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THE STRONG TREE PROPERTY AT SUCCESSORS OF SINGULAR CARDINALS

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Abstract. An inaccessible cardinal is strongly compact if, and only if, it satisfies the strong tree property. We prove that if there is a model of ZFC with infinitely many supercompact cardinals, then there is a model of ZFC where $\aleph_{\omega+1}$ has the strong tree property. Moreover, we prove that every successor of a singular limit of strongly compact cardinals has the strong tree property.

§1. Introduction.

The strong tree property is a strong generalization of the usual tree property. Given a regular cardinal κ, we say that κ has the tree property when every κ-tree (i.e., every tree of height κ with levels of size less than κ) has a branch of length κ. König's Lemma establishes that the tree property holds at $\aleph_0$. On the other hand, $\aleph_1$ does not satisfy the tree property, and for larger regular cardinals whether or not they satisfy the tree property is independent from ZFC. It is well known that the tree property provides a combinatorial characterization of weak compactness.

THEOREM 1.1 (Erdös and Tarski [3]). *Assume κ is an inaccessible cardinal, then κ is weakly compact if and only if it satisfies the tree property.*

Strongly compact and supercompact cardinals admit similar characterizations.

THEOREM 1.2. *If κ is an inaccessible cardinal, then*

1. *κ is strongly compact if and only if it satisfies the strong tree property (Jech* [6], *Di Prisco and Zwicker* [2] *and Weiss* [16]);
2. *κ is supercompact if and only if it satisfies the super tree property (Jech* [6], *Magidor* [10] *and Weiss* [16]).

The strong and super tree properties generalize the usual tree property to the combinatorics of $[\lambda]^{<\kappa}$, in fact they concern special structures known as (κ, λ)-*trees* that can be seen as "trees over $[\lambda]^{<\kappa}$" whose "levels" have size less than κ (this notion will be defined in §3). The super tree property implies the strong tree property, that entails the usual tree property in its turn. While the previous characterizations date

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back to the early 1970s, a systematic study of the strong and the super tree properties has only recently been undertaken by Weiss[1] who worked on these properties in his Ph.D thesis [16] and proved that even small cardinals can consistently satisfy the strong and the super tree properties, if we assume large cardinals.

There is a huge literature concerning the construction of models of set theory in which several distinct regular cardinals satisfy the usual tree property. We list a few classical results of that sort.

(1) (Mitchell [12]) Let τ be a regular cardinal such that $\tau^{<\tau} = \tau$. Assume there is a model of ZFC with a weakly compact cardinal, then there is a model of ZFC where τ^{++} has the tree property. In particular, the consistency of the tree property at $\aleph_2$ can be obtained from a weakly compact cardinal.
(2) (Cummings and Foreman [1]) Assume there is a model of ZFC with infinitely many supercompact cardinals, then there is a model of ZFC where every cardinal of the form $\aleph_n$ with $2 \leq n < \omega$ has the tree property.
(3) (Magidor and Shelah [11]) Assume there is a model of ZFC with an increasing sequence $\langle \lambda_n \rangle_{n<\omega}$ such that
 (a) if $\lambda = \sup_{n \geq 0} \lambda_n$, then λ_n is λ^+-supercompact, for all $n > 0$;
 (b) λ_0 is the critical point of an embedding $j : V \to M$ where $j(\lambda_0) = \lambda_1$ and ${}^{\lambda^+}M \subseteq M$.
 Then there is a model of ZFC where $\aleph_{\omega+1}$ has the tree property.
(4) (Sinapova [14]) Assume there is a model of ZFC with infinitely many supercompact cardinals, then there is a model of ZFC where $\aleph_{\omega+1}$ has the tree property.
(5) (Neeman [13]) Assume there is a model of ZFC with infinitely many supercompact cardinals, then there is a model of ZFC where the tree property holds at every $\aleph_n$ with $n \geq 2$ and at $\aleph_{\omega+1}$.

All these results were oriented toward the construction of a model where the tree property holds simultaneously at every regular cardinal—whether such a model can be found is still an open question. Some of these theorems can be generalized to the strong or the super tree property. In fact, Weiss proved that for every integer $n \geq 2$, if we force with Mitchell's forcing over a supercompact cardinal, we get a model of set theory where even the super tree property holds at $\aleph_n$. The author [4, 5] (and independently Unger [15]) proved that Weiss result can be generalized to get a model, where all cardinals of the form $\aleph_n$ with $2 \leq n < \omega$ simultaneously satisfy the super tree property, starting from infinitely many supercompact cardinals. Indeed, a forcing construction by Cummings and Foreman produces a model where all the $\aleph_n$'s satisfy the super tree property. We are going to prove from large cardinals that even $\aleph_{\omega+1}$ can consistently satisfy the strong tree property. More precisely we will prove the following theorem.

THEOREM 1.3. *If there is a model of ZFC with infinitely many supercompact cardinals, then there is a model of ZFC where $\aleph_{\omega+1}$ has the strong tree property.*

[1] In Weiss' terminology, the strong tree property at a regular cardinal κ corresponds to the property (κ, λ)-TP for all $\lambda \geq \kappa$, while the super tree property corresponds to (κ, λ)-ITP for all $\lambda \geq \kappa$.

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The proof of such theorem is motivated by Neeman's paper [13]. By generalizing a theorem by Magidor and Shelah [11], we will also prove the following result.

THEOREM 1.4. *If ν is a singular limit of strongly compact cardinals, then the strong tree property holds at ν^+.*

Moreover, we will weaken the hypothesis of the latter theorem by using a partition property satisfied by strongly compact cardinals.

§2. Preliminaries and notation.

It may be useful to recall some terminology. The main reference for basic set theory is [7], while we will refer to [8] for large cardinals notions and to [9] for the forcing technique.

Given a forcing $\mathbb{P}$ and conditions $p, q \in \mathbb{P}$, we use $p \leq q$ in the sense that p is stronger than q. Assume that $\mathbb{P}$ is a forcing notion in a model V, we will use $V^{\mathbb{P}}$ to denote the class of $\mathbb{P}$-names. If $G \subseteq \mathbb{P}$ is a generic filter over V, then $V[G]$ denotes the generic extension of V determined by G. If $a \in V^{\mathbb{P}}$ and $G \subseteq \mathbb{P}$ is generic over V, then a^G denotes the interpretation of a in $V[G]$. Every element x of the ground model V is represented in a canonical way by a name $\check{x}$. However, to simplify the notation, we will use just x instead of $\check{x}$ in forcing formulas.

DEFINITION 2.1. *Given a forcing $\mathbb{P}$ and a cardinal κ, we say that*

(1) *$\mathbb{P}$ is κ-*closed *if and only if every decreasing sequence of conditions of $\mathbb{P}$ of size less than κ has an infimum*;
(2) *$\mathbb{P}$ is κ-*c.c. *when every antichain of $\mathbb{P}$ has size less than κ*;
(3) *$\mathbb{P}$ has the κ-*covering property *if $\mathbb{P}$ preserves κ as a cardinal and for every filter $G \subseteq \mathbb{P}$ generic over V, every set $X \subseteq V$ in $V[G]$ of cardinality less than κ is contained in a set $Y \in V$ of cardinality less than κ in V.*

We denote by $\mathrm{Coll}(\kappa, \lambda)$ the usual Levy collapse.

We will assume familiarity with the theory of large cardinals and elementary embeddings, as developed for example in [8]. In particular, we will use repeatedly and without comments Silver's technique for lifting embeddings (see Lemma 2.2).

LEMMA 2.2 (Silver). *Let $j : M \to N$ be an elementary embedding between inner models of* ZFC. *Let $\mathbb{P} \in M$ be a forcing and suppose that G is $\mathbb{P}$-generic over M, H is $j(\mathbb{P})$-generic over N, and $j[G] \subseteq H$. Then there is a unique $j^* : M[G] \to N[H]$ such that $j^* \restriction M = j$ and $j^*(G) = H$.*

§3. The strong and the super tree properties.

In this section, we introduce the strong and super tree properties. Although the main results presented in this paper do not concern the super tree property (just the strong tree property), for the sake of completeness we include the definition of this property as well. In order to define the strong and the super tree properties, we need to introduce the notion of (κ, λ)-tree.[2]

DEFINITION 3.1. *Given a regular cardinal $\kappa \geq \omega_2$ and an ordinal $\lambda \geq \kappa$, a (κ, λ)-*tree *is a set F satisfying the following properties*:

[2]In Weiss Ph.D-thesis [16] (κ, λ)-trees were called $\mathscr{P}_\kappa\lambda$-thin lists.

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(1) *for every* $f \in F$, $f : X \to 2$, *for some* $X \in [\lambda]^{<\kappa}$;
(2) *for all* $f \in F$, *if* $X \subseteq \mathrm{dom}(f)$, *then* $f \restriction X \in F$;
(3) *the set* $\mathrm{Lev}_X(F) := \{f \in F;\ \mathrm{dom}(f) = X\}$ *is nonempty, for all* $X \in [\lambda]^{<\kappa}$;
(4) $|\mathrm{Lev}_X(F)| < \kappa$, *for all* $X \in [\lambda]^{<\kappa}$.

The elements of a (κ, λ)-tree are called *nodes*. Note that, despite the name, a (κ, λ)-tree is not a tree. In fact for a given node f on some level Lev_X, the set of all its *predecessors* is $\{f \restriction Y;\ Y \subseteq X\}$ and it is not well ordered. So the main difference between a κ-tree and a (κ, λ)-tree is that in the former the levels are indexed by ordinals which are well ordered, while in the latter we have a level for every set in $[\lambda]^{<\kappa}$ which is not even linearly ordered. As usual, when there is no ambiguity, we will simply write Lev_X instead of $\mathrm{Lev}_X(F)$.

Definition 3.2. *Given a regular* $\kappa \geq \omega_2$, *an ordinal* $\lambda \geq \kappa$ *and a* (κ, λ)*-tree* F,

(1) *a* cofinal branch *for* F *is a function* $b : \lambda \to 2$ *such that* $b \restriction X \in \mathrm{Lev}_X(F)$, *for all* $X \in [\lambda]^{<\kappa}$;
(2) *an* F-level sequence *is a function* $D : [\lambda]^{<\kappa} \to F$ *such that for every* $X \in [\lambda]^{<\kappa}$, $D(X) \in \mathrm{Lev}_X(F)$;
(3) *given an* F*-level sequence* D, *an* ineffable branch *for* D *is a cofinal branch* $b : \lambda \to 2$ *such that* $\{X \in [\lambda]^{<\kappa};\ b \restriction X = D(X)\}$ *is stationary.*

Definition 3.3. *Given a regular cardinal* $\kappa \geq \omega_2$ *and an ordinal* $\lambda \geq \kappa$,

(1) (κ, λ)-TP *holds if every* (κ, λ)*-tree has a cofinal branch*;
(2) (κ, λ)-ITP *holds if for every* (κ, λ)*-tree* F *and for every* F*-level sequence* D, *there is an an ineffable branch for* D;
(3) *we say that* κ *satisfies the* strong tree property *if* (κ, μ)-TP *holds, for all* $\mu \geq \kappa$;
(4) *we say that* κ *satisfies the* super tree property *if* (κ, μ)-ITP *holds, for all* $\mu \geq \kappa$;

We prove a simple result that will be used repeatedly.

Lemma 3.4. *Let* κ *be a regular cardinal and* $\lambda \geq \kappa$. *For every* $\lambda^* > \lambda$, *every* (κ, λ)*-tree with no cofinal branches, can be extended to a* (κ, λ^*)*-tree with no cofinal branches.*

Proof. Let F be a (κ, λ)-tree with no cofinal branches and let $\lambda^* > \lambda$. We define a (κ, λ^*)-tree F^* as follows: for every $X \in [\lambda^*]^{<\kappa}$, we let

$$f : X \to 2 \in F^* \iff_{def} f \restriction (X \cap \lambda) \in F \text{ and for every } \alpha \in X \setminus \lambda,\ f(\alpha) = 0.$$

It is clear that F^* extends F, i.e., for every $X \in [\lambda]^{<\kappa}$, $\mathrm{Lev}_X(F) = \mathrm{Lev}_X(F^*)$. We check that F^* is a (κ, λ)-tree. Conditions 1 and 3 of Definition 3.1 are trivially satisfied. Condition 2 is easily proved: if $f : X \to 2$ is in F^* and $Y \subseteq X$, then by definition $f \restriction (X \cap \lambda) \in F$, hence $f \restriction (Y \cap \lambda) \in F$. Moreover, for every $\alpha \in Y \setminus \lambda$, we have $f \restriction Y(\alpha) = f(\alpha) = 0$. Therefore $f \restriction Y \in F^*$. It remains to prove that for every $X \in [\lambda^*]^{<\kappa}$, the level $\mathrm{Lev}_X(F^*)$ has size less than κ, but the function $f \mapsto f \restriction \lambda$ defines a bijection of $\mathrm{Lev}_X(F^*)$ into $\mathrm{Lev}_{(X\cap\lambda)}(F)$, so $\mathrm{Lev}_X(F^*)$ has size less than κ. If F^* has a cofinal branch $b^* : \lambda^* \to 2$, then $b^* \restriction \lambda$ is a cofinal branch for F as well, because for every $X \in [\lambda]^{<\kappa}$, $b^* \restriction X \in \mathrm{Lev}_X(F^*) = \mathrm{Lev}_X(F)$. Since F has no cofinal branches, F^* has no cofinal branches as required. ⊣

§4. The strong tree property at successors of singular cardinals.

To prove the consistency of the usual tree property at $\aleph_{\omega+1}$, Magidor and Shelah first proved a more general result concerning the tree property at successors of singular cardinals.

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THEOREM 4.1 (Magidor and Shelah [11]). *Assume ν is a singular limit of strongly compact cardinals, then ν^+ has the tree property.*

In this section, we prove that under the same assumptions, even the *strong* tree property is satisfied at ν^+. The structure of the proof is very close to Magidor and Shelah's proof of the previous theorem, although we will prove that to get the strong tree property at ν^+, it is enough for ν to be a singular limit of cardinals satisfying a nice partition property. This result is very important in the following, since the proof of the consistency of the strong tree property at $\aleph_{\omega+1}$, will mimic the proof of this theorem.

NOTATION 4.2. *Let μ be a regular cardinal and let $\lambda \geq \mu$ be any ordinal. For every cofinal set $I \subseteq [\lambda]^{<\mu}$ we denote by $[[\ I\ ]]^2$ the set of all pairs $(X, Y) \in I \times I$ such that $X \subseteq Y$.*

DEFINITION 4.3. *Let $\mu > \kappa$ be two regular cardinals and let $S \subseteq [\lambda]^{<\mu}$ be a cofinal set and $c : [[\ S\ ]]^2 \to \gamma$ a function such that $\gamma < \kappa$. We say that a cofinal set $H \subseteq S$ is a* quasi homogenous *set of color $i < \gamma$ iff for every $X, Y \in H$ there is $W \supseteq X, Y$ in H such that $c(X, W) = i = c(Y, W)$.*

DEFINITION 4.4. *Given two regular cardinals $\nu \geq \kappa$, we say that the principle $\varphi(\kappa, \nu)$ holds when for every $\lambda \geq \nu$ if $S \subseteq [\lambda]^{<\nu}$ is a stationary set, then every function $c : [[\ S\ ]]^2 \to \gamma$ with $\gamma < \kappa$ has a quasi-homogenous set H which is also stationary.*

We now prove that strongly compact cardinals satisfy φ everywhere.

THEOREM 4.5. *Let κ be a strongly compact cardinal, then $\varphi(\kappa, \nu)$ holds for every regular $\nu \geq \kappa$.*

PROOF. Fix $\lambda \geq \nu$ and a function $c : [[\ S\ ]]^2 \to \gamma$ where $\gamma < \kappa$, and let $S \subseteq [\lambda]^{<\nu}$ be a stationary set. Consider all the sets of the form $C \cap S$ where $C \subseteq [\lambda]^{<\nu}$ is a club; they form a κ-complete family. Since κ is strongly compact, there exists a κ-complete ultrafilter U that contains all these sets. Note that every set in U is stationary. In fact if $H \in U$ and C is a club, then by definition $C \cap S \in U$, hence $H \cap C \cap S$ is in U as well, and it is nonempty. First, we show that for every $X \in S$, there is $i_X < \gamma$ and a set $H_X \subseteq S$ in U such that for every Y in H_X we have $c(X, Y) = i_X$. Assume for a contradiction that for every $i < \gamma$, the set $K_i := \{Y \in S;\ Y \supseteq X \text{ and } c(X, Y) \neq i\} \in U$ then, by the κ-completeness of U, the intersection $\bigcap_{i<\gamma} K_i$ is in U and it is empty, a contradiction. A similar argument proves that the function $X \mapsto i_X$ is constant on a set $H \in U$; let i be such that $i = i_X$, for every $X \in H$. Now, it is easy to see that H is quasi-homogenous of color i. Indeed, if $X, Y \in H$, then $H_X \cap H_Y \cap H$ belongs to U and it is, therefore, nonempty. Let $Z \in H_X \cap H_Y \cap H$, then we have $c(X, Z) = i = c(Y, Z)$ as required. ⊣

THEOREM 4.6. *Let ν be a singular cardinal such that $\nu = \lim_{i<cof(\nu)} \kappa_i$ where every κ_i is an uncountable cardinal satisfying $\varphi(\kappa_i, \nu^+)$. Then ν^+ has the strong tree property.*

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Proof. To simplify the notation we will assume that ν has countable cofinality, so $\nu = \lim_{n<\omega} \kappa_n$. Suppose without loss of generality that $\langle \kappa_n \rangle_{n<\omega}$ is increasing. Let $\mu \geq \nu^+$ and let F be a (ν^+, μ)-tree. For every $X \in [\mu]^{<\nu^+}$, let $\{f_i^X\}_{i<|\mathrm{Lev}_X(F)|}$ be an enumeration of $\mathrm{Lev}_X(F)$. First, we "shrink" the tree as follows.

Lemma 4.7. *There exists $n < \omega$ and a stationary set $S \subseteq [\mu]^{<\nu^+}$, such that for all $X, Y \in S$, there are $\zeta, \eta < \kappa_n$ such that $f_\zeta^X \restriction (X \cap Y) = f_\eta^Y \restriction (X \cap Y)$.*

Proof. Given a function $f \in Lev_X$, we write $\#f = i$ for $i < \nu$, when $f = f_i^X$. Define $c : [[\, [\mu]^{<\nu^+}]]^2 \to \omega$ by $c(X, Y) = \min\{i;\ \#(f_0^Y \restriction X) < \kappa_i\}$. By hypothesis, $\varphi(\kappa_0, \nu^+)$ holds, hence there is a stationary quasi homogenous set $S \subseteq [\mu]^{<\nu^+}$ of color $n < \omega$. Then, for every $X, Y \in S$, there is $Z \supseteq X, Y$ in S such that $c(X, Z) = n = c(Y, Z)$. This means that $\#(f_0^Z \restriction X),\ \#(f_0^Z \restriction Y) < \kappa_n$, namely there are $\zeta, \eta < \kappa_n$ such that $f_0^Z \restriction X = f_\zeta^X$ and $f_0^Z \restriction Y = f_\eta^Y$. So we have

$$f_\zeta^X \restriction (X \cap Y) = f_0^Z \restriction (X \cap Y) = f_\eta^Y \restriction (X \cap Y),$$

as required. That completes the proof of the lemma. ⊣

Let n and S be as above, we prove the following fact.

Lemma 4.8. *There is a cofinal $S' \subseteq S$ and an ordinal $\zeta < \kappa_n$ such that for all $X, Y \in S'$, we have $f_\zeta^X \restriction (X \cap Y) = f_\zeta^Y \restriction (X \cap Y)$ (the set S' is even stationary).*

Proof. For every $(X, Y) \in [[\, S\,]]^2$, we define $\bar{c}(X, Y)$ as the minimum couple $(\zeta, \eta) \in \kappa_n \times \kappa_n$, in the lexicografical order, such that $f_\eta^Y \restriction X = f_\zeta^X$—the function is well defined by definition of n and S. We can apply $\varphi(\kappa_{n+1}, \nu^+)$ to $\bar{c}$ as this can be seen as a function from $[[\, S\,]]^2$ into κ_n—take any bijection $h : \kappa_n \times \kappa_n \to \kappa_n$ and apply $\varphi(\kappa_{n+1}, \nu^+)$ to $\bar{c} \circ h : [[\, S\,]]^2 \to \kappa_n$. So, there exists a quasi homogenous stationary set S' of color $(\zeta, \eta) \in \kappa_n \times \kappa_n$, hence, for every $X, Y \in S'$, there is $Z \supseteq X, Y$ in S' such that $\bar{c}(X, Z) = (\zeta, \eta) = \bar{c}(Y, Z)$. This means that $f_\eta^Z \restriction X = f_\zeta^X$ and $f_\eta^Z \restriction Y = f_\zeta^Y$. It follows that

$$f_\zeta^X \restriction (X \cap Y) = f_\eta^Z \restriction (X \cap Y) = f_\zeta^Y \restriction (X \cap Y).$$

That completes the proof of the lemma. ⊣

Now we conclude the proof of the theorem by defining a cofinal branch. Let $b := \bigcup_{X \in S'} f_\zeta^X$, by the previous lemma b is a function. Moreover, for every $Y \in S'$ we have

$$b \restriction Y = \bigcup_{X \in S'} f_\zeta^X \restriction Y = \bigcup_{X \in S'} f_\zeta^X \restriction (X \cap Y) = \bigcup_{X \in S'} f_\zeta^Y \restriction (X \cap Y) = f_\zeta^Y.$$

It follows that b is a cofinal branch for F. ⊣

Corollary 4.9. *Let ν be a singular limit of strongly compact cardinals, then ν^+ has the strong tree property.*

Proof. Apply Theorems 4.5 and 4.6. ⊣

Whether such result can be generalized to the *super* tree property is still an open problem. Based on the analogy between supercompact cardinals and the super tree property, we can conjecture that the successor of a singular limit of *supercompact* cardinals satisfy the super tree property.

We conclude this section by proving the following fact.

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PROPOSITION 4.10. *Given a regular cardinal κ, if $\varphi(\kappa, \kappa)$ holds, then κ has the strong tree property.*

PROOF. Let F be a (κ, λ)-tree where $\lambda \geq \kappa$. For every $X \in [\lambda]^{<\kappa}$, let $\{f_i^X\}_{i<\gamma_X}$ be an enumeration of $\mathrm{Lev}_X(F)$. We can assume without loss of generality that $\lambda^{<\kappa}$ is large enough, so that γ_X (the size of $|\mathrm{Lev}_X(F)|$) is constant on a stationary set $S \subseteq [\lambda]^{<\kappa}$—indeed, if it is not the case we can take a larger λ^* and use Lemma 3.4. So let $\gamma < \kappa$ be such that $\gamma_X = \gamma$ for every $X \in S$. We define a function $c : [[\, S\,]]^2 \to \gamma \times \gamma$ by letting $c(X, Y)$ be the minimum couple (i, j) in the lexicografical order such that $f_j^Y \restriction X = f_i^X$. The function c can be seen as a function from $[[\, S\,]]^2$ into γ, so there exists a quasi-homogenous and stationary set $H \subseteq S$. Assume H has color $(i, j) \in \gamma \times \gamma$, we let $b := \bigcup_{X \in H} f_i^X$ and we prove that b is a cofinal branch. Given $X, Y \in H$, there is $Z \in H$ such that $X, Y \subseteq Z$ and $c(X, Z) = (i, j) = c(Y, Z)$. By definition of c, we have

(1) $f_j^Z \restriction X = f_i^X$;
(2) $f_j^Z \restriction Y = f_i^Y$.

It follows that $f_i^X \restriction (X \cap Y) = f_i^Y \restriction (X \cap Y)$. Therefore b is a function and for every $Y \in H$, we have

$$b \restriction Y = \bigcup_{X \in H} f_i^X \restriction Y = \bigcup_{X \in H} f_i^X \restriction (X \cap Y) = \bigcup_{X \in H} f_i^Y \restriction (X \cap Y) = f_i^Y,$$

so b is a cofinal branch. ⊣

§5. **Systems.** To prove the consistency of the strong tree property at $\aleph_{\omega+1}$, we will work with a special structure that we call a "system". To understand this notion, suppose we are in the following situation. Let ν be a cardinal in a model V and let $\mathbb{P}$ be a forcing notion with the ν^+-covering property—so that for every $\lambda \geq \nu^+$, the set $([\lambda]^{<\nu^+})^V$ is cofinal in the $[\lambda]^{<\nu^+}$ of the generic extension. Assume $\dot{F} \in V^{\mathbb{P}}$ is a name for a (ν^+, λ)-tree and for every $X \in [\lambda]^{<\nu^+}$, $\dot{e}_X$ is a $\mathbb{P}$-name for a an enumeration of $\mathrm{Lev}_X(\dot{F})$ (i.e., $\Vdash_{\mathbb{P}} \dot{e}_X : \nu \to \mathrm{Lev}_X(\dot{F})$ is onto). For every $p \in \mathbb{P}$, we define a binary relation S_p over the pairs (X, ζ), where $X \in [\lambda]^{<\nu^+}$ and $\zeta < \nu$:

$$(X, \zeta)\ S_p\ (Y, \eta) \iff_{def}\ p \Vdash\ \dot{e}_X(\zeta) = \dot{e}_Y(\eta) \restriction X.$$

In other words, we have $(X, \zeta)\ S_p\ (Y, \eta)$ when p forces that the η-th function on level Y extends the ζ-th function on level X. The family $\{S_p\}_{p \in \mathbb{P}}$ satisfies the following definition.

DEFINITION 5.1. *Given an ordinal $\lambda \geq \nu^+$, a cofinal set $D \subseteq [\lambda]^{<\nu^+}$ and a family $\mathcal{S} := \{S_i\}_{i \in I}$ of transitive, reflexive binary relations over $D \times \nu$, we say that $\mathcal{S}$ is a* system *if the following holds*:

(1) *if $(X, \zeta)\ S_i\ (Y, \eta)$ and $(X, \zeta) \neq (Y, \eta)$, then $X \subsetneq Y$;*
(2) *for every $X \subseteq Y$, if both $(X, \zeta)\ S_i\ (Z, \theta)$ and $(Y, \eta)\ S_i\ (Z, \theta)$, then $(X, \zeta)\ S_i\ (Y, \eta)$;*
(3) *for every $X, Y \in D$, there is $Z \supseteq X, Y$ and $\zeta_X, \zeta_Y, \eta \in \nu$ such that for some $i \in I$ we have $(X, \zeta_X)\ S_i\ (Z, \eta)$ and $(Y, \zeta_Y)\ S_i\ (Z, \eta)$ (in particular, if $X \subseteq Y$, then $(X, \zeta_X)\ S_i\ (Y, \zeta_Y)$).*

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To prove the consistency of the strong tree property at $\aleph_{\omega+1}$, we will have to deal with a system similar to the one defined above. In this section, we analyze some properties of these structures.

The elements of $D \times \nu$ are called *nodes of the system*. Given two nodes u and v, we say that they are S_i*-incompatible,* for some $i \in I$, if there is no $w \in D \times \nu$ such that $u\ S_i\ w$ and $v\ S_i\ w$. We will say that a node u belongs to a *level* X if the first coordinate of u is X (i.e., $u = (X, \zeta)$, for some $\zeta \in \nu$).

Definition 5.2. *Let $\{S_i\}_{i\in I}$ be a system on $D \times \nu$ and let $b : D \to \nu$ be a partial function.*

(1) *We say that b is an* S_i-branch *for some $i \in I$, if the following holds. For every $X \in \mathrm{dom}(b)$ and for every $Y \in D$ such that $Y \subseteq X$, we have $Y \in \mathrm{dom}(b)$ iff there exists $\zeta < \nu$ such that $(Y, \zeta)\ S_i\ (X, b(X))$, and $b(Y)$ is the unique ζ witnessing this.*
(2) *We say that b is a* cofinal branch *for the system if it is an S_i-branch for some $i \in I$, and $X \in \mathrm{dom}(b)$ for cofinally many X's in D.*

We will often work with families of branches satisfying specific conditions.

Definition 5.3. *Let $\{S_i\}_{i\in I}$ be a system on $D \times \nu$, a* system of branches *is a family $\{b_j\}_{j\in J}$ such that*

(1) *every b_j is an S_i-branch for some $i \in I$;*
(2) *for every $X \in D$, there is $j \in J$ such that $X \in \mathrm{dom}(b_j)$.*

A lemma by Silver establishes that whenever we force with a forcing that has enough closure, it cannot add cofinal branches to a given tree.

Lemma 5.4 (Silver). *Let τ, κ be regular cardinals, and suppose $\tau < \kappa \leq 2^\tau$. Let $\mathbb{P}$ be a τ^+-closed forcing in a model V and let T be a κ-tree. Then for every generic extension $V[G]$ by $\mathbb{P}$, every branch of T in $V[G]$ is in fact a member of V.*

Proof. We may assume that τ is minimal with $2^\tau \geq \kappa$. Let $\dot{b}$ be a $\mathbb{P}$-name for a new branch. We build by induction for each $s \in {}^{\leq\tau}2$ conditions p_s and points x_s of T such that

(1) if $t \sqsubseteq s$, then $p_s \leq p_t$ and $x_s >_T x_t$;
(2) $p_s \Vdash x_s \in \dot{b}$;
(3) for each α, the nodes $\{x_s;\ s \in {}^{\alpha}2\}$ are all on the same level η_α;
(4) for each $s \in {}^{<\tau}2$, the nodes $x_{s\frown 0}$ and $x_{s\frown 1}$ are incompatible.

By minimality of τ, for every $\alpha < \tau$, the set $\{x_s;\ s \in {}^{\alpha}2\}$ has size less than κ, so we can choose $\eta_{\alpha+1}$. The closure of $\mathbb{P}$ guarantees that the construction works at limit stages. In the end we have a contradiction, because the level η_τ must have fewer than κ many nodes, yet we have constructed 2^τ many distinct ones. ⊣

Now we want to generalize Silver's lemma to systems. More precisely, we are going to prove that if a κ-closed forcing adds a system of branches through a "small" system, then a cofinal branch must already exist in the ground model (Theorem 5.6 below). Such a result generalizes a lemma by Sinapova (see [14] Preservation Lemma) and will be used to prove the consistency of the strong tree property at $\aleph_{\omega+1}$. First, we prove the following lemma that provides a useful "splitting argument".

Lemma 5.5 (Splitting lemma). *Let ν be a singular cardinal of countable cofinality and let $\lambda \geq \nu^+$. Let $\{R_i\}_{i \in I}$ be a system on $D \times \tau$ (with $D \subseteq [\lambda]^{<\nu^+}$ cofinal) and let $\mathbb{P}$ be a forcing notion such that*:

(1) $\max(|I|, \tau) < \nu$;
(2) *$\mathbb{P}$ is κ-closed for some regular κ between $\max(|I|, \tau)^+$ and ν;*
(3) *for some $p \in \mathbb{P}$, $\dot{b} \in V^{\mathbb{P}}$ and $i \in I$, we have $p \Vdash \dot{b}$ is a cofinal R_i-branch.*

If V has no cofinal branches for the system, then for all $\eta < \kappa$, we can find a sequence $\langle v_\zeta;\ \zeta < \eta \rangle$ of pairwise R-incompatible elements of $D \times \tau$ such that for every $\zeta < \eta$, there exists $q \leq p$ that forces $v_\zeta \in \dot{b}$.

Proof. It might be helpful to point out that if G is a generic filter containing p, then in $V[G]$ the domain of $\dot{b}^G$ is a cofinal set in $([\lambda]^{<\nu^+})^V$. We work in V. Let $R := R_i$ and let $E := \{u \in D \times \tau;\ \exists q \leq p (q \Vdash u \in \dot{b})\}$. First remark that, since p forces that $\dot{b}$ is cofinal, the set $\{X \in D;\ \exists \zeta \in \tau\ (X, \zeta) \in E\}$ is cofinal. As V has no cofinal branches for the system, we can find, for all $v \in E$ two R-incompatible nodes $w_1, w_2 \in E$ such that $v\ R\ w_1$, $v\ R\ w_2$.

We inductively define for all $\zeta < \eta$ two nodes $u_\zeta, v_\zeta \in E$ and a condition $p_\zeta \leq p$ such that:

(1) u_ζ and v_ζ are R-incompatible;
(2) for all $\varepsilon < \zeta$, $u_\varepsilon\ R\ u_\zeta$ and $u_\varepsilon\ R\ v_\zeta$;
(3) $p_\zeta \Vdash u_\zeta \in \dot{b}$;
(4) the sequence $\langle p_\varepsilon;\ \varepsilon \leq \zeta \rangle$ is decreasing.

Let u be any node in E. From the remark above, there are $u_0, v_0 \in E$ which are R-incompatible and both $u\ R\ u_0$ and $u\ R\ v_0$ hold. By definition of E, there is a condition $p_0 \leq p$ such that $p_0 \Vdash u_0 \in \dot{b}$.

Let $\zeta > 0$ and assume that $u_\varepsilon, v_\varepsilon, p_\varepsilon$ are defined for every $\varepsilon < \zeta$. Let q be stronger than every condition in $\{p_\varepsilon;\ \varepsilon < \zeta\}$. By the inductive hypothesis (Claim 5), the nodes $\langle u_\varepsilon;\ \varepsilon < \zeta \rangle$ form an R-chain whose levels are sets in $[\lambda]^{<\nu^+}$. The union of the levels of these nodes is a set X in $[\lambda]^{<\nu^+}$ and since $\dot{b}$ is forced to be a cofinal R-branch we can find a node h of level above X and a condition $q^* \leq q$ such that $q^* \Vdash h \in \dot{b}$. It follows that $u_\varepsilon\ R\ h$, for all $\varepsilon < \zeta$. Since there is no cofinal branch in V for the system, we can find two R-incompatible nodes $u_\zeta, v_\zeta \in E$ and a condition $p_\zeta \leq q^*$ such that $h\ R\ u_\zeta$, $h\ R\ v_\zeta$ and $p_\zeta \Vdash u_\zeta \in \dot{b}$. That completes the construction.

The sequence $\langle v_\zeta;\ \zeta < \eta \rangle$ is as required: for if $\zeta' < \zeta < \eta$, then by definition $u_{\zeta'}$ and $v_{\zeta'}$ are R-incompatible, and $u_{\zeta'}\ R\ v_\zeta$, hence $v_{\zeta'}$ and v_ζ are R-incompatible as well. ⊣

Theorem 5.6 (Preservation theorem). *In a model V, we let ν be a singular cardinal of countable cofinality and let $\lambda \geq \nu^+$. Let $\{R_i\}_{i \in I}$ be a system on $D \times \tau$ (with $D \subseteq [\lambda]^{<\nu^+}$ cofinal), let $\mathbb{P}$ be a forcing notion and let $G \subseteq \mathbb{P}$ a generic filter over V. Assume that*

(1) $\max(|I|, \tau) < \nu$;
(2) *$\mathbb{P}$ is κ-closed for some regular κ between $\max(|I|, \tau)^+$ and ν;*
(3) *in $V[G]$ there is a system of branches $\{b_j\}_{j \in J}$ through $\{R_i\}_{i \in I}$ such that*
    (a) *$J \in V$ and $|J|^+ < \kappa$;*
    (b) *for some $j \in J$, the branch b_j is cofinal.*

Then, for some $i \in I$, there exists in V a cofinal R_i-branch.

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PROOF. Suppose for contradiction that V has no cofinal branches for the system $\{R_i\}_{i\in I}$. Let $\{\dot b_j\}_{j\in J}$ be $\mathbb{P}$-names for the branches of the system of branches in the generic extension. The idea of the proof is similar to the proof of Silver's lemma above and it follows three steps.

(1) We consider just the $\dot b_j$'s that are forced to be cofinal and for every such $\dot b_j$, we use the Splitting Lemma to build η many incompatible nodes that are forced to belong to $\dot b_j$, where η is a cardinal between $\max(|J|, |I|, \tau)$ and κ.
(2) By using the κ-closure of $\mathbb{P}$ and the fact that there are less than κ many possible cofinal branches, we find a name $\dot b$ for a R-branch and η many R-incompatible nodes $\langle u_\gamma;\ \gamma < \eta\rangle$ that are forced by "nice conditions" to belong to $\dot b$.
(3) As $\eta < \nu^+$, all these nodes are below some level $X \in D$ and we can find a node w on a level above X which is forced by those conditions to belong to $\dot b$ as well. Then we have a contradiction, as w stands in the relation R with R-incompatible nodes below it.

We work in V. Fix, for every $j \in J$ a condition p_j deciding whether or not $\dot b_j$ is cofinal. We can choose the p_j's so that they form a decreasing sequence, then by the κ-closure of $\mathbb{P}$ (recall $|J| < \kappa$) there exists a condition p deciding, for every $j \in J$ whether or not b_j is cofinal. We let $B := \{j \in J;\ p \Vdash \dot b_j \text{ is not cofinal}\}$. For every $j \in B$, fix $X_j \in [\lambda]^{<\nu^+}$ such that p forces that $\mathrm{dom}(\dot b_j)$ has empty intersection with every $Y \supseteq X_j$. Since B has size less than ν, the set $X^* := \bigcup_{j\in B} X_j$ is in $[\lambda]^{<\nu^+}$. Let $C^* := \{Z \in D;\ X^* \subseteq Z\}$. Define $A := \{j \in J;\ p \Vdash \dot b_j \text{ is cofinal}\}$, then by hypothesis A is nonempty (Claim 5.6). Moreover, by strengthening p if necessary, we can assume

$$p \Vdash \forall X \in C^* \exists j \in A(X \in \mathrm{dom}(\dot b_j)) \tag{1}$$

(use condition 2 of Definition 5.3 and the definition of C^*). As the size of A is less than the closure of the poset, we can fix for every $a \in A$, a relation R_a in the system such that $p \Vdash \dot b_a$ is an R_a-branch. Fix a regular cardinal η between $\max(|J|, |I|, \tau)$ and κ, we prove the following claim.

CLAIM 5.7. *Let $\lhd$ be a well ordering (strict) of A. For every $a \in A$, we can define $\langle q^a_\gamma;\ \gamma < \eta\rangle$ and $\langle u^a_\gamma;\ \gamma < \eta\rangle$ such that*

(1) *for all $\gamma < \eta$, $q^a_\gamma \leq p$ and $q^a_\gamma \Vdash u^a_\gamma \in \dot b_a$;*
(2) *the nodes $\langle u^a_\gamma;\ \gamma < \eta\rangle$ are pairwise R_a-incompatible;*
(3) *for all $\gamma < \eta$, the sequence $\langle q^c_\gamma;\ c \lhd a\rangle$ is decreasing, i.e., if $b \lhd c$, then $q^c_\gamma \leq q^b_\gamma$.*

PROOF. We proceed by induction on the ordering $\lhd$. Assume that the sequences have been defined up to $a \in A$ (i.e., for every $c \lhd a$). For every $\gamma < \eta$, let r_γ be stronger than every condition in the set $\{q^c_\gamma;\ c \lhd a\}$ (the sequence $\langle q^c_\gamma;\ c \lhd a\rangle$ is decreasing by Claim 5.7) and let $E_\gamma := \{u \in D \times \tau;\ \exists q \leq r_\gamma(q \Vdash u \in \dot b_a)\}$. For all $\gamma < \eta$, there exists $\langle v^\gamma_\zeta;\ \zeta < \eta\rangle$ like in the conclusion of Lemma 5.5 applied to r_γ and $\dot b_a$. Let $X_\gamma \in [\lambda]^{<\nu^+}$ be such that the level of each v^γ_ζ is below X_γ and let $X^* \supsetneq \bigcup_{\gamma<\eta} X_\gamma$ in D. We want to define the sequence $\langle u^a_\gamma;\ \gamma < \eta\rangle$ with each $u^a_\gamma \in E_\gamma$

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belonging to a level above X^*. We proceed by induction: suppose we have defined $\langle u^a_\gamma;\ \gamma < \delta\rangle$ for some $\delta < \eta$. For every $\gamma < \delta$, there is at most one $\zeta < \eta$ such that $v^\delta_\zeta\ R_a\ u^a_\gamma$ (because the v^δ_ζ's are pairwise R_a-incompatible), let ζ_γ be that unique index if it exists and let ζ_γ be 0 otherwise. Choose $\zeta \in \eta \setminus \{\zeta^\delta_\gamma;\ \gamma < \delta\}$, then for all $\gamma < \delta$, the nodes v^δ_ζ and u^a_γ are R_a-incompatible. Let $u^a_\delta \in E_\delta$ be such that $v^\delta_\zeta\ R_a\ u^a_\delta$. Then, for all $\gamma < \delta$, the nodes u^a_γ and u^a_δ are R_a-incompatible. Since for every $\gamma < \eta$, we have $u^a_\gamma \in E_\gamma$, we can find a condition $q^a_\gamma \leq r_\gamma$ such that $q^a_\gamma \Vdash u^a_\gamma \in \dot{b}_a$. That completes the proof of the claim. ⊣

We return to the proof of the theorem. Condition 3 above guarantees that for every $\gamma < \eta$, the sequence $\langle q^a_\gamma;\ a \in A\rangle$ is decreasing. Since A has size less than κ, we can find for every $\gamma < \eta$, a condition p_γ stronger than all the conditions $\langle q^a_\gamma;\ a \in A\rangle$ and there is $Y_\gamma \in D$ such that the nodes in $\{u^a_\gamma;\ a \in A\}$ belong to levels below Y_γ. Let $Y^* \in C^*$ be such that $Y^* \supseteq \bigcup_\gamma Y_\gamma$. For all $\gamma < \eta$, we fix p^*_γ, w_γ and a_γ such that $p^*_\gamma \leq p_\gamma$, w_γ is a node on level Y^*, $a_\gamma \in A$ and $p^*_\gamma \Vdash w_\gamma \in \dot{b}_{a_\gamma}$ (use Equation 1). Since $|A|, \tau < \eta$, there is w^* on level Y^* and $a^* \in A$ such that $w_\gamma = w^*$, $a_\gamma = a^*$, for almost all $\gamma < \eta$. Let $b^* := \dot{b}_{a^*}$. Given two distinct $\gamma, \delta < \eta$ large enough, if $u := u^{a^*}_\gamma$ and $v := u^{a^*}_\delta$, then the following hold:

(1) $p^*_\gamma \Vdash u \in b^*$, $p^*_\gamma \Vdash w^* \in b^*$;
(2) $p^*_\delta \Vdash v \in b^*$, $p^*_\delta \Vdash w^* \in b^*$.

It follows that $u\ R_{a^*} w$ and $v\ R_{a^*} w$. However, u and v are R_{a^*}-incompatible by definition, and that leads to a contradiction. ⊣

§6. The strong tree property at $\aleph_{\omega+1}$. Now we are ready to prove the consistency of the strong tree property at $\aleph_{\omega+1}$. The structure of the proof of this theorem is motivated by Neeman [13].

Theorem 6.1. *Let $\langle\kappa_n\rangle_n < \omega$ be an increasing sequence of indestructibly supercompact cardinals. There is a strong limit cardinal $\mu < \kappa_0$ of cofinality ω such that by forcing over V with the poset*

$$\mathrm{Coll}(\omega, \mu) \times \mathrm{Coll}(\mu^+, < \kappa_0) \times \Pi_{n<\omega}\mathrm{Coll}(\kappa_n, < \kappa_{n+1}),$$

one gets a model where the strong tree property holds at $\aleph_{\omega+1}$.

Proof. Let κ denote κ_0, for every $\mu < \kappa$ we let

(1) $\mathbb{R}(\mu) := \mathrm{Coll}(\omega, \mu) \times \mathrm{Coll}(\mu^+, < \kappa_0) \times \Pi_{n<\omega}\mathrm{Coll}(\kappa_n, < \kappa_{n+1})$;
(2) $\mathbb{L}(\mu) := \mathrm{Coll}(\omega, \mu) \times \mathrm{Coll}(\mu^+, < \kappa_0)$;
(3) $\mathbb{C} := \Pi_{n<\omega}\mathrm{Coll}(\kappa_n, < \kappa_{n+1})$.

Assume that $\nu = \sup_n \kappa_n$. Note that for every $\mu < \kappa$, the forcing $\mathbb{R}(\mu)$ produces a model where $\aleph_{\omega+1} = \nu^+$. Fix $H := \Pi_{n<\omega} H_n \subseteq \mathbb{C}$ generic over V. We work in $W := V[H]$. Assume for a contradiction that in every extension of W by $\mathbb{L}(\mu)$ with $\mu < \kappa$, strong limit of cofinality ω, the strong tree property fails at ν^+. For every such μ, let λ_μ and $\dot{F}(\mu) \in W^{\mathbb{L}(\mu)}$ be a name for a (ν^+, λ_μ)-tree with no cofinal branches. Let $\lambda = \sup_{\mu<\kappa} \lambda_\mu$, without loss of generality we can assume that $\lambda_\mu = \lambda$ for every μ, since a (ν^+, λ_μ)-tree with no cofinal branches can be extended to a (ν^+, λ)-tree with no cofinal branches (by Lemma 3.4). Note that for every μ the poset $\mathbb{L}(\mu)$ has the ν^+-covering property since it is κ_0-c.c. Therefore, $([\lambda]^{<\nu^+})^W$ is

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cofinal in the $[\lambda]^{<\nu^+}$ of any generic extension of W by $\mathbb{L}(\mu)$. Given $X, Y \in [\lambda]^{<\nu^+}$ and $\zeta, \eta < \nu$, we will write $\Vdash_{\mathbb{L}(\mu)} (X, \zeta) <_{\dot{F}_\mu} (Y, \eta)$ when

$$\Vdash_{\mathbb{L}(\mu)} \text{ the } \eta\text{-th function on level } Y \text{ extends the } \zeta\text{-th function on level } X$$

(i.e., for every μ and X, we fix an $\mathbb{L}(\mu)$-name $\dot{e}^\mu_X$ for an enumeration of the level of X into at most ν elements, then we write $\Vdash_{\mathbb{L}(\mu)} (X, \zeta) <_{\dot{F}_\mu} (Y, \eta)$ when $\Vdash_{\mathbb{L}(\mu)} \dot{e}^\mu_X(\zeta) = \dot{e}^\mu_Y(\eta) \restriction X$). Consider the following set

$$I := \{(a, b, \mu);\ \mu < \kappa \text{ is strong limit of cof } \omega \text{ and } (a, b) \in \mathbb{L}(\mu)\}.$$

We define a system $\mathscr{S} = \{S_i\}_{i \in I}$ on $[\lambda]^{<\nu^+} \times \nu$ as follows. Given $i = (a, b, \mu) \in I$, for every $X, Y \in [\lambda]^{<\nu^+}$ and for every $\zeta, \eta < \nu$, we let

$$(X, \zeta)\ S_i\ (Y, \eta) \iff_{def} (a, b) \Vdash (X, \zeta) <_{\dot{F}_\mu} (Y, \eta).$$

First, we prove that we can shrink the system.

Lemma 6.2. *There is, in W, an integer $n < \omega$ and a cofinal set $D \subseteq [\lambda]^{<\nu^+}$ such that $\{ S_i \restriction D \times \kappa_n\}_{i \in I}$ is a system.*

Proof. κ is indestructibly supercompact, so we can fix $j : W \to W^*$ a σ-supercompact elementary embedding with critical point κ, where σ is large enough for the argument that follows. We have $a^* := j[\lambda] \in W^* \cap [j(\lambda)]^{<j(\nu^+)}$. Let F^* be the name $j(\dot{F})(\nu)$, where $\dot{F}$ is the map $\mu \mapsto \dot{F}(\mu)$. We denote by $\ll \lambda \gg^{<\nu^+}$ the set of all the strictly increasing sequences from an ordinal $\alpha < \nu^+$. into λ. For every $s \in \ll \lambda \gg^{<\nu^+}$, the image of s is a subset of $[\lambda]^{<\nu^+}$. We define a sequence $\langle (p_s, q_s, \zeta_s, n_s);\ s \in \ll \lambda \gg^{<\nu^+} \rangle$ such that

(1) $(p_s, q_s) \in \text{Coll}(\omega, \nu) \times \text{Coll}(\nu^+, < j(\kappa))$, $n_s < \omega$, and $\zeta_s < j(\kappa_{n_s})$;
(2) $(p_s, q_s) \Vdash (j[Im(s)], \zeta_s) <_{F^*} (a^*, 0)$;
(3) for every $t \sqsubseteq s$ in $\ll \lambda \gg^{<\nu^+}$, we have $q_s \leq q_t$.

The sequence is inductively defined as follows. Let $s : \alpha \to \lambda$ be a strictly increasing sequence, assume by inductive hypothesis that

$$\langle (p_s, q_s, \zeta_s, n_s);\ s \in \ll \lambda \gg^{<\alpha} \rangle$$

is defined. By condition (3), the sequence $\langle q_{s \restriction \beta};\ \beta < \alpha \rangle$ is decreasing. Moreover, $\text{Coll}(\nu^+, < j(\kappa))$ is ν^+-closed, so there exists a lower bound $\bar{q}_s$ for $\langle q_{s \restriction \beta};\ \beta < \alpha \rangle$. The set $j[Im(s)]$ is in $[j(\lambda)]^{<j(\nu^+)}$, so there exists $p_s \in \text{Coll}(\omega, \nu)$, $q_s \leq \bar{q}_s$ in $\text{Coll}(\nu^+, < j(\kappa))$ and $\zeta_s < j(\nu)$ such that

$$(p_s, q_s) \Vdash (j[Im(s)], \zeta_s) <_{F^*} (a^*, 0).$$

If we let n_s be the minimum integer such that $\zeta_s < j(\kappa_{n_s})$, then p_s, q_s, ζ_s and n_s satisfy conditions 1, 2 and 3 for the sequence s. That completes the definition.

For every $X \in [\lambda]^{<\nu^+}$ we denote by s_X the unique strictly increasing sequence whose image is X (i.e., $s_X : o.t.(X) \to \lambda$ and $Im(s_X) := X$). As $\text{Coll}(\omega, \nu)$ has size less than $\lambda^{<\nu^+}$, there is a condition p and a cofinal set $D \subseteq [\lambda]^{<\nu^+}$ such that for every $X \in D$, we have $p = p_{s_X}$. By shrinking D, we can also assume that there exists $n < \omega$ such that $n = n_{s_X}$, for every $X \in D$.

Claim 6.3. *$\{ S_i \restriction D \times \kappa_n\}_{i \in I}$ is a system.*

Proof. We just have to prove that it satisfies condition (3) of Definition 5.1. Fix $X, Y \in D$, by construction we have

(1) $(p, q_{s_X}) \Vdash (j[X], \zeta_X) <_{F^*} (a^*, 0)$,
(2) $(p, q_{s_Y}) \Vdash (j[Y], \zeta_Y) <_{F^*} (a^*, 0)$.

Take any set Z in D such that $s_Z \sqsupseteq s_X, s_Y$ (in particular $Z \supseteq X, Y$), then q_Z is stronger than both q_X and q_Y. Therefore, the condition (p, q_Z) forces that:

(i) $(j[X], \zeta_X) <_{F^*} (a^*, 0)$;
(ii) $(j[Z], \zeta_Z) <_{F^*} (a^*, 0)$;
(iii) $(j[Y], \zeta_Y) <_{F^*} (a^*, 0)$.

From (i) and (ii) follows $(p, q_Z) \Vdash (j[X], \zeta_X) <_{F^*} (j[Z], \zeta_Z)$; from (ii) and (iii) follows $(p, q_Z) \Vdash (j[Y], \zeta_Y) <_{F^*} (j[Z], \zeta_Z)$. Then, by elementarity, there exists $\mu < \kappa$ and $(\bar{p}, \bar{q}) \in \mathbb{L}(\mu)$ and $\bar{\zeta}_X, \bar{\zeta}_Y, \bar{\zeta}_Z < \kappa_n$ such that

$$(\bar{p}, \bar{q}) \Vdash (X, \bar{\zeta}_X) <_{\dot{F}_\mu} (Z, \bar{\zeta}_Z) \text{ and } (Y, \bar{\zeta}_Y) <_{\dot{F}_\mu} (Z, \bar{\zeta}_Z).$$

If $i = (\bar{p}, \bar{q}, \mu)$, then we just proved $(X, \bar{\zeta}_X)\ S_i\ (Z, \bar{\zeta}_Z)$ and $(Y, \bar{\zeta}_Y)\ S_i\ (Z, \bar{\zeta}_Z)$. ⊣

That completes the proof of the lemma. ⊣

To simplify the notation, we define $R_i := S_i \upharpoonright D \times \kappa_n$, for every $i \in I$.

Let $m = n + 2$, by the indestructibility of κ_{m+1} forcing over $W = V[H]$ with $\mathrm{Coll}(\kappa_m, \gamma)^V$ for sufficiently large γ, adds an elementary embedding $\pi : V[H] \to M[H^*]$ with critical point κ_{m+1} and $\pi(\kappa_{m+1}) > \sup \pi[\lambda]$ (use standard arguments for extending embeddings).

Lemma 6.4. *There is in $V[H^*]$ a system of branches $\{b_j\}_{j \in J}$ for the system $\{R_i\}_{i \in I}$ with $J = I \times \kappa_n$, such that for some $j \in J$, the branch b_j is cofinal.*

Proof. First note that since $\kappa_n, |I| < cr(\pi)$, we may assume that $\pi(I) = I$ and $\pi(\{R_i\}_{i \in I}) = \{\pi(R_i)\}_{i \in I}$. This is a system on $\pi(D) \times \kappa_n$. Let a^* be a set in $\pi(D)$ such that $\pi[\lambda] \subseteq a^*$. For every $(i, \delta) \in I \times \kappa_n$, let $b_{i,\delta}$ be the partial map sending each $X \in D$ to the unique $\zeta < \kappa_n$ such that $(\pi[X], \zeta)\ \pi(R_i)\ (a^*, \delta)$ if such ζ exists. By elementarity, every $b_{i,\delta}$ is an R_i-branch. Condition (2) of Definition 5.3 is satisfied as well: indeed, if $X \in D$, then by condition (3) of Definition 5.1, there exists $\zeta, \eta < \kappa_n$ and $i \in I$ such that $(\pi[X], \zeta)\ \pi(R_i)\ (a^*, \eta)$, hence $X \in \mathrm{dom}(b_{i,\eta})$. It remains to prove that for some $j \in J$, b_j is cofinal. For every $X \in D$, we fix i_X, δ_X such that $X \in \mathrm{dom}(b_{i_X, \delta_X})$. The set I has size less than κ_m in W, moreover $\mathrm{Coll}(\kappa_m, \gamma)^V$ is κ_m-closed in $V[H_m \times H_{m+1} \times \dots]$ and $W = V[H]$ is a κ_m-c.c. forcing extension of $V[H_m \times H_{m+1} \times \dots]$, so I has size $< \kappa_m$ even in $V[H^*]$. On the other hand $|D| \geq \kappa_m$, so there exists a cofinal $D' \subseteq D$ and i, δ in $V[H^*]$ such that $i = i_X$ and $\delta = \delta_X$, for every $X \in D'$. This means that $X \in \mathrm{dom}(b_{i,\delta})$ for every $X \in D'$, namely $b_{i,\delta}$ is a cofinal branch. ⊣

$V[H^*]$ is a κ_m-closed forcing extension of $V[H] = W$, so we can apply the Preservation Theorem. Therefore a cofinal R_i-branch b exists in W, for some $i \in I$. Assume that $i = (a, c, \mu)$, for every $X \subseteq Y$ in $\mathrm{dom}(b)$, we have

$$(a, c) \Vdash (X, b(X)) <_{\dot{F}_\mu} (Y, b(Y)).$$

If $G_0 \times G_1 \subseteq \mathbb{L}(\mu)$ is any generic filter containing the condition (a, c), then the branch b determines a cofinal branch for $\dot{F}_\mu^{G_0 \times G_1}$ in $W[G_0 \times G_1]$ contradicting the

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fact that $\dot{F}_\mu$ is a name for a (ν^+, λ)-tree with no cofinal branches. This completes the proof of the theorem. ⊣

§7. Conclusions. We proved that if infinitely many supercompact cardinals exist in a model V, then there is a forcing extension of V where $\aleph_{\omega+1}$ has the strong tree property. We do not know whether $\aleph_{\omega+1}$ can consistently satisfy even the *super* tree property.

We also know (see [5]) that from infinitely many supercompact cardinals, one can build a model where the super tree property (hence, in particular the strong tree property) holds at every cardinal of the form $\aleph_{n+2}$, where $n < \omega$. Then, it is natural to ask whether it is possible to combine the two consistency results and prove from infinitely many supercompact cardinals, the consistency of the strong tree property "up to" $\aleph_{\omega+1}$, i.e., at every regular cardinal $\leq \aleph_{\omega+1}$ (above $\aleph_1$). These problems remain open.

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