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Fully nonlinear electrostatic waves in electron–positron plasmas

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Abstract. Fully nonlinear electrostatic waves in a plasma containing electrons, positrons, and ions are investigated by solving the governing equations exactly. It is found that both smooth and spiky quasistationary waves exist, and large-amplitude waves necessarily have large-phase velocities, but small-amplitude waves can be both fast and slow.

1. Introduction

Electron-positron (EP) plasmas can appear due to pair production in the early Universe, galactic nuclei, pulsar magnetospheres, microquasars, interstellar jet outflows, cosmological fireballs, as well as the center of our galaxy (e.g. Hardee and Rose 1978; Rao et al. 1986; Gallant et al. 1992; Liang et al. 1998; Cowan 1999; Nakashima and Cowan 2002; Cattaert et al. 2005). Significant quantities of positrons are also produced in the laboratory under high-energy conditions, such as in inertial confinement fusion schemes where ultra-intense laser pulses are employed, as well as in advanced tokamaks because of collisions of high-energy runaway electrons (e.g. Stenflo et al. 1985; Rao et al. 1986; Rizzato 1988; Berezhiani et al. 1992; Grieves and Surko 1997; Liang et al. 1998; Shukla et al. 2003; Cattaert et al. 2005; Marklund et al. 2005). An EP plasma is basically a multispecies plasma containing two oppositely charged equal-mass light-particle species. Many phenomena in classical plasmas, such as linear and nonlinear waves, often find counterparts in EP plasmas.

Most investigations on nonlinear waves in fluids and plasmas are based on perturbation schemes, where one starts with a steady homogeneous or nearly homogeneous state and considers the nonlinear evolution of the linear, or normal, modes of that state. Since the perturbation schemes necessarily involve small-amplitude expansion and/or renormalization, the nonlinear waves studied are mainly a result of the interaction of the linear waves with the background plasma, themselves, or other normal modes. Since many linear normal modes in plasmas have physically similar dispersion, dissipation, and nonlinear properties, their nonlinear evolution can be described by certain paradigm evolution equations, such as the Korteweg-deVries and Burgers equations, the Kadomtsev–Petviashvili equation, the nonlinear Schrödinger equation, etc., whose derivation and solutions are now well understood (e.g. Davidson 1972; Dodd 1982). On the other hand, most ultrahigh energy systems are far from equilibrium. That is, they can exist as fully dynamic states that are not near any steady state. In this case one cannot apply perturbation techniques in solving the governing equations of motion and the latter must be solved exactly. Only a few exact solutions of the equations describing a plasma, even a cold one, exist (e.g. Yu 1976; Popel et al. 1995; Yu et al. 2001; Karimov et al. 2009). In this paper, we investigate fully nonlinear electrostatic waves in a plasma containing electrons, positrons, and ions. It is found that under specific conditions, exact propagating wave solutions exist. Both smooth and spiky waves are found.

2. Governing equations

We consider a plasma consisting of electrons (e), positrons (p), and ions (i). We are interested in phenomena on the time scale of the light particles, so that the heavy ions are assumed to be stationary with density n_{i0}.

We shall assume that the electron and positron fluids are adiabatic. The continuity, momentum, and energy equations are

$$\partial_t n_s + \partial_x(n_s u_s) = 0, \tag{1}$$

$$(\partial_t + u_s\partial_x)u_s = -(q_s/m)\partial_x\phi - (1/n_s m)\partial_x P_s, \tag{2}$$

$$(\partial_t + u_s\partial_x)P_s/n_s^3 = 0, \tag{3}$$

where, m, $q_s\,(= \mp e)$, P_s, n_s, and $u_s$are the mass, charge, pressure, density, and velocity, respectively, of the species $s(= e, p)$. Since the electrons and positrons have equal mass and charge, it is reasonable to assume that the adiabatic constant is 3 for both species. The electrostatic potential ϕ is given by the Poisson's equation

$$\partial_x^2\phi = -4\pi(q_p n_p + q_e n_e + q_i n_{i0}), \tag{4}$$

where n_{i0} is the constant ion density. Without loss of generality, we shall assume that the ions are singly charged ($q_i = +e$).

We are interested in exact solutions of the governing equations. The boundary conditions can be arbitrary. The plasma also does not have to be neutral or quasineutral. However, for definitiveness and convenience in comparing with existing results, we shall assume charge neutrality $n_{i0} + n_{p0} = n_{e0}$ and pressure balance ($P_{p0} = P_{e0} \equiv P_0$) at infinity, where the physical quantities are denoted by the subscript 0.

For simplicity we shall consider quasisteady wave states. Accordingly, we set $\partial_t = 0$ and add the boundary condition $u_{s0} = -V$. Integrating (1) and applying the boundary conditions, one easily obtains the velocities of the electrons and positrons in terms of their densities:

$$u_e = -M/n_e, \tag{5}$$

$$u_p = -\alpha M/n_p, \tag{6}$$

where x, n_s, and u_s (thus also V) have been normalized by $\lambda = \sqrt{T_{e0}/4\pi n_{e0} e^2}$, n_{e0}, $v_T = \sqrt{T_{e0}/m}$, respectively. We have defined $M = V/v_T$, $T_{e0} = 3P_0/2n_{e0}$, and $\alpha = n_{p0}/n_{e0}$. Integrating the momentum equation (2) and applying the boundary conditions, one then obtains for the electron and positron densities:

$$2n_e^4 - n_e^2(M^2 + 2 + 2\phi) + M^2 = 0, \tag{7}$$

$$2n_p^4 - n_p^2\alpha^2(\alpha M^2 + 2 - 2\alpha\phi) + \alpha^5 M^2 = 0, \tag{8}$$

where ϕ has been normalized by T_{e0}/e. After normalization, the Poisson's equation can be written as

$$d_x^2\phi = n_e - n_p - (1 - \alpha). \tag{9}$$

Combining (5)–(9), we obtain

$$d_x^2\phi = \frac{1}{2}Q_- - \frac{\alpha}{2}Q_+ - (1 - \alpha), \tag{10}$$

where $Q_\pm = \sqrt{Y_\pm - A_\pm}$, $Y_- = (M^2 + 2 + 2\phi)$, $Y_+ = (\alpha M^2 + 2 - 2\alpha\phi)$, $A_- = \sqrt{Y_-^2 - 8M^2}$, and $A_+ = \sqrt{Y_+^2 - 8\alpha M^2}$. The corresponding equation for an electron plasma can be recovered by setting $\alpha = 0$, and that for a pure electron-positron plasma can be obtained by setting $\alpha = 1$. In the latter case the electron and positron motion in the waves are exactly out of phase. In the presence of the ion background, this phase relationship is broken because of the presence of α in the positron term Q_+. In the small-amplitude limit, one obtains

$$\partial_x^2\phi = -\left(\frac{1}{M^2 - 2} + \frac{\alpha^2}{\alpha M^2 - 2}\right)\phi,$$

which for $\alpha M^2 \to \alpha\omega^2/k^2v_T^2 \gg 2$ leads to the linear dispersion relation $\omega^2 = (1 + \alpha)\omega_{pe}^2 + 2k^2v_T^2$ (in the original units) for plasma waves in an electron-positron-ion plasma with $\gamma = 3$ for the light particles.

3. The Sagdeev potential

Multiplying both sides of (10) by $d_x\phi$, we can integrate it into the quadrature

$$\frac{1}{2}(d_x\phi)^2 = -V(\phi) + H, \tag{11}$$

$$V = -\frac{1}{6}\left[(2Y_- + A_-)\sqrt{Y_- - A_-} + (2Y_+ + A_+)\sqrt{Y_+ - A_+}\right] - \phi(1 - \alpha), \tag{12}$$

where H is the integration constant and $V(\phi)$is the Sagdeev potential (Chen 1983). If one also assumes $d_x\phi = 0$ at the boundary point, one obtains $H = -M^2(1+\alpha) - 4/3$, but for the present exact system this is not necessary in general. Equation (11) is in the form of the energy integral of a classical particle with total energy H moving in a potential field. One can therefore analyze the latter to see the behavior of the system. On the right-hand side of (12), the first, second, and third terms are the integrals over ϕ of the electron, positron, and ion densities, respectively.

Equations (7) and (8) show that in order for the solutions be real, a sufficient condition is $M > \sqrt{2/\alpha}$ for $0 < \alpha < 1$ and $M > \sqrt{2}$ for $\alpha = 0$. Figure 1 shows the Sagdeev potential $V(\phi)$ for different values of α and M. For clarity, in the figure the segments of the curves not satisfying the sufficient condition (corresponding

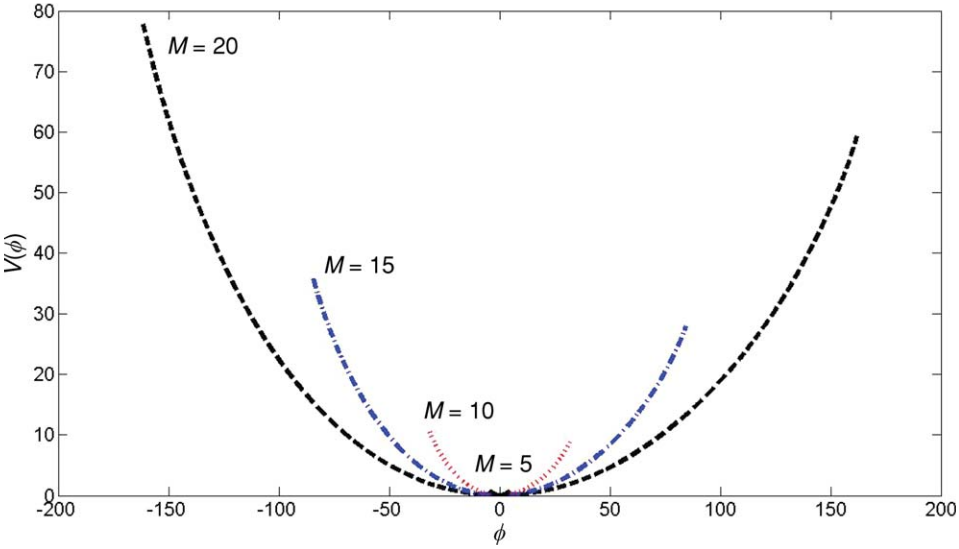

Figure 1. (Color online) Typical profiles of $V(\phi)$, for $M = 20$, 15, 10, 5, and $\alpha = 0.5$. For clarity, only the curve segments satisfying the sufficient condition (for real solutions) are shown.

to $M < \sqrt{2/\alpha}$) have been omitted. Accordingly, for any valid H (a horizontal line in the figure), there exists a nonlinear wave solution with phase speed M and the maximum and minimum values of ϕ given by the intersection points (Chen 1983). One can also see from the profile of $V(\phi)$ that, unlike the case where the electrons are isothermal (Yu 1976; Popel et al. 1995), there is no soliton solution. Closer inspection of (7)–(8) shows that the segment of the $V(\phi)$ profile containing a local maximum for finite ϕ (which would lead to a soliton solution) falls inside the excluded range. This conclusion is also valid for a pure electron plasma ($\alpha = 0$) with the same adiabatic constant and thus contradicts that of Yu (1976). The reason can be attributed to an algebraic error in (9b) and thus the corresponding curves in Fig. 1 of the latter, which placed the local maximum inside the valid range of the Sagdeev potential. Moreover, we can see that large-amplitude waves necessarily have large-phase velocities (large M values), but small-amplitude waves (if exist) can be both fast and slow. This result can be expected since only energetic particle motion can support large-amplitude waves.

4. Numerical results

To search for exact wave solutions, we have also directly integrated (7)–(9) numerically for different initial values $\phi(0)$ and $d_x\phi(0)$, corresponding to assigning different values of the total energy H. Different types of wave solutions are found.

Figure 2 for $\alpha = 0.02$ (small positron content) and $M = 4$ shows a typical nearly sinusoidal smooth wave solution. Because of the small positron concentration, the positron oscillations are of very small amplitude. If the phase speed of the wave is decreased to $M = 3$, the electron density oscillations becomes spiky, although the positron density and potential oscillations remain smooth, as shown in Fig. 3. This behavior appears because the electron fluid compression is too high, although not

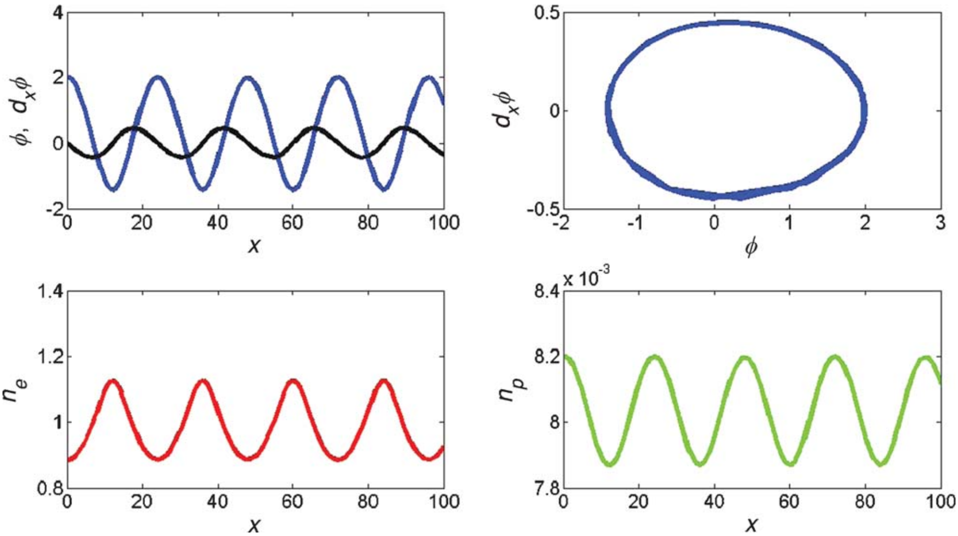

Figure 2. (Color online) Nonlinear plasma waves for $\alpha = 0.02$ and $M = 4$. The initial conditions are $\phi = 2$ and $d_x\phi = 0.015$ at $x = 0$, which also distinguish the two curves in the first subfigure.

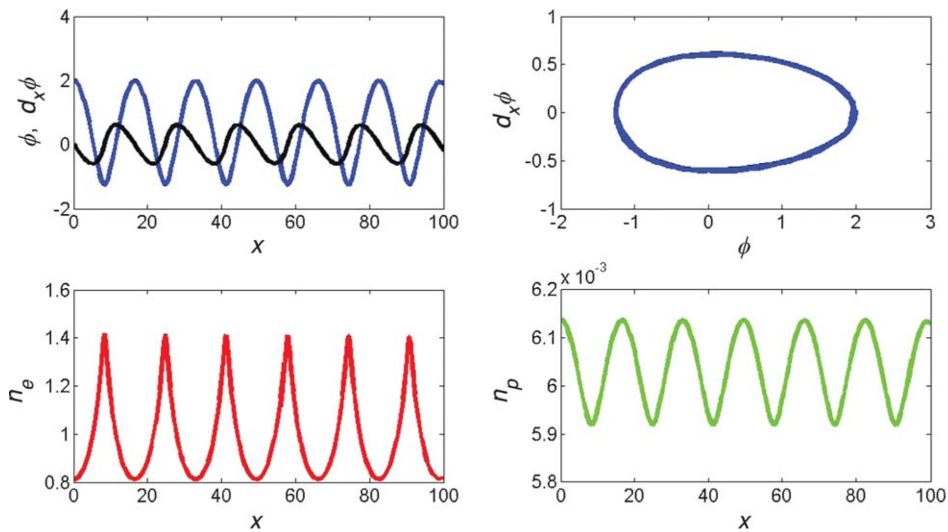

Figure 3. (Color online) Spiky plasma waves for $\alpha = 0.02$ and $M = 3$. The initial conditions are the same as in Fig. 2.

high enough (because of the adiabatic nature of the flow) to form solitons. However, when the phase speed is further increased, the wave again becomes smooth due to reduction of the amplitude of the electron oscillations, as shown in Fig. 4 for $M = 16$. In fact, we see that the wave is almost linear in this case. Plasma waves with smooth potential and electron density oscillations and spiky positron density oscillations are also possible, as demonstrated in Fig. 5 for $\alpha = 0.5$ and $M = 4.001$.

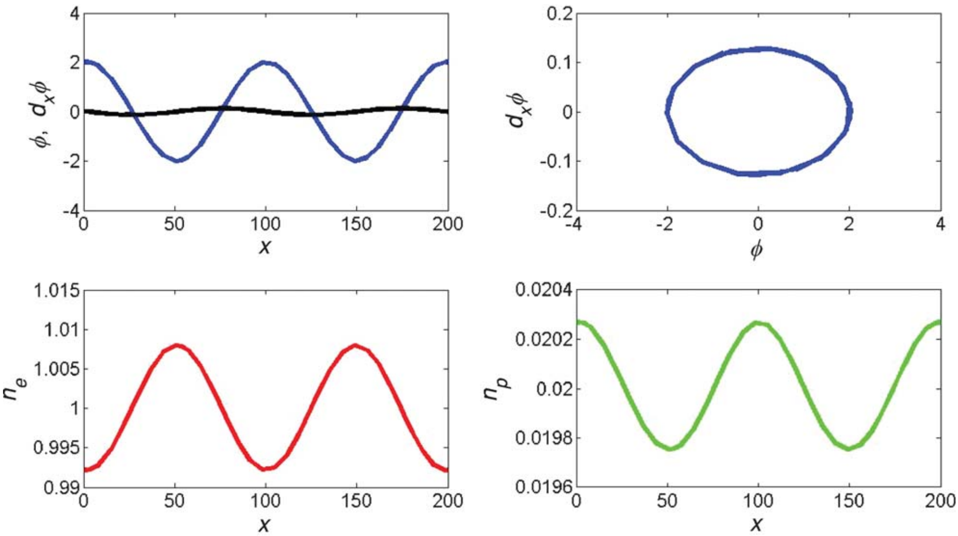

Figure 4. (Color online) Plasma waves for $\alpha = 0.02$ and $M = 16$. The initial conditions are the same as in Fig. 2.

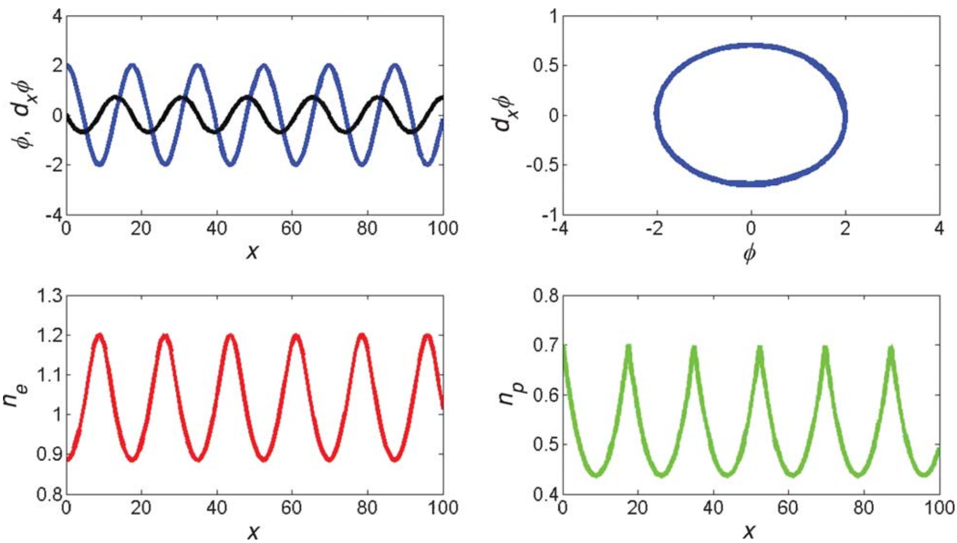

Figure 5. (Color online) Plasma waves for $\alpha = 0.5$ and $M = 4.001$. The initial conditions are the same as in Fig. 2.

There are also solutions with spiky oscillations in both the electron and positron densities. Figure 6, which is for $\phi(0) = 20$, $d_x\phi(0) = -30$, $\alpha = 0.7$, and $M = 41.5$ shows such a solution. We note that here the density oscillations are of large-amplitude and large-phase speed.

Since the governing equations are solved exactly, we have also looked for very large wavelength and/or high-speed waves. They can indeed exist, as shown in Fig. 7 for $\phi_0 = 10$, $d_x\phi_0 = 5$, $\alpha = 0.9$, and $M = 100$. However, although the wave here has

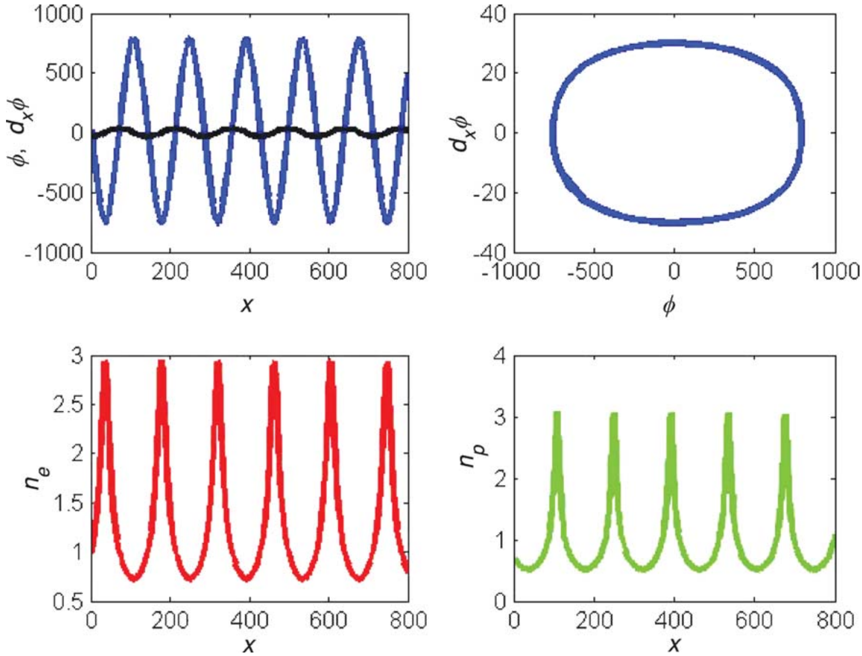

Figure 6. (Color online) Large-amplitude spiky plasma waves, for $\phi(0) = 20$, $d_x\phi(0) = -30$, $\alpha = 0.7$, and $M = 41.5$.

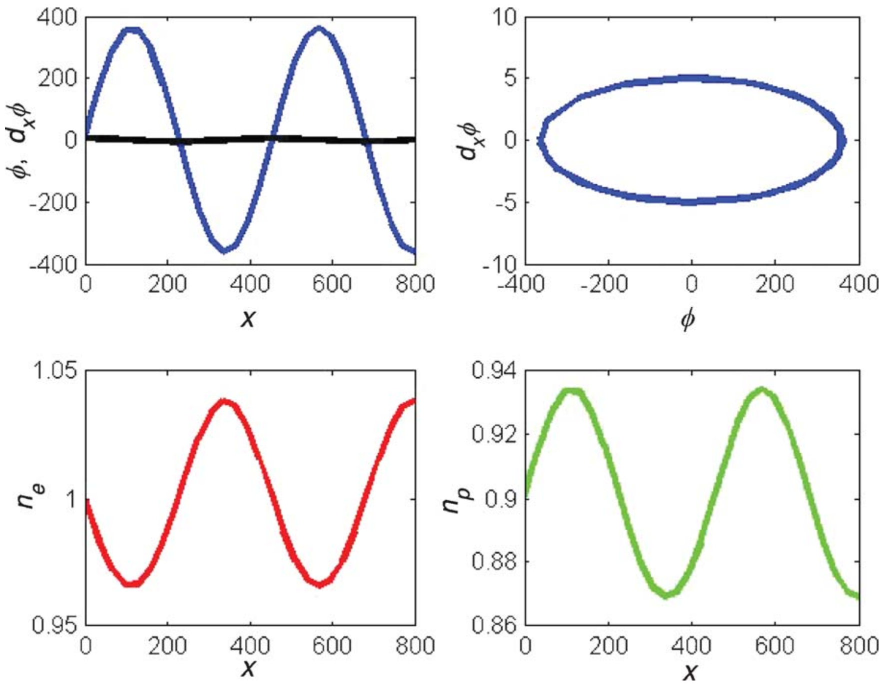

Figure 7. (Color online) Plasma waves for $\phi(0) = 10$, $d_x\phi(0) = 5$, $\alpha = 0.9$, and $M = 100$.

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high-phase speed, the amplitudes of the electron, and positron density oscillations remain fairly small.

5. Conclusion

We have solved the equations governing the electron-positron plasma with stationary ions exactly. The equations are integrated to a quadrature, which involves a singularity. It is shown numerically that the plasma system can have a variety of dynamic quasistationary states in the form of waves, including that with spiky electron and/or positron density structures. Large-amplitude waves necessarily have large-phase velocities, but small-amplitude waves can be of high as well as low speeds. Since the waves here satisfy the governing equations exactly, they are inherently stable against small perturbations within the validity of the latter. However, sufficiently large perturbations can drive a wave to a neighboring state.

The results here may be useful for interpreting nonstationary astrophysical and laboratory phenomena involving pair production or positron emission. Large-amplitude spiky periodic structures with controllable (for example, by varying the positron concentration, etc.) phases may also be useful as plasma gratings for modulating intense optical pulses (Sheng et al. 2003; Wu et al. 2005; Liu et al. 2009).

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