Finite simple groups with
no elements of order six

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The aim of this paper is to completely classify the finite simple
groups with no elements of order 6 . The proof is by induction
and involves an analysis of the structure of the 2-local
subgroups of a counterexample of minimum order. A recent result
of Glauberman plays an essential role.

The object of this paper is to completely classify the groups of the
title. The result has recently been obtained independently by Fletcher,
Stellmacher, Stewart [4], but the author feels that the present proof has
independent interest.

THEOREM. *Let G be a finite simple group which has no elements of
order six. Then G is isomorphic to one of the following groups:*

   (i)   PSL$(2, q)$, $q \not\equiv 1, 11 \bmod 12$;

   (ii) PSL$(3, 2^n)$, $n \geq 2$, $n \not\equiv 0 \bmod 6$;

   (iii) PSU$(3, 2^n)$, $n \geq 2$, $n \not\equiv 3, 5 \bmod 6$;

   (iv) $Sz(2^{2n+1})$, $n \geq 1$.

The centralizer of any 3-element in G has odd order whence the main
theorem could be rephrased to give an "odd characterization" of the simple
groups listed.

Notation is essentially standard (see [5], [6], [9]). For a set V ,

$o(V)$ will denote the cardinality of V . In particular, $o(H)$ is the
order of a group H and if $x \in H$, $o(x) = o(\langle x \rangle)$. If H is a finite
group then $\mathrm{sol}(H)$ is the maximal soluble normal subgroup of H , $\mathrm{soc}(H)$
is the product of the minimal normal subgroups of H , and $L(H)$ is the
maximal normal semisimple subgroup of H .

For the remainder of the paper, G will denote a counterexample to
the theorem of minimum order. We commence with an important observation.
The hypothesis of the theorem is inductive to subgroups and sections and
thus the only simple groups involved in G are those listed in the
theorem. Moreover, if t is an involution of G then $C_G(t)$ is soluble
or involves only Suzuki groups and so G is balanced ([16], p. 285). An
inspection of the list of Gorenstein and Harada [11, Corollary A] shows
that $SCN_3(2)$ is non-empty for G . It follows that $m(G) \geq 3$ and G is
connected. By [5, 2.63], $O\big(C_G(t)\big) = 1$ for any involution of G . This
will be of particular importance in the final stages of the proof.

The following lemma is useful for a study of the 2-local subgroups.

LEMMA 1. *Suppose* $Q \triangleleft H$, $Q \in S_2(H)$, $R \in S_r(H)$, $o(R) = r$, *and
that* H *has no elements of order* $2r$.

If $P \in \mathsf{H}(R, 2')$, *then* $[P, R] \subseteq O(H)$.

Proof. Suppose the lemma is false and let H be a counterexample of
minimum order.

First, $O(H) = 1$. Otherwise setting $\overline{H} = H/O(H)$, $\overline{H}, \overline{P}, \overline{R}$ satisfy
the hypothesis of the lemma, whence $[\overline{P}, \overline{R}] \subseteq O(\overline{H}) = \overline{O(H)} = 1$ giving
$[P, R] \subseteq O(H)$.

If R centralizes all Sylow subgroups of P the result is trivial.
By the minimality of $o(H)$, we may assume P is a p-group for some prime
p and $H = QPR$. Put $R = \langle \phi \rangle$. If P is abelian then
$P = C_P(R) \times [P, R]$ and ϕ acts fixed-point-freely on $[P, R]$ and hence
on $Q[P, R]$. It follows that $Q[P, R] = Q \times [P, R]$, whence $[P, R] = 1$
contradicting our choice of R .

By [9, Theorem 5.3.8], P is a non-abelian special p-group with ϕ
acting irreducibly on $P/\Phi(P)$ and trivially on $\Phi(P)$. Further, $\Phi(Q) = 1$

by Burnside's Lemma and the minimality of $o(H)$. Thus $Q = Q_1 \oplus \ldots \oplus Q_s$ is elementary abelian with Q_i irreducible under PR .

Put $P_i = C_{PR}(Q_i) = C_p(Q_i)$. As P_i is ϕ-invariant, $P_i(P)/\Phi(P)$ is a ϕ-invariant subspace of $P/\Phi(P)$; ϕ acts irreducibly on $P/\Phi(P)$, whence $P_i\Phi(P) = \Phi(P)$ or P .

Assume $P_i \subseteq \Phi(P)$ for some i . Then PR/P_i has a faithful irreducible representation on Q_i and so $Z(PR/P_i)$ is cyclic. Since $Z(PR/P_i) = Z(P/P_i)$, P/P_i is extra-special of order p^{2r+1} , say. By [9, Exercises, p. 371-372], Q has a minimum polynomial of $X^r - 1$ on Q_i or $p^t = r - 1$. As p and r are odd, the latter is impossible, and so ϕ has an eigenvalue of 1 in its representation on Q_i . Then ϕ fixes an involution, contrary to hypothesis.

We are left with $P_i\Phi(P) = P$ for each i , whence $P_i = P$ for each i . Thus P centralizes Q : a final contradiction which establishes the lemma.

Recently Glauberman ([7], [8]) has classified all simple groups which do not involve Σ_4 , the symmetric group on four letters. His work guarantees the existence of an element of order 3 in some 2-local subgroup in any counterexample to our theorem. To exploit this the following lemma is basic.

LEMMA 2. *Let* H *be a finite group with* $H/O_2(H) \simeq \Sigma_3$. *Suppose* H *has no elements of order* 6 , $P \in S_2(H)$, *and* ϕ *is an element of order* 3 *in* H . *Then*

(i) $\mathrm{cl}(O_2(H)) \le 2$,

(ii) if $R \le O_2(H)$ *is abelian,* $\langle R, R^\phi \rangle$ *is also abelian,*

(iii) if $o(O_2(H)) > 4$ *then the class of* $O_2(H)$ *is less than the class of any other subgroup of* P *of index* 2 ,

(iv) if $R \leq O_2(H)$ is ϕ-invariant and $s \in P \backslash O_2(H)$, then

$$o\big(C_R(S)\big) \leq o(R)^{\frac{1}{2}} .$$

Proof. Explicitly given in [20] as Proposition 2.2, although attributed, by Higman, to Burnside.

LEMMA 3. *A non-soluble proper subgroup of G has order prime to 3 iff it involves a Suzuki group.*

Proof. By induction any non-soluble proper subgroup of G with order prime to 3 must involve a Suzuki group.

For the converse, let H be a proper subgroup of G which involves a Suzuki group. Suppose H has a chief series $H = H_0 \rhd H_1 \rhd \ldots \rhd H_n = 1$ with $H_i/H_{i+1} = \overline{H}_i = \overline{S}_1 \times \overline{S}_2 \times \ldots \times \overline{S}_r$, a direct product of isomorphic Suzuki groups. Let $R \in S_3(H_{i+1})$. If $N(R)$ does not contain a four-group then a Sylow 2-subgroup of $N(R)$ is cyclic, generalized quaternion, or trivial. By the Frattini argument $N(R)$ involves a Suzuki group, whence $N(R)$ contains a subgroup VR with V a four-group. As

$R = \prod_{x \in V^{\#}} C_R(x)$ and G has no elements of order 6 , it follows that

$R = 1$, and it is sufficient to show that $\overline{H} = H/H_{i+1}$ contains no elements of order 3 .

Let the kernel of the action of $\overline{H}$ on the components of $\overline{H}_i$ be $\overline{K}$, and suppose that ϕ is an element of order 3 in $\overline{H}$. If $\phi \in \overline{H} \backslash \overline{K}$, then ϕ acts non-trivially on $\overline{S}_1 \times \overline{S}_2 \times \ldots \times \overline{S}_r$. For some $\overline{S}_i$ and some

involution $t \in \overline{S}_i$ the subgroup $\overline{S}_i \times \overline{S}_i^{\phi} \times \overline{S}_i^{\phi^2}$ contains the involution

$t.t^{\phi}.t^{\phi^2}$, which is centralized by ϕ . If $\phi \in \overline{K}$, it normalizes each component, permuting the involutions in each one. The number of involutions in any component is clearly prime to 3 , and so ϕ centralizes at least one (in fact ϕ centralizes a Suzuki group). In either case we have an element of order 6 , contrary to hypothesis.

Suppose that G has a non-soluble 2-local subgroup which contains an element of order 3 . In the following lemmas we investigate the structure

of such a subgroup, eventually contradicting its existence. Thus choose N
to be a non-soluble 2-local subgroup with $3 \mid o(N)$. Further choose N
such that $o\bigl(O_2(N)\bigr)$ is maximal and put $Q = O_2(N)$.

LEMMA 4. *N has a section M such that $M/O_2(M)$ is isomorphic to
one of $L_2(q)$, $L_3(q)$, and $U_3(q)$, $O_2(M) \simeq O_2(N)$, and a Sylow 2-subgroup
of M is isomorphic to a Sylow 2-subgroup of N .*

Proof. Let $K = \mathrm{sol}(N)$. If $R \in S_3(K)$, then $N(R)$ is non-soluble.
Since R is cyclic, $C(R)$ is non-soluble and G contains an element of
order 6 . Thus R is trivial.

If $P \in S_2(K)$ then by the Frattini argument again, $N(P)$ is non-
soluble and contains a 3-element. By the maximality of Q , $P = Q$ and
$K = O_{2,2'}(N)$. Since we seek a section of N , we may assume $O(N) = 1$.

Let H/K be a minimal normal subgroup of M/K . By Lemma 3, H/K is
isomorphic to one of $L_2(q), L_3(q)$, and $U_3(q)$. Put $\overline{H} = H/O_2(H)$. Let
ϕ be an element of order 3 in H , L a ϕ-invariant complement of
$O_2(H)$ in $O_{2,2'}(H)$. Applying Lemma 1 to $QL\langle\phi\rangle$, $[L, \phi] \subseteq O(H) = 1$,
whence $C_{\overline{H}}(\overline{L}).\overline{L} > \overline{L}$ forcing $\overline{H} = C_{\overline{H}}(\overline{L}).\overline{L}$. As each of $L_2(q), L_3(q)$,
and $U_3(q)$ has a multiplier which is, at worst, a $\{2, 3\}$-group [2],
$\overline{H} = \overline{M} \times \overline{L}$, where $\overline{M} \simeq L_2(q), L_3(q)$, or $U_3(q)$, and $\overline{L} = O(\overline{H})$. As $\overline{M}$ has
an element of order 3 , so $C_{\overline{N}}(\overline{M})$ has odd order and thus equals $\overline{L}$. It
follows that $\overline{H}/\overline{L} \le \mathrm{aut}(\overline{M})$. All outer automorphisms of $\overline{M}$ of even order
fix elements of order 3 in $\overline{M}$. Thus $[\overline{N} : \overline{M}]$ is odd and the required
properties of M follow easily.

LEMMA 5. $M/O_2(M) \not\cong L_3(q)$ *or* $U_3(q)$.

Proof. Put $\overline{M} = M/O_2(M)$. Let $\varepsilon = 1$ (respectively -1) if
$\overline{M} \simeq L_3(q)$ $\bigl($respectively $U_3(q)\bigr)$ and $\nu = (3, q-\varepsilon)$. $\overline{M}$ contains a cyclic
Hall subgroup $\overline{D}$, $o(\overline{D}) = \bigl(q^2 + \varepsilon q + 1\bigr)/\nu$ with $[N_{\overline{M}}(\overline{D}) : \overline{D}] = 3$ ($\overline{D}$ is a Singer-
cycle [17], Satz 7.3), and $\bigl(o(\overline{D}), 3\bigr) = 1$. If ϕ is an element of order 3
in $N_{\overline{M}}(\overline{D})$ then $\overline{D} = C_{\overline{D}}(\phi) \times [\overline{D}, \phi]$ and ϕ acts fixed-point-freely on

$[\overline{D}, \phi]$ and so on $[\overline{D}, \phi]O_2(M)$. It follows that $[D, \phi]O_2(M)$ is nilpotent and $C_{\overline{M}}(O_2(M))$ is non-trivial. Thus $C_{\overline{M}}(O_2(M)) = \overline{M}$ and we have an element of order 6 : a contradiction.

LEMMA 6 . $M/O_2(M) \not\cong L_2(q)$, q odd.

Proof . By $[20]$, Proposition 3.1.

LEMMA 7 . $M/O_2(M) \not\cong L_2(2^n)$.

Proof . By $[20]$, Proposition 4.1, if $P \in S_2(M)$, then $\text{cl}(P) = 2$. If $o(O_2(M)) > 2^n$ or $n \geq 3$ then $O_2(M)$ char P , ($[20]$, Proposition 4.1, 5.4). It follows that $P \in S_2(G)$, G has Sylow 2-subgroups of class 2 . The classification of all such groups has been completed ($[5, 6]$) and no counter-example exists in their list.

Thus we may assume $M/O_2(M) \simeq L_2(4) \simeq A_5$, $O_2(M) \simeq E_{16}$. Let $P \in S_2(M)$, ϕ a 3-element in $N(P)$, and t an involution in $P\backslash O_2(M)$ (otherwise $O_2(M) = \Omega_1(P)$ char P and we are finished as above). Now t and t^{ϕ} commute, so P is the split extension of E_{16} by A_5 . Only two such groups exist $[19]$, and one contains elements of order 6 , whence M is completely determined. A Sylow 2-subgroup of M and hence of N is isomorphic to that of $\text{PSL}(3, 4)$: we need only note that it contains precisely two elementary abelian normal subgroups of order 16 , one of which is $O_2(M) = Q$, which contain all the involutions of the Sylow 2-subgroup.

Let $P \in S_2(N)$, $R \in S_2(N(P))$. If $P = R$ then $P \in S_2(G)$, $\text{cl}(P) = 2$, and G is known. As R acts on the two elementary abelian subgroups of P of order 16 , with kernel of the action in $N(Q)$, we have $[R : P] = 2$. As $3 | o(N(P))$ we can choose $R < X \leq N(P)$ with $X/P \simeq \Sigma_3$ and the extension splits by a Sylow argument. By Lemma 2 *(iii)*, P char R , whence $R \in S_2(G)$. If t is an involution in $R\backslash P$, t interchanges Q and Q^t . Now $Q \cap Q^t = Z(P) = \Phi(P)$ admits a 3-element

in its normalizer, and by Lemma 2 *(iv)*, $\left| C_{Z(P)}(t) \right| \leq \left| Z(P) \right|^{\frac{1}{2}} = 2$. Thus t interchanges two involutions in $Z(P)$. Checking the outer automorphism group of P , we see that R is now completely determined. The arguments of Gorenstein and Harada [*10*, pp. 354-355] quickly produce an element of order 6 and finish the proof.

COROLLARY 7. *If a 2-local subgroup has order divisible by* 3 *, it is soluble.*

LEMMA 8. *Let* $S \in S_2(G)$. *Then* S *contains a characteristic subgroup* S_0 *of index* 2 *with* $\mathrm{cl}(S_0) \leq 2$. *Moreover,* $3 | o(N(S_0))$.

Proof. By Glauberman, some 2-group of G has an element of order 3 in its normalizer. Let Q be such a 2-group with maximal order and put $N = N(Q)$. By the Corollary 7, N is soluble and by the maximality of $o(Q)$ and the Frattini argument $Q = O_2(N)$. Let $R \in S_3(O_{2,2'}(N))$, $K = O_{2,2',2}(N)$. Then $K = O_{2,2'}(N).N_K(R)$. As R is cyclic and non-trivial, $N_K(R)$ has a Sylow 2-subgroup of order 2 and hence a subgroup of index 2 . It is immediate that $O_{2,2',2,2'}(N) = O_{2,2',2}(N) = N$. Moreover there exists $H \leq N$ with $H/Q \simeq \Sigma_3$. If $o(Q) = 4$, then Q is elementary abelian and self-centralizing in any Sylow 2-subgroup of G in which it appears. The Sylow 2-subgroups of G are then dihedral or semi-dihedral [*21*] and G is known [*1*] and [*13*, *14*, *15*]. No counterexample to our theorem lies in their list, whence $o(Q) > 4$. If $P \in S_2(N)$, then Q char P by Lemma 2 *(iii)*. Thus $P \in S_2(G)$, and the lemma follows from Lemma 2 *(ii)*, *(iii)*.

COROLLARY 8. *There are no perfect central extensions of* $Sz(8)$ *involved in* G .

Proof. [*2*] gives generators and relations for the perfect central extensions of $Sz(8)$. In all cases the Sylow 2-subgroup has class 3 and no subgroup of index 2 has class 2 .

REMARK. For the remainder of the proof, if $S \in S_2(G)$ then S_0 will denote the subgroup given in Lemma 8. Moreover, when no confusion is possible, if $X \leq S$ then X_0 will denote $X \cap S_0$.

LEMMA 9. $N(Q)/C(Q)$ *is soluble for any* 2-*group* Q *of* G .

Proof. Suppose that Q is a 2-group of G maximal subject to $N(Q)/C(Q)$ being non-soluble. Put $T = \mathrm{sol}\bigl(N \bmod C(Q)\bigr)$, where $N = N(Q)$ and let $K = \mathrm{soc}(N \bmod T)$. Let $Q_1 \in S_2(T)$. Then $N/T \simeq N\bigl(Q_1\bigr)/N_T\bigl(Q_1\bigr)$ is non-soluble, $C\bigl(Q_1\bigr) \subseteq N_T\bigl(Q_1\bigr)$ and by the maximality of Q , $Q = O_2(N) \in S_2(T)$, and K/T is a Suzuki group. It follows that $C(Q)$ is soluble and N/K has odd order.

Suppose that $Q \subset P \subseteq S$, with $P \in S_2(N)$, $S \in S_2(G)$. Now $Q_0 \subset P_0$ and $P_0' \subseteq Z\bigl(P_0\bigr)$, whence $Q_0 \subseteq C\bigl(P_0'\bigr)$ and $\bigl[Q : C_Q\bigl(P_0'\bigr)\bigr] \leq 2$. Choose another Sylow 2-subgroup P^* of N such that $\langle P, P^*\rangle$ involves a Suzuki group. Suppose that $P^* \subseteq S^* \in S_2(G)$, and set $P_0^* = S_0^* \cap P^*$. As above $\bigl[Q : C_Q\bigl(P_0^{*\prime}\bigr)\bigr] \leq 2$ and $\langle P_0', P_0^{*\prime}\rangle$ still involves a Suzuki group and centralizes $C_Q\bigl(P_0'\bigr) \cap C_Q\bigl(P_0^{*\prime}\bigr)$. But this latter group has index at most 4 in Q , whence $C_G(Q)$ involves a Suzuki group: a contradiction.

Gorenstein and Lyons have recently classified all simple groups with soluble 2-local subgroups [12]. Consequently G contains some non-soluble 2-local subgroups and, by Lemma 9, an involution with a non-soluble centralizer.

Let A be an elementary abelian 2-group of G maximal subject to

(*)   there exists $a \in A$ such that $C_G(\dot{a})$ involves a Suzuki group.

For the remainder of the proof we fix the following notation: $H = C_G(a)$, $T \in S_2(H)$, $S \in S_2(G)$, $A \leq T \leq S$, and ϕ is an element of order 3 in $N\bigl(S_0\bigr)$.

LEMMA 10. *We can choose* a, A *satisfying* (*) *and* A *maximal and have* $A \leq T_0$ *and* A ϕ-*invariant. In particular* $3 \mid o\bigl(N(A)\bigr)$.

Proof. If $A \leq T_0$ then $\langle A, A^\phi\rangle$ is abelian by Lemma 2 *(ii)*, and hence elementary abelian. By the maximality of $o(A)$, $A = \langle A, A^\phi\rangle$ is ϕ-invariant. Thus we may assume $A > A_0$.

Suppose we can choose $a \in A_0$. Then $\left\langle A_0, A_0^\phi \right\rangle$ is elementary abelian

and the maximality of $r(A)$ gives $r(A_0) \leq r\left(A_0 \quad A_0^\phi \right) + 1$. $A_0 \cap A_0^\phi$ is

ϕ-invariant and centralized by any $t \in A \backslash A_0$. By Lemma 2 *(iv)*,

$A_0 \cap A_0^\phi = 1$, $o(A_0) = 2$, $o(A) = 4$. As $o\left(\left\langle A_0, A_0^\phi \right\rangle \right) = 4$, we can change

our choice of a, A to satisfy the lemma.

Arguing by contradiction we may assume that there is no choice of a

such that $a \in A_0$. Now $\langle a \rangle \triangleleft S$, $S_0 \triangleleft S$, whence $S = \langle a \rangle \times S_0$ and

$H = \langle a \rangle \times H_0$, say. $L(H_0)$ is a central product of Suzuki groups. By [2],

the only Suzuki group which admits a perfect central extension is $Sz(8)$,

and by Corollary 8 this does not appear. Thus $L(H_0)$ is a direct product.

Moreover $A = B \times \langle a \rangle$ where $S_0 \geq B \in M(H_0)$. Let $t \in S_2 \backslash L(H_0)$ and

suppose t interchanges two components K_1 and K_2 of $L(H_0)$. Choosing

$x, y \in R \in S_2(K_1)$ with $[x, y] \neq 1$, we have $[x, y, t] \neq 1$ contrary to

$cl(S_0) = 2$. Thus S_0 normalizes each component of $L(H_0)$. Moreover all

outer automorphisms of Suzuki groups have odd order, whence $S_0 = U \times V$,

with $U \in S_2(C_{H_0}(L(H_0)))$ and $V \in S_2(L(H_0))$. Thus B is a direct

product of a member of $E(U)$ and a direct product of centres of Sylow

2-subgroups of Suzuki groups. If U is non-trivial we could choose

$a \in U < A$ to satisfy the lemma and similarly if $L(H_0)$ has more than one

component. Since $O(H) = 1$ by an earlier remark, $C_G(a) = H = \langle a \rangle \times K$,

where $K \leq aut(Sz(q))$. The result of Dempwolff [3] completes the proof.

LEMMA 11. *(i)* $L(H)C_H(L(H)) = L(H) \times C_H(L(H))$.

(ii) $L(H)$ *is a direct product of Suzuki groups,*

$$K_1 \times K_2 \times \ldots \times K_r .$$

(iii) Each K_i is A-invariant.

(iv) $A \cap K_i \in M(K_i)$, that is $A \cap K_i = Z(R_i)$, $R_i \in S_2(K_i)$.

Proof. The arguments of Lemma 10 give *(i)* and *(ii)*.

Choosing $t \in A$ and knowing $t \in T_0$ we can argue as in Lemma 10 with K_i and K_i^t to prove *(iii)*.

Now $A \in M(H)$ whence $A \in M(AK_i)$. From $AK_i = C_A(K_i) \times K_i$ it follows immediately that $A \cap K_i \in M(K_i)$ and *(iv)* is true.

Completion of the proof. By Lemma 10, $\phi \in N_G(A) \backslash C_G(A)$ whence $\phi \in N_G(S)$ for some $S \in S_2(C_G(A))$.

Let K be a component of H with $R \in S_2(K)$, $R > Z(R) = A \cap K$. Let $t \in C_G(A)$. Then $K \cap K^t \geq A \cap K$, whence $K = K^t$, $R = R^t$, and $R \leq S$. Moreover $S \subseteq C_G(A) \leq N_H(K)$ and, as $|\operatorname{aut} K : K|$ is odd, $S \subseteq C_H(K).K = C_H(K) \times K$. If we write $S = S_1 \times S_2 \times \ldots \times S_q$ with S_i indecomposable, then we may assume by the above argument that $R = S_1$. By the Krull-Schmidt Theorem, ϕ permutes the derived groups S_i' . If ϕ fixes S_1', it permutes the $2^{2n+1} - 1$ involutions in the group and so fixes at least one. On the other hand, choosing $t \in S_1'$,

$$S_1' \times (S_1')^\phi \times (S_1')^{\phi^2}$$ contains the involution $t.t^\phi.t^{\phi^2}$ fixed by ϕ .

In either case we have an element of order 6 contrary to assumption.

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