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TRANSMISSION OF VERIFICATION

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Abstract. This paper clarifies, revises, and extends the account of the transmission of truthmakers by core proofs that was set out in chap. 9 of Tennant (2017). Brauer provided two kinds of example making clear the need for this. Unlike Brouwer's counterexamples to excluded middle, the examples of Brauer that we are dealing with here establish the *need* for appeals to excluded middle when applying, to the problem of truthmaker-transmission, the already classical metalinguistic theory of model-relative evaluations.

§1. Introduction. In chap. 9 of Tennant (2017), an account was offered of the logical consequence established by any core proof of a sequent $\Delta : \varphi$, in terms of transforming (model-relative) truthmakers for its premises Δ into a truthmaker for its conclusion φ.

In this paper we describe two shortcomings of that account, and propose remedies to ensure that the revised account applies as generally as was originally intended. One of those remedies has as a corollary that one can extend the methods of truthmaker-transformation so as to deal with the logical consequences that are established by *classical* core proofs.

§2. Preliminaries. All chapter and page references, unless otherwise indicated, are to Tennant (2017). This paper relies on the reader's familiarity with the main concepts deployed in that work. For reasons of space we refrain from reprising any of their lengthy inductive definitions. Instead, we confine ourselves by giving page references for all of the definitions that the reader might require.

1. Rules of inference for Core Logic (pp. 118–121 for natural deduction; pp. 128–131 for sequent calculus).[1]
2. The inductively defined notion $\mathcal{P}(\Pi, \varphi, \Delta)$, using those rules. In words: Π is a (core) proof of the conclusion φ from the set Δ of premises (i.e., undischarged assumptions).
3. Model-relative rules of verification and of falsification (pp. 70–72).[2]
4. The co-inductively defined notions

$$\mathcal{V}(\Pi, \varphi, M, D)$$

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[1] The reader will also find these rules in Tennant (2012, 2015a, 2015b).
[2] The reader may also consult Tennant (2010, 2018) for these rules.

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— Π is a verification of φ with respect to model M with domain D; and

$$\mathcal{F}(\Pi, \varphi, M, D)$$

— Π is a falsification of φ with respect to model M with domain D, using those rules. (p. 59 *ff.*)

We shall freely use the synonyms 'truthmaker' and 'falsitymaker' for 'verification' and 'falsification' respectively. Both kinds of construction are called *evaluations*.[3]

5. Conditional M-relative constructs (p. 253). These are tree-like constructs which include, as extremal cases, M-relative evaluations at one end, and, at the other end, completely formal natural deductions in Core Logic.
6. The inductively defined notion

$$[\,\Pi, \Sigma\,]^M,$$

where Π is an M-relative truthmaker (typically, for a premise of Σ) and the argument Σ is a conditional M-relative construct. The reduct in question is intended to be an M-relative truthmaker for the conclusion of Σ. (See chapter 9.) The process of determining the reduct $[\Pi, \Sigma]^M$ is analogous to the process of normalizing natural deductions (or eliminating cuts in sequent proofs).

Main results proved in the book, which will be relied upon here, include the following:

1. Metatheorem 9: *Modulo* a metatheory which contains the mathematics of $\overline{\overline{D}}$-furcating trees of finite depth, we have, for all models M with domain D,

$$\exists \Pi\ \mathcal{V}(\Pi, \varphi, M, D) \quad \Leftrightarrow \quad \varphi \text{ is true in } M$$

where the right-hand side is in the sense of Tarski (p. 211).
2. Metatheorem 10 (*Bivalence of Evaluation*): for all φ, for all M interpreting the primitives in φ, either there exists an M-relative verification of φ or there exists an M-relative falsification of φ (p. 220).

In light of Metatheorem 9 just mentioned above, we can abbreviate *both* sides of its biconditional in the same way. The notation

$$M \Vdash \varphi$$

was chosen for this purpose, in order to reserve the symbol $\models$ for the relation of logical consequence, which holds between a set Δ of sentences (the premises) and a single sentence φ (the conclusion). With this choice of notation for 'model M makes sentence φ true', and with the usual Frobenian understanding of $M \Vdash \Delta$, one can then define classical logical consequence $\models$ in the standard way as follows:

Definition 1. $\Delta \models \psi \ \equiv_{df} \forall M (M \Vdash \Delta \Rightarrow M \Vdash \psi)$.

[3] There is no need to cover any of the existing literature on other kinds of truthmakers than the formal kind at issue here, which has been explained in Tennant (2010, 2018).

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§3. Background. The foregoing definition of logical consequence makes it amount to preservation of truth (in all models) from premises to conclusion. In any model, if the premises are all true, then so is the conclusion.

Suppose now that we are dealing with only finitely many premises (as would be the case with any proof). Let the statement of logical consequence under consideration accordingly be

$$\varphi_1, \dots, \varphi_n \models \psi.$$

With truth-in-M consisting in the *existence* of an M-relative truthmaker for the sentence in question, the quantificational structure of the definiens for logical consequence (the right-hand side of Definition 1 above) is the following:

$$\forall M((\exists\Pi_1\mathcal{V}(\Pi_1, \varphi_1, M, D) \wedge \cdots \wedge \exists\Pi_n\mathcal{V}(\Pi_n, \varphi_n, M, D) \Rightarrow \exists\Pi\mathcal{V}(\Pi, \psi, M, D)).$$

Note the existentials. There is no express relation between any witness (V, say), to the concluding existential ($\exists\Pi$), and witnesses ($V_1, \dots, V_n$, say) to the existentials ($\exists\Pi_1, \dots, \exists\Pi_n$) attesting to the truth-in-M of the premises $\varphi_1, \dots, \varphi_n$ respectively. With our analysis of truth as consisting in the existence of a truthmaker, however, the question now arises: might it be possible to characterize the extent to which a witness to the concluding existential can be concocted out of witnesses to the existentials expressing the truth of the premises? Put another way, using terminology familiar to the mathematical logician: might we be able to *Skolemize* the definiens (for logical consequence) as follows?

$$\begin{aligned}\forall M\,\exists f\,\forall\Pi_1 \dots \forall\Pi_n((\mathcal{V}(\Pi_1, \varphi_1, M, D) \wedge \cdots \wedge \mathcal{V}(\Pi_n, \varphi_n, M, D))\\ \Rightarrow \mathcal{V}(f(\Pi_1, \dots, \Pi_n), \psi, M, D)).\end{aligned}$$

The overarching aim of chapter 9 was to reveal the extent to which such Skolemization can be effected, if one is in possession of a *core proof* of the conclusion ψ from premises $\varphi_1, \dots, \varphi_n$. The aim was to show how, given—

- a *core proof* (not: *classical* core proof) Σ of the conclusion ψ from the premises $\varphi_1, \dots, \varphi_n$;[4]
- a model M for the extralogical vocabulary involved; and
- respective M-relative verifications $\Pi_1, \dots, \Pi_n$ of the premises $\varphi_1, \dots, \varphi_n$

—one can (*eventually*) 'calculate' an M-relative verification for the conclusion ψ of Σ. This was the import of Metatheorem 12 (pp. 258–259).

§4. The problem to be addressed. There is a lacuna in the proof of Metatheorem 12 of Tennant (2017), and it is the aim of this study to fill it. It is located at the fourth line of the list of Distribution conversions given on p. 256, in the lengthy inductive definition of the reduct $[\Pi, \Sigma]^M$ (the superscript M subsequently being systematically suppressed in the statements of conversions). Here is the culprit line:

$$[\Pi, \rightarrow_I\Sigma_1] = \{\rightarrow_I\}[\Pi, \Sigma_1].$$

[4] We shall see by the end of this study, however, that the methods to be developed to handle the second troublesome kind of example afford an extension to *classical* core proofs of the recipe for transforming truthmakers for premises into truthmakers for conclusions.

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We shall call this particular Distribution conversion the *inadequate conversion*. It is inadequate because it does not deal with the possibility that the antecedent of the conditional conclusion of Σ *might be an undischarged assumption of the immediate subproof* Σ_1. And for this case the expression $\{\rightarrow_I\}[\Pi, \Sigma_1]$ is not defined. The required *adequate* conversion needs to deal with this possibility. Otherwise that undischarged assumption of Σ_1 goes begging for a verification yet to be supplied. But if a falsification is available instead (for, remember, evaluations here are with respect to a particular model M), then it needs to be used for a terminal application of $\rightarrow$-Verification via the falsity (in M) of the antecedent. Details will emerge below.

It is important to stress for the reader, in advance, that all that is needed for the required 'fix' of the proof of Metatheorem 12 is that the inadequate conversion be replaced by an adequate one. This replacement does not occasion any further need for alterations or additions to the original proof. For this reason, we shall avoid reprising here any parts of that proof.

The driving idea behind the treatment of truthmaker transmission was the 'Skolemizing' one: that an M-relative verification for the conclusion ψ of a core proof Σ would be intimately and organically determined by the given M-relative verifications of the premises $\varphi_1, \ldots, \varphi_n$ of Σ, as well as by Σ itself. There could and would, in general, be yet other M-relative verifications of the conclusion—indeed, a great many of them—but these other M-relative verifications would not fall within the purview of possible outcomes of the 'calculation' of an M-relative verification for the conclusion from the given M-relative verifications of the premises of Σ, along with Σ.

The motivation was to show how certain ways of 'seeing that the premises are true in M'—that is, certain truthmakers for the premises—could be drawn upon, or exploited, by the proof Σ so that they would be able to be transformed into a closely related way of seeing that the conclusion of Σ is also true in M.

That one's calculation according to the transformations given in chapter 9 will *always, in general*, result in an M-relative verification for the conclusion of Σ turns out to be overly optimistic, as an example of Brauer will in due course show. (The example, as already intimated, involves the conditional.) The original treatment in chapter 9 succeeds in its aim if (but only if) the conditional is not taken as a primitive in the object language, and the core proof Σ has no zero-premise subproofs (including itself). Omitting the conditional from the language, of course, in the case of Core Logic, would reduce the expressive power of the object language. Here we shall replace the overly brief and inadequate treatment of the conditional with one that is more detailed, and that draws on the appropriate resources.

It was stressed in Observation 11 at p. 253 that model-relative verifications (and falsifications), on the one hand, and core proofs, on the other hand, were extremal special cases of constructions more generally (and somewhat ploddingly) called, in the text, 'conditional M-relative constructs'.[5] We shall repeat that Observation here:

> *M-relative truthmakers, M-relative falsitymakers, and core proofs are all conditional M-relative constructs.*

Conditional M-relative constructs that lie in the 'middle range'—ones that are neither M-relative evaluations (i.e., verifications or falsifications) nor core proofs—

[5] It would be good to have a snappier and more suggestive label. But at least 'conditional M-relative construct' is descriptively adequate.

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are similar to *evaluations* in containing applications of M-relative verification and/or falsification rules, but also similar to certain *core proofs* in having undischarged assumptions (premises) for which no M-relative evaluation (neither verification nor falsification) is yet in hand.

The ‘calculation’ referred to above would be carried out by applying the relevant transformations involved in the inductive definition of the notion

$$[\,\Pi, \Sigma\,]^{M},$$

where Π is an M-relative truthmaker (typically, for a premise of Σ) and the argument Σ is a conditional M-relative construct. It should be noted that, in the full language that contains the conditional as a primitive, *and with the inadequate conversion uncorrected*, the final output *could itself be a conditional M-relative construct with undischarged assumptions*, and therefore not (yet) a genuine or ‘thoroughbred’ truthmaker for its conclusion. This is what Brauer’s example clearly reveals.

§5. Two troublesome examples. There are two kinds of examples occasioning the need for extension and revision of the original treatment of how one converts truthmakers for premises into a truthmaker for the conclusion of a core proof. §5.1 deals with the first kind of example, and §5.3 with the second.

5.1. Examples where a conditional is inferred, with discharge of assumption. We can proceed at the propositional level. Let A and B be atoms interpreted (i.e., made true or false) by M. Suppose A is true in M. Consider the following (simplest possible) M-relative verification Π of A, and core proof Σ.

$$\Pi : \quad \overline{A}^{\,M} \qquad\qquad \Sigma : \quad \dfrac{\dfrac{A \qquad \overline{B}^{\,(1)}}{A \wedge B}}{B \to (A \wedge B)}{\scriptstyle (1)}$$

The defective definition of the binary operation $[\,,]^{M}$ (defective because of the inadequate conversion) that was given in chap. 9 of Tennant (2017) yields the result that

$$[\,\Pi, \Sigma\,]^{M} \;=\; \left[\; \overline{A}^{\,M} \;,\; \dfrac{\dfrac{A \qquad \overline{B}^{\,(1)}}{A \wedge B}}{B \to (A \wedge B)}{\scriptstyle (1)} \;\right]^{M} \;=\; \dfrac{\dfrac{\overline{A}^{\,M} \qquad B}{A \wedge B}}{B \to (A \wedge B)}$$

Note that this last reduct ends with an application of $\to$-Verification. *But* the assumption B remains undischarged, which makes it the case that the reduct, though final, is *not* an M-relative verification of its (conditional) conclusion.

In chapter 9, the superscript M in the notation $[\,,]^{M}$ was suppressed at a certain stage, to make for less clutter. But it remains the case that the determination of $[\Pi, \Sigma]^{M}$ could in general have recourse to facts about M. The main interest, of course, is *how*, and *to what extent*.

The final reduct on the right in the last display is not (for the reason just explained) an M-relative verification of $B \to (A \wedge B)$, but rather a conditional M-relative construct (because conditional upon B) with $B \to (A \wedge B)$ as its conclusion. This untoward result is owing to the inadequate conversion

$$[\Pi, \rightarrow_I \Sigma_1] \quad = \quad \{\rightarrow_I\}[\Pi, \Sigma_1]$$

in the definition of the binary operation $[\,,]^M$, on p. 256 of Tennant (2017).

In any truthmaker for a conditional, the final step is an application of the appropriate half of the rule $\rightarrow\mathcal{V}$ of $\rightarrow$-*Verification*:

$$\rightarrow\mathcal{V} \qquad \frac{\psi}{\varphi \rightarrow \psi} \qquad \dfrac{\begin{array}{c} \Box\,\overline{}^{(i)} \\ \varphi \\ \vdots \\ \bot \end{array}}{\varphi \rightarrow \psi}{\scriptstyle (i)}$$

Note that this is a full statement of $\rightarrow$-Verification. Conspicuously absent from it is any analogue of that part of the *deductive* rule of $\rightarrow$-*Introduction* (conventionally called Conditional Proof) that allows for the discharge of the assumption φ upon deriving ψ $(\neq \bot)$ from it:

$$\dfrac{\begin{array}{c} \overline{}^{(i)} \\ \varphi \\ \vdots \\ \psi \end{array}}{\varphi \rightarrow \psi}{\scriptstyle (i)}$$

In the foregoing version of Brauer's example, both A and B have been assumed to be atoms. More generally now, let us consider possibly complex sentences φ and ψ, all of whose logical vocabulary is completely interpreted by M. We have to consider an M-relative verification Π of φ, along with the 'same' core proof Σ, only now involving the sentences φ and ψ in place of A and B respectively.

$$\Pi \;:\; \begin{array}{c} V \\ \varphi \end{array} \qquad\qquad \Sigma \;:\; \dfrac{\dfrac{\varphi \qquad \overline{\psi}^{(1)}}{\varphi \wedge \psi}}{\psi \rightarrow (\varphi \wedge \psi)}{\scriptstyle (1)}$$

There are two possible general forms for the outcome in calculating the reduct

$$\big[\;\Pi, \Sigma\;\big]^M, \text{ i.e., } \big[\;V, \Sigma\;\big]^M,$$

if it is to be an M-relative *truthmaker* for $\psi \rightarrow (\varphi \wedge \psi)$. These are

$$\dfrac{\dfrac{\begin{array}{c} V \\ \varphi \end{array} \qquad \begin{array}{c} V' \\ \psi \end{array}}{\varphi \wedge \psi}}{\psi \rightarrow (\varphi \wedge \psi)}{\scriptstyle (1)}$$

and

$$\dfrac{\begin{array}{c} \overline{}^{(1)} \\ \psi \\ F \\ \bot \end{array}}{\psi \rightarrow (\varphi \wedge \psi)}{\scriptstyle (1)}$$

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depending, respectively, on whether ψ is true (courtesy of V'), or false (courtesy of F), in M. We say 'general forms' of outcomes for the calculation, because, although (*ex hypothesi*) the M-relative verification V of φ is assumed given, we have no guarantee that (in the first case) the choice of M-relative verification V' for ψ will be unique; nor do we have any guarantee that (in the second case) the choice of M-relative falsification F for ψ will be unique. In general, there may be many distinct M-relative truthmakers (falisitymakers, respectively) for a given sentence that is true (false) in M. Each of these represents a distinctive *way* in which the sentence enjoys the truth value that it does in M. All we know (see below) is that *there exists at least one* M-relative evaluation of ψ (as true, or as false, in M), and that any such evaluation can be deployed appropriately as just indicated.

The binary 'operation'

$$\left[\ \Pi, \Sigma\ \right]^M$$

is therefore not (yet) single-valued. We could, however, impose uniqueness of choice for M-relative evaluations of ψ by resorting to some lexicographic ordering of primitives, along with a well-ordering of the domain of M, and using those orderings to define a well-ordering of saturated terms and formulae, and thereafter of evaluations of ψ. If we were to do this systematically whenever an M-relative evaluation of any sentence needed to be determined, we would end up with a binary operation $\left[\ \Pi, \Sigma\ \right]^M$ that *is* single-valued.

5.2. The fix for the first class of examples. In this section we amend the definition of [,] given in chapter 9 to deal with Brauer's first class of examples.

Definition 2. *If $M \Vdash \varphi$, then let V^M_φ be the first M-relative verification of φ in the well-ordering of M-relative verifications.*

If $M, \varphi \Vdash \bot$, then let F^M_φ be the first M-relative falsification of φ in the well-ordering of M-relative falsifications.

(If M is decidable, then V^M_φ and F^M_φ can be found effectively.)

We now *restrict* applications of the inadequate distribution conversion

$$[\Pi, \rightarrow_I \Sigma_1] \quad = \quad \{\rightarrow_I\}[\Pi, \Sigma_1]$$

on p. 256 of Tennant (2017) to cases of the following two kinds:

(1) the terminal step of $\rightarrow$-Introduction in Σ inferring $\psi \rightarrow \theta$ involves no discharge of the antecedent ψ as an assumption in the subordinate proof Σ_1 of the consequent θ, i.e.

$$[\Pi, \rightarrow_I \Sigma_1] \ \text{ is } \left[\ \Pi \ ,\ \begin{array}{c}\Gamma\\ \Sigma_1\\ \rightarrow\text{-I}\ \dfrac{\theta}{\psi\rightarrow\theta}{\scriptstyle(i)}\end{array}\right];$$

(2) the terminal step of $\rightarrow$-Introduction in Σ inferring $\psi \rightarrow \theta$ has as its subordinate proof Σ_1 a *refutation* of the antecedent ψ:[6]

[6] Note that in case (2) the immediate subconstruction for the terminal step of $\rightarrow_\mathcal{V}$ in the reduct on the right (once truthmakers have eventually been [,]-applied on the left of the comma for every premise in Γ) will be an M-relative *falsification* of ψ.

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$$[\Pi, \rightarrow_I \Sigma_1] \quad \text{is} \quad \left[\ \Pi\ ,\ \begin{array}{c} \underbrace{\Gamma\ ,\ \overline{\psi}^{(i)}}_{\textstyle \Sigma_1} \\ \dfrac{\bot}{\psi \rightarrow \theta}{\scriptstyle (i)} \end{array}\ \right].$$

Cases (1) and (2), however, are not exhaustive. It remains to specify what $[\Pi, \rightarrow_I \Sigma_1]$ is to be in the important remaining case (overlooked in chapter 9) where the terminal step of $\rightarrow$-Introduction in Σ inferring $\psi \rightarrow \theta$ has θ as the conclusion of its subordinate proof Σ_1 *and discharges the antecedent* ψ, which is a premise of Σ_1. For this remaining case, we stipulate that

$$[\Pi, \rightarrow_I \Sigma_1]\ ,\ \text{i.e.} \left[\ \Pi\ ,\ \begin{array}{c} \underbrace{\Gamma\ ,\ \overline{\psi}^{(i)}}_{\textstyle \Sigma_1} \\ \dfrac{\theta}{\psi \rightarrow \theta}{\scriptstyle (i)} \end{array}\ \right],\ \text{is} \begin{cases} [V^M_\psi, \{\rightarrow_\nu\}[\Pi, \Sigma_1]] & \text{if } M \Vdash \psi \\[2ex] \begin{array}{c} \overline{\psi}^{(i)} \\ F^M_\psi \\ {\scriptstyle \rightarrow_\nu}\dfrac{\bot}{\psi \rightarrow \theta}{\scriptstyle (i)} \end{array} & \text{if } M, \psi \Vdash \bot. \end{cases}$$

The well-founded relation (the compound-argument complexity ordering) on which the definition of [,] is now based will accord highest priority to reducing the complexity of the embedded argument Σ. This will ensure that the proposed way of applying the 'new' truthmaker-argument V^M_ψ in the reduct in the case where $M \Vdash \psi$ will not obstruct the inductive definability of [,] overall. Note that by exploiting V^M_ψ in this way, we sustain the theme that the truthmakers supplied are 'intimately and organically' contributing to the determination of a final M-relative truthmaker for the conclusion of the original proof Σ. It is just that these 'newly supplied' truthmakers need not be among those originally given for the premises of Σ. To the extent that we might need to invoke such truthmakers, however, we make them (as far as is possible) contribute to the final determination of an M-relative truthmaker for the conclusion of Σ in the same way as do the truthmakers originally given for the premises of Σ.

5.3. Examples involving a logical theorem, i.e. a conclusion being inferred from no assumptions. Let us now assume, as we go forward, that we have secured in the manner described above the single-valuedness of the operation $\left[\ \Pi, \Sigma\ \right]^M$. We shall then be in a position to address another problem that Brauer has raised—this time arising from the possibility that Σ might include a subproof with no undischarged assumptions. The following core proof Σ illustrates this kind of problem.

$$\Sigma \quad : \quad \dfrac{\begin{array}{c} \emptyset \\ \Xi \\ \varphi \end{array} \qquad \psi}{\varphi \wedge \psi}$$

Here, as indicated by the empty-set symbol $\emptyset$, the subproof Ξ establishes φ from *no premises at all*.

Suppose now that one is given an M-relative verification Π of the sole premise ψ of Σ. How does one obtain from it an M-relative verification of $\varphi \wedge \psi$? That is, what

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should be the output in this case of the operation $\left[\, \Pi, \Sigma \,\right]^M$? The answer should be reasonably obvious. It will be an M-relative verification of the form

$$\dfrac{\begin{matrix} V & \Pi \\ \varphi & \psi \end{matrix}}{\varphi \wedge \psi}$$

where V is an M-relative verification of φ.

The choice of V could be determined by the well-ordering method we are assuming has been adopted so as to be able to make such choices in general. All we need, of course, is the assurance that *there exists* an M-relative verification of ψ. But we have such assurance. First, note the following.

LEMMA 1 (Soundness of Core Logic). *Let M be a model for a language L. Let Δ be a finite set of sentences in L, and let φ be a sentence in L. Then for all Π, if $\mathcal{P}(\Pi, \varphi, \Delta)$, then if $M \Vdash \Delta$, then $M \Vdash \varphi$.*

Proof. Taken for granted, since Core Logic is a subsystem of Classical Logic, which we know to be sound. □

So the rules of Core Logic are sound, i.e. validity-preserving. Suppose now that φ is the conclusion of a core proof from the empty set of premises (whence $\Delta = \emptyset$). It follows by Lemma 1 that φ is true in any model interpreting its extralogical vocabulary. Hence by Metatheorem 9, there exists an M-relative verification of φ.

We can, however, improve somewhat on this rather brute method of determining V. In §6 we shall prove a result about truthmakers for logical theorems that reveals more detail about their structures. This affords a fix for the second troublesome kind of examples.

§6. Truthmakers for logical theorems. In this section we will prove a result guaranteeing the existence of truthmakers for logical theorems. This result forms the basis of our solution to Brauer's second class of problems.

Metatheorem 1 in Tennant (2017), at p. 74 has as an easy corollary the following:

Noncontradictoriness of Evaluation. For any coherent model M interpreting the extralogical vocabulary of φ, it is not the case both that there exists an M-relative verification of φ and that there exists an M-relative falsification of φ. Hence also: for no such sentence φ is there an M-relative verification of φ and an M-relative verification of $\neg\varphi$.

The following result about Core Logic is easy to extend to Classical Core Logic; but the more modest form (that for just Core Logic) is all we need for present purposes.

METATHEOREM 1. *Let M be a model for a language L. Let φ be a sentence in L. Then for all core proofs Π:*

(i) *if $\mathcal{P}(\Pi, \varphi, \emptyset)$, then there is an M-relative verification of φ; and*
(ii) *if $\mathcal{P}(\Pi, \bot, \{\varphi\})$, then there is an M-relative falsification of φ.*

Proof. By induction on the complexity of φ.

Basis: φ is an atomic sentence A. There is no core proof or disproof of any atomic sentence. So the antecedents of the conditionals (i) and (ii) are false. Hence (i) and (ii) are true.

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Inductive hypothesis: Suppose (i) and (ii) hold for cases less complex than the one being dealt with.

Inductive step: We show by cases that (i) and (ii) hold for φ of the forms $\neg\psi$, $\psi \wedge \theta$, $\psi \vee \theta$, $\psi \to \theta$, $\exists x\psi$, and $\forall x\psi$.

Case (1). φ is $\neg\psi$.

Ad (i). Suppose $\mathcal{P}(\Pi, \varphi, \emptyset)$, i.e. $\mathcal{P}(\Pi, \neg\psi, \emptyset)$. Then the proof Π has the form

$$\dfrac{\begin{array}{c}\overline{\psi}^{\,(i)}\\ \Pi_1\\ \bot\end{array}}{\neg\psi}\,(i)$$

By IH(ii), there is an M-relative falsification of ψ. Call it F. Then

$$\dfrac{\begin{array}{c}\overline{\psi}^{\,(i)}\\ F\\ \bot\end{array}}{\neg\psi}\,(i)$$

is an M-relative verification of $\neg\psi$. Its final step is an application of the rule $\neg\mathcal{V}$.

Ad (ii). Suppose $\mathcal{P}(\Pi, \bot, \{\varphi\})$, i.e. $\mathcal{P}(\Pi, \bot, \{\neg\psi\})$. Then either

(a) Π has the form $$\dfrac{\neg\psi \qquad \begin{array}{c}\emptyset\\ \Pi_1\\ \psi\end{array}}{\bot}$$

or

(b) Π has the form $$\dfrac{\neg\psi \qquad \begin{array}{c}\neg\psi\\ \Pi_1\\ \psi\end{array}}{\bot}$$

First consider case (a). By IH(i) there is an M-relative verification of ψ. Call it V. Then

$$\dfrac{\neg\psi \qquad \begin{array}{c}V\\ \psi\end{array}}{\bot}$$

is an M-relative falsification of $\neg\psi$. Its final step is an application of the rule $\neg\mathcal{F}$.

Now consider case (b). Suppose there is an M-relative falsification of ψ. Call it F. Then

$$\dfrac{\begin{array}{c}\overline{\psi}^{\,(i)}\\ F\\ \bot\end{array}}{\neg\psi}\,(i)$$

is an M-relative verification of $\neg\psi$. Its final step is an application of the rule $\neg\mathcal{V}$. It follows by Lemma 1 applied to Π_1 that there is an M-relative verification of ψ. By Noncontradictoriness of Evaluation, this contradicts our supposition, since M

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is coherent. So there is no M-relative falsification of ψ. It follows by Bivalence of Evaluation that there is an M-relative verification of ψ. Call it V. Then

$$\dfrac{\neg\psi \qquad \begin{matrix} V \\ \psi \end{matrix}}{\bot}$$

is an M-relative falsification of $\neg\psi$. Its final step is an application of the rule $\neg\mathcal{F}$.

Case (2). φ is $\psi \wedge \theta$.

Ad (i). Suppose $\mathcal{P}(\Pi, \varphi, \emptyset)$, i.e. $\mathcal{P}(\Pi, \psi \wedge \theta, \emptyset)$. Then the proof Π has the form

$$\dfrac{\begin{matrix} \emptyset \\ \Pi_1 \\ \psi \end{matrix} \qquad \begin{matrix} \emptyset \\ \Pi_2 \\ \theta \end{matrix}}{\psi \wedge \theta}$$

By IH(i) applied to Π_1 and Π_2, there are M-relative verifications of ψ and of θ. Call them V_1 and V_2 respectively. Then

$$\dfrac{\begin{matrix} V_1 \\ \psi \end{matrix} \qquad \begin{matrix} V_2 \\ \theta \end{matrix}}{\psi \wedge \theta}$$

is an M-relative verification of $\psi \wedge \theta$. Its final step is an application of the rule $\wedge\mathcal{V}$.

Ad (ii). Suppose $\mathcal{P}(\Pi, \bot, \{\varphi\})$, i.e. $\mathcal{P}(\Pi, \bot, \{\psi \wedge \theta\})$. Then the proof Π has the form

$$\dfrac{\psi \wedge \theta \qquad \begin{matrix} \underbrace{{}^{(i)}\overline{\psi},\ \overline{\theta}^{(i)}} \\ \Pi_1 \\ \bot \end{matrix}}{\bot}\,(i)$$

By Lemma 1 applied to Π_1, it is not the case both that there is an M-relative verification of ψ and that there is an M-relative verification of θ. It follows by the Bivalence of Evaluation in the object language and by bivalence in the metalanguage that either (a) there is an M-relative falsification of ψ or (b) there is an M-relative falsification of θ.

In case (a), let the falsification of ψ be called F. Then

$$\dfrac{\psi \wedge \theta \qquad \begin{matrix} \overline{\psi}^{(i)} \\ F \\ \bot \end{matrix}}{\bot}\,(i)$$

is an M-relative falsification of $\psi \wedge \theta$. Its final step is an application of the rule $\wedge\mathcal{F}$.

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In case (b), let the falsification of θ likewise be called F. Then

$$\dfrac{\psi \wedge \theta \qquad \begin{array}{c} \overline{\theta}^{\,(i)} \\ F \\ \bot \end{array}}{\bot}\,(i)$$

is an M-relative falsification of $\psi \wedge \theta$. Its final step is an application of the rule $\wedge\mathcal{F}$.

Case (3). φ is $\psi \vee \theta$.

Ad (i). Suppose $\mathcal{P}(\Pi, \varphi, \emptyset)$, i.e. $\mathcal{P}(\Pi, \psi \vee \theta, \emptyset)$. Then the proof Π has one of the following two forms:

$$\text{(a)} \quad \dfrac{\begin{array}{c} \emptyset \\ \Pi_1 \\ \psi \end{array}}{\psi \vee \theta} \qquad\qquad \text{(b)} \quad \dfrac{\begin{array}{c} \emptyset \\ \Pi_1 \\ \theta \end{array}}{\psi \vee \theta}$$

In case (a), by IH(i) applied to Π_1, there is an M-relative verification V, say, of ψ. Then

$$\dfrac{\begin{array}{c} V \\ \psi \end{array}}{\psi \vee \theta}$$

is an M-relative verification of $\psi \vee \theta$. Its final step is an application of the rule $\vee\mathcal{V}$.

Similarly in case (b), by IH(i) applied to Π_1, there is an M-relative verification V, say, of θ. Then

$$\dfrac{\begin{array}{c} V \\ \theta \end{array}}{\psi \vee \theta}$$

is an M-relative verification of $\psi \vee \theta$. Its final step is an application of the rule $\vee\mathcal{V}$.

Ad (ii). Suppose $\mathcal{P}(\Pi, \bot, \{\varphi\})$, i.e. $\mathcal{P}(\Pi, \bot, \{\psi \vee \theta\})$. Then Π has the form

$$\dfrac{\psi \vee \theta \qquad \begin{array}{c} \overline{\psi}^{\,(i)} \\ \Pi_1 \\ \bot \end{array} \qquad \begin{array}{c} \overline{\theta}^{\,(i)} \\ \Pi_2 \\ \bot \end{array}}{\bot}\,(i)$$

By IH(ii) applied to both Π_1 and Π_2, there are M-relative falsifications F_1 of ψ and F_2 of θ, respectively, that can feature thus:

$$\dfrac{\psi \vee \theta \qquad \begin{array}{c} \overline{\psi}^{\,(i)} \\ F_1 \\ \bot \end{array} \qquad \begin{array}{c} \overline{\theta}^{\,(i)} \\ F_2 \\ \bot \end{array}}{\bot}\,(i)$$

in an M-relative falsification of $\psi \vee \theta$ whose final step is an application of the rule $\vee\mathcal{F}$.

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Case (4). φ is $\psi \to \theta$.

Ad (i). Suppose $\mathcal{P}(\Pi, \varphi, \emptyset)$, i.e. $\mathcal{P}(\Pi, \psi \to \theta, \emptyset)$. Then the proof Π has one of the following three forms:

$$\text{(a)}\quad \dfrac{\begin{matrix}\emptyset\\ \Pi_1\\ \theta\end{matrix}}{\psi\to\theta} \qquad\qquad \text{(b)}\quad \dfrac{\begin{matrix}\overline{\psi}^{\,(i)}\\ \Pi_1\\ \bot\end{matrix}}{\psi\to\theta}{\scriptstyle (i)} \qquad\qquad \text{(c)}\quad \dfrac{\begin{matrix}\overline{\psi}^{\,(i)}\\ \Pi_1\\ \theta\end{matrix}}{\psi\to\theta}{\scriptstyle (i)}$$

In case (a), by IH(i) applied to Π_1, there is an M-relative verification, V say, of θ, which can feature thus:

$$\dfrac{\begin{matrix}V\\ \theta\end{matrix}}{\psi\to\theta}$$

in an M-relative verification of $\psi \to \theta$ whose final step is an application of the rule $\to\mathcal{V}$.

In case (b), by IH(ii) applied to Π_1, there is an M-relative falsification, F say, of ψ, which can feature thus:

$$\dfrac{\begin{matrix}\overline{\psi}^{\,(i)}\\ F\\ \bot\end{matrix}}{\psi\to\theta}{\scriptstyle (i)}$$

in an M-relative verification of $\psi \to \theta$ whose final step is an application of the rule $\to\mathcal{V}$.

In case (c), suppose on one hand that there is an M-relative falsification, F say, of ψ. Then use F as in case (b) to obtain an M-relative of $\psi \to \theta$.

Now suppose on the other hand that there is no M-relative falsification of ψ. Then by Bivalence of Evaluation there is an M-relative verification of ψ. By Lemma 1 applied to Π_1, it follows that there is an M-relative verification, V say, of θ. Then use V as in case (a) to obtain an M-relative verification of $\psi \to \theta$.

Ad (ii). Suppose $\mathcal{P}(\Pi, \bot, \{\varphi\})$, i.e. $\mathcal{P}(\Pi, \bot, \{\psi \to \theta\})$. Then the proof Π has the form

$$\dfrac{\psi\to\theta \qquad \begin{matrix}\emptyset\\ \Pi_1\\ \psi\end{matrix} \qquad \begin{matrix}\overline{\psi}^{\,(i)}\\ \Pi_2\\ \bot\end{matrix}}{\bot}{\scriptstyle (i)}$$

By IH(i) applied to Π_1 there is an M-relative verification, V say, of ψ. By IH(ii) applied to Π_2 there is an M-relative falsification, F say, of ψ. Then V and F can feature thus:

$$\dfrac{\psi\to\theta \qquad \begin{matrix}V\\ \psi\end{matrix} \qquad \begin{matrix}\overline{\psi}^{\,(i)}\\ F\\ \bot\end{matrix}}{\bot}{\scriptstyle (i)}$$

in an M-relative falsification of $\psi \to \theta$ whose final step is an application of the rule $\to\mathcal{F}$.

Case (5). φ is $\exists x\psi$.

Ad (i). Suppose $\mathcal{P}(\Pi, \varphi, \emptyset)$, i.e. $\mathcal{P}(\Pi, \exists x\psi, \emptyset)$. Then the proof Π has the following form:

$$\begin{array}{c}\emptyset\\ \Pi_1\\ \psi^x_t \\ \hline \exists x\psi\end{array}$$

Let α be the denotation, in M, of the term t. Then $(\Pi_1)^t_\alpha$ is a saturated proof to which IH(i) applies. Thus there exists an M-relative verification, V say, of ψ^x_α. Then V can feature thus:

$$\begin{array}{c}V\\ \psi^x_\alpha \\ \hline \exists x\psi\end{array}$$

in an M-relative verification of $\exists x\psi$ whose final step is an application of the rule $\exists\mathcal{V}$.

Ad (ii). Suppose $\mathcal{P}(\Pi, \bot, \{\varphi\})$, i.e. $\mathcal{P}(\Pi, \bot, \{\exists x\psi\})$. Then the proof Π has the form

$$\dfrac{\exists x\psi \qquad \begin{array}{c}\overline{\psi^x_a}^{(i)}\\ \Pi_1\\ \bot\end{array}}{\bot}{\scriptstyle (i)}$$

For each α in the domain D of M, consider the saturated disproof

$$\begin{array}{c}\psi^x_\alpha\\ (\Pi_1)^a_\alpha\\ \bot\end{array}$$

By IH(ii) there is an M-relative falsification F_α, say, of ψ^x_α. These falsifications F_α can be used thus:

$$\dfrac{\exists x\psi \qquad \left\{\begin{array}{c}\overline{\psi^x_\alpha}^{(i)}\\ F_\alpha\\ \bot\end{array}\right\}_{\alpha\in D}}{\bot}{\scriptstyle (i)}$$

to obtain an M-relative falsification $\exists x\psi$. Its final step is an application of the rule $\exists\mathcal{F}$.

Case (6). φ is $\forall x\psi$.

Ad (i). Suppose $\mathcal{P}(\Pi, \varphi, \emptyset)$, i.e. $\mathcal{P}(\Pi, \forall x\psi, \emptyset)$. Then the proof Π has the form

$$\begin{array}{c}\emptyset\\ \Pi_1\\ \psi^x_a \\ \hline \forall x\psi\end{array}$$

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For each α in the domain D of M, consider the saturated proof

$$\begin{array}{c} \emptyset \\ (\Pi_1)^a_\alpha \\ \psi^x_\alpha \end{array}$$

By IH(i) there is an M-relative verification V_α, say, of ψ^x_α. These verifications V_α can be used thus:

$$\frac{\left\{ \begin{array}{c} V_\alpha \\ \psi^x_\alpha \end{array} \right\}_{\alpha \in D}}{\forall x \psi}$$

to obtain an M-relative verification of $\forall x \psi$. Its final step is an application of the rule $\forall \mathcal{V}$.

Ad (ii). Suppose $\mathcal{P}(\Pi, \bot, \{\varphi\})$, i.e. $\mathcal{P}(\Pi, \bot, \{\forall x \psi\})$. Then the proof Π has the form

$$\dfrac{\forall x \psi \qquad \begin{array}{c} \underbrace{\overline{\psi^x_{t_1}}^{(i)}, \ldots, \overline{\psi^x_{t_n}}^{(i)}} \\ \Pi_1 \\ \bot \end{array}}{\bot}{\scriptstyle (i)}$$

Let the denotations of $t_1, \ldots, t_n$ in D be $\alpha_1, \ldots, \alpha_n$ respectively. By Lemma 1 applied to Π_1, there cannot be verifications for all of $\psi^x_{\alpha_1}, \ldots, \psi^x_{\alpha_n}$. It follows by the Bivalence of Evaluation in the object language and by bivalence in the metalanguage that for some k $(1 \leq k \leq n)$ there is an M-relative falsification, F_{α_k} say, of $\psi^x_{\alpha_k}$. We can use F_{α_k} thus:

$$\dfrac{\forall x \psi \qquad \begin{array}{c} \overline{\psi^x_{\alpha_k}}^{(i)} \\ F_{\alpha_k} \\ \bot \end{array}}{\bot}{\scriptstyle (i)}$$

to obtain an M-relative falsification of $\forall x \psi$ whose final step is an application of the rule $\forall \mathcal{F}$. □

We are now in a position to take care of Brauer's second class of examples. All that is called for is more careful consideration of the basis, or 'grounding' cases, in the inductive definition of $[\Pi, \Sigma]^M$. One simply adds a *new grounding case*. (Its noninclusion in the book, at p. 254, is precisely what gives rise to Brauer's second class of examples.) Consider the case where $c\Pi \notin p\Sigma$ (i.e., where the conclusion of Π is not a premise of Σ). What the book failed to note was that this might be because $p\Sigma = \emptyset$. In such a case, $c\Sigma$ is a logical theorem. Take for the sought truthmaker $[\Pi, \Sigma]^M$ the truthmaker for $c\Sigma$ that is determined by the well-ordering method (explained at the end of §5.1) that we are (safely) assuming has been adopted so as to be able to make such choices in general. The recursion imposed by the definition of $[\Pi, \Sigma]^M$ ensures that every zero-premise subproof of a proof Σ that *has* a premise will be dealt with by this provision, when such a subproof comes up for consideration 'as a Σ' in its own right—that is, as the second argument in an application of the binary function [,].

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We conclude this section by revisiting the topic to which we referred in footnote 4. The binary function $[\Pi, \Sigma]^M$ can now be extended (in its domain) so as to apply to *classical* core proofs Σ as the process of normalization begins. This is effected by the simple stratagem of classicizing by means of the rule of Double Negation Elimination:

$$\frac{\neg\neg\varphi}{\varphi}$$

Any truthmaker for $\neg\neg\varphi$ contains a falsitymaker for $\neg\varphi$; and this in turn contains a truthmaker for φ.

§7. Conclusion. According to the Tarskian conception, logical consequence is a matter of truth preservation. The motivating idea for the treatment in chapter 9 of Tennant (2017) was that logical consequence can alternatively be thought of as a matter of truthmaker-transformation. That work aimed to show that there exists a uniform, general method of using a core proof of conclusion φ with premises Δ to transform M-relative truthmakers for the premises in Δ into an M-relative truthmaker for φ. As the examples above reveal, however, there were two oversights in the treatment of chapter 9 that prevented it from achieving that aim in full generality. The procedure $[\ ,\ ]^M$ defined there could fail to terminate in a truthmaker for φ when the core proof of φ from Δ either (i) contains an application of $\rightarrow$-I that discharges an assumption, or (ii) contains a subproof that has no undischarged premises. This paper makes up for both of those oversights.

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