y) \frac{\partial^2\eta}{\partial r^2} y^{N-1}\,\mathrm{d}y \\
&= -\tfrac{1}{3}\varpi\tau(x)A_1 + O(\varepsilon) + O\biggl(\exp\biggl(-\frac{1}{4\varepsilon^{1-\alpha}}\biggr)\biggr).
\end{aligned} \tag{A 9}$$

Similarly, we find

$$\begin{aligned}
\frac{1}{\varepsilon^N}\int_{S^{N-1}} \mathrm{d}\zeta \int_0^1 \varphi_\varepsilon^2 U^2 \frac{\partial\eta}{\partial r} r^{N-2}\,\mathrm{d}r
&= \int_{S^{N-1}} \mathrm{d}\zeta \int_0^{1/\varepsilon} \psi^2 U_0^2(-\tfrac{1}{3}\rho_x(\zeta) + O(\varepsilon y)) y^{N-1}\,\mathrm{d}y \\
&= -\tfrac{1}{3}\varpi\tau(x)A_1 + O(\varepsilon).
\end{aligned} \tag{A 10}$$

Noting that $\partial\psi/\partial r = 0$ on $[0, \frac{1}{4}\varepsilon^\alpha]$, we also have, by (2.14),

$$\begin{aligned}
\frac{1}{\varepsilon^N}\int_{S^{N-1}} \mathrm{d}\zeta \int_0^1 \varphi_\varepsilon \frac{\partial\varphi_\varepsilon}{\partial r} U^2 \frac{\partial\eta}{\partial r} r^{N-1}\,\mathrm{d}r
&= \frac{1}{\varepsilon^N}\int_{S^{N-1}} \mathrm{d}\zeta \int_{1/4\varepsilon^{1-\alpha}}^1 \psi \frac{\partial\psi}{\partial r} U^2 \frac{\partial\eta}{\partial r} r^{N-1}\,\mathrm{d}r = O\biggl(\exp\biggl(-\frac{1}{4\varepsilon^{1-\alpha}}\biggr)\biggr).
\end{aligned}$$

Then combining (A 9) and (A 10) with the equality above, we find that (A 7) holds. Next we see that (A 8) holds. Again noting that $\partial\psi/\partial r = 0$ on $[0, \frac{1}{4}\varepsilon^\alpha]$ and that (2.14) holds, by setting $r = \varepsilon y$, we have that

$$\begin{aligned}
\frac{1}{\varepsilon^N}\int_M |\nabla_g\tilde{\varphi}\cdot\nabla_g\tilde{U}|^2\,\mathrm{d}\mu_g
&= \frac{1}{\varepsilon^N}\int_{S^{N-1}} \mathrm{d}\zeta \int_{\varepsilon^\alpha/4}^1 \biggl(\frac{\partial\psi}{\partial r}\frac{\partial U}{\partial r}\biggr)^2 \eta r^{N-1}\,\mathrm{d}r \\
&= \int_{S^{N-1}} \mathrm{d}\zeta \int_{1/4\varepsilon^{1-\alpha}}^{1/\varepsilon} \biggl(\frac{\partial\psi}{\partial r}(2\varepsilon^{1-\alpha}y)\biggr)^2 \biggl(\frac{\partial U_0}{\partial r}(y)\biggr)^2 \eta y^{N-1}\,\mathrm{d}y \\
&= O\biggl(\exp\biggl(-\frac{1}{4\varepsilon^{1-\alpha}}\biggr)\biggr).
\end{aligned}$$

By a parallel argument, we have

$$\frac{1}{\varepsilon^N}\int_M |\tilde{U}\Delta_g\tilde{\varphi}|^2\,\mathrm{d}\mu_g = O\biggl(\exp\biggl(-\frac{1}{4\varepsilon^{1-\alpha}}\biggr)\biggr).$$

□

LEMMA A.3. *For each* $(\varepsilon, x) \in (0, \varepsilon_0) \times M$,

$$\frac{1}{\varepsilon^N}\int_{S^{N-1}}\mathrm{d}\zeta\int_0^1(\tfrac{1}{2}\varphi_\varepsilon^2 f(\varphi_\varepsilon U_\varepsilon)U_\varepsilon - F(\varphi_\varepsilon U_\varepsilon))\eta_x r^{N-1}\,\mathrm{d}r = c_0 - \tfrac{1}{6}\varepsilon^2\tau(x)A_0 + O(\varepsilon^3). \tag{A 11}$$

Proof. Let $(\varepsilon, x) \in (0, \varepsilon_0) \times M$. We put $\psi = \varphi_1$, $U = U_\varepsilon$ and $\eta = \eta_x$. Then

$$\begin{aligned}
\frac{1}{\varepsilon^N}&\int_{S^{N-1}}\mathrm{d}\zeta\int_0^1(\tfrac{1}{2}\varphi_\varepsilon^2 f(\varphi_\varepsilon U)U - F(\varphi_\varepsilon U))\eta r^{N-1}\,\mathrm{d}r \\
&= \int_{S^{N-1}}\mathrm{d}\zeta\int_0^\infty(\tfrac{1}{2}f(U_0)U_0 - F(U_0))\eta y^{N-1}\,\mathrm{d}y \\
&\quad + \int_{1/4\varepsilon^{1-\alpha}}^\infty((\psi^2-1)f(U_0)U_0 + \psi^2(f(\psi U_0) - f(U_0)) \\
&\qquad\qquad - (F(\psi U_0) - F(U_0)))\eta y^{N-1}\,\mathrm{d}y.
\end{aligned}$$

From (f2) and (2.13), noting $p > 2$, one can see that

$$\begin{aligned}
\int_{1/4\varepsilon^{1-\alpha}}^\infty&((\psi^2-1)f(U_0)U_0 \\
&+ \psi^2(f(\psi U_0) - f(U_0)) - (F(\psi U_0) - F(U_0)))\eta y^{N-1}\,\mathrm{d}y \\
&= O\biggl(\exp\biggl(-\frac{1}{4\varepsilon^{1-\alpha}}\biggr)\biggr).
\end{aligned}$$

Then we have

$$\begin{aligned}
\frac{1}{\varepsilon^N}&\int_{S^{N-1}}\mathrm{d}\zeta\int_0^1(\tfrac{1}{2}\varphi_\varepsilon^2 f(U)U - F(\varphi_\varepsilon U))\eta r^{N-1}\,\mathrm{d}r \\
&= \int_{S^{N-1}}\mathrm{d}\zeta\int_0^\infty(\tfrac{1}{2}f(U_0)U_0 - F(U_0))\eta y^{N-1}\,\mathrm{d}y + O\biggl(\exp\biggl(-\frac{1}{4\varepsilon^{1-\alpha}}\biggr)\biggr).
\end{aligned}$$

From (2.6) and (2.19), we have

$$\begin{aligned}
\frac{1}{\varepsilon^N}&\int_{S^{N-1}}\mathrm{d}\zeta\int_0^1(\tfrac{1}{2}\varphi_\varepsilon^2 f(U)U - F(\varphi_\varepsilon U))\eta r^{N-1}\,\mathrm{d}r \\
&= \int_{S^{N-1}}\mathrm{d}\zeta\int_0^\infty(\tfrac{1}{2}f(U_0)U_0 - F(U_0))(1 - \tfrac{1}{6}\varepsilon^2\rho_x(\zeta)y^2 + O(\varepsilon^3y^3))y^{N-1}\,\mathrm{d}r \\
&\quad + O\biggl(\exp\biggl(-\frac{1}{4\varepsilon^{1-\alpha}}\biggr)\biggr) \\
&= c_0 - \tfrac{1}{6}\varepsilon^2\varpi\tau(x)A_0 + O(\varepsilon^3).
\end{aligned}$$

This completes the proof. □

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Lemma A.4. *There exists $\bar{m} > 0$ such that for each $(\varepsilon, x) \in (0, \varepsilon_0) \times M$ and $w \in H$,*

$$|\langle -\varepsilon^2 \Delta_g W_{\varepsilon,x} + W_{\varepsilon,x} - f(W_{\varepsilon,x}), w\rangle_\varepsilon| \leqslant \bar{m}\varepsilon^{1+\alpha}\|w\|_\varepsilon.$$

Proof. Let $(\varepsilon, x) \in (0, \varepsilon_0) \times M$ and $w \in H$. Setting $W = W_{\varepsilon,x}$, $\psi = \varphi_1$ and $U = U_\varepsilon$, we have by (3.3) that

$$\begin{aligned}
&\langle -\varepsilon^2 \Delta_g W + W - f(W), w\rangle_\varepsilon \\
&\quad = \langle \varphi_\varepsilon f(U) - f(\varphi_\varepsilon U), w\rangle_\varepsilon \\
&\qquad - \frac{\varepsilon^2}{\varepsilon^N}\int_{S^{N-1}} \mathrm{d}\zeta \int_0^1 \varphi_\varepsilon \frac{\partial U}{\partial r}\frac{\partial \eta}{\partial r} w r^{N-1}\,\mathrm{d}r + O\biggl(\exp\biggl(-\frac{1}{8\varepsilon^{1-\alpha}}\biggr)\|w\|_\varepsilon\biggr).
\end{aligned}$$

By the definition of φ_ε and (2.13), we have

$$\begin{aligned}
&\frac{1}{\varepsilon^N}\int_M |\varphi_\varepsilon f(U) - f(\varphi_\varepsilon U)|^2 \mu_g \\
&\qquad \leqslant \int_{S^{N-1}} \mathrm{d}\zeta \int_{1/4\varepsilon^{1-\alpha}}^{1/\varepsilon} |\psi f(U_\varepsilon) - f(\psi U_\varepsilon)|^2 \eta y^{N-1}\,\mathrm{d}y \\
&\qquad = O\biggl(\exp\biggl(-\frac{1}{4\varepsilon^{1-\alpha}}\biggr)\biggr) \qquad \text{(A 12)}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\varepsilon^2}{\varepsilon^N}\int_{S^{N-1}} \mathrm{d}\zeta \int_0^1 \varphi_\varepsilon \frac{\partial U}{\partial r}\frac{\partial \eta}{\partial r} w r^{N-1}\,\mathrm{d}r \\
&\qquad = \frac{\varepsilon^2}{\varepsilon^N}\int_{S^{N-1}} \mathrm{d}\zeta \int_0^{\varepsilon^\alpha/4} \varphi_\varepsilon \frac{\partial U}{\partial r}\frac{\partial \eta}{\partial r} w r^{N-1}\,\mathrm{d}r \\
&\qquad\quad + \frac{\varepsilon^2}{\varepsilon^N}\int_{S^{N-1}} \mathrm{d}\zeta \int_{\varepsilon^\alpha/4}^1 \varphi_\varepsilon \frac{\partial U}{\partial r}\frac{\partial \eta}{\partial r} w r^{N-1}\,\mathrm{d}r \\
&\qquad \leqslant \frac{\varepsilon^2}{\varepsilon^N}\int_{S^{N-1}} \mathrm{d}\zeta \int_0^{\varepsilon^\alpha/4} \frac{\partial U}{\partial r}\frac{\partial \eta}{\partial r} w r^{N-1}\,\mathrm{d}r + O(\varepsilon^2\|w\|_\varepsilon) \\
&\qquad \leqslant \varepsilon\biggl(\frac{1}{\varepsilon^N}\int_{S^{N-1}} \mathrm{d}\zeta \int_0^{\varepsilon^\alpha/4} \biggl(\frac{\partial \eta}{\partial r} w\biggr)^2 r^{N-1}\,\mathrm{d}r\biggr)^{1/2} \|U\|_{\mathbb{R}^N,\varepsilon} \\
&\qquad \leqslant \varepsilon^{1+\alpha}|w|_{\mathbb{R}^N,2,\varepsilon}\|U\|_{\mathbb{R}^N,\varepsilon}.
\end{aligned}$$

Therefore, the assertion follows. □

Lemma A.5. *Let $(\varepsilon, x) \in (0, \varepsilon_0) \times M$.*

(i) *For $v \in L^{3/2}(B_g(x, \varepsilon^\alpha))$ and $w \in L^3(B_g(x, \varepsilon^\alpha))$,*

$$|\langle v, w\rangle_\varepsilon - \langle \tilde{v}, \tilde{w}\rangle_{\mathbb{R}^N,\varepsilon}| = O(\varepsilon^{2\alpha}|v|_{3/2,\varepsilon}|w|_{3,\varepsilon}). \qquad \text{(A 13)}$$

(ii) *For $v, w \in H_0^1(B_g(x, \varepsilon^\alpha))$,*

$$|\langle -\varepsilon^2\Delta_g v, w\rangle_\varepsilon - \langle -\varepsilon^2\Delta\tilde{v}, \tilde{w}\rangle_{\mathbb{R}^N,\varepsilon}| = O(\varepsilon^{1+\alpha}\|v\|_\varepsilon\|w\|_\varepsilon), \qquad \text{(A 14)}$$

where $\tilde{v}(z) = v(\exp_x(z))$ and $\tilde{w}(z) = w(\exp_x(z))$ for $z \in B_1(0)$.

(iii) *For* $v \in H_0^1(B_g(x,\varepsilon^\alpha)) \cap E_{\varepsilon,x}^\perp \cap \mathcal{N}_\varepsilon$ *with* $\|v\|_\varepsilon < \delta_0$, *there exists* $v' \in H^1(\mathbb{R}^N) \cap (T_0S)^\perp \cap \mathcal{N}$ *such that*

$$\|v' - \tilde{v}\|_{\mathbb{R}^N,\varepsilon} = O(\varepsilon^{2\alpha}\|v\|_\varepsilon),$$

where $d_2 > 0$ *is a constant independent of* ε.

Proof. Let $(\varepsilon, x) \in (0, \varepsilon_0) \times M$, $v \in L^{3/2}(B_g(x,\varepsilon^\alpha))$ and $w \in L^3(B_g(x,\varepsilon^\alpha))$. Since $|g_x^{jk}(z) - \delta_{jk}| = O(|z|^2)$ for $1 \leqslant j,k \leqslant N$ and $z \in \mathbb{R}^N$ with $|z|$ sufficiently small, we have

$$\begin{aligned}|\langle \tilde{v}, \tilde{w}\rangle_{\mathbb{R}^N,\varepsilon} - \langle v, w\rangle_\varepsilon| &= \frac{1}{\varepsilon^N}\bigg|\int_{B_1(0)} \tilde{v}(y)\tilde{w}(y)(\sqrt{|g_x|} - 1)\,\mathrm{d}y_1\,\mathrm{d}y_2\cdots\mathrm{d}y_N\bigg| \\ &= O(\varepsilon^{2\alpha}|\tilde{v}|_{,\mathbb{R}^N,3/2,\varepsilon}|\tilde{w}|_{\mathbb{R}^N,3,\varepsilon}) \\ &= O(\varepsilon^{2\alpha}|v|_{3/2,\varepsilon}|w|_{3,\varepsilon}).\end{aligned}$$

On the other hand, recalling that $\Delta - \Delta_g = \delta_{ij}\partial_i\partial_j - (\sqrt{|g_x|})^{-1}\partial_i(\sqrt{|g_x|}\partial_j)$ and $|\nabla g_x(z)| = O(|z|)$, we have

$$\begin{aligned}|\langle \varepsilon^2(\Delta - \Delta_g)v, w\rangle_\varepsilon| &= \varepsilon\bigg|\int_{R^N} \varepsilon\nabla\tilde{v}(\varepsilon z)\tilde{w}(\varepsilon z)\frac{\nabla\sqrt{g_x}(\varepsilon z)}{2\sqrt{|g_x|}}\,\mathrm{d}z_1\,\mathrm{d}z_2\cdots\mathrm{d}z_N\bigg| \\ &= O(\varepsilon^{1+\alpha}\|\tilde{v}\|_{\mathbb{R}^N,\varepsilon}\|\tilde{w}\|_{\mathbb{R}^N,\varepsilon}) \quad \text{for } v, w \in H_0^1(B_g(x,\varepsilon^\alpha)).\end{aligned}$$

Since we have $|\langle \Delta v, w\rangle_{\mathbb{R}^N,\varepsilon} - \langle \Delta v, w\rangle_\varepsilon| = O(\varepsilon^{2\alpha}\|v\|_\varepsilon\|w\|_\varepsilon)$, by the same argument as above we obtain (A 13).

Let $v \in H_0^1(B_g(x,\varepsilon^\alpha)) \cap E_{\varepsilon,x}^\perp \cap \mathcal{N}_\varepsilon$ with $\|v\|_\varepsilon < \delta_0$. Since

$$\left\langle\!\!\left\langle v, \frac{\partial W_{\varepsilon,x}}{\partial x_i}\right\rangle\!\!\right\rangle_\varepsilon = 0$$

by definition,

$$\left\langle\!\!\left\langle \tilde{v}, \frac{\partial U_\varepsilon}{\partial x_i} - \frac{\partial W_\varepsilon}{\partial x_i}\right\rangle\!\!\right\rangle_{\mathbb{R}^N,\varepsilon} = O\bigg(\exp\left(\frac{-1}{4\varepsilon^\alpha}\right)\|\tilde{v}\|_{\mathbb{R}^N,\varepsilon}\bigg)$$

by (2.14), and

$$\left|\left\langle\!\!\left\langle v, \frac{\partial W_{\varepsilon,x}}{\partial x_i}\right\rangle\!\!\right\rangle_\varepsilon - \left\langle\!\!\left\langle \tilde{v}, \frac{\partial W_\varepsilon}{\partial x_i}\right\rangle\!\!\right\rangle_{\mathbb{R}^N,\varepsilon}\right| = O(\varepsilon^{1+\alpha}\|v\|_\varepsilon)$$

by (A 14), we find

$$\begin{aligned}\left\langle\!\!\left\langle \tilde{v}, \frac{\partial U_\varepsilon}{\partial x_i}\right\rangle\!\!\right\rangle_{\mathbb{R}^N,\varepsilon} &= \left\langle\!\!\left\langle \tilde{v}, \frac{\partial U_\varepsilon}{\partial x_i} - \frac{\partial W_\varepsilon}{\partial x_i}\right\rangle\!\!\right\rangle_{\mathbb{R}^N,\varepsilon} \\ &\quad + \bigg(\left\langle\!\!\left\langle \tilde{v}, \frac{\partial W_\varepsilon}{\partial x_i}\right\rangle\!\!\right\rangle_{\mathbb{R}^N,\varepsilon} - \left\langle\!\!\left\langle v, \frac{\partial W_{\varepsilon,x}}{\partial x_i}\right\rangle\!\!\right\rangle_\varepsilon\bigg) + \left\langle\!\!\left\langle v, \frac{\partial W_{\varepsilon,x}}{\partial x_i}\right\rangle\!\!\right\rangle_\varepsilon \\ &= O(\varepsilon^{1+\alpha}\|v\|_\varepsilon).\end{aligned}$$

Here we set

$$\bar{v} = \tilde{v} - \sum_{i=1}^N \left\langle\!\!\left\langle \tilde{v}, \frac{\partial U_\varepsilon}{\partial x_i}\right\rangle\!\!\right\rangle_{\mathbb{R}^N,\varepsilon}\frac{\partial U_\varepsilon}{\partial x_i}.$$

Then $\bar{v} \in (T_0 S_\varepsilon)^\perp$ and $\|\bar{v} - \tilde{v}\|_{\mathbb{R}^N,\varepsilon} = O(\varepsilon^{1+\alpha}\|v\|_\varepsilon)$. On the other hand, noting that $\|W_{\varepsilon,x} + v\|_\varepsilon^2 - \langle f(W_{\varepsilon,x} + v), W_{\varepsilon,x} + v\rangle_\varepsilon = 0$, we have by (A 14) and (A 13) that

$$\|W_\varepsilon + \bar{v}\|_{\mathbb{R}^N,\varepsilon}^2 - \int_{R^N} f(W_\varepsilon + \bar{v})(W_\varepsilon + \bar{v})\,\mathrm{d}\mu = O(\varepsilon^{1+\alpha}\|v\|_\varepsilon).$$

Let $\tau \in \mathbb{R}$ such that $(1+\tau)\bar{v} \in \mathcal{N}_{\mathbb{R}^N,\varepsilon}$. Then from the equality above, we have that $\tau = O(\varepsilon^{1+\alpha}\|v\|_\varepsilon)$. Here we set $v' = (1+\tau)\bar{v}$. Then we obtain that $\|\tilde{v} - v'\|_{\mathbb{R}^N,\varepsilon} = O(\varepsilon^{1+\alpha}\|v\|_\varepsilon)$. This completes the proof. □

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(*Issued 17 February 2012*)