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# ASYMPTOTIC EXPANSION FOR THE TRANSITION DENSITIES OF STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY THE GAMMA PROCESSES

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In this paper, enlightened by the asymptotic expansion methodology developed by Li [(2013). Maximum-likelihood estimation for diffusion processes via closed-form density expansions. *Annals of Statistics* 41: 1350–1380] and Li and Chen [(2016). Estimating jump-diffusions using closed-form likelihood expansions. *Journal of Econometrics* 195(1): 51–70], we propose a Taylor-type approximation for the transition densities of the stochastic differential equations (SDEs) driven by the gamma processes, a special type of Lévy processes. After representing the transition density as a conditional expectation of Dirac delta function acting on the solution of the related SDE, the key technical method for calculating the expectation of multiple stochastic integrals conditional on the gamma process is presented. To numerically test the efficiency of our method, we examine the pure jump Ornstein–Uhlenbeck model and its extensions to two jump-diffusion models. For each model, the maximum relative error between our approximated transition density and the benchmark density obtained by the inverse Fourier transform of the characteristic function is sufficiently small, which shows the efficiency of our approximated method.

**Keywords:** asymptotic expansion, gamma process, stochastic differential equation, transition density

## 1. INTRODUCTION

It is known that Lévy-driven stochastic differential equations (SDEs) have been discussed in detail [2,20,21]. The jump-diffusion SDE driven by the gamma process, as one important type of the Lévy-driven SDEs, has been widely used in finance. For instance, the Ornstein–Uhlenbeck (OU)-type SDEs driven by the gamma processes were applied for modeling the short rate [8] and the returns of S&P 500 index [14]. The various sensitivity indices for the asset price dynamics driven by the gamma processes were discussed in Kawai and Takeuchi [17,18]. Note that the gamma process is a pure jump increasing Lévy process [2,5,30]. Starting from the gamma process, the variance gamma process was defined [24,25].

For the financial applications mentioned above, the transition densities of the related SDEs play a vital role [4,28]. However, except for some special cases, the transition densities or even characteristic functions of the SDEs usually do not admit closed-form formulas, which bring difficulties for related applications. In this paper, enlightened by the asymptotic expansion method presented in Li [22] and Li and Chen [23], we propose a Taylor-type closed-form expansion for the transition density of the jump-diffusion SDE driven by the gamma process.

For the jump-diffusion SDE driven by the gamma process, we start from representing its transition density as a conditional expectation of a Dirac delta function acting on the solution of the related SDE by applying the theory of Malliavin calculus [9,12,16,21]. The main challenge in our method is to calculate the expectation of the product of the values of a gamma process at different intermediate times, conditional on the value of this gamma process at the terminal time. Consequently, through the distributional property of gamma bridge discussed in Ribeiro and Webber [27], we express this type of conditional expectation as a polynomial function of the value of this gamma process at the terminal time. In this context, the expansion term of the transition density for any finite order can be analytically calculated in an efficient manner.

To illustrate the efficiency of our method, we conduct numerical analyses through three examples of the SDEs driven by the gamma processes, that is, the pure jump OU model, along with its extensions to the constant diffusion and the square-root diffusion models. For each model, we compare the true transition density obtained by the inverse Fourier transform of its characteristic function with the approximated density obtained by our proposed asymptotic expansion method. The numerical results show that our approximated transition density can be efficiently calculated and converge rapidly to the true density.

The rest of this paper is organized as follows. Section 2 lays our model setup and gives the general expression of the asymptotic expansion. Section 3 provides detailed procedures for explicitly representing the expansion terms. Section 4 exhibits the numerical performance of our expansion method through three concrete examples. Section 5 concludes the paper. The appendix gives the explicit expressions of the Sobolev norms and the dual Sobolev norms along with the definition of the convergences under these norms, which will be used in this paper.

## 2. THE MODEL SETUP AND APPROXIMATION METHODOLOGY

### 2.1. Preliminaries of Dirac Delta function

Before introducing our asymptotic expansion methodology, we first give a brief introduction of the Dirac delta function (see Hayashi [9], Ishikawa [12], Kanwal [16], and Kunita [21] for more details).

For ease of exposition later, we introduce the following notations and concepts. Denote by $\mathcal{S}'(\mathbb{R})$ the set of all real-valued tempered distributions. According to Section 6.2 in Kanwal [16], the Dirac delta function denoted as $\delta(\cdot)$ and its associated derivative operators $d^\ell\delta(\cdot)/dx^\ell$ for $\ell \geq 1$ belong to $\mathcal{S}'(\mathbb{R})$. Here, for each $\ell \geq 1$, the derivative operator $d^\ell\delta(\cdot)/dx^\ell$ is defined through an inner product with an infinitely differentiable function $f(\cdot)$ with compact support on $\mathbb{R}$, that is,

$$\left\langle \frac{d^\ell \delta(x-y)}{dx^\ell}, f(x) \right\rangle_x = (-1)^\ell \left\langle \delta(x-y), \frac{d^\ell f}{dx^\ell}(x) \right\rangle_x \tag{1}$$

for a fixed $y \in \mathbb{R}$, where the inner product $\langle f(x), g(x)\rangle_x := \int_{-\infty}^{\infty} f(x)g(x)\,dx$ (see Section 2.6 in Kanwal [16] for more details).

Let $D_\infty(\mathbb{R})$ be the set of all real-valued smooth Wiener–Poisson functionals and $D'_\infty(\mathbb{R})$ be the set of all real-valued generalized Wiener–Poisson functionals [21]. According to Theorem 5.12.1 and equation (5.175) in Kunita [21], for a tempered distribution $\Phi \in \mathcal{S}'(\mathbb{R})$, a regular nondegenerate Wiener–Poisson functional $F \in D_\infty(\mathbb{R})$ and a smooth Wiener–Poisson functional $G \in D_\infty(\mathbb{R})$, the generalized expectation $\mathbb{E}[\cdot]$ is defined as

$$\mathbb{E}[\Phi(F) \cdot G] := \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-ivx} \Phi(x) E[G e^{ivF}] \, dx \, dv, \tag{2}$$

where $F$ and $G$ can be treated as the random variables on the Wiener–Poisson space and the expectation in the right-hand side of Eq. (2) is the usual expectation in common sense. Hereafter, the notations $\mathbb{E}[\cdot]$ and $E[\cdot]$ represent the generalized expectation and usual expectation, respectively.

For a fixed $y \in \mathbb{R}$, when we take $\Phi(\cdot) = \delta(\cdot - y)$ and $G \equiv 1$ in Eq. (2), from the equation

$$\int_{-\infty}^{+\infty} e^{-ivx} \delta(x - y) \, dx = e^{-ivy}, \tag{3}$$

we obtain that

$$\mathbb{E}[\delta(F - y)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivy} E[e^{ivF}] \, dv. \tag{4}$$

Here, the functional $F$ in Eq. (4) can be taken as the strong solution of a homogeneous jump-diffusion SDE satisfying the nondegenerate-bounded (NDB) condition (see Sections 3.5–3.6 in Ishikawa [12] for more details). Especially, taking $F \equiv 0$ in Eq. (4), we can obtain that

$$\delta(y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ivy} \, dv. \tag{5}$$

Let $F(\epsilon)$ be a regular Wiener–Poisson functional in $D_\infty(\mathbb{R})$ for some parameter $\epsilon \in (0, 1]$. For example, $F(\epsilon)$ can be taken as the strong solution of some jump-diffusion SDEs with a parameter $\epsilon \in (0, 1]$. In the remaining of this section, for a fixed $y \in \mathbb{R}$, we introduce some theoretical results about the asymptotic expansion of $\delta(F(\epsilon) - y)$ with respect to the parameter $\epsilon$ (see [12 Sect. 4.1]). We assume that the functional $F(\epsilon)$ satisfies the uniformly nondegenerate condition (see [12 Definition 4.1]) and has an expansion

$$F(\epsilon) = \sum_{j=0}^{\infty} f_j \epsilon^j \tag{6}$$

with respect to the Sobolev norms in $D_\infty(\mathbb{R})$, where $f_0, f_1, f_2, \ldots$ are smooth Wiener–Poisson functionals. See Appendix A for more details about the explicit expressions of the Sobolev norms in $D_\infty(\mathbb{R})$ and the definition of the convergence in $D_\infty(\mathbb{R})$.

According to Theorem 4.1 of Ishikawa [12], for each fixed $y \in \mathbb{R}$, $\delta(F(\epsilon) - y)$ belongs to $D'_\infty(\mathbb{R})$ and has an asymptotic expansion

$$\delta(F(\epsilon) - y) = \sum_{m=0}^{M} \Phi_m(y) \epsilon^m + \mathcal{O}(\epsilon^{M+1}) \tag{7}$$

with respect to the dual Sobolev norms in $D'_\infty(\mathbb{R})$, where $M \in \mathbb{N}$ denotes an arbitrary order of the expansion. See Appendix B for more details about the explicit expressions of the dual Sobolev norms in $D'_\infty(\mathbb{R})$ and the definition of the convergence in $D'_\infty(\mathbb{R})$. Given the

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functionals $\{f_0, f_1, f_2, \ldots\}$ defined by Eq. (6), the coefficients $\Phi_m(y) \in D'_\infty(\mathbb{R})$ for $m \geq 0$ can be expressed as

$$\Phi_0(y) = \delta(f_0 - y) \quad \text{and} \quad \Phi_m(y) = \sum_{(\ell,(j_1,j_2,\cdots,j_\ell)) \in \mathcal{S}_m} \frac{1}{\ell!} \frac{d^\ell \delta(f_0 - y)}{dx^\ell} \prod_{i=1}^{\ell} f_{j_i} \quad \text{for } m \geq 1, \tag{8}$$

where the index set $\mathcal{S}_m$ is defined as

$$\mathcal{S}_m := \{(\ell, \mathbf{j}(\ell)) \,|\, \ell = 1, 2, \ldots,\ \mathbf{j}(\ell) = (j_1, j_2, \ldots, j_\ell) \text{ with } j_1, j_2, \ldots, j_\ell \geq 1 \\ \text{and } j_1 + j_2 + \cdots + j_\ell = m\}. \tag{9}$$

For example, the coefficients $\Phi_1(y)$ and $\Phi_2(y)$ are given by

$$\Phi_1(y) = f_1 \frac{d\delta(f_0 - y)}{dx} \quad \text{and} \quad \Phi_2(y) = f_2 \frac{d\delta(f_0 - y)}{dx} + \frac{1}{2} f_1^2 \frac{d^2\delta(f_0 - y)}{dx^2}.$$

According to Section 4 in Ishikawa [12], by taking the generalized expectation defined in Eq. (2) on both sides of Eq. (7), we can obtain that

$$\mathbb{E}[\delta(F(\epsilon) - y)] = \sum_{m=0}^{M} \mathbb{E}[\Phi_m(y)]\epsilon^m + \mathcal{O}(\epsilon^{M+1}). \tag{10}$$

For example, the terms $\mathbb{E}[\Phi_0(y)]$ and $\mathbb{E}[\Phi_1(y)]$ in Eq. (10) can be evaluated via Eq. (1), Eqs. (2) and (3) as

$$\mathbb{E}[\Phi_0(y)] = \mathbb{E}[\delta(f_0 - y)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ivy} E[e^{ivf_0}] dv$$

and

$$\mathbb{E}[\Phi_1(y)] = \mathbb{E}\left[f_1 \frac{d\delta(f_0 - y)}{dx}\right] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} iv\, e^{-ivy} E[f_1\, e^{ivf_0}]\, dv.$$

In the subsequent calculation of the expansion terms for the transition density, by using Eq. (2), we will transform some specific generalized expectations like $\mathbb{E}[\Phi_m(y)]$ into the usual expectations.

### 2.2. The Model Setup

In this paper, we consider the following homogeneous jump-diffusion SDE driven by a gamma process

$$dX(t) = \mu(X(t); \boldsymbol{\theta})\, dt + \sigma(X(t); \boldsymbol{\theta})\, dW(t) + dL(t), \quad X(0) = x_0, \tag{11}$$

where the functions $\mu(x; \boldsymbol{\theta})$ and $\sigma(x; \boldsymbol{\theta})$ are assumed to depend on some parameter vector $\boldsymbol{\theta}$ belonging to an open bound set $\Theta$, $\{W(t), t \geq 0\}$ is a Brownian motion and $\{L(t), t \geq 0\}$ is a gamma process. Moreover, we assume that the gamma process $\{L(t), t \geq 0\}$ starts at $L(0) = 0$ with the density function

$$p_{L(t)}(x) = \frac{b^{at} x^{at-1}\, e^{-bx}}{\Gamma(at)}, \quad x \geq 0 \tag{12}$$

at time $t > 0$, where $a$ and $b$ are positive constants, and $\Gamma(\cdot)$ denotes the gamma function. The two processes $\{W(t), t \geq 0\}$ and $\{L(t), t \geq 0\}$ are independent. The characteristic function of the gamma process $L(t)$ is calculated as

$$Ee^{i\lambda L(t)} = \left(1 - \frac{i\lambda}{b}\right)^{-at} \triangleq \exp[t\psi_L(\lambda)], \quad \textbf{(13)}$$

where $\psi_L(\lambda) = -a\log(1 - i\lambda/b)$ is the characteristic exponent of $\{L(t), t \geq 0\}$.

To guarantee the existence and uniqueness of the strong solution $X(t)$ of SDE (11) and obtain other desirable properties for implementing our method, the following standard and technical assumptions are assumed in this paper:

Assumption 1: *The diffusion function $\sigma(x;\boldsymbol{\theta})$ satisfies that $\inf_{x\in\mathbb{R}} \sigma(x;\boldsymbol{\theta}) > 0$ for any $\boldsymbol{\theta} \in \Theta$.*

Assumption 2: *For each $k \in \mathbb{N}_+$, the $k$thorder partial derivatives in $x$ of $\mu(x;\boldsymbol{\theta})$ and $\sigma(x;\boldsymbol{\theta})$ are uniformly bounded for any $(x,\boldsymbol{\theta}) \in \mathbb{R} \times \Theta$.*

Assumption 3: *The functions $\mu(x;\boldsymbol{\theta})$ and $\sigma(x;\boldsymbol{\theta})$ satisfy the linear growth conditions*

$$|\mu(x;\boldsymbol{\theta})| \leq c_1(1 + |x|) \quad \textit{and} \quad |\sigma(x;\boldsymbol{\theta})| \leq c_2(1 + |x|),$$

*for some $c_1, c_2 \in \mathbb{R}_+$ and any $(x,\boldsymbol{\theta}) \in \mathbb{R} \times \Theta$.*

Assumptions 1 and 2 guarantee the NDB condition and the uniformly nondegenerate condition for justifying the validity and convergence of our proposed asymptotic expansion method, which will be shown in Section 2.3. Assumptions 2 and 3 guarantee the existence and uniqueness of the strong solution $X(t)$ of SDE (11).

## 2.3. The Expansion of the Transition Density

For the jump-diffusion SDE (11), by the time-homogeneity nature, the transition density of $X(t+\Delta)$ given $X(t) = x_0$ can be expressed as

$$\mathbb{P}(X(t+\Delta) \in dx \,|\, X(t) = x_0;\boldsymbol{\theta}) = p_{X(\Delta)}(x \,|\, x_0;\boldsymbol{\theta})\, dx, \quad \textbf{(14)}$$

where $\Delta$ denotes the time interval. For most SDEs defined in (11), their transition densities do not admit closed-form expressions. Even for some special cases with closed-form conditional characteristic functions, the inversion to transition densities may not be easy, especially for the pure-jump processes [4,28]. In the following, we propose a closed-form expansion for approximating the transition density $p_{X(\Delta)}(x|x_0;\boldsymbol{\theta})$ of the jump-diffusion SDE (11).

To start with, we parameterize the dynamics of $X(t)$ in (11) via a parameter $\epsilon \in (0,1]$ as

$$dX(\epsilon,t) = \epsilon[\mu(X(\epsilon,t);\boldsymbol{\theta})\, dt + \sigma(X(\epsilon,t);\boldsymbol{\theta})dW(t) + dL(t)], \quad X(\epsilon,0) = x_0. \quad \textbf{(15)}$$

Note that the solution $X(\epsilon,t)$ of (15) satisfies $X(\epsilon,t)|_{\epsilon=1} = X(t)$. By regarding $\epsilon \in (0,1]$ as an extra element of the parameter vector, we see that the SDE (15) still satisfies Assumptions 2 and 3, which implies the existence and uniqueness of the strong solution $X(\epsilon,t)$ [26]. The transition density of $X(\epsilon,t)$ in (15) can be expressed as

$$\mathbb{P}(X(\epsilon,t+\Delta) \in dx \,|\, X(\epsilon,t) = x_0;\boldsymbol{\theta}) = p_{X(\epsilon,\Delta)}(x \,|\, x_0;\boldsymbol{\theta})\, dx. \quad \textbf{(16)}$$

Once we obtain an asymptotic expansion of $p_{X(\epsilon,\Delta)}(x \,|\, x_0;\boldsymbol{\theta})$ as a series of $\epsilon$, the transition density $p_{X(\Delta)}(x \,|\, x_0;\boldsymbol{\theta})$ in Eq. (14) can be obtained by letting $\epsilon = 1$.

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To derive the asymptotic expansion of $p_{X(\epsilon,\Delta)}(x|x_0;\boldsymbol{\theta})$ in Eq. (16), we first claim that $X(\epsilon,t)$ satisfies the NDB condition introduced in Section 2.1, which will justify the representation of the transition density $p_{X(\epsilon,\Delta)}(x\,|\,x_0;\boldsymbol{\theta})$ as a conditional expectation shown below. Note that for $X(\epsilon,t)$, the NDB condition introduced from Definition 3.5 in Ishikawa [12] is transformed to the condition that $\sigma(x;\boldsymbol{\theta})\neq 0$ for any $(x,\boldsymbol{\theta})\in\mathbb{R}\times\Theta$, which is guaranteed by Assumption 1.

Based on the NDB condition and the time-homogeneity nature of $X(\epsilon,t)$, we represent $p_{X(\epsilon,\Delta)}(x\,|\,x_0;\boldsymbol{\theta})$ as a conditional expectation of Dirac delta function acting on $X(\epsilon,\Delta)-x$ by

$$p_{X(\epsilon,\Delta)}(x\,|\,x_0;\boldsymbol{\theta})=\mathbb{E}[\delta(X(\epsilon,\Delta)-x)\,|\,X(\epsilon,0)=x_0;\boldsymbol{\theta}]. \quad \textbf{(17)}$$

The validity of Eq. (17) will be verified in Remark 2 below in detail. For brevity, we omit the initial condition $X(\epsilon,0)=x_0$ and drop the dependence of $\boldsymbol{\theta}$ in the SDEs (11) and (15) hereafter, unless especially noted. According to Proposition 4.4 in Ishikawa [12], we give the expansion of $X(\epsilon,\Delta)$ in the following proposition. Here, we recall the Sobolev norms in $D_\infty(\mathbb{R})$ introduced after Eq. (6).

PROPOSITION 1: *The solution $X(\epsilon,\Delta)$ of the SDE (15) can be expressed as a pathwise Taylor-type expansion*[1]

$$X(\epsilon,\Delta)=\sum_{m=0}^{M}X_m(\Delta)\epsilon^m+\mathcal{O}(\epsilon^{M+1})\ \textit{for}\,\epsilon\in(0,1) \quad \textbf{(18)}$$

*with respect to the Sobolev norms in $D_\infty(\mathbb{R})$, where $M\in\mathbb{N}$ denotes an arbitrary order of expansion,*

$$X_0(\Delta)=X(0,\Delta)\equiv x_0, \quad \textbf{(19)}$$

$$X_1(\Delta)=\mu(x_0)\Delta+\sigma(x_0)W(\Delta)+L(\Delta), \quad \textbf{(20)}$$

*and*

$$X_m(\Delta)=\int_0^{\Delta}\mu_{m-1}(s)\,ds+\int_0^{\Delta}\sigma_{m-1}(s)\,dW(s)\quad \textit{for } m\geq 2, \quad \textbf{(21)}$$

*in which*

$$\mu_m(s):=\frac{1}{m!}\left.\frac{d^m\mu(X(\epsilon,s))}{d\epsilon^m}\right|_{\epsilon=0}=\sum_{(\ell,(j_1,j_2,\ldots,j_\ell))\in\mathcal{S}_m}\frac{1}{\ell!}\frac{d^\ell\mu(x_0)}{dx^\ell}\prod_{i=1}^{\ell}X_{j_i}(s) \quad \textbf{(22)}$$

*and*

$$\sigma_m(s):=\frac{1}{m!}\left.\frac{d^m\sigma(X(\epsilon,s))}{d\epsilon^m}\right|_{\epsilon=0}=\sum_{(\ell,(j_1,j_2,\ldots,j_\ell))\in\mathcal{S}_m}\frac{1}{\ell!}\frac{d^\ell\sigma(x_0)}{dx^\ell}\prod_{i=1}^{\ell}X_{j_i}(s) \quad \textbf{(23)}$$

*with the index set $\mathcal{S}_m$ defined by (9).*

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[1] According to the proof of Proposition 7.8 in Hayashi and Ishikawa [10], the remainder term $\mathcal{O}(\epsilon^{M+1})$ of the expansion for $X(\epsilon,\Delta)$ in (18) can be explicitly expressed as

$$\frac{\epsilon^{m+1}}{m!}\int_0^1(1-\theta)^m\left(\int_0^{\Delta}\left.\frac{d^m\mu(X(\alpha,t))}{d\alpha^m}\right|_{\alpha=\epsilon\theta}dt+\int_0^{\Delta}\left.\frac{d^m\sigma(X(\alpha,t))}{d\alpha^m}\right|_{\alpha=\epsilon\theta}dW(t)\right)d\theta.$$

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Proof: By Proposition 4.4 in Ishikawa [12], we can obtain Eqs. (19), (20), and

$$X_m(\Delta) = \int_0^\Delta \frac{1}{(m-1)!} \left.\frac{d^{m-1}\mu(X(\epsilon,s))}{d\epsilon^{m-1}}\right|_{\epsilon=0} ds + \int_0^\Delta \frac{1}{(m-1)!} \left.\frac{d^{m-1}\sigma(X(\epsilon,s))}{d\epsilon^{m-1}}\right|_{\epsilon=0} dW(s) \quad \textbf{(24)}$$

for $m \geq 2$. Further, from the expansion of $X(\epsilon,\Delta)$ in (18), we can obtain Eqs. (22) and (23) with the index set $\mathcal{S}_m$ defined by (9) and the condition $X(0,t) \equiv x_0$. Plugging Eqs. (22) and (23) into Eq. (24), we can obtain Eq. (21). ■

Next, we illustrate the expansion of $p_{X(\epsilon,\Delta)}(x \,|\, x_0; \boldsymbol{\theta})$ in Eq. (17) as a convergent series of $\epsilon$. To do this, we standardize $X(\epsilon,\Delta)$ into

$$Y(\epsilon,\Delta) = \frac{X(\epsilon,\Delta) - x_0}{\sigma(x_0)\sqrt{\Delta}\epsilon}, \quad \textbf{(25)}$$

from which the transition density $p_{X(\epsilon,\Delta)}(x \,|\, x_0; \boldsymbol{\theta})$ in Eq. (17) can be represented in terms of $Y(\epsilon,\Delta)$ as

$$p_{X(\epsilon,\Delta)}(x \,|\, x_0; \boldsymbol{\theta}) = \frac{1}{\sigma(x_0)\sqrt{\Delta}\epsilon} \mathbb{E}[\delta(Y(\epsilon,\Delta) - y)]\Big|_{y=\frac{x-x_0}{\sigma(x_0)\sqrt{\Delta}\epsilon}}. \quad \textbf{(26)}$$

According to Hayashi and Ishikawa [10] and Definition 4.1 in Ishikawa [12], $Y(\epsilon,\Delta)$ in Eq. (25) satisfies the uniformly nondegenerate condition which is guaranteed by Assumptions 1 and 2. Then, the expectation in the right-hand side of Eq. (26) admits a convergent series of $\epsilon$, which is summarized into the following conclusion. Here, we recall the dual Sobolev norms in $D'_\infty(\mathbb{R})$ introduced in the analysis after Eq. (7) and give the following proposition.

Proposition 2: *The generalized expectation* $\mathbb{E}[\delta(Y(\epsilon,\Delta) - y)]$ *in the right side of Eq. (26) admits the following expansion*

$$\mathbb{E}[\delta(Y(\epsilon,\Delta) - y)] = \sum_{m=0}^{M} \Omega_m(y)\epsilon^m + \mathcal{O}(\epsilon^{M+1}) \quad \textit{for } \epsilon \in (0,1) \quad \textbf{(27)}$$

*with respect to the dual Sobolev norms in* $D'_\infty(\mathbb{R})$*, where* $\Omega_m(y) := \mathbb{E}\Phi_m(y)$ *for* $m \geq 0$ *is the generalized expectation with*

$$\Phi_0(y) = \delta(Y_0(\Delta) - y) \quad \textbf{(28)}$$

*and*

$$\Phi_m(y) = \sum_{(\ell,(j_1,j_2,\ldots,j_\ell))\in\mathcal{S}_m} \frac{1}{\ell!} \frac{1}{(\sigma(x_0)\sqrt{\Delta})^\ell} \frac{d^\ell \delta(Y_0(\Delta) - y)}{dx^\ell} \prod_{i=1}^{\ell} X_{j_i+1}(\Delta) \quad \textbf{(29)}$$

*for* $m \geq 1$*, in which the index set* $\mathcal{S}_m$ *is defined in (9),* $X_i(\Delta), i \geq 1$ *are defined in Eq. (18) and* $Y_0(\Delta) = X_1(\Delta)/\sigma(x_0)\sqrt{\Delta}$.

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PROOF: To obtain an expansion of $\mathbb{E}[\delta(Y(\epsilon,\Delta)-y)]$ in Eq. (26) with respect to $\epsilon$, we notice from Eqs. (18), (25), and $X_0(\Delta)\equiv x_0$ that

$$Y(\epsilon,\Delta)=\sum_{m=0}^{M} Y_m(\Delta)\epsilon^m+\mathcal{O}(\epsilon^{M+1}), \tag{30}$$

where

$$Y_m(\Delta)=\frac{X_{m+1}(\Delta)}{\sigma(x_0)\sqrt{\Delta}},\quad \text{for } m=0,1,2,\ldots. \tag{31}$$

Since the functional $\delta(\cdot-y)$ belongs to $\mathcal{S}'(\mathbb{R})$, according to Eq. (7), we obtain a Taylor-type expansion of $\delta(Y(\epsilon,\Delta)-y)$ as

$$\delta(Y(\epsilon,\Delta)-y)=\sum_{m=0}^{M}\Phi_m(y)\epsilon^m+\mathcal{O}(\epsilon^{M+1}) \tag{32}$$

for any $M\in\mathbb{N}$. Here in Eq. (32), it follows from Eqs. (8), (30), and (31) that $\Phi_m(y)$ for $m\geq 0$ can be given by Eqs. (28) and (29) for $m\geq 1$. Further, based on Eq. (10), we take the generalized expectation on both sides of Eq. (32) to obtain the expansion Eq. (27). ■

For simplicity, we name $\Omega_0(y)$ and $\Omega_m(y)$ for $m\geq 1$ in Eq. (27) the leading term and the higher-order terms, respectively. Combining Eqs. (26) and (27) by letting $\epsilon=1$, the approximated transition density of $X(\Delta)$ up to the $M$thorder is proposed as

$$p^{(M)}_{X(\Delta)}(x\,|\,x_0;\boldsymbol{\theta}):=\frac{1}{\sigma(x_0)\sqrt{\Delta}}\sum_{m=0}^{M}\Omega_m\left(\frac{x-x_0}{\sigma(x_0)\sqrt{\Delta}}\right). \tag{33}$$

Consequently, to approximate the transition density $p_{X(\Delta)}(x\,|\,x_0;\boldsymbol{\theta})$ in Eq. (14) up to any finite order, it suffices to specify the functions $\Omega_0(y)$ and $\Omega_m(y)$ for $m\geq 1$ in Eq. (33), which will be investigated in Section 2.4.

REMARK 1: *To the best of our knowledge, we do not find any rigorous theoretical results about the convergence of the expansion series for $X(\epsilon,\Delta)$ by letting $\epsilon=1$. Nevertheless, in Section 4 of Ishikawa [12], the authors point out that the convergence of $X(\epsilon,\Delta)$ for $\epsilon\in(0,1)$ given by Proposition 4.4 therein can be extended to hold for $\epsilon=1$. Further, in Li [22] and Li and Chen [23] , they approximate the transition densities of the parameterized solutions $X(\epsilon,\Delta)$ of the considered SDEs by letting $\epsilon=1$ in a similar manner, and their numerical results exhibit the order-by-order convergence. Inherited from the above facts, in our model, we also choose to take $\epsilon=1$ in the theoretical approximation. Numerically, as will be seen in Section 4, the results also exhibit the order-by-order convergence for the expansions of the considered transition densities.*

REMARK 2: *Equation (17) can be verified as follows. According to Section 6.4 in Kunita [21], under the initial condition $X(\epsilon,0)=x_0$, the strong solution $X(\epsilon,t)$ of SDE (15) is uniquely tied to a stochastic flow, which is a regular Wiener–Poisson functional belonging to the set $D_\infty(\mathbb{R})$ ([21 Sect. 3.1]). Thus, by noting that $X(\epsilon,t)$ satisfies the NDB condition, under the initial condition $X(\epsilon,0)=x_0$, we take $F=X(\epsilon,\Delta)$ in Eq. (4) to obtain that*

$$\begin{aligned}&\mathbb{E}[\delta(X(\epsilon,\Delta)-x)\,|\,X(\epsilon,0)=x_0;\boldsymbol{\theta}]\\&=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-ivx}E[e^{ivX(\epsilon,\Delta)}\,|\,X(\epsilon,0)=x_0;\boldsymbol{\theta}]\,dv=p_{X(\epsilon,\Delta)}(x\,|\,x_0;\boldsymbol{\theta}),\end{aligned}$$

*which verifies Eq. (17).*

Remark 3: *The above closed-form expansion method can also be applied to the special case of SDE (11) with $\sigma(X(t);\boldsymbol{\theta}) \equiv 0$. In this context, we only adjust the above algorithm to standardize $X(\epsilon,\Delta)$ defined by (15) into*

$$Y(\epsilon,\Delta) = \frac{X(\epsilon,\Delta) - x_0}{\epsilon}$$

*instead of Eq. (25), which implies that*

$$\mathbb{E}[\delta(X(\epsilon,\Delta) - x)] = \frac{1}{\epsilon}\mathbb{E}[\delta(Y(\epsilon,\Delta) - y)]\bigg|_{y=\frac{x-x_0}{\epsilon}}.$$

*The remaining procedures are performed in a similar manner. Therefore, the approximated transition density of $X(\Delta)$ up to the $M$thorder can be obtained by*

$$p^{(M)}_{X(\Delta)}(x \mid x_0;\boldsymbol{\theta}) = \sum_{m=0}^{M} \Omega_m(x - x_0), \tag{34}$$

*where $\Omega_m(y) = E\Phi_m(y)$ and the terms $\Phi_m(y)$ are calculated by*

$$\Phi_0(y) = \delta(Y_0(\Delta) - y)$$

*and*

$$\Phi_m(y) = \sum_{(\ell,(j_1,j_2,\ldots,j_\ell))\in\mathcal{S}_m} \frac{1}{\ell!}\frac{d^\ell\delta(Y_0(\Delta) - y)}{dx^\ell}\prod_{i=1}^{\ell} X_{j_i+1}(\Delta)$$

*for $m \geq 1$, with the index set $\mathcal{S}_m$ defined in (9) and $Y_0(\Delta) = X_1(\Delta)$.*

### 2.4. General Expressions of the Leading Term and High-order Terms

In this part, we give the explicit expression of the leading term $\Omega_0(y)$ and the general representations of the higher-order terms $\Omega_m(y)$ for $m \geq 1$ defined in Eq. (33). Throughout this section, we denote by $\phi(\cdot)$ the density function of a standard normal variable and recall that $p_{L(t)}(\cdot)$ is the density function of the gamma process $L(t)$ given by Eq. (12).

From Eq. (28), the leading term $\Omega_0(y)$ is expressed as

$$\Omega_0(y) = \mathbb{E}[\delta(Y_0(\Delta) - y)], \tag{35}$$

which is exactly the density function of $Y_0(\Delta)$ evaluated at $y$. The explicit expression of $\Omega_0(y)$ is given in the following proposition.

Proposition 3: *The leading term $\Omega_0(y)$ in Eq. (33) admits the following explicit expression*

$$\Omega_0(y) = \int_0^{+\infty} \phi\left(y - \frac{\mu(x_0)\Delta + u}{\sigma(x_0)\sqrt{\Delta}}\right) p_{L(\Delta)}(u)\,du,$$

*where $p_{L(\Delta)}(\cdot)$ is the density function of the gamma process $L(\Delta)$ given by Eq. (12).*

PROOF: From Eqs. (4) and (35), we obtain that

$$\begin{aligned}\Omega_0(y) &= \mathbb{E}[\delta(Y_0(\Delta)-y)] \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ivy} E[e^{ivY_0(\Delta)}]\,dv \\ &= E\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ivy} E[e^{ivY_0(\Delta)} \mid L(\Delta)] dv\right]. \end{aligned} \tag{36}$$

We notice from Eqs. (20) and (31) that $Y_0(\Delta)$ in Eq. (36) can be represented as

$$Y_0(\Delta) = \frac{X_1(\Delta)}{\sigma(x_0)\sqrt{\Delta}} = \frac{W(\Delta)}{\sqrt{\Delta}} + \frac{\mu(x_0)\Delta + L(\Delta)}{\sigma(x_0)\sqrt{\Delta}}. \tag{37}$$

Here, conditioned on the jump term $L(\Delta)$, the variable $Y_0(\Delta)$ in Eq. (37) follows a normal distribution. Therefore, the inner term of the expectation in the last equation of Eq. (36) can be calculated as

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ivy} E[e^{ivY_0(\Delta)} \mid L(\Delta)]\,dv \\ &= E\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ivy} e^{iv\left(\frac{W(\Delta)}{\sqrt{\Delta}} + \frac{\mu(x_0)\Delta+L(\Delta)}{\sigma(x_0)\sqrt{\Delta}}\right)}\,dv \,\middle|\, L(\Delta)\right] \\ &= \int_{-\infty}^{\infty} E\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ivy}\, e^{iv\left(x+\frac{\mu(x_0)\Delta+L(\Delta)}{\sigma(x_0)\sqrt{\Delta}}\right)}\,dv \,\middle|\, L(\Delta)\right]\phi(x)\,dx \\ &= E\left[\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ivy}\, e^{iv\left(x+\frac{\mu(x_0)\Delta+L(\Delta)}{\sigma(x_0)\sqrt{\Delta}}\right)}\,dv\right)\phi(x)\,dx \,\middle|\, L(\Delta)\right]. \end{aligned} \tag{38}$$

By the relation Eq. (5), we obtain that

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ivy}\, e^{iv\left(x+\frac{\mu(x_0)\Delta+L(\Delta)}{\sigma(x_0)\sqrt{\Delta}}\right)}\,dv = \delta\left(x + \frac{\mu(x_0)\Delta + L(\Delta)}{\sigma(x_0)\sqrt{\Delta}} - y\right).$$

Then plugging the above equation into Eq. (38) by noting the definition of the Dirac delta function, we obtain that

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ivy} E[e^{ivY_0(\Delta)} \mid L(\Delta)]\,dv \\ &= E\left[\int_{-\infty}^{\infty}\delta\left(x + \frac{\mu(x_0)\Delta + L(\Delta)}{\sigma(x_0)\sqrt{\Delta}} - y\right)\phi(x)\,dx \,\middle|\, L(\Delta)\right] \\ &= E\left[\phi\left(y - \frac{\mu(x_0)\Delta + L(\Delta)}{\sigma(x_0)\sqrt{\Delta}}\right) \,\middle|\, L(\Delta)\right]. \end{aligned} \tag{39}$$

Plugging Eq. (39) into (36), the leading term $\Omega_0(y)$ can be finally calculated as

$$\begin{aligned}\Omega_0(y) &= E\left[E\left[\phi\left(y - \frac{\mu(x_0)\Delta + L(\Delta)}{\sigma(x_0)\sqrt{\Delta}}\right)\,\middle|\, L(\Delta)\right]\right] \\ &= \int_0^{+\infty}\phi\left(y - \frac{\mu(x_0)\Delta + u}{\sigma(x_0)\sqrt{\Delta}}\right) p_{L(\Delta)}(u)\,du. \end{aligned}$$

■

To calculate the higher-order terms $\Omega_m(y)$ for $m \geq 1$, we introduce the following notations. For $\ell \geq 1$ and $\mathbf{j} = (j_1, j_2, \ldots, j_\ell)$ with $j_i \geq 1$, we define

$$K_{(\ell,\mathbf{j})}(z_1, z_2) := E\left(\prod_{i=1}^{\ell} X_{j_i+1}(\Delta) \,\middle|\, W(\Delta), L(\Delta)\right)\Bigg|_{W(\Delta)=z_1\sqrt{\Delta}, L(\Delta)=z_2}. \tag{40}$$

Here, we omit the dependent variate $\Delta$ in the notation $K_{(\ell,\mathbf{j})}(z_1, z_2)$ when there is no ambiguity. Meanwhile, for any bivariate differentiable function $u(x, y)$ defined on $\mathbb{R}^2$, we introduce the following partial differential operators on the first variable:

$$\begin{aligned} &\mathcal{D}_1^{(1)}(u(x,y)) := \frac{\partial u(x,y)}{\partial x} - xu(x,y) \\ \text{and} \quad &\mathcal{D}_1^{(n)}(u(x,y)) := \mathcal{D}_1^{(1)}(\mathcal{D}_1^{(n-1)}(u(x,y))) \quad \text{for } n \geq 2. \end{aligned} \tag{41}$$

The representations of $\Omega_m(y)$ for $m \geq 1$ are given in the following theorem.

THEOREM 1: *For any integer $m \geq 1$, the high-order term $\Omega_m(y)$ in Eq. (33) admits the following expression:*

$$\Omega_m(y) = \sum_{(\ell,(j_1,j_2,\ldots,j_\ell))\in\mathcal{S}_m} \frac{(-1)^\ell}{\ell!} \frac{1}{(\sigma(x_0)\sqrt{\Delta})^\ell} \int_0^{+\infty} \mathcal{D}_1^{(\ell)}(K_{(\ell,\mathbf{j})}(z_1, z_2)) \cdot \phi(z_1) \cdot p_{L(\Delta)}(z_2)\, dz_2, \tag{42}$$

*where the index set $\mathcal{S}_m$ is defined in (9), $p_{L(\Delta)}(\cdot)$ is the density function of the gamma process $L(\Delta)$ given by Eq. (12) and*

$$z_1 = y - \frac{\mu(x_0)\Delta + z_2}{\sigma(x_0)\sqrt{\Delta}}. \tag{43}$$

PROOF: We see from the definition of $\Phi_m(y)$ in Eq. (29) that

$$\begin{aligned} \Omega_m(y) = \mathbb{E}\Phi_m(y) &= \sum_{(\ell,(j_1,j_2,\ldots,j_\ell))\in\mathcal{S}_m} \mathbb{E}\left[\frac{1}{\ell!}\frac{1}{(\sigma(x_0)\sqrt{\Delta})^\ell}\frac{d^\ell\delta(Y_0(\Delta)-y)}{dx^\ell}\prod_{i=1}^{\ell} X_{j_i+1}(\Delta)\right] \\ &= \sum_{(\ell,(j_1,j_2,\ldots,j_\ell))\in\mathcal{S}_m} \frac{1}{\ell!}\frac{1}{(\sigma(x_0)\sqrt{\Delta})^\ell}\mathbb{E}\left[\frac{d^\ell\delta(Y_0(\Delta)-y)}{dx^\ell}\prod_{i=1}^{\ell} X_{j_i+1}(\Delta)\right]. \end{aligned} \tag{44}$$

For the generalized expectation in the last line of Eq. (44), according to Eq. (2), we have

$$\begin{aligned} &\mathbb{E}\left[\frac{d^\ell\delta(Y_0(\Delta)-y)}{dx^\ell}\prod_{i=1}^{\ell} X_{j_i+1}(\Delta)\right] \\ &\quad = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-ivx}\frac{d^\ell\delta(x-y)}{dx^\ell} E\left[\prod_{i=1}^{\ell} X_{j_i+1}(\Delta)\cdot e^{ivY_0(\Delta)}\right] dx\, dv \\ &\quad = \int_{-\infty}^{+\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ivx}\frac{d^\ell\delta(x-y)}{dx^\ell}\, dx\right) E\left[\prod_{i=1}^{\ell} X_{j_i+1}(\Delta)\cdot e^{ivY_0(\Delta)}\right] dv. \end{aligned}$$

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By the relations Eqs. (1) and (3), the above equation is further calculated as

$$\begin{aligned}
&\mathbb{E}\left[\frac{d^\ell \delta(Y_0(\Delta)-y)}{dx^\ell}\prod_{i=1}^{\ell} X_{j_i+1}(\Delta)\right] \\
&\quad = \int_{-\infty}^{+\infty}\left(\frac{1}{2\pi}(-1)^\ell \int_{-\infty}^{+\infty}\delta(x-y)\frac{d^\ell e^{-ivx}}{dx^\ell}dx\right) E\left[\prod_{i=1}^{\ell} X_{j_i+1}(\Delta)\cdot e^{ivY_0(\Delta)}\right] dv \\
&\quad = \int_{-\infty}^{+\infty}\frac{1}{2\pi}(iv)^\ell e^{-ivy}\cdot E\left[\prod_{i=1}^{\ell} X_{j_i+1}(\Delta)\cdot e^{ivY_0(\Delta)}\right] dv. \qquad (45)
\end{aligned}$$

Then following the definition of $Y_0(\Delta)$ in Eq. (37) and using the independence between Brownian motion $W(t)$ and the gamma process $L(t)$, the expectation in the last line of Eq. (45) is calculated as

$$\begin{aligned}
&E\left[\prod_{i=1}^{\ell} X_{j_i+1}(\Delta)\cdot e^{ivY_0(\Delta)}\right] \\
&\quad = E\left[\prod_{i=1}^{\ell} X_{j_i+1}(\Delta)\cdot e^{iv\left(\frac{W(\Delta)}{\sqrt{\Delta}}+\frac{\mu(x_0)\Delta+L(\Delta)}{\sigma(x_0)\sqrt{\Delta}}\right)}\right] \\
&\quad = \int_0^\infty\int_{-\infty}^\infty E\left[\prod_{i=1}^{\ell} X_{j_i+1}(\Delta)\cdot e^{iv\left(z_1+\frac{\mu(x_0)\Delta+z_2}{\sigma(x_0)\sqrt{\Delta}}\right)}\,\middle|\, W(\Delta),L(\Delta)\right]\Bigg|_{W(\Delta)=z_1\sqrt{\Delta},L(\Delta)=z_2} \\
&\quad\times \phi(z_1)p_{L(\Delta)}(z_2)\,dz_1\,dz_2. \qquad (46)
\end{aligned}$$

By the definition of $K_{(\ell,\mathbf{j})}(z_1,z_2)$ in Eq. (40), the expectation calculated in Eq. (46) can be further expressed as

$$E\left[\prod_{i=1}^{\ell} X_{j_i+1}(\Delta)\cdot e^{ivY_0(\Delta)}\right] = \int_0^\infty\int_{-\infty}^\infty e^{iv\left(z_1+\frac{\mu(x_0)\Delta+z_2}{\sigma(x_0)\sqrt{\Delta}}\right)}K_{(\ell,\mathbf{j})}(z_1,z_2)\phi(z_1)p_{L(\Delta)}(z_2)\,dz_1\,dz_2. \qquad (47)$$

Plugging Eq. (47) into (45), we obtain that

$$\begin{aligned}
&\mathbb{E}\left[\frac{d^\ell \delta\left(Y_0(\Delta)-y\right)}{dx^\ell}\prod_{i=1}^{\ell} X_{j_i+1}(\Delta)\right] \\
&\quad = \int_{-\infty}^{+\infty}\frac{1}{2\pi}(iv)^\ell e^{-ivy}\int_0^\infty\int_{-\infty}^\infty e^{iv\left(z_1+\frac{\mu(x_0)\Delta+z_2}{\sigma(x_0)\sqrt{\Delta}}\right)}K_{(\ell,\mathbf{j})}(z_1,z_2)\phi(z_1)p_{L(\Delta)}(z_2)\,dz_1\,dz_2\,dv \\
&\quad = \int_0^{+\infty}\int_{-\infty}^\infty\frac{1}{2\pi}\,e^{-ivy}p_{L(\Delta)}(z_2)\left(\int_{-\infty}^\infty\frac{\partial^\ell\, e^{iv(z_1+\frac{\mu(x_0)\Delta+z_2}{\sigma(x_0)\sqrt{\Delta}})}}{\partial z_1^\ell}K_{(\ell,\mathbf{j})}(z_1,z_2)\phi(z_1)\,dz_1\right)dv\,dz_2. \qquad (48)
\end{aligned}$$

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Using integration by parts, the last line of Eq. (48) can be further calculated as

$$\int_0^{+\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi}e^{-ivy}p_{L(\Delta)}(z_2)\int_{-\infty}^{\infty}(-1)^\ell e^{iv(z_1+\frac{\mu(x_0)\Delta+z_2}{\sigma(x_0)\sqrt{\Delta}})}\frac{\partial^\ell(K_{(\ell,\mathbf{j})}(z_1,z_2)\phi(z_1))}{\partial z_1^\ell}dz_1dvdz_2$$
$$=\int_0^{+\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\frac{1}{2\pi}e^{-ivy}e^{iv\left(z_1+\frac{\mu(x_0)\Delta+z_2}{\sigma(x_0)\sqrt{\Delta}}\right)}dv\right)$$
$$\times(-1)^\ell\frac{\partial^\ell(K_{(\ell,\mathbf{j})}(z_1,z_2)\phi(z_1))}{\partial z_1^\ell}p_{L(\Delta)}(z_2)\,dz_1\,dz_2. \quad (49)$$

By the relation Eq. (5), it follows that

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-ivy}e^{iv\left(z_1+\frac{\mu(x_0)\Delta+z_2}{\sigma(x_0)\sqrt{\Delta}}\right)}dv=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{iv\left(z_1+\frac{\mu(x_0)\Delta+z_2}{\sigma(x_0)\sqrt{\Delta}}-y\right)}dv$$
$$=\delta\left(z_1+\frac{\mu(x_0)\Delta+z_2}{\sigma(x_0)\sqrt{\Delta}}-y\right),$$

Plugging the above equation into Eq. (49), we obtain that

$$\mathbb{E}\left[\frac{d^\ell\delta\left(Y_0(\Delta)-y\right)}{dx^\ell}\prod_{i=1}^{\ell}X_{j_i+1}(\Delta)\right]$$
$$=\int_0^{+\infty}\int_{-\infty}^{\infty}\delta\left(z_1+\frac{\mu(x_0)\Delta+z_2}{\sigma(x_0)\sqrt{\Delta}}-y\right)(-1)^\ell\frac{\partial^\ell(K_{(\ell,\mathbf{j})}(z_1,z_2)\phi(z_1))}{\partial z_1^\ell}p_{L(\Delta)}(z_2)\,dz_1\,dz_2$$
$$=\int_0^{+\infty}(-1)^\ell\left.\frac{\partial^\ell(K_{(\ell,\mathbf{j})}(z_1,z_2)\phi(z_1))}{\partial z_1^\ell}\right|_{z_1=y-\frac{\mu(x_0)\Delta+z_2}{\sigma(x_0)\sqrt{\Delta}}}\cdot p_{L(\Delta)}(z_2)\,dz_2. \quad (50)$$

Thus, plugging Eq. (50) into Eq. (44), we obtain that

$$\Omega_m(y)=\sum_{(\ell,(j_1,j_2,\ldots,j_\ell))\in\mathcal{S}_m}\frac{(-1)^\ell}{\ell!}\frac{1}{(\sigma(x_0)\sqrt{\Delta})^\ell}$$
$$\times\int_0^{+\infty}\left.\frac{\partial^\ell(K_{(\ell,\mathbf{j})}(z_1,z_2)\phi(z_1))}{\partial z_1^\ell}\right|_{z_1=y-\frac{\mu(x_0)\Delta+z_2}{\sigma(x_0)\sqrt{\Delta}}}p_{L(\Delta)}(z_2)\,dz_2. \quad (51)$$

From the definition (41), for any bivariate differentiable function $u(z_1,z_2)$, we have

$$\frac{\partial}{\partial z_1}(u(z_1,z_2)\phi(z_1))=\left(\frac{\partial u(z_1,z_2)}{\partial z_1}-z_1u(z_1,z_2)\right)\phi(z_1)\equiv\mathcal{D}_1^{(1)}(u(z_1,z_2))\cdot\phi(z_1),$$

and

$$\frac{\partial^\ell}{\partial z_1^\ell}(u(z_1,z_2)\phi(z_1))=\mathcal{D}_1^{(\ell)}(u(z_1,z_2))\cdot\phi(z_1)$$

for any $\ell\geq 1$, from which we obtain that

$$\frac{\partial^\ell(K_{(\ell,\mathbf{j})}(z_1,z_2)\phi(z_1))}{\partial z_1^\ell}=\mathcal{D}_1^{(\ell)}(K_{(\ell,\mathbf{j})}(z_1,z_2))\cdot\phi(z_1). \quad (52)$$

Plugging Eq. (52) into (51), we obtain the formula Eq. (42). ■

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According to Eq. (42) in Theorem 1, to calculate the high-order terms $\Omega_m(y)$ for $m \geq 1$, it suffices to derive the bivariate function $\mathcal{D}_1^{(\ell)}(K_{(\ell,\mathbf{j})}(z_1, z_2))$ with $K_{(\ell,\mathbf{j})}(z_1, z_2)$ defined in Eq. (40), which will be shown in Section 3.

## 3. EXPLICIT CALCULATION OF $\mathcal{D}_1^{(\ell)}(K_{(\ell,\mathbf{J})}(Z_1, Z_2))$

In this section, we explicitly derive the function $\mathcal{D}_1^{(\ell)}(K_{(\ell,\mathbf{j})}(z_1, z_2)$ for every fixed $\ell \geq 1$ and $\mathbf{j} = (j_1, j_2, \ldots, j_\ell)$ with $j_i \geq 1$ in Eq. (42) and thus $\Omega_m(y)$ for $m \geq 1$ in Eq. (42), from which we obtain the approximated transition density $p_{X(\Delta)}^{(M)}(x \,|\, x_0; \boldsymbol{\theta})$ in Eq. (33). Moreover, for illustration purpose, we exhibit the first several expansion terms of $\{\Omega_m(y),\ m \geq 0\}$ in Eq. (33) below for three examples of SDE (11), that is, the pure jump OU model, constant diffusion model, and square-root diffusion model.

To present our methodology for calculating the function $\mathcal{D}_1^{(\ell)}(K_{(\ell,\mathbf{j})}(z_1, z_2))$, we introduce the following notations. For fixed integer $h \geq 1$ and arbitrary index $\mathbf{n} = (n_1, n_2, \ldots, n_h)$ with nonnegative integers $n_1, n_2, \ldots, n_h$, we define the $h$-dimensional vector

$$\mathbf{L}^{\mathbf{n}}(t) := (L^{n_1}(t), L^{n_2}(t), \ldots, L^{n_h}(t)) \tag{53}$$

by using the gamma process $L(\cdot)$. For example, when $\mathbf{n} = (1)$, $\mathbf{L}^{\mathbf{n}}(t) = (L(t))$, and when $\mathbf{n} = (0, 1)$, $\mathbf{L}^{\mathbf{n}}(t) = (1, L(t))$.

For any index $\mathbf{i} = (i_1, i_2, \ldots, i_h)$ with $i_1, i_2, \ldots, i_h \in \{0, 1\}$ and $h$-dimensional vector $\mathbf{L}^{\mathbf{n}}(t)$ in (53), we define an iterated Itô integral as

$$\mathbf{I}_{\mathbf{i},\mathbf{L}^{\mathbf{n}}}(\Delta) := \int_0^\Delta \int_0^{s_h} \cdots \int_0^{s_2} L^{n_1}(s_1) \cdots L^{n_{h-1}}(s_{h-1}) L^{n_h}(s_h)\, dW_{i_1}(s_1) \cdots dW_{i_{h-1}}(s_{h-1})\, dW_{i_h}(s_h), \tag{54}$$

where $W_0(t) := t$ and $W_1(t) := W(t)$. For example, we have

$$\mathbf{I}_{(0),\mathbf{L}^{(0)}}(\Delta) = \Delta, \mathbf{I}_{(1),\mathbf{L}^{(0)}}(\Delta) = W(\Delta), \quad \mathbf{I}_{(0),\mathbf{L}^{(1)}}(\Delta) = \int_0^\Delta L(s_1)\, ds_1,$$
$$\mathbf{I}_{(1),\mathbf{L}^{(1)}}(\Delta) = \int_0^\Delta L(s_1)\, dW(s_1),$$
$$\mathbf{I}_{(0,0),\mathbf{L}^{(0,0)}}(\Delta) = \int_0^\Delta \int_0^{s_2} ds_1\, ds_2, \mathbf{I}_{(0,1),\mathbf{L}^{(0,0)}}(\Delta) = \int_0^\Delta \int_0^{s_2} ds_1\, dW(s_2),$$

and

$$\mathbf{I}_{(1,0),\mathbf{L}^{(0,1)}}(\Delta) = \int_0^\Delta \int_0^{s_2} L(s_2)\, dW(s_1)\, ds_2, \mathbf{I}_{(1,1),\mathbf{L}^{(1,0)}}(\Delta) = \int_0^\Delta \int_0^{s_2} L(s_1)\, dW(s_1)\, dW(s_2).$$

We notice that the iterated Itô integral $\mathbf{I}_{\mathbf{i},\mathbf{L}^{\mathbf{n}}}(\Delta)$ defined by Eq. (54) involves two independent processes, that is, the Brownian motion $W(\cdot)$ and the gamma process $L(\cdot)$. Such independence will simplify the calculation related to $\mathbf{I}_{\mathbf{i},\mathbf{L}^{\mathbf{n}}}(\Delta)$ as seen below.

In order to clarify the procedures of calculating $\mathcal{D}_1^{(\ell)}(K_{(\ell,\mathbf{j})}(z_1, z_2))$ in Eq. (42), we give a general methodology below with the details elaborated subsequently.

*Methodology 1*: The procedures of calculating $\mathcal{D}_1^{(\ell)}\left(K_{(\ell,\mathbf{j})}(z_1,z_2)\right)$ in Eq. (42).

Step 1 Convert the multiplication of the expansion terms in $K_{(\ell,\mathbf{j})}(z_1,z_2)$, that is, $\prod_{i=1}^{\ell} X_{j_i+1}(\Delta)$, into a linear combination of iterated Itô integrals $\mathbf{I}_{\mathbf{i},\mathbf{L}^{\mathbf{n}}}(\Delta)$ as defined in Eq. (54) with the coefficients depending on $\mu(x_0)$, $\sigma(x_0)$ and their higher-order derivatives evaluated at $x_0$, via the relation Eq. (21) and the Itô product formula.

Step 2 Calculate the conditional expectation

$$E(\mathbf{I}_{\mathbf{i},\mathbf{L}^{\mathbf{n}}}(\Delta)\,|\,W(\Delta),L(\Delta))|_{W(\Delta)=z_1\sqrt{\Delta},L(\Delta)=z_2} \tag{55}$$

as a linear combination of the terms uniformly represented as

$$z_1^m\cdot\int_0^{\Delta}\int_0^{s_h}\cdots\int_0^{s_2}E[L^{n_1}(s_1)\cdots L^{n_{h-1}}(s_{h-1})L^{n_h}(s_h)|L(\Delta)]\,ds_1\cdots ds_{h-1}\,ds_h \tag{56}$$

for some $h\geq 1$, $0\leq m\leq h$, $0<s_1<s_2<\cdots<s_h<\Delta$, and nonnegative integers $n_1,n_2,\ldots,n_h$ via Brownian bridge.

Step 3 Compute the conditional expectation in Eq. (56) as a power function of $L(\Delta)$.

Step 4 Express the function $\mathcal{D}_1^{(\ell)}(K_{(\ell,\mathbf{j})}(z_1,z_2))$ in Eq. (42) as

$$\mathcal{D}_1^{(\ell)}(K_{(\ell,\mathbf{j})}(z_1,z_2))=\sum_{n_1,n_2\geq 0}P_{(\ell,\mathbf{j})}^{\mu,\sigma}(n_1,n_2)z_1^{n_1}z_2^{n_2}, \tag{57}$$

where the coefficient functions $P_{(\ell,\mathbf{j})}^{\mu,\sigma}(n_1,n_2), n_1,n_2\geq 0$ depend on the functions $\mu(x_0)$, $\sigma(x_0)$ and their higher-order derivatives evaluated at $x_0$.

In the following sections 3.1–3.4, we give the detailed descriptions of Steps 1, 2, 3, and 4 in the above methodology, respectively. The calculating procedures can be implemented by traditional symbolic softwares, and we employ Wolfram Mathematica in this paper. In Section 3.5, we consider three examples of SDE (11) for illustrations.

### 3.1. Conversion of the Multiplication $\prod_{i=1}^{\ell} X_{j_i+1}(\Delta)$ as a Linear Combination of Iterated Itô Integrals

First, we show that the multiplication of iterated Itô integrals defined in Eq. (54) can be converted into a linear combination of the iterated Itô integrals taking the same form as in Eq. (54).

Given an index $\mathbf{i}=(i_1,i_2,\ldots,i_h)$, we denote by $\mathbf{i}-$ the index obtained from deleting the last element of index $\mathbf{i}$, that is,

$$\mathbf{i}-:=(i_1,i_2,\ldots,i_{h-1}).$$

Similarly, we denote by

$$\mathbf{L}^{\mathbf{n}-}(t):=(L^{n_1}(t),L^{n_2}(t),\ldots,L^{n_{h-1}}(t))$$

the $(h-1)$-dimensional vector obtained from deleting the last element of $\mathbf{L}^{\mathbf{n}}(t)$ in (53). Consequently, the iterated Itô Integral $\mathbf{I}_{\mathbf{i}-,\mathbf{L}^{\mathbf{n}-}}(\Delta)$ can be defined as

$$\begin{aligned}\mathbf{I}_{\mathbf{i}-,\mathbf{L}^{\mathbf{n}-}}(\Delta):=&\int_0^{\Delta}\int_0^{s_{h-1}}\cdots\int_0^{s_2}L^{n_1}(s_1)\cdots L^{n_{h-2}}(s_{h-2})\\&\times L^{n_{h-1}}(s_{h-1})\,dW_{i_1}(s_1)\cdots dW_{i_{h-2}}(s_{h-2})\,dW_{i_{h-1}}(s_{h-1}).\end{aligned}$$

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For two fixed positive integers $h, q$ and the gamma process $L(\cdot)$, we consider the $h$-dimensional vector $\mathbf{L}^{\mathbf{n}}(t)$ and $q$-dimensional vector $\mathbf{L}^{\mathbf{m}}(t)$ written as

$$\mathbf{L}^{\mathbf{n}}(t) = (L^{n_1}(t), L^{n_2}(t), \ldots, L^{n_h}(t)) \quad \text{and} \quad \mathbf{L}^{\mathbf{m}}(t) = (L^{m_1}(t), L^{m_2}(t), \ldots, L^{m_q}(t))$$

for some indices $\mathbf{n} = (n_1, n_2, \ldots, n_h)$ and $\mathbf{m} = (m_1, m_2, \ldots, m_q)$ with nonnegative integers $n_1, n_2, \ldots, n_h, m_1, m_2, \ldots, m_q$. Then for two indices $\mathbf{i} = (i_1, i_2, \ldots, i_h)$ and $\mathbf{k} = (k_1, k_2, \ldots, k_q)$ with $i_1, i_{2,} \ldots, i_h,\ k_1,\ k_2, \ldots, k_q \in \{0, 1\}$, the product of two iterated Itô integrals $\mathbf{I}_{\mathbf{i},\mathbf{L}^{\mathbf{n}}}(\Delta)$ and $\mathbf{I}_{\mathbf{k},\mathbf{L}^{\mathbf{m}}}(\Delta)$ satisfies the following iterative relation

$$\begin{aligned}
&\mathbf{I}_{\mathbf{i},\mathbf{L}^{\mathbf{n}}}(\Delta)\mathbf{I}_{\mathbf{k},\mathbf{L}^{\mathbf{m}}}(\Delta) \\
&\quad = \left[\int_0^{\Delta} \mathbf{I}_{\mathbf{i}-,\mathbf{L}^{\mathbf{n}-}}(s_1) \cdot L^{n_h}(s_1)\, dW_{i_h}(s_1)\right] \cdot \left[\int_0^{\Delta} \mathbf{I}_{\mathbf{k}-,\mathbf{L}^{\mathbf{m}-}}(s_1) \cdot L^{m_q}(s_1)\, dW_{k_q}(s_1)\right] \\
&\quad = \int_0^{\Delta} \mathbf{I}_{\mathbf{i},\mathbf{L}^{\mathbf{n}}}(s_1)\mathbf{I}_{\mathbf{k}-,\mathbf{L}^{\mathbf{m}-}}(s_1) \cdot L^{m_q}(s_1)\, dW_{k_q}(s_1) \\
&\qquad + \int_0^{\Delta} \mathbf{I}_{\mathbf{i}-,\mathbf{L}^{\mathbf{n}-}}(s_1)\mathbf{I}_{\mathbf{k},\mathbf{L}^{\mathbf{m}}}(s_1) \cdot L^{n_h}(s_1)\, dW_{i_h}(s_1) \\
&\qquad + \int_0^{\Delta} \mathbf{I}_{\mathbf{i}-,\mathbf{L}^{\mathbf{n}-}}(s_1)\mathbf{I}_{\mathbf{k}-,\mathbf{L}^{\mathbf{m}-}}(s_1) \cdot L^{n_h+m_q}(s_1) \cdot 1_{\{i_h=k_q=1\}}\, ds_1, \qquad (58)
\end{aligned}$$

where the second equation follows from the Itô product formula

$$\begin{aligned}
&\int_0^{\Delta} f(s_1)\, dW_{i_1}(s_1) \cdot \int_0^{\Delta} g(s_1)\, dW_{j_1}(s_1) \\
&\quad = \int_0^{\Delta}\int_0^{s_1} f(s_2)\, dW_{i_1}(s_2) g(s_1)\, dW_{j_1}(s_1) + \int_0^{\Delta}\int_0^{s_1} g(s_2)\, dW_{j_1}(s_2) f(s_1)\, dW_{i_1}(s_1) \\
&\qquad + \int_0^{\Delta} f(s_1)g(s_1) 1_{\{i_1=j_1=1\}}\, ds_1
\end{aligned}$$

and $1_{\{i_h=k_q=1\}}$ is the indicator function defined as

$$1_{\{i_h=k_q=1\}} = \begin{cases} 1, & \text{if } i_h = k_q = 1, \\ 0, & \text{otherwise.} \end{cases}$$

By iterative applications of the relation Eq. (58), the product of $I_{\mathbf{i},\mathbf{L}^{\mathbf{n}}}(\Delta)$ and $I_{\mathbf{k},\mathbf{L}^{\mathbf{m}}}(\Delta)$ can be expressed as a linear combination of the iterated Itô integrals defined by Eq. (54).

Next, we show that the expansion terms $X_{j_1+1}(\Delta), X_{j_2+1}(\Delta), \ldots, X_{j_\ell+1}(\Delta)$ in Eq. (40) can be expressed as a linear combination of iterated Itô integrals $I_{\mathbf{i},\mathbf{L}^{\mathbf{n}}}(\Delta)$ defined in Eq. (54), with coefficients depending on $\mu(x_0)$, $\sigma(x_0)$, and their higher-order derivatives evaluated at $x_0$. Based on this, it follows from Eq. (58) that the multiplication $\prod_{i=1}^{\ell} X_{j_i+1}(\Delta)$ can be converted into a linear combination of $I_{\mathbf{i},\mathbf{L}^{\mathbf{n}}}(\Delta)$ defined in Eq. (54). To do this, in what follows, we illustrate that $X_m(\Delta)$ admits the aforementioned linear combination form for

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$m \geq 1$. By using Eq. (54), $X_1(\Delta)$ in Eq. (20) can be written as

$$X_1(\Delta) = \mu(x_0)\mathbf{I}_{(0),\mathbf{L}^{(0)}}(\Delta) + \sigma(x_0)\mathbf{I}_{(1),\mathbf{L}^{(0)}}(\Delta) + L(\Delta), \tag{59}$$

which admits the linear combination form. For $m \geq 1$, we notice from Eq. (21) that

$$X_{m+1}(\Delta) = \int_0^\Delta \mu_m(s)\,ds + \int_0^\Delta \sigma_m(s)\,dW(s), \tag{60}$$

with $\mu_m(s)$ and $\sigma_m(s)$ defined by Eqs. (22)–(23), from which we observe that both $\mu_m(s)$ and $\sigma_m(s)$ are linear combinations of the products of the terms from $\{X_1(s), X_2(s), \ldots, X_m(s)\}$. Then, by iterative applications of Eqs. (58)–(60), we can also derive $X_{m+1}(\Delta)$ for $m \geq 1$ as a linear combination of iterated Itô integrals $\mathbf{I}_{\mathbf{i},\mathbf{L}^\mathbf{n}}(\Delta)$ for $h \leq m+1$ formed as Eq. (54), where the coefficients depend on $\mu(x_0)$, $\sigma(x_0)$, and their higher-order derivatives evaluated at $x_0$.

Therefore, to calculate $K_{(\ell,\mathbf{j})}(z_1, z_2)$ in Eq. (40) for every fixed $\ell \geq 1$ and $\mathbf{j} = (j_1, j_2, \ldots, j_\ell)$ with $j_i \geq 1$, it suffices to focus on the type of conditional expectation (55).

### 3.2. Simplification of the Conditional Expectation (55) via Brownian Bridge

Starting from this part, we focus on calculating the following conditional expectation

$$\begin{aligned} &E(\mathbf{I}_{\mathbf{i},\mathbf{L}^\mathbf{n}}(\Delta)\,|\,W(\Delta), L(\Delta))|_{W(\Delta)=z_1\sqrt{\Delta}, L(\Delta)=z_2} \\ &\quad = E\Bigg(\int_0^\Delta \int_0^{s_h} \cdots \int_0^{s_2} L^{n_1}(s_1)\cdots L^{n_{h-1}}(s_{h-1})L^{n_h}(s_h) \\ &\qquad \times dW_{i_1}(s_1)\cdots dW_{i_{h-1}}(s_{h-1})dW_{i_h}(s_h)|W(\Delta), L(\Delta)\Bigg)\Bigg|_{W(\Delta)=z_1\sqrt{\Delta}, L(\Delta)=z_2}, \end{aligned} \tag{61}$$

where the iterated Itô integral $\mathbf{I}_{\mathbf{i},\mathbf{L}^\mathbf{n}}(\Delta)$ is defined by Eq. (54) with $i_1, i_2, \ldots, i_h \in \{0, 1\}$, $W_0(t) = t$, and $W_1(t) = W(t)$.

To simplify the conditional expectation in Eq. (61), we utilize the following representation of Brownian bridge, that is,

$$(W(s)\,|\,W(\Delta) = z_1\sqrt{\Delta}) \overset{d}{=} B^{z_1}(s) := B(s) - \frac{s}{\Delta}B(\Delta) + \frac{s}{\sqrt{\Delta}}z_1 \tag{62}$$

for $0 \leq s \leq \Delta$, where the symbol "$\overset{d}{=}$" means distributional identity and $B(\cdot)$ is a one-dimensional standard Brownian motion. Then, by the independence between $W(\cdot)$ and $L(\cdot)$, the conditional expectation in Eq. (61) can be equivalently expressed as

$$\begin{aligned} &E(\mathbf{I}_{\mathbf{i},\mathbf{L}^\mathbf{n}}(\Delta)\,|\,W(\Delta), L(\Delta))|_{W(\Delta)=z_1\sqrt{\Delta}, L(\Delta)=z_2} \\ &\quad = E\Bigg(\int_0^\Delta \int_0^{s_h} \cdots \int_0^{s_2} L^{n_1}(s_1)\cdots L^{n_{h-1}}(s_{h-1}) \\ &\qquad \times L^{n_h}(s_h)\,dB^{z_1}_{i_1}(s_1)\cdots dB^{z_1}_{i_{h-1}}(s_{h-1})\,dB^{z_1}_{i_h}(s_h)\,|\,L(\Delta)\Bigg)\Bigg|_{L(\Delta)=z_2}, \end{aligned} \tag{63}$$

where $B_1^{z_1}(s) := B^{z_1}(s)$ and $B_0^{z_1}(s) := s$. Therefore, we only need focus on the conditional expectation

$$E\left(\int_0^{\Delta}\cdots\int_0^{s_2} L^{n_1}(s_1)\cdots L^{n_h}(s_h)\,dB_{i_1}^{z_1}(s_1)\cdots dB_{i_h}^{z_1}(s_h)\,\middle|\, L(\Delta)\right), \tag{64}$$

from which the conditional expectation in Eq. (63) can be obtained by letting $L(\Delta) = z_2$. For the sake of simplicity, we denote by $E_L(\cdot) := E(\cdot|L(\Delta))$ the conditional expectation given $L(\Delta)$ hereafter. By plugging (62) into (64), we obtain that

$$\begin{aligned}
&E\left(\int_0^{\Delta}\int_0^{s_h}\cdots\int_0^{s_2} L^{n_1}(s_1)\cdots L^{n_{h-1}}(s_{h-1})L^{n_h}(s_h)\,dB_{i_1}^{z_1}(s_1)\cdots dB_{i_{h-1}}^{z_1}(s_{h-1})\,dB_{i_h}^{z_1}(s_h)\,\middle|\, L(\Delta)\right)\\
&= E_L\left(\int_0^{\Delta}\int_0^{s_h}\cdots\int_0^{s_2} L^{n_1}(s_1)\,dB_{i_1}^{z}(s_1)\cdots L^{n_{h-1}}(s_{h-1})\,dB_{i_{h-1}}^{z_1}(s_{h-1})L^{n_h}(s_h)\,dB_{i_h}^{z_1}(s_h)\right)\\
&= E_L\left(\int_0^{\Delta}\int_0^{s_h}\cdots\int_0^{s_2} L^{n_1}(s_1)\left\{1_{\{i_1=1\}}\left(dB(s_1)-\frac{B(\Delta)}{\Delta}\,ds_1+\frac{z_1}{\sqrt{\Delta}}\,ds_1\right)+1_{\{i_1=0\}}\,ds_1\right\}\right.\\
&\quad\times\cdots\times L^{n_{h-1}}(s_{h-1})\left\{1_{\{i_{h-1}=1\}}\left(dB(s_{h-1})-\frac{B(\Delta)}{\Delta}ds_{h-1}+\frac{z_1}{\sqrt{\Delta}}\,ds_{h-1}\right)+1_{\{i_{h-1}=0\}}\,ds_{h-1}\right\}\\
&\quad\left.\times L^{n_h}(s_h)\left\{1_{\{i_h=1\}}\left(dB(s_h)-\frac{B(\Delta)}{\Delta}\,ds_h+\frac{z_1}{\sqrt{\Delta}}\,ds_h\right)+1_{\{i_h=0\}}\,ds_h\right\}\right).
\end{aligned} \tag{65}$$

In order to derive the explicit expression of Eq. (65), for any index $\mathbf{i} = (i_1, i_2, \ldots, i_h)$ with $i_1, i_2, \ldots, i_h \in \{0, 1\}$ and $h$-dimensional vector $\mathbf{L^n}(t)$ in (53), we define an iterated Itô integral

$$\begin{aligned}
\mathbf{J}_{\mathbf{i},\mathbf{L^n}}(\Delta) := \int_0^{\Delta}\int_0^{s_h}\cdots\int_0^{s_2} & L^{n_1}(s_1)\cdots L^{n_{h-1}}(s_{h-1})L^{n_h}(s_h)\,dB_{i_1}(s_1)\cdots\\
& B_{i_{h-1}}(s_{h-1})\,dB_{i_h}(s_h),
\end{aligned} \tag{66}$$

where $B_0(t) := t$ and $B_1(t) := B(t)$, with the Brownian motion $B(t)$ introduced in (62). Then, we fully expand the product of the differential forms in the last equation of Eq. (65) and find that in order to calculate Eq. (65), it suffices to calculate the following two kinds of conditional expectations

$$\left(\frac{z_1}{\sqrt{\Delta}}\right)^{k_1} E_L(\mathbf{J}_{\mathbf{i},\mathbf{L^n}}(\Delta)) \tag{67}$$

and

$$\left(\frac{z_1}{\sqrt{\Delta}}\right)^{k_2} E_L(B(\Delta)^{k_3}\cdot\mathbf{J}_{\mathbf{i},\mathbf{L^n}}(\Delta)), \tag{68}$$

where the integers $k_1, k_2, k_3 \in \{0, 1, \ldots, h\}$ satisfy the condition $k_2 + k_3 \le h$ and $\mathbf{J}_{\mathbf{i},\mathbf{L^n}}(\Delta)$ is defined in Eq. (66). To precede, we notice the following relation

$$\begin{aligned}
B(\Delta)\mathbf{J}_{\mathbf{i},\mathbf{L^n}}(\Delta) = &\sum_{m=1}^{h+1}\mathbf{J}_{(i_1,\ldots,i_{m-1},1,i_m,\ldots,i_h),(L^{n_1}(\Delta),\ldots,L^{n_{m-1}}(\Delta),1,L^{n_m}(\Delta),\ldots,L^{n_h}(\Delta))}(\Delta)\\
&+\sum_{m=1}^{h}1_{\{i_m=1\}}\mathbf{J}_{(i_1,\ldots,i_{m-1},0,i_{m+1},\ldots,i_h),\mathbf{L^n}}(\Delta),
\end{aligned} \tag{69}$$

which can be verified similarly as in Proposition 5.2.3 of Kloeden and Platen [19]. By iterative applications of Eq. (69), the conditional expectation (68) can be converted into a linear combination of the conditional expectations uniformly represented as in (67). Then from the martingale property of stochastic integrals and the independence between the gamma process and Brownian motion, the conditional expectation (67) equals to zero if there exists some integer $m \in \{1, 2, \ldots, h\}$ such that $i_m = 1$ in $\mathbf{i}$. Therefore, the conditional expectation (64) can be finally derived as a linear combination of the terms uniformly represented as (56). Based on this, to calculate (64), it suffices to derive the conditional expectation in (56).

### 3.3. Calculating the Conditional Expectation in (56)

In this part, we focus on the following conditional expectation

$$E[L^{n_1}(s_1)L^{n_2}(s_2)\cdots L^{n_h}(s_h)|L(\Delta)] \tag{70}$$

appeared in (56), for some $0 < s_1 < s_2 < \cdots < s_h < \Delta$ and nonnegative integers $n_1, n_2, \ldots, n_h$. The expectation (70) involves the product of values of the gamma process $L(\cdot)$ evaluated at different intermediate times conditional on the value of $L(\cdot)$ at the terminal time $\Delta$ and can be represented as a power function of $L(\Delta)$ by the following theorem.

THEOREM 2: *For $h \geq 1$, $0 < s_1 < s_2 < \cdots < s_h < \Delta$ and nonnegative integers $n_1, n_2, \ldots, n_h$, we have*

$$\begin{aligned} &E[L^{n_1}(s_1)L^{n_2}(s_2)\cdots L^{n_h}(s_h)\,|\,L(\Delta)] \\ &\quad = \frac{\prod_{r=0}^{m_1-1}(as_1+r)\prod_{r=m_1}^{m_2-1}(as_2+r)\cdots\prod_{r=m_{h-1}}^{m_h-1}(as_h+r)}{\prod_{r=0}^{m_h-1}(a\Delta+r)}L^{m_h}(\Delta), \end{aligned}$$

*where $m_k = n_1 + n_2 + \cdots + n_k$ for $k = 1, 2, \ldots, h$, and the parameter $a$ is defined through the density function of $L(\cdot)$ in Eq. (12).*

PROOF: We first notice a fact that for the gamma process $L(\cdot)$ with density function given by Eq. (12) and $t_0 < t_1 < t_2$, conditional on $L(t_0) = v_0$ and $L(t_2) = v_2$, we have (cf. [27])

$$L(t_1) \overset{d}{=} v_0 + p(v_2 - v_0), \tag{71}$$

where $p$ is a random variable following Beta distribution as $p \sim \mathcal{B}(a(t_1 - t_0), a(t_2 - t_1))$.

Now we return to the proof of this proposition. For $0 < s_1 < s_2 < \cdots < s_h < \Delta$, by the property of iterated expectation and $L(0) = 0$, we can get

$$\begin{aligned} &E[L^{n_1}(s_1)L^{n_2}(s_2)\cdots L^{n_h}(s_h)\,|\,L(\Delta)] \\ &\quad = E\{E[L^{n_1}(s_1)L^{n_2}(s_2)\cdots L^{n_h}(s_h)\,|\,L(s_2), \ldots, L(s_h), L(\Delta)]|L(\Delta)\} \\ &\quad = E\{L^{n_2}(s_2)\cdots L^{n_h}(s_h)E[L^{n_1}(s_1)\,|\,L(s_2), \ldots, L(s_h), L(\Delta)]|L(\Delta)\} \\ &\quad = E\{L^{n_2}(s_2)\cdots L^{n_h}(s_h)E[L^{n_1}(s_1)\,|\,L(s_2)]|L(\Delta)\}, \end{aligned} \tag{72}$$

where the last equality follows from the harness property of general Lévy process (see, for example, [15 Sect. 11.2.7]). To calculate $E[L^{n_1}(s_1)|L(s_2)]$ in the last line of Eq. (72), we see from (71) that given $L(s_2)$,

$$L(s_1) \stackrel{d}{=} p_1 L(s_2), \quad \text{where } p_1 \sim \mathcal{B}(as_1, a(s_2 - s_1)),$$

from which $E[L^{n_1}(s_1)|L(s_2)] = L^{n_1}(s_2)E[p_1^{n_1}]$ and Eq. (72) can be further calculated as

$$E[L^{n_1}(s_1)L^{n_2}(s_2)\cdots L^{n_h}(s_h)\,|\,L(\Delta)] = E[p_1^{n_1}]E[L^{n_1+n_2}(s_2)L^{n_3}(s_3)\cdots L^{n_h}(s_h)\,|\,L(\Delta)]. \tag{73}$$

Similarly, in the right-hand side of Eq. (73), we notice that

$$\begin{aligned} &E[L^{n_1+n_2}(s_2)L^{n_3}(s_3)\cdots L^{n_h}(s_h)\,|\,L(\Delta)] \\ &\quad = E\{L^{n_3}(s_3)\cdots L^{n_h}(s_h)E[L^{n_1+n_2}(s_2)\,|\,L(s_3)]\,|\,L(\Delta)\} \\ &\quad = E[p_2^{n_1+n_2}]E[L^{n_1+n_2+n_3}(s_3)L^{n_4}(s_4)\cdots L^{n_h}(s_h)\,|\,L(\Delta)], \end{aligned}$$

where $p_2 \sim \mathcal{B}(as_2, a(s_3 - s_2))$, so that

$$\begin{aligned} &E[L^{n_1}(s_1)L^{n_2}(s_2)\cdots L^{n_h}(s_h)\,|\,L(\Delta)] \\ &\quad = E[p_1^{n_1}]E[p_2^{n_1+n_2}]E[L^{n_1+n_2+n_3}(s_3)L^{n_4}(s_4)\cdots L^{n_h}(s_h)\,|\,L(\Delta)]. \end{aligned}$$

Continuing the above procedure in a similar manner, for $h \geq 2$, we deduce that

$$\begin{aligned} &E[L^{n_1}(s_1)L^{n_2}(s_2)\cdots L^{n_h}(s_h)|L(\Delta)] \\ &\quad = E[p_1^{n_1}]E[p_2^{n_1+n_2}]\cdots E[p_h^{n_1+n_2+\cdots+n_h}]L^{n_1+n_2+\cdots+n_h}(\Delta) \\ &\quad \triangleq E[p_1^{m_1}]E[p_2^{m_2}]\cdots E[p_h^{m_h}]L^{m_h}(\Delta), \end{aligned} \tag{74}$$

where

$$p_k \sim \mathcal{B}(as_k, a(s_{k+1} - s_k)), \quad \text{for } 1 \leq k \leq h-1 \tag{75}$$

and

$$p_h \sim \mathcal{B}(as_h, a(\Delta - s_h)), \tag{76}$$

with $m_k = n_1 + n_2 + \cdots + n_k$ for $1 \leq k \leq h$.

To evaluate the expectation $E[p_k^{m_k}]$ for $k = 1, 2, \ldots, h$ in Eq. (74), we notice that for a random variable $X \sim \mathcal{B}(\alpha, \beta)$,

$$E[X^k] = \frac{\alpha^{(k)}}{(\alpha+\beta)^{(k)}} := \prod\nolimits_{r=0}^{k-1} \frac{\alpha + r}{\alpha + \beta + r}$$

holds for any positive integer $k$. Then, it follows from (75) and (76) that

$$E[p_k^{m_k}] = \prod_{r=0}^{m_k-1} \frac{as_k + r}{as_{k+1} + r} = \frac{\prod_{r=0}^{m_k-1}(as_k + r)}{\prod_{r=0}^{m_k-1}(as_{k+1} + r)}, \quad \text{for } 1 \leq k \leq h-1,$$

and

$$E[p_h^{m_h}] = \prod_{r=0}^{m_h-1} \frac{as_h + r}{a\Delta + r} = \frac{\prod_{r=0}^{m_h-1}(as_h + r)}{\prod_{r=0}^{m_h-1}(a\Delta + r)}.$$

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Plugging the above two equations into Eq. (74), we obtain that

$$
\begin{aligned}
&E[L^{n_1}(s_1)L^{n_2}(s_2)\cdots L^{n_h}(s_h)\,|\,L(\Delta)]\\
&=\frac{\prod_{r=0}^{m_1-1}(as_1+r)}{\prod_{r=0}^{m_1-1}(as_2+r)}\frac{\prod_{r=0}^{m_2-1}(as_2+r)}{\prod_{r=0}^{m_2-1}(as_3+r)}\cdots\frac{\prod_{r=0}^{m_{h-1}-1}(as_{h-1}+r)}{\prod_{r=0}^{m_{h-1}-1}(as_h+r)}\frac{\prod_{r=0}^{m_h-1}(as_h+r)}{\prod_{r=0}^{m_h-1}(a\Delta+r)}L^{m_h}(\Delta)\\
&=\frac{\prod_{r=0}^{m_1-1}(as_1+r)\prod_{r=m_1}^{m_2-1}(as_2+r)\cdots\prod_{r=m_{h-1}}^{m_h-1}(as_h+r)}{\prod_{r=0}^{m_h-1}(a\Delta+r)}L^{m_h}(\Delta),
\end{aligned}
$$

which concludes the proof. ■

### 3.4. The Explicit Expressions of $\mathcal{D}_1^{(\ell)}(K_{(\ell,\mathbf{j})}(z_1,z_2))$ and $\Omega_m(y)$

Based on the previous calculations in Sections 3.1–3.3, the function $K_{(\ell,\mathbf{j})}(z_1,z_2)$ defined by Eq. (40) can be established as a linear combination of $z_1^{n_1}z_2^{n_2}$ for $n_1,n_2\ge 0$, denoted by

$$
K_{(\ell,\mathbf{j})}(z_1,z_2)=\sum_{n_1,n_2\ge 0}H_{(\ell,\mathbf{j})}^{\mu,\sigma}(n_1,n_2)z_1^{n_1}z_2^{n_2},
$$

where the coefficient functions $H_{(\ell,\mathbf{j})}^{\mu,\sigma}(n_1,n_2), n_1,n_2\ge 0$ depend on the functions $\mu(x_0)$, $\sigma(x_0)$, and their higher-order derivatives evaluated at $x_0$. According to the definition of the partial differential operators $\mathcal{D}_1^{(\ell)}(\cdot)$ for $\ell\ge 1$ in (41), the expression of $\mathcal{D}_1^{(\ell)}(K_{(\ell,\mathbf{j})}(z_1,z_2))$ can take the form as in Eq. (57). Then, it follows from Eqs. (42) and (43) that $\Omega_m(y)$ can be finally represented as

$$
\begin{aligned}
\Omega_m(y)=&\sum_{(\ell,(j_1,j_2,\ldots,j_\ell))\in\mathcal{S}_m}\frac{(-1)^\ell}{\ell!}\frac{1}{(\sigma(x_0)\sqrt{\Delta})^\ell}\sum_{n_1,n_2\ge 0}P_{(\ell,\mathbf{j})}^{\mu,\sigma}(n_1,n_2)\\
&\times\int_0^{+\infty}z_1^{n_1}z_2^{n_2}\phi(z_1)p_{L(\Delta)}(z_2)dz_2,
\end{aligned}
$$

where $z_1=y-(\mu(x_0)\Delta+z_2)/\sigma(x_0)\sqrt{\Delta}$ and $y=(x-x_0)/\sigma(x_0)\sqrt{\Delta}$, the index set $\mathcal{S}_m$ is defined in (9), $\mu(\cdot)$ and $\sigma(\cdot)$ are defined through the SDE (11), $\phi(\cdot)$ is the standard normal density function and $p_{L(\Delta)}(\cdot)$ is the density function of $L(\Delta)$ in Eq. (12).

Remark 4: *For the special case of the SDE (11) with $\sigma(X(t);\boldsymbol{\theta})\equiv 0$ in Remark 3, to calculate $\Omega_m(y)$ for $m=0,1,2,\ldots$ in Eq. (34), we skip the procedures in Section 3.2. The remaining procedures are performed in a similar manner.*

### 3.5. Examples

In this section, we consider three examples of SDE (11), that is, the pure jump OU model, the constant diffusion model, and the square-root diffusion model introduced below, to give the concrete expressions of the first several expansion terms of $\{\Omega_m(y), m\ge 0\}$ in Eq. (33).

The first model below is the pure jump OU process which is a special case of the non-Gaussian OU processes proposed by Barndorff-Nielsen and Shephard [3]. The pure jump OU process is widely used in finance and statistical analysis, for example, to specify the stochastic volatility driving the dynamics of asset prices. We refer Barndorff-Nielsen and Shephard [3], Cont and Tankov [5] and Schoutens [28] for more details of the non-Gaussian OU processes.

Model 1 (Pure-jump OU model). By taking $\boldsymbol{\theta} = \{\kappa, \theta\}$ and letting $\mu(x;\boldsymbol{\theta}) = \kappa(\theta - x)$ and $\sigma(x;\boldsymbol{\theta}) \equiv 0$ in (11), we obtain the pure jump model

$$dX(t) = \kappa(\theta - X(t))dt + dL(t), \quad X(0) = x_0. \tag{77}$$

The function $\Omega_m(y)$ for $m = 0, 1, 2, 3$ in Eq. (34) can be calculated as

$$\begin{aligned}
\Omega_0(y) &= \frac{b^{a\Delta}(y-\eta\Delta)^{a\Delta-1}\, e^{-b(y-\eta\Delta)}}{\Gamma(a\Delta)},\\
\Omega_1(y) &= -\frac{b^{a\Delta}(y-\eta\Delta)^{a\Delta-2}\, e^{-b(y-\eta\Delta)}}{2\Gamma(a\Delta)}\kappa\Delta[(by-a\Delta)y+\eta\Delta(1-by)],\\
\Omega_2(y) &= \frac{b^{a\Delta}(y-\eta\Delta)^{a\Delta-3}\, e^{-b(y-\eta\Delta)}}{24(1+a\Delta)\Gamma(a\Delta)}\kappa^2\Delta^2[b^2(y-\eta\Delta)^2(y^2(4+3a\Delta)-2y\eta\Delta+\eta^2\Delta^2)\\
&\quad - 2b(1+a\Delta)(y-\eta\Delta)(y^2(2+3a\Delta)-6\eta\Delta y+\kappa^2\eta^2)\\
&\quad + (1+a\Delta)(3a^2y^2\Delta^2+2(2-5a\Delta)\eta\Delta y+(2+a\Delta)\eta^2\Delta^2)],
\end{aligned}$$

and

$$\begin{aligned}
\Omega_3(y) = &-\frac{b^{a\Delta}(y-\eta\Delta)^{a\Delta-4}\, e^{-b(y-\eta\Delta)}}{48(1+a\Delta)\Gamma(a\Delta)}\kappa^3\Delta^3 \times \{[b^3(2+a\Delta)y^3 - b^2(6+a\Delta(8+3a\Delta))y^2\\
&+ b(1+a\Delta)(2+a\Delta(4+3a\Delta))y - a^3\Delta^3(1+a\Delta)]y^3\\
&- \eta\Delta[b^3(8+3a\Delta)y^3 - b^2(26+3a\Delta(9+2a\Delta))y^2 + b(1+a\Delta)(6+a\Delta(20+3a\Delta))y\\
&- (1+a\Delta)(2+a\Delta(-6+7a\Delta))]y^2 + \eta^2\Delta^2[b^3(13+3a\Delta)y^3\\
&- b^2(40+3a\Delta(11+a\Delta))y^2\\
&+ b(1+a\Delta)(14+19a\Delta)y - (1+a\Delta)(-4+a\Delta(6+a\Delta))]y\\
&- \eta^3\Delta^3[b^3(11+a\Delta)y^3 - b^2(27+17a\Delta)y^2 + 3b(1+a\Delta)(4+a\Delta)y - a\Delta(1+a\Delta)]\\
&+ b\eta^4\Delta^4[by(-8-3a\Delta+5by)+2(1+a\Delta)] + b^2\eta^5\Delta^5(1-by)\},
\end{aligned}$$

where $\eta := \kappa(\theta - x_0)$.

The following model introduces an extra innovation driven by the Brownian motion, which generalize the Vasicek model in Vasicek [29].

Model 2 (Constant diffusion model). By taking $\boldsymbol{\theta} = \{\kappa, \theta, \sigma\}$ and letting $\mu(x;\boldsymbol{\theta}) = \kappa(\theta - x)$ and $\sigma(x;\boldsymbol{\theta}) \equiv \sigma > 0$ in (11), we obtain the constant diffusion model

$$dX(t) = \kappa(\theta - X(t))\, dt + \sigma\, dW(t) + dL(t), \quad X(0) = x_0. \tag{78}$$

The function $\Omega_m(y)$ for $m = 0, 1, 2, 3$ in Eq. (33) can be calculated as

$$\begin{aligned}
\Omega_0(y) &= S_0(y),\\
\Omega_1(y) &= \frac{\kappa\Delta^{1/2}}{2\sigma}\{yS_1(y) + [\eta\Delta y + \sigma\Delta^{1/2}(1-y^2)]S_0(y)\},
\end{aligned}$$

$$\Omega_2(y) = \frac{\kappa^2}{24\sigma^4(1+a\Delta)}$$
$$\times \{S_4(y) + 2(\eta\Delta - \sigma\Delta^{1/2}y)S_3(y) + [\eta^2\Delta - 2\eta\sigma\Delta^{1/2}y$$
$$+ \sigma^2(a\Delta + (4+3a\Delta)y^2)]\Delta S_2(y)$$
$$+ 2\sigma^2\Delta^{3/2}(1+a\Delta)[\eta(1+3y^2)\Delta^{1/2} + 3\sigma(1-y^2)y]S_1(y)$$
$$\sigma^2\Delta^2(1+a\Delta)[\eta^2(1+3y^2)\Delta + 6\eta\sigma\Delta^{1/2}(1-y^2)y + \sigma^2(1-10y^2+3y^4)]S_0(y)\}$$

and

$$\Omega_3(y) = \frac{\kappa^3\Delta^{1/2}}{336\sigma^5(1+a\Delta)} \times \{7yS_5(y) + [21\eta\Delta y - \sigma\Delta^{1/2}(-4+3a\Delta+21y^2)]S_4(y)$$
$$+ [21\eta^2\Delta y + \eta\sigma\Delta^{1/2}(5-9a\Delta-42y^2) + \sigma^2(-19+16a\Delta+7(4+a\Delta)y^2)y]\Delta S_3(y)$$
$$+ [7\eta^3\Delta^{3/2}y - \eta^2\sigma\Delta(2+9a\Delta+21y^2) + \eta\sigma^2\Delta^{1/2}y(4+39a\Delta+21(2+a\Delta)y^2)$$
$$+ \sigma^3(9+16a\Delta-7(4+3a\Delta)y^4+(33+5a\Delta)y^2)]\Delta^{3/2}S_2(y)$$
$$+ \sigma\Delta^2(1+a\Delta)[-3\eta^3\Delta^{3/2} + 3\eta^2\sigma\Delta(10+7y^2)y + \eta\sigma^2\Delta^{1/2}(23+19y^2-42y^4)$$
$$+ \sigma^3(-16-67y^2+21y^4)y]S_1(y) + 7\sigma^2\Delta^{5/2}(1+a\Delta)$$
$$\times [\eta^3\Delta^{3/2}(1+y^2)y + \eta^2\sigma\Delta(1+2y^2-3y^4)$$
$$+ \eta\sigma^2\Delta^{1/2}(-1-10y^2+3y^4)y - \sigma^3(1+5y^2-7y^4+y^6)]S_0(y)\},$$

where $\eta := \kappa(\theta - x_0)$ and for $m = 0, 1, 2, \ldots$,

$$S_m(y) := \frac{1}{2\sqrt{2\pi}} \frac{b^{a\Delta}}{\Gamma(a\Delta)} A^{-1-r_m/2} \exp\left[-\frac{1}{2}(y - \kappa(\theta - x_0)\Delta^{1/2}/\sigma)^2\right]$$
$$\times \left[B\Gamma\left(1+\frac{r_m}{2}\right){}_1F_1\left(1+\frac{r_m}{2}, \frac{3}{2}; \frac{B^2}{4A}\right) + \sqrt{A}\Gamma\left(\frac{1+r_m}{2}\right){}_1F_1\left(\frac{1+r_m}{2}, \frac{1}{2}; \frac{B^2}{4A}\right)\right]$$

with $r_m := m + a\Delta - 1$, $A := 1/(2\sigma^2\Delta)$, $B := y/\sigma\Delta^{1/2} - \kappa(\theta - x_0)/\sigma^2 - b$, and

$${}_1F_1(a, b; z) := \sum_{k=0}^{\infty} ((a)_k/(b)_k) z^k/k!$$

as the Kummer confluent hypergeometric function.

The following jump-diffusion model is a generalization of the square-root diffusion model proposed by Cox *et al.* [6]. For this model, Assumption 2 is not satisfied. Nevertheless, it deserves noting that Assumption 2 is a sufficient condition instead of a necessary condition to guarantee the convergence of the approximation theoretically. Although we have not given a theoretical justification for the approximation in the square-root model, we still consider this model as an example in this paper since its wide applications in finance (see, e.g., [6,7,11]). At the same time, we also want to show that for some models which do not satisfy all of Assumptions 1–3, the approximation method in this paper still works.

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Model 3 (Square-root diffusion model). By taking $\boldsymbol{\theta} = \{\kappa, \theta, \sigma\}$ and letting $\mu(x; \boldsymbol{\theta}) = \kappa(\theta - x)$ and $\sigma(x; \boldsymbol{\theta}) \equiv \sigma\sqrt{x}$ with $\sigma > 0$ in (11), we obtain the square-root diffusion model

$$dX(t) = \kappa(\theta - X(t))\, dt + \sigma\sqrt{X(t)}\, dW(t) + dL(t), \quad X(0) = x_0. \tag{79}$$

The function $\Omega_m(y)$ for $m = 0, 1, 2$ in Eq. (33) can be calculated as

$$\begin{aligned}
\Omega_0(y) &= S_0(y),\\
\Omega_1(y) &= \frac{1}{4\sigma x_0^2\sqrt{\Delta}}\{yS_2(y) + [2\kappa\theta\Delta y + 2\sigma(x_0 - y^2)\sqrt{\Delta}]S_1(y)\\
&\qquad [\kappa\eta(\theta + x_0)\Delta^2 y + 2\kappa\theta\sigma(x_0 - y^2)\Delta^{3/2} + \sigma^2(-3x_0 + y^2)\Delta y]S_0(y)\},
\end{aligned}$$

and

$$\begin{aligned}
\Omega_2(y) = {} & \frac{1}{480\sigma^4 x_0^4\Delta^2(1 + a\Delta)}\{5S_6(y) + 20(\kappa\theta\Delta - \sigma\sqrt{\Delta}y)S_5(y)\\
& + [10\kappa^2(3\theta^2 - x_0^2)\Delta^2 - 60\kappa\theta\sigma\Delta^{3/2}y + (15(3 + a\Delta)y^2 + (-12 + 17a\Delta)x_0)\sigma^2\Delta]S_4(y)\\
& + 20\kappa^3\theta(\theta^2 - x_0^2)\Delta^3 - 20\kappa^2\sigma(3\theta^2 - x_0^2)\Delta^{5/2}y\\
& - 2\kappa\sigma^2((7 + 6a\Delta)x_0^2 + \theta(x_0(1 - 28a\Delta) - 30(2 + a\Delta)y^2))\Delta^2\\
& + 2\sigma^3(x_0(38 + 9a\Delta) - 10(4 + 3a\Delta)y^2)\Delta^{3/2}y]S_3(y)\\
& + [5\kappa^4(\theta^2 - x_0^2)^2\Delta^2 - 20\kappa^3\sigma\theta(\theta^2 - x_0^2)\Delta^{3/2}y\\
& + (\eta^2(30y^2(4 + 3a\Delta) + x_0(37 + 66a\Delta)) + 12\kappa\eta x_0(5y^2(4 + 3a\Delta) + x_0(4 + 9a\Delta))\\
& + 20\kappa^2 x_0^2(x_0 a\Delta + (4 + 3a\Delta)y^2))\sigma^2\Delta + 2\sigma^3\eta(-10y^2(10 + 9a\Delta)\\
& + x_0(77 + 48a\Delta))\sqrt{\Delta}y + 4\kappa\sigma^3 x_0(-5y^2(10 + 9a\Delta) + x_0(48 + 33a\Delta))\sqrt{\Delta}y\\
& - \sigma^4(-5(19 + 18a\Delta)y^4 + (281 + 252a\Delta)x_0 y^2\\
& + (9 + 23a\Delta)x_0^2)]\Delta^2 S_2(y)\\
& + \sigma^2(1 + a\Delta)\Delta^{5/2}[-4\kappa^3(\theta^2 - x_0^2)(3x_0^2 - 8\theta x_0 - 15\theta y^2)\Delta^{3/2}\\
& + 6\sigma(\eta^2(23x_0 - 30y^2) + 4\kappa\eta x_0(13x_0 - 15y^2) + 20\kappa^2 x_0^2(x_0 - y^2))\Delta y\\
& + \sigma^2(\eta(7x_0^2 - 552x_0 y^2 + 180y^4) + 36\kappa x_0(x_0^2 - 16x_0 y^2 + 5y^4))\sqrt{\Delta}\\
& + \sigma^3(-241x_0^2 + 382x_0 y^2 - 60y^4)y]S_1(y)\\
& + 5\sigma^2(1 + a\Delta)\Delta^3\kappa^2\eta^2(x_0 + 3y^2)(\theta + x_0)^2\Delta^2 + 12\kappa^2\theta\sigma\eta(x_0 - y^2)(\theta + x_0)\Delta^{3/2}y\\
& + 2\sigma^2\Delta(x_0^2 - 10x_0 y^2 + 3y^4)(3\eta^2 + 6\kappa\eta x_0 + 2\kappa^2 x_0^2)\\
& - 4\kappa\theta\sigma^3(15x_0^2 - 20x_0 y^2 + 3y^4)\sqrt{\Delta}y + 3\sigma^4(-3x_0^3 + 21x_0^2 y^2 - 11x_0 y^4 + y^6)]S_0(y)\},
\end{aligned}$$

where $\eta := \kappa(\theta - x_0)$ and for $m = 0, 1, 2, \ldots,$

$$\begin{aligned}
S_m(y) := {} & \frac{1}{2\sqrt{2\pi}}\frac{b^{a\Delta}}{\Gamma(a\Delta)}A^{-1-r_m/2}\exp\left[-\frac{1}{2}\left(y - \frac{\kappa(\theta - x_0)\Delta^{1/2}}{\sigma\sqrt{x_0}}\right)^2\right]\\
& \times\left[B\Gamma\left(1 + \frac{r_m}{2}\right){}_1F_1\left(1 + \frac{r_m}{2}, \frac{3}{2}; \frac{B^2}{4A}\right) + \sqrt{A}\Gamma\left(\frac{1 + r_m}{2}\right){}_1F_1\left(\frac{1 + r_m}{2}, \frac{1}{2}; \frac{B^2}{4A}\right)\right]
\end{aligned}$$

with $r_m := m + a\Delta - 1$, $A := 1/(2\sigma^2 x_0\Delta)$, and $B := y/\sigma\sqrt{x_0}\Delta^{1/2} - \kappa(\theta - x_0)/\sigma^2 x_0 - b$.

## 4. NUMERICAL PERFORMANCE

In this section, we demonstrate the performance of the approximations for transition densities via the previous introduced pure-jump OU model, constant diffusion model, and square-root diffusion model in Section 3.5. To test the accuracy of the asymptotic expansion methodology, we calculate the true transition density by inverse Fourier transform of its known characteristic function as the benchmark for each of the above three models. Here, we use the numerical inverse Fourier transform method proposed by Abate and Whitt [1], which has been proved to be efficient and accurate. By comparing the approximated transition densities using our asymptotic expansion method with the true densities obtained by inverse Fourier transform, we show that the approximation errors decrease quickly as the approximation order $M$ in Eq. (33) increases.

For each example, the true transition density of $X(\Delta)$ can be obtained via its characteristic function $\phi(\Delta;\omega) := Ee^{i\omega X(\Delta)}$ by

$$\begin{aligned} p_{X(\Delta)}(x \mid x_0;\theta) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ix\omega}\phi(\Delta;\omega)\,d\omega \\ &= \frac{1}{\pi}\int_{0}^{+\infty}[\cos(x\omega)\,\mathrm{Re}(\phi)(\Delta;\omega)+\sin(x\omega)\,\mathrm{Im}(\phi)(\Delta;\omega)]\,d\omega, \end{aligned} \quad \textbf{(80)}$$

which can be efficiently approximated via the following Euler summation as

$$E(m,n,x) = \sum_{k=1}^{m}\binom{m}{k}2^{-m}s_{n+k}(x),$$

where the truncated series is defined by

$$s_n(x) := \frac{h}{2\pi} + \frac{h}{\pi}\sum_{k=1}^{n}[\mathrm{Re}(\phi)(\Delta;kh)\cos(khx) + \mathrm{Im}(\phi)(\Delta;kh)\sin(khx)].$$

We refer to Abate and Whitt [1] for more technical details.

By using Eq. (13), we can derive the explicit expressions of the characteristic functions for the above three models as follows. For the pure jump OU model, we have

$$\phi(t;\omega) = \exp\left\{i\omega[e^{-\kappa t}x_0 + \theta(1-e^{-\kappa t})] - \frac{a}{\kappa}\left[\mathrm{Li}_2\left(\frac{i\omega e^{-\kappa t}}{b}\right) - \mathrm{Li}_2\left(\frac{i\omega}{b}\right)\right]\right\},$$

where $\mathrm{Li}_s(z) := \sum_{k=1}^{+\infty} z^k/k^s$ is the polylogarithm function. For the constant diffusion model, we have

$$\begin{aligned} \phi(t;\omega) = \exp\Bigg\{ & i\omega[e^{-\kappa t}x_0 + \theta(1-e^{-\kappa t})] - \frac{\omega^2\sigma^2(1-e^{-2\kappa t})}{4\kappa} \\ & - \frac{a}{\kappa}\left[\mathrm{Li}_2\left(\frac{i\omega e^{-\kappa t}}{b}\right) - \mathrm{Li}_2\left(\frac{i\omega}{b}\right)\right]\Bigg\}. \end{aligned}$$

For the square-root diffusion model, we have

$$\phi(t;\omega) = \mathbb{E}[e^{i\omega X(t)}|X(0)=x_0] = e^{\alpha(t)+\beta(t)x_0},$$

where $\beta(t) = 2i\omega\kappa/(2\kappa e^{\kappa t} + i\omega\sigma^2(1 - e^{\kappa t}))$ and

$$\alpha(t) = \frac{1}{\sigma^2}\left\{2\kappa^2\theta t + a\sigma^2 t\log(b) - (2\kappa\theta + a\sigma^2 t)\log\left[1 - e^{\kappa t}\left(1 - \frac{2\kappa}{i\omega\sigma^2}\right)\right] + 2\kappa\theta\log\left(\frac{2\kappa}{i\omega\sigma^2}\right)\right.$$
$$\left. + a\sigma^2 t\left[\log\left(1 - \frac{be^{\kappa t}(2\kappa - i\omega\sigma^2)}{i\omega(2\kappa - b\sigma^2)}\right) - \log\left(b - \frac{2i\omega\kappa}{i\omega\sigma^2(1 - e^{\kappa t}) + 2\kappa e^{\kappa t}}\right)\right]\right\}$$
$$+ \frac{a}{\kappa}\left[\mathrm{Li}_2\left(1 - \frac{2\kappa}{i\omega\sigma^2}\right) - \mathrm{Li}_2\left(e^{\kappa t}\left(1 - \frac{2\kappa}{i\omega\sigma^2}\right)\right)\right.$$
$$\left. - \mathrm{Li}_2\left(\frac{b(2\kappa - i\omega\sigma^2)}{i\omega(2\kappa - b\sigma^2)}\right) + \mathrm{Li}_2\left(\frac{be^{\kappa t}(2\kappa - i\omega\sigma^2)}{i\omega(2\kappa - b\sigma^2)}\right)\right].$$

For the numerical comparison, according to Barndorff-Nielsen and Shephard [3], James *et al.* [13] and Li and Chen [23], we set the parameters of the above three examples as follows. For the pure jump OU model in (77), we set $\kappa = 0.6$, $\theta = 0.02$, $a = 100$, and $b = 10$. For the constant diffusion model in (78), we set $\kappa = 0.6$, $\theta = 0.02$, $\sigma = 0.3$, $a = 100$, and $b = 10$. For the square-root diffusion model in (79), we set $\kappa = 0.6$, $\theta = 0.02$, $\sigma = 0.3$, $a = 100$, and $b = 10$. In each model, we set the initial value $x_0 = 0.3$.

For each model, given the true transition density $p_{X(\Delta)}(x|x_0;\theta)$ in Eq. (80) and the approximated density $p^{(M)}_{X(\Delta)}(x \mid x_0;\theta)$ derived by Eq. (33) or Eq. (34), we denote by

$$e_M(\Delta, x \mid x_0;\theta) = p_{X(\Delta)}(x \mid x_0;\theta) - p^{(M)}_{X(\Delta)}(x \mid x_0;\theta) \tag{81}$$

the $M$thorder approximation error and define the maximum relative error as

$$\max_{x\in\mathcal{D}}\left|\frac{e_M(\Delta, x \mid x_0;\theta)}{p_{X(\Delta)}(x \mid x_0;\theta)}\right|$$

over a region $\mathcal{D}$.

We consider monthly, weekly, and daily monitoring frequencies ($\Delta = 1/12, 1/52, 1/252$)[2] and plot the maximum relative errors of order $M = 0, 1, 2, 3$ for each model in Figure 1. It is easy to observe that the maximum relative errors decrease quickly as the order of expansion increases. For example, when we choose $\Delta = 1/252$ and the order of $M = 2$, the maximum relative error of each model can attain the level as $10^{-5}$. Besides, when the monitoring frequency rises, that is, the time interval $\Delta$ becomes smaller, the maximum relative error will decrease correspondingly for each model.

The CPU time used to compute the explicit expansion terms of the various orders by Mathematica is given in Table 1. Here, the column of $M = 2$ (respectively $M = 3$) means that the approximated density in each of the three models employs the first three expansion terms $\Omega_0$, $\Omega_1$ , and $\Omega_2$ (respectively the first four terms $\Omega_0$, $\Omega_1$, $\Omega_2$, and $\Omega_3$), and the CPU time includes the total time for computing all the three (respectively four) expansion terms. Meanwhile, for each model given different $\Delta$, we give the CPU time used to compute the approximated densities with approximation order $M = 3$ by Matlab, as shown in Table 2. Here, for each case, we choose the initial point as $x_0 = 0.3$ and calculate 800 values of the transition density function to evaluate the maximum relative error.

[2] For the pure jump OU model, the region of the forward variable $x$ is $[0.3, 1.3]$ for $\Delta = 1/12, 1/52, 1/252$. For the constant diffusion model, when $\Delta = 1/12, 1/52$ ,and $1/252$, the regions of the forward variable $x$ are $[0.1, 2.5]$, $[0.1, 1.1]$, and $[0.22, 0.39]$, respectively. For the square-root diffusion model, when $\Delta = 1/12, 1/52$, and $1/252$, the regions of the forward variable $x$ are $[0.3, 2.3]$, $[0.20, 1.14]$, and $[0.22, 0.62]$, respectively.

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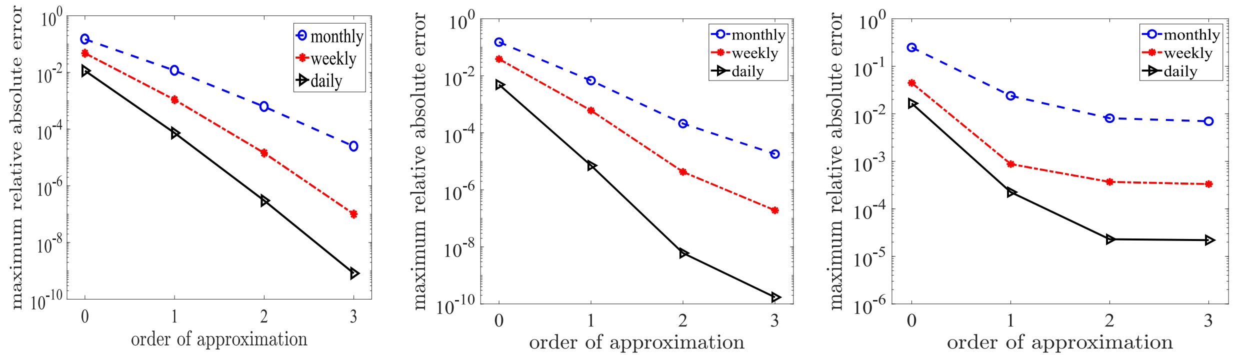
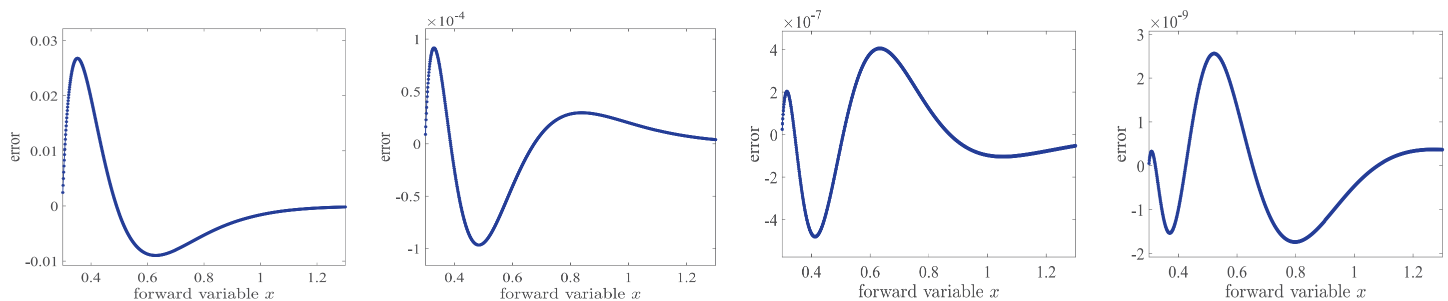


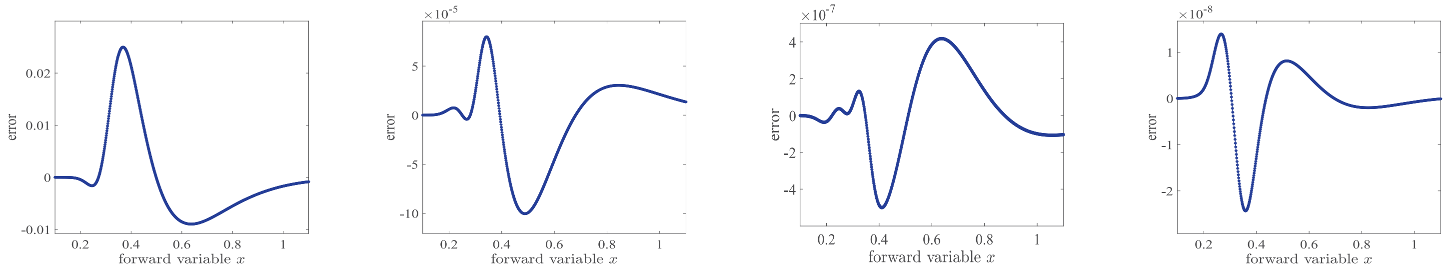

(a) Pure jump OU model. (b) Constant diffusion model. (c) Square-root diffusion model.

FIGURE 1. Maximum relative absolute errors of density approximation for Models 1, 2, and 3 with orders $M = 0, 1, 2, 3$. (a) Pure jump OU model, (b) constant diffusion model, and (c) square-root diffusion model.



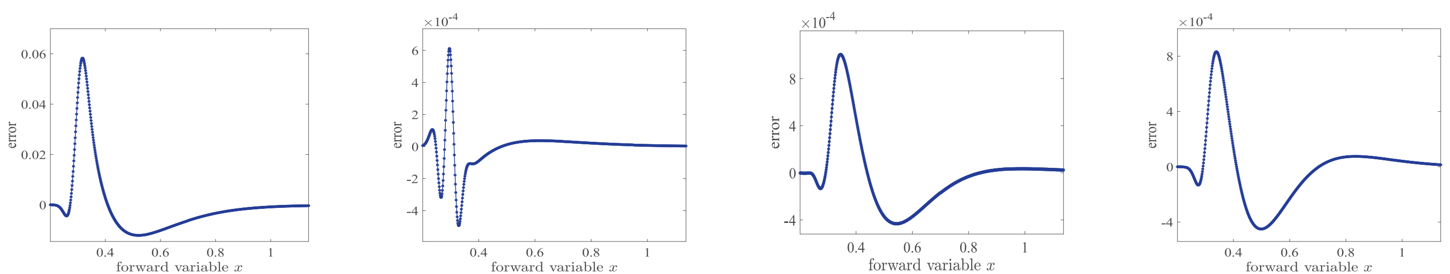

(a) $e_0$. (b) $e_1$. (b) $e_2$. (c) $e_3$.

FIGURE 2. Errors of density approximation for weekly monitoring frequency ($\Delta = 1/52$) in the pure jump OU model, that is, $e_0, e_1, e_2, e_3$ as the approximation errors with respect to the expansion orders $M = 0, 1, 2, 3$, respectively. The region of the forward variable $x$ in this model is $[0.3, 1.3]$. (a) $e_0$, (b) $e_1$, (c) $e_2$, and (d) $e_3$.




(a) $e_0$. (b) $e_1$. (b) $e_2$. (c) $e_3$.

FIGURE **3.** Errors of density approximation for weekly monitoring frequency ($\Delta = 1/52$) in the constant diffusion model, that is, $e_0, e_1, e_2, e_3$ as the approximation errors with respect to the expansion orders $M = 0, 1, 2, 3$, respectively. The region of the forward variable $x$ in this model is $[0.1, 1.1]$. (a) $e_0$, (b) $e_1$, (c) $e_2$, and (d) $e_3$.




(a) $e_0$. (b) $e_1$. (b) $e_2$. (c) $e_3$.

FIGURE **4.** Errors of density approximation for weekly monitoring frequency ($\Delta = 1/52$) in the square-root diffusion model, that is, $e_0, e_1, e_2, e_3$ as the approximation errors with respect to the expansion orders $M = 0, 1, 2, 3$, respectively. The region of the forward variable $x$ in this model is $[0.20, 1.14]$. (a) $e_0$, (b) $e_1$, (c) $e_2$, and (d) $e_3$.

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TABLE 1. CPU time for computing the various orders of approximations by Mathematica

| | $M = 2$ | $M = 3$ |
|---|---|---|
| Pure jump OU model | 6.4s | 24.3s |
| Constant diffusion model | 3.9s | 25.7s |
| Square-root diffusion model | 35.3s | 643.5s |

TABLE 2. CPU time for computing the approximated densities for different $\Delta$ with $M = 3$ by Matlab

| | $\Delta = 1/252$ | $\Delta = 1/52$ | $\Delta = 1/12$ |
|---|---|---|---|
| Pure jump OU model | 0.464s | 0.654s | 0.491s |
| Constant diffusion model | 123.6s | 112.2s | 127.8s |
| Square-root diffusion model | 241.2s | 247.8s | 265.2s |

In Figures 2–4, we plot the series of approximation errors defined by (81) for the three models, respectively. For each model, we consider the case of $\Delta = 1/52$ and denote by $e_0, e_1, e_2, e_3$ the abbreviation of $e_M(\Delta, x|x_0; \theta)$ in Eq. (81) for $M = 0, 1, 2, 3$, respectively. We observe from Figures 2–4 that the approximation errors decrease quickly and consistently as the order of expansion increases.

The advantages of our asymptotic expansion method over the method of inverse Fourier transform can be summarized as follows:

(1) When the solution of SDE (11) does not admit an explicit expression of $X(t)$ or characteristic function $\phi(t; \omega)$, our method can still be used to approximate the transition density.

(2) The approximation errors decrease quickly as the approximation order $M$ in Eq. (33) increases; thus, it suffices to use the first several expansion terms for the approximation.

(3) Since the SDE driven by the gamma process involves the fat-tail characteristic, the characteristic function $\phi(t; \omega)$ for the pure jump SDE decreases slowly when $\omega \to +\infty$, which induces heavy computation burden in the inverse Fourier transform method. In contrast, our expansion terms for the pure jump SDE can achieve quick convergence and be evaluated in a few seconds for any rational initial value of $X_0$.

## 5. CONCLUDING REMARKS

In this paper, we propose a closed-form asymptotic expansion to approximate the transition density of the jump-diffusion SDE driven by the gamma process. We employ three examples with known characteristic functions for numerical illustrations and comparisons. Compared with the method of calculating the transition density by inverse Fourier transform of the characteristic function, our method is more efficient while achieving low approximation errors. In terms of the applications in financial engineering, our approximation method can be directly applied for option pricing and hedging to obtain analytically tractable results, which is left for further study.

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## References

1. Abate, J. & Whitt, W. (1992). The Fourier-series method for inverting transforms of probability distributions. *Queueing Systems* 10: 5–88.
2. Applebaum, D. (2009). *Lévy processes and stochastic calculus*. New York, NY: Cambridge University Press.
3. Barndorff-Nielsen, O.E. & Shephard, N. (2001). Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics. *Journal of the Royal Statistical Society, B (Statistical Methodology)* 63: 167–241.
4. Barndorff-Nielsen, O.E., Mikosch, T. & Resnick, S.I. (2001). *Lévy processes: Theory and applications*. Boston: Birkhäuser.
5. Cont, R. & Tankov, P. (2004). *Financial modelling with jump processes*. London: Chapman and Hall.
6. Cox, J.C., Ingersoll, Jr J.E., & Ross, S.A. (1985). A theory of the term structure of interest rates. *Econometrica* 53(2): 385–408.
7. Duffie, D., Pan, J., & Singleton, K. (2000). Transform analysis and asset pricing for affine jump-diffusions. *Econometrica* 68(6): 1343–1376.
8. Eberlein, E., Madan, D., Pistorius, M., & Yor, M. (2013). A simple stochastic rate model for rate equity hybrid products. *Applied Mathematical Finance* 20: 461–488.
9. Hayashi, M. (2008). Asymptotic expansions for functionals of a Poisson random measure. *Journal of Mathematics of Kyoto University* 48(1): 91–132.
10. Hayashi, M. & Ishikawa, Y. (2012). Composition with distributions of Wiener-Poisson variables and its asymptotic expansion. *Mathematische Nachrichten* 285: 619–658.
11. Heston, S.L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The Review of Financial Studies* 6(2): 327–343.
12. Ishikawa, Y. (2013). *Stochastic calculus of variations for jump processes*. Berlin: Walter De Gruyter.
13. James, L.F., Kim, D., & Zhang, Z. (2013). Exact simulation pricing with gamma processes and their extensions. *Journal of Computational Finance* 17: 3–39.
14. James, L.F., Müller, G., & Zhang, Z. (2017). Stochastic volatility models based on OU-gamma time change: Theory and estimation. *Journal of Business & Economic Statistics* 36: 1–13.
15. Jeanblanc, M., Yor, M., & Chesney, M. (2009). *Mathematical methods for financial markets*. London, UK: Springer-Verlag.
16. Kanwal, R.P. (2004). *Generalized functions: Theory and applications*, 3rd ed. Boston: Birkhäuser.
17. Kawai, R. & Takeuchi, A. (2011). Greeks formulas for an asset price model with gamma processes. *Mathematical Finance* 21(4): 723–742.
18. Kawai, R. & Takeuchi, A. (2010). Sensitivity analysis for averaged asset price dynamics with gamma processes. *Statistics & Probability Letters* 80(1): 42–49.
19. Kloeden, P. & Platen, E. (1992). *Numerical solutions of stochastic differential equations*. Berlin: Springer-Verlag.
20. Kohatsu-Higa, A. & Takeuchi, A. (2019). *Jump SDEs and the study of their densities*. Singapore: Springer.
21. Kunita, H. (2019). *Stochastic flows and jump-diffusions*. Singapore: Springer.
22. Li, C. & Chen, D. (2016). Estimating jump-diffusions using closed-form likelihood expansions. *Journal of Econometrics* 195(1): 51–70.
23. Li, C. (2013). Maximum-likelihood estimation for diffusion processes via closed-form density expansions. *Annals of Statistics* 41: 1350–1380.
24. Madan, D.B. & Seneta, E. (1990). The variance gamma (VG) model for share market returns. *Journal of Business* 63(4): 511–524.
25. Madan, D.B., Carr, P.P., & Chang, E.C. (1998). The variance gamma process and option pricing. *European Finance Review* 2(1): 79–105.
26. Platen, E. & Bruti-Liberati, N. (2010). *Numerical solution of stochastic differential equations with jumps in finance*. Berlin: Springer-Verlag.
27. Ribeiro, C. & Webber, N. (2004). Valuing path-dependent options in the variance-gamma model by monte carlo with a gamma bridge. *Journal of Computational Finance* 7(2): 81–100.

28. Schoutens, W. (2003). *Lévy processes in finance: Pricing financial derivatives*. New York: Wiley.
29. Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics* 5: 177–188.
30. Yor, M. (2007). Some remarkable properties of gamma processes. In Fu, M.C., Jarrow, R.A., Yen, J.-Y.J. and Elliott, R.J. (eds.). *Advances in mathematical finance*. Boston, MA: Springer, pp. 37–47.

## APPENDIX

In this part, we give a brief introduction of the Sobolev norms in $D_\infty(\mathbb{R})$ and the dual Sobolev norms in $D'_\infty(\mathbb{R})$(see Hayashi and Ishikawa [10], Ishikawa [12], and Kunita [21] for more details).

## APPENDIX A. THE SOBOLEV NORMS IN $D_\infty(\mathbb{R})$

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the Wiener–Poisson space with $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ denote the probability measures on the Wiener space and the Poisson space, respectively. Denote by $H$ the collection of random variables on $(\Omega, \mathcal{F})$, which can be expressed as a function of finite one-dimensional Wiener integrals and finite one-dimensional integrals with respect to the Poisson random measure. Then, in Hayashi and Ishikawa [10], the authors define $D_{k,l,p}(\mathbb{R}) := \bar{H}^{\|\cdot\|_{k,l,p}}$ for $k, l \in \mathbb{N}$ and $p \geq 2$, which is the completion of $H$ under the Sobolev norm $\|\cdot\|_{k,l,p}$ defined by

$$\|F\|_{k,l,p} := \|F\|_{k,l,p,1},$$

where for $\rho \in \mathbb{N}$, they define

$$\|F\|_{k,l,p,\rho} := \left( \|F\|_{0,l,p}^p + \sum_{k'=1}^{k} \sum_{l'=0}^{l} E_{\mathcal{P}} \left[ \int_{A(\rho)^{k'}} \left( \int_{T^{l'}} \left| \frac{D_t^{l'} \tilde{D}_u^{k'} F}{\gamma(u)} \right|^2 dt \right)^{p/2} \hat{M}(du) \right] \right)^{1/p}$$

and

$$\|F\|_{0,l,p}^p := E_{\mathcal{P}_1}(|F|^p) + \sum_{j=1}^{l} E_{\mathcal{P}_1} \left[ \left( \int_{T^j} |D_t^j F|^2 \, dt \right)^{p/2} \right],$$

with the set $A(\rho) := \{u \in \mathbb{R} : \gamma(u) \leq \rho\}$. Here, $D^l$, $\tilde{D}_u$, $E_{\mathcal{P}}(\cdot)$, and $E_{\mathcal{P}_1}(\cdot)$ are the $l$th differential operator, the difference operator, the expectation with respect to $\mathcal{P}$, and the expectation with respect to $\mathcal{P}_1$ , respectively. Define

$$D_\infty(\mathbb{R}) := \bigcap_{k,l=0}^{\infty} \bigcap_{p \geq 2} D_{k,l,p}(\mathbb{R}).$$

According to Ishikawa [12], the functional $F(\epsilon)$ in Eq. (6) can be expanded as a convergent series of $\epsilon$ with respect to the Sobolev norms in $D_\infty(\mathbb{R})$, if for any $m \geq 1$, $k, l \in \mathbb{N}$, and $p \geq 2$, the condition

$$\limsup_{\epsilon \to 0} \frac{1}{\epsilon^{m+1}} \left\| F(\epsilon) - \sum_{j=0}^{m} f_j \epsilon^j \right\|_{k,l,p} < +\infty$$

holds.

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## APPENDIX B. THE DUAL SOBOLEV NORMS IN $D'_\infty(\mathbb{R})$

Denote by $D'_{k,l,p}(\mathbb{R})$ the analytic adjoint space of $D_{k,l,p}(\mathbb{R})$, i.e., the normed space with the dual Sobolev norm $\|\cdot\|'_{k,l,p}$ given by

$$\|\Phi\|'_{k,l,p} = \sup_{\|G\|_{k,l,p}=1} |\mathbb{E}[\Phi \cdot G]|,$$

where the generalized expectation $\mathbb{E}[\cdot]$ is introduced in Eq. (2). Define

$$D'_\infty(\mathbb{R}) := \bigcup_{k,l=0}^{\infty} \bigcup_{p\geq 2} D'_{k,l,p}(\mathbb{R}).$$

According to Ishikawa [12], the functional $\delta(F(\epsilon) - y)$ can be expanded as a convergent series of $\epsilon$ in Eq. (7) with respect to the dual Sobolev norms in $D'_\infty(\mathbb{R})$, if for any $m = 1, 2, \ldots$, there exists $k, l \in \mathbb{N}$ and $p \geq 2$, such that

$$\limsup_{\epsilon\to 0} \frac{1}{\epsilon^{m+1}} \left\| \delta(F(\epsilon) - y) - \sum_{m+0}^{M} \Phi_m(y)\epsilon^m \right\|_{k,l,p} < +\infty.$$

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