Notes

106.31 What proportion of square-free numbers are divisible by 2? or by 30 but not by 7?

In 1885 Gegenbauer proved that the natural density of the set of square-free integers, i.e., the proportion of natural numbers which are square-free, is $6/\pi^2$ [1, Theorem 333; reference on page 272]. In 2008 J. A. Scott conjectured that the proportion of natural numbers which are odd square-free numbers is $4/\pi^2$ or, equivalently, the proportion of natural numbers which are square-free and divisible by 2 is $2/\pi^2$ [2]. The conjecture was proven in 2010 by G. J. O. Jameson, in an argument adapted from one computing the natural density of the set of all square-free numbers [3]. In this note we give a simple elementary argument which uses the classical result for all square-free numbers to reprove Jameson's result and in fact to generalise it:

Theorem: Let $p_1, p_2, \ldots, p_k$ be distinct primes and $0 \leqslant i \leqslant k$. Then the proportion of all natural numbers which are square-free and for all $1 \leqslant j \leqslant k$ are divisible by p_j if, and only if, $j \leqslant i$ is

$$\frac{6}{\pi^2} \prod_{1 \leqslant j \leqslant k} \frac{1}{p_j + 1} \prod_{k \geqslant j > i} p_j.$$

The term *numbers* will always refer to positive integers. Empty products, such as $\prod_{k > j > k} p_j$, are understood to equal 1.

Example 1: Set $k = 1$ and $p_1 = 2$. Setting $i = 0$ in the theorem we see that the natural density of the set of odd square-free numbers is $\frac{6}{\pi^2}\frac{2}{2+1} = \frac{4}{\pi^2}$; taking $i = 1$ we see that the natural density of the set of even square-free numbers is $\frac{6}{\pi^2}\frac{1}{2+1} = \frac{2}{\pi^2}$. Thus one third of the square-free numbers are even and two thirds are odd. (These are Jameson's results of course.)

Example 2: Set $p_1, p_2, \ldots, p_k = 2, 3, 5, 7$ and $i = 3$ in the theorem. Then the theorem says that the natural density of the set of square-free numbers divisible by 30 but not by 7 is $\frac{6}{\pi^2}\frac{1}{2+1}\frac{1}{3+1}\frac{1}{5+1}\frac{7}{7+1}$, so the proportion of square-free numbers which are divisible by 30 but not by 7 is $\frac{1}{2+1}\frac{1}{3+1}\frac{1}{5+1}\frac{7}{7+1} = \frac{7}{576}$.

For any real number x and set B of numbers, we let $B[x]$ denote the number of elements t of B with $t \leqslant x$. Recall that if $\lim_{x \to \infty} \frac{B[x]}{x}$ exists, then it is by definition the *natural density* of B [4, Definition 11.1].

Let A denote the set of square-free numbers. Suppose r and s are relatively prime square-free numbers. Then we let $A(r, s)$ denote the set of

https://doi.org/10.1017/mag.2022.123 Published online by Cambridge University Press

elements of A which are divisble by r and relatively prime to s (so, for example, $A = A(1, 1)$). The set of square-free numbers analysed in the theorem is $A(p_1 p_2 \dots p_i, p_{i+1} p_{i+2} \dots p_k)$.

Lemma 1: For r, s and x as above, we have $A(r, s)[x] = A(rs, 1)[xs]$.

Proof: This is immediate from the fact that multiplication by s gives a bijection from the set of elements of $A(r, s)$ less than or equal to x to the set of elements of $A(rs, 1)$ less than or equal to xs.

This lemma implies that the calculation of the natural density of the sets $A(r, s)$ reduces to the calculation of the natural density of the sets of the form $A(t, 1)$. More precisely we have

Lemma 2: If the set $A(rs, 1)$ has natural density D, then the set $A(r, s)$ has natural density sD.

Proof: The previous lemma tells us that

$$\frac{A(r, s)[x]}{x} = s\frac{A(rs, 1)[xs]}{xs}.$$

The lemma follows from taking the limit as x (and hence xs) goes to infinity.

The theorem itself will follow from the previous lemma if we can prove the theorem in the case of sets of the form $A(t, 1)$. The theorem in this case is proved by induction on k; thus the next lemma completes the proof of the theorem since it gives the required induction step.

Lemma 3: Let p be a prime number not dividing the square-free number t. If the set $A(t, 1)$ has natural density D, then the set $A(pt, 1)$ has natural density $\frac{1}{p+1}D$.

Proof: For any real number x we set $E(x) = A(pt, 1)[x]$. Let $\varepsilon > 0$. The lemma says that

$$\lim_{x \to \infty} \frac{E(x)}{x} = \frac{1}{p + 1}D.$$

Therefore it suffices to show for all choices of ε above that, for all sufficiently large x (depending on ε),

$$\left|\frac{E(x)}{x} - \frac{1}{p + 1}D\right| < \varepsilon.$$

Note that $A(t, 1)$ is the disjoint union of $A(tp, 1)$ and $A(t, p)$. Hence by Lemma 1 for any real number x,

$$A(t, 1)\left[\frac{x}{p}\right] = E\left(\frac{x}{p}\right) + E(x)$$

and so by the definition of D there exists a number M such that if $x > M$

https://doi.org/10.1017/mag.2022.123 Published online by Cambridge University Press

then

$$\left|\frac{E(x)}{x/p} + \frac{E(x/p)}{x/p} - D\right| < \frac{\varepsilon}{3}.$$

We next pick an even integer k such that $\frac{1}{p^k} < \frac{\varepsilon}{3}$. Then

$$\left|E\left(\frac{x}{p^k}\right)\right| \leqslant \frac{x}{p^k} < \frac{\varepsilon}{3}x \tag{1}$$

and also (using the usual formula for summing a geometric series)

$$\left|-Dx\sum_{i=1}^{k}\left(-\frac{1}{p}\right)^i - Dx\frac{1}{p+1}\right| = Dx\left|\frac{\left(-\frac{1}{p}\right) - \left(-\frac{1}{p}\right)^{k+1}}{1 - \left(-\frac{1}{p}\right)} + \frac{1}{p+1}\right| \tag{2}$$

$$= Dx\left|\frac{-1 + \frac{1}{p^k}}{p+1} + \frac{1}{p+1}\right| < \frac{1}{p^k}Dx < \frac{\varepsilon}{3}Dx < \frac{\varepsilon}{3}x.$$

Now suppose that $x > p^k M$. Then for all $i \leqslant k$ we have $x/p^i > M$ and hence (applying the choice of M above),

$$\left|E(x) + E\left(\frac{x}{p}\right) - D\frac{x}{p}\right| < \frac{\varepsilon}{3}\frac{x}{p} \tag{3}$$

and similarly

$$\left|-E\left(\frac{x}{p}\right) - E\left(\frac{x}{p^2}\right) + D\frac{x}{p^2}\right| < \frac{\varepsilon}{3}\frac{x}{p^2}$$

and

$$\left|E\left(\frac{x}{p^2}\right) + E\left(\frac{x}{p^3}\right) - D\frac{x}{p^3}\right| < \frac{\varepsilon}{3}\frac{x}{p^3}$$

and

$$\left|-E\left(\frac{x}{p^3}\right) - E\left(\frac{x}{p^4}\right) + D\frac{x}{p^4}\right| < \frac{\varepsilon}{3}\frac{x}{p^4}$$

$$\vdots$$

$$\left|-E\left(\frac{x}{p^{k-1}}\right) - E\left(\frac{x}{p^k}\right) + D\frac{x}{p^k}\right| < \frac{\varepsilon}{3}\frac{x}{p^k}. \tag{4}$$

Using the triangle inequality to combine the inequalities (1) and (2) together with all those between (3) and (4) (inclusive) and dividing through by x, we can conclude that

$$\left|\frac{E(x)}{x} - \frac{1}{p+1}D\right| < \frac{\varepsilon}{3}\left(\sum_{i=1}^{k}\frac{1}{p^i}\right) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon,$$

which was to be proved.

https://doi.org/10.1017/mag.2022.123 Published online by Cambridge University Press

Remark 1: In the language of probability theory, the theorem says that the probability that a number in A is divisible by a prime p is $1/(p+1)$ (so the probability that it is not is $p/(p+1)$) and, moreover, for any finite set S of primes not equal to p, being divisible by p is independent of being divisible by all of the elements of S.

Remark 2: If we only know that A has a natural density (but not its value $6/\pi^2$), then the arguments above can still be used to compute the proportion of square-free numbers which are in $A(r, s)$ for any numbers r, s which are relatively prime.

Remark 3: One can generalise the Theorem and give a simple formula for the natural density of the set of square-free numbers which are divisible by each of the elements in a finite set of primes and not divisible by any of the elements in a possibly infinite second set of primes (disjoint from the first of course). For example, the formula would imply that the set of square-free numbers which are not divisible by any of the infinite number of primes congruent to 1 modulo 4 has natural density 0 [5].

References

1. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers* (5th edn.) Oxford University Press (1979).
2. J. A. Scott, Square-free integers once again, *Math. Gaz.* **92** (March 2008) pp. 70-71.
3. G. J. O. Jameson, Even and odd square-free numbers, *Math. Gaz.* **94** (March 2010) pp. 123-127.
4. I. Niven and H. S. Zuckerman, *An Introduction to the Theory of Numbers* (4th edn.) Wiley (1980).
5. R. Brown, The natural density of some sets of square-free numbers, *Integers* **21** (2021)#A81

10.1017/mag.2022.123

RON BROWN
Department of Mathematics,
University of Hawaii,
2565 McCarthy Mall, Honolulu, Hawaii 96822
e-mail: *ron@math.hawaii.edu*

106.32 Pentagonal numbers and their relationships to other figurate numbers

Proposition: Where P_n, T_n, h_n and S_n stand for the nth pentagonal number, the nth triangular number, the nth hexagonal number, and the nth star number, respectively; for $n \in N$, the following identities hold:

$$2P_n = h_n + (2n - 1) \qquad P_n + P_{n+1} + n = h_{n+1} \qquad 3P_n = T_{3n-1}$$
$$4P_n + n = P_{2n} \qquad 6P_n = h_{2n} - h_n \qquad 12P_n = S_{2n} - S_n$$