

THE IMPLICITLY CONSTRUCTIBLE UNIVERSE

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Abstract. We answer several questions posed by Hamkins and Leahy concerning the *implicitly constructible universe* Imp , which they introduced in [5]. Specifically, we show that it is relatively consistent with ZFC that $Imp \models \neg CH$, that $Imp \neq HOD$, and that $Imp \models V \neq Imp$, or in other words, that $(Imp)^{Imp} \neq Imp$.

§1. Introduction. The *implicitly constructible universe*, denoted Imp , was defined by Hamkins and Leahy [5]:

DEFINITION 1.1. For a transitive set X , a subset $S \subseteq X$ is *implicitly definable* over X if for some formula $\varphi(x_1, \dots, x_n)$ in the language of ZFC with an additional one-place predicate symbol, and some parameters $a_1, \dots, a_n \in X$, the set S is the unique subset of X such that

$$(X, \in, S) \models \varphi(a_1, \dots, a_n).$$

DEFINITION 1.2. Imp is defined by iteratively applying the implicitly definable power set operation as follows.

$$\begin{aligned} Imp_0 &= \emptyset; \\ Imp_{\alpha+1} &= \{S \mid S \text{ is implicitly definable over } Imp_\alpha\} \quad (Imp_1 = \{\emptyset\}); \\ Imp_\lambda &= \bigcup_{\alpha < \lambda} Imp_\alpha \text{ for limit } \lambda. \\ Imp &= \bigcup_{\alpha \in OR} Imp_\alpha. \end{aligned}$$

Hamkins and Leahy showed the following facts.

PROPOSITION 1.3 (Hamkins and Leahy [5]). Imp is an inner model of ZF, with $L \subseteq Imp \subseteq HOD$.

If ZF is consistent, so is ZFC + $(Imp \neq L)$.

For $\alpha < \omega_1^L$, as a consequence of Shoenfield absoluteness, $Imp_\alpha = (Imp_\alpha)^L$; thus, $Imp_{\omega_1^L} = (Imp_{\omega_1^L})^L = L_{\omega_1^L}$.

In this article, we answer some questions posed by Hamkins and Leahy [5]. These questions aim to separate Imp from L and from HOD , both literally (Hamkins and Leahy showed that we may have $Imp \neq L$, and we show here that we may have

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$Imp \neq HOD$) and in terms of their properties. In particular, we show that (given the consistency of ZF):

1. It is consistent that $Imp \models \neg CH$. (Theorem 2.3.)
2. It is consistent that $Imp \neq HOD$. (Theorem 3.6.)
3. It is consistent that $Imp \models V \neq Imp$ (that is, $(Imp)^{Imp} \neq Imp$). (Theorem 4.11.)

Imp is defined level-by-level, inductively, as is the constructible universe L . An important distinction is that, given L_α and $S \subset L_\alpha$, whether $S \in L_{\alpha+1}$ depends only on L_α , while given Imp_α and $S \subset Imp_\alpha$, whether $S \in Imp_{\alpha+1}$ depends on the power set of Imp_α . By the above results, despite the similarity of the definitions of Imp and L , we see that Imp is less like L and more like HOD .

We leave open, among other things, the question of whether $Imp \models \neg AC$ is consistent. As $(V = Imp) \implies AC$, Theorem 4.11 is relevant to this question.

§2. Preliminary results and notation. The following facts were noted by Hamkins and Leahy:

PROPOSITION 2.1 (Hamkins and Leahy [5]). *Suppose $M \models ZFC + V = L$. If \mathbb{P} is a forcing poset in M , and G is the unique element of $M[G]$ that is \mathbb{P} -generic over M , then in $M[G]$ we must have $G \in Imp$, and therefore $(Imp)^{M[G]} = M[G]$.*

PROPOSITION 2.2 (Hamkins and Leahy [5]). *If G is \mathbb{P} -generic over M , where \mathbb{P} is an almost-homogeneous notion of forcing in M , then $M[G] \models Imp \subseteq M$.*

Hamkins and Leahy use Proposition 2.1 in the proof of the consistency of $Imp \neq L$ [5].

By playing off rigidity (unique generics) against (almost) homogeneity, we can control which sets belong to Imp in generic extensions and in their submodels.

We begin with the following proposition:

THEOREM 2.3. *If ZF is consistent, so is $ZFC + (Imp \models \neg CH)$.*

PROOF. Abraham's model [1] and Groszek's model [3] for a minimal failure of CH are each produced by forcing over a model M of $V = L$ with a poset \mathbb{P} such that G is the unique element of $M[G]$ that is \mathbb{P} -generic over M , and $M[G] \models \neg CH$. Since, by Proposition 2.1, in each of these models $Imp = M[G]$, it follows that in each of these models $Imp \models \neg CH$. \dashv

To prove the main theorems of this article, we will employ the technique of Groszek [3] to produce unique generics. This entails coding a generic sequence of reals into the degrees of constructibility of the generic extension, by combining products and iterations of Sacks forcing.

For the remainder of this article, we will use the following notation.

We identify a set with its characteristic function, so if $x \in 2^\alpha$, we may say $\beta \in x$ rather than $x(\beta) = 1$.

We let M be a model of $ZFC + V = L$, and define forcing partial orders in M .

In general, we will blur the distinction between objects in a generic extension and names for those objects in the ground model. This is to avoid the confusion of, for example, introducing notation for the name in the ground model of the name in an intermediate extension of an object in a further extension.

\mathbb{S} will denote Sacks forcing. Forcing with \mathbb{S} over a model of $V = L$ adds a generic real $g \subseteq 2^\omega$ of minimal (nonzero) L -degree [6].

\mathbb{Q} will always denote a countable-support iteration $\langle \mathbb{Q}_\beta \mid \beta < \alpha \rangle$ of some countable length α , such that each \mathbb{Q}_β is forced to be either \mathbb{S} or $\mathbb{S} \times \mathbb{S}$, and $\mathbb{Q}_0 = \mathbb{S}$. Whether \mathbb{Q}_β is \mathbb{S} or $\mathbb{S} \times \mathbb{S}$ may depend on the generic sequence below β .

The following proposition follows from earlier work on Sacks forcing (for example, see Baumgartner and Laver [2], and Groszek [3]).

PROPOSITION 2.4. *Forcing with \mathbb{Q} preserves ω_1 , and if G is \mathbb{Q} -generic over a model M of $V = L$, then in $M[G]$ the L -degrees of reals are exactly:*

1. a well-ordered sequence $\langle d_\beta \mid \beta \leq \alpha \rangle$, where d_0 is the degree of \emptyset , $d_{\beta+1}$ is the degree of the \mathbb{Q}_β -generic, and for limit β , d_β is the degree of the sequence of generic reals $\langle d_\gamma \mid \gamma < \beta \rangle$; and
2. for each $\beta < \alpha$ such that \mathbb{Q}_β is $\mathbb{S} \times \mathbb{S}$, a pair of incomparable degrees $d_{\beta,0}$ and $d_{\beta,1}$ between d_β and $d_{\beta+1}$.

\mathbb{P} will always denote a countable-support product $\prod_{i \in I} \mathbb{P}_i$, where each \mathbb{P}_i has the form \mathbb{Q} described above.

We will denote the \mathbb{P} -generic sequence by $G = \langle G_i \mid i \in I \rangle$, where G_i is \mathbb{P}_i -generic.

Each G_i is equivalent to a sequence of generic reals of countable length α_i ; we will denote the join of these reals (relative to some fixed counting of α_i) as g_i .

An important technical lemma is the following.

LEMMA 2.5. *Suppose $M \models V = L$, the poset $\mathbb{P} \in M$ is as described above, and G is \mathbb{P} -generic over M , where $G = \langle G_i \mid i \in I \rangle$. If x is a real in $M[G]$ and for all $i \in I$ we have $x \notin M[G_i]$, then the L -degree of x lies above at least two minimal (nonzero) L -degrees of reals.*

This lemma is proven using a fusion construction of the sort common to Sacks forcing arguments. Since the proof is not especially illuminating of any new ideas, we defer it to Section 5, at the end of the article. At the end of Section 5 we state a more general result about degrees of constructibility of reals in generic extensions by forcing notions built from Sacks forcing.

§3. Separating *Imp* from *HOD*. In this section, we produce a model N in which $Imp \neq HOD$.

DEFINITION 3.1. Let M be a model of $V = L$, and in M , let $\mathbb{P} = \prod_{i \in I} \mathbb{P}_i$ be the countable-support product defined by letting I be $\omega_1 \times \omega_1$ and $\mathbb{P}_{(\alpha,\beta)}$ be the length α iteration of Sacks forcing \mathbb{S} .

By Proposition 2.4, each $\mathbb{P}_{(\alpha,\beta)}$ adds an initial segment of degrees of constructibility of reals of order type $\alpha + 1$, which we will call a tower of height $\alpha + 1$, with top point $deg(g_{(\alpha,\beta)})$. (By the conventions stated after Proposition 2.4, $g_{(\alpha,\beta)}$ denotes the join of the sequence of reals added by $\mathbb{P}_{(\alpha,\beta)}$. Since $\mathbb{P}_{(0,\beta)}$ is trivial, we let $g_{(0,\beta)}$ denote the zero real, or \emptyset .) Furthermore, in $M[G]$, the only well-ordered initial segments of the degrees of constructibility of reals are these towers and their initial segments. (To see this, suppose x is a real whose degree is not in one of these towers. Then for all $i \in \omega_1 \times \omega_1$, we have $x \notin M[G_i]$. Therefore, by the technical lemma

(Lemma 2.5), x lies above at least two minimal (nonzero) L -degrees of reals, so its degree is not in any well-ordered tower of L -degrees.) Hence, each of these towers is maximal. In our argument later, we will code information into certain submodels of the forcing extension by controlling the ordinals α for which there is a unique such maximal tower.

In $M[G]$ there are also Cohen subsets of ω_1 , that is, subsets of ω_1 that are generic over M for the forcing $Add(\omega_1, 1)$ whose conditions are countable partial functions from ω_1 to 2. One such element is x , defined by $x(\alpha) = g_{(\alpha,0)}(0)$.

Define the model N by $N = M[H]$, where

$$H = \langle G_{(\alpha,\beta)} \mid (\gamma \in Lim \cup \{0\} \ \& \ n \in \omega \ \& \ \alpha = \gamma + 2n \ \& \ x(\gamma + n) = 0) \Rightarrow \beta = 0 \rangle.$$

That is, for $\gamma \in Lim \cup \{0\}$, if $x(\gamma + n) = 0$, then we omit from N all but one maximal tower of height $\gamma + 2n + 1$.

CLAIM 3.2. *If $G_{(\alpha,\beta)}$ is not an entry in the sequence H , then no real whose (nonzero) degree is in the tower added by $G_{(\alpha,\beta)}$ is in N .*

Therefore, the maximal towers in N are precisely those added by the $G_{(\alpha,\beta)}$, where for some $\gamma \in Lim \cup \{0\}$ and $n \in \omega$, we have either $\alpha = \gamma + 2n \ \& \ (x(\gamma + n) = 0 \Rightarrow \beta = 0)$ or $\alpha = \gamma + 2n + 1$.

PROOF. Suppose p forces that $G(\alpha, \beta)$ is not an entry in H . That is, for some $\gamma \in Lim \cup \{0\}$ and some $n \in \omega$, we have $\alpha = \gamma + 2n$ and $\beta \neq 0$ and p forces that $x(\gamma + n) = 0$ (that is, $p \Vdash g_{\alpha,0}(0) = 0$). Since p forces that $g_{\alpha,0}(0) = 0$, then p forces that H is an element of

$$M' = M[\langle G_{\alpha',\beta'} \mid \alpha' = \alpha \Rightarrow \beta' = 0 \rangle].$$

But since $\langle G_{\alpha',\beta'} \mid \alpha' = \alpha \Rightarrow \beta' = 0 \rangle$ is generic over M for $\prod_{i \in J} \mathbb{P}_i$, where $J \subset I$ is in M and $(\alpha, \beta) \notin J$, it follows by standard results about product forcing that no element of $M[G_{(\alpha,\beta)}] \setminus M$ is in M' . Since $M' \supseteq N$, this completes the proof. \dashv

CLAIM 3.3. *In N , x is ordinal definable*

PROOF. For $\gamma \in Lim \cup \{0\}$, recalling that $G_{(\alpha,\beta)}$ adds a maximal tower of height $\alpha + 1$, we have $x(\gamma + n) = 0$ iff there is a unique maximal tower of L -degrees of reals of height $\gamma + 2n + 1$. (If $x(\gamma + n) = 1$, there are ω_1 -many maximal towers of height $\gamma + 2n + 1$. There are always ω_1 -many maximal towers of height $\gamma + 2n + 2$.) \dashv

Furthermore, in N , there is G' that is \mathbb{P} -generic over M , defined by, for $\gamma, \rho \in Lim \cup \{0\}$ and $n, m \in \omega$,

$$G'(\gamma + 2n, \rho + m) = G(\gamma + 2n + 1, \rho + 2m) \upharpoonright \gamma + 2n;$$

$$G'(\gamma + 2n + 1, \rho + m) = G(\gamma + 2n + 1, \rho + 2m + 1);$$

where $G(\alpha, \beta) \upharpoonright \delta$, for $\delta < \beta$, is $\{p \upharpoonright \delta \mid p \in G(\alpha, \beta)\}$. That is, we are recovering the existence of many mutually generic maximal towers of L -degrees of reals of height $\alpha + 1$ for all α , by cutting half the towers of height $\gamma + 2n + 2$ down to height $\gamma + 2n + 1$.

Now we have $M[G'] \subset N \subset M[G]$.

CLAIM 3.4. $(Imp)^N = M$.

REMARK 3.5. The following proof generalizes to show:

Suppose $M \subseteq M' \subseteq N \subseteq M''$ are models of set theory. Suppose further that $Imp^{M'} \subseteq M$, $Imp^{M''} \subseteq M$, and M' and M'' satisfy the same sentences with parameters from M . (This will follow if M' and M'' are generic extensions of M via the same almost homogeneous forcing notion in M .) Then

$$Imp^{M'} = Imp^N = Imp^{M''}.$$

PROOF. The forcing \mathbb{P} is almost homogeneous, so by Proposition 2.2, $(Imp)^{M[G']} = (Imp)^{M[G]} = M$. In fact, for all α , we have

$$(Imp_\alpha)^{M[G']} = (Imp_\alpha)^{M[G]} = \{y \in M \mid \Vdash_{\mathbb{P}} (y \in Imp_\alpha)\},$$

and so we may define $I_\alpha = (Imp_\alpha)^{M[G']} = (Imp_\alpha)^{M[G]} \in M$.

Show inductively that, for all α ,

$$(Imp_\alpha)^N = I_\alpha.$$

Assume as inductive hypothesis that $(Imp_\alpha)^N = I_\alpha$.

First, suppose that $S \in I_{\alpha+1} = (Imp_{\alpha+1})^{M[G]}$. Then for some formula φ and $a_1, \dots, a_n \in I_\alpha$, we have that S is the unique subset of I_α in $M[G]$ such that $(I_\alpha, \in, S) \models \varphi(a_1, \dots, a_n)$. But since $S \in M$, we have $S \in N$, and since $N \subset M[G]$, we have that S is also unique in N . Hence $S \in (Imp_{\alpha+1})^N$.

Conversely, suppose that $S \in (Imp_{\alpha+1})^N$. Then, for some formula φ and $a_1, \dots, a_n \in I_\alpha$, we have that S is the unique subset of I_α in N such that $(I_\alpha, \in, S) \models \varphi(a_1, \dots, a_n)$. Since $N \subset M[G]$, we have $S \in M[G]$, and so by the almost-homogeneity of \mathbb{P} , we have

$$\Vdash_{\mathbb{P}} (\exists Z \subseteq I_\alpha) ((I_\alpha, \in, Z) \models \varphi(a_1, \dots, a_n)).$$

But then

$$M[G'] \models (\exists Z \subseteq I_\alpha) ((I_\alpha, \in, Z) \models \varphi(a_1, \dots, a_n)).$$

Since S was unique in N and $M[G'] \subseteq N$, then the only possible such Z in $M[G']$ is S . Therefore $S \in (Imp_{\alpha+1})^{M[G']} = I_{\alpha+1}$. -1

Now we have that $x \notin M = (Imp)^N$, but $x \in (HOD)^N$. This proves the following theorem.

THEOREM 3.6. *If ZF is consistent, then so is ZFC + (Imp ≠ HOD).*

§4. Separating $(Imp)^{Imp}$ from Imp . In this section, we produce a model N in which $(Imp)^{Imp} \neq Imp$.

DEFINITION 4.1. The forcing poset $\mathbb{S}\mathbb{C}_n$ is the countable-support length ω iteration of $(\mathbb{S}_k)_{k < \omega}$ where (letting g denote the join of the generic reals g_k for \mathbb{S}_k , which we may call the generic real for $\mathbb{S}\mathbb{C}_n$),

$$\mathbb{S}_k = \begin{cases} \mathbb{S} & k \leq n \text{ or } (k = n + 2 + j \ \& \ j \notin g); \\ \mathbb{S} \times \mathbb{S} & k = n + 1 \text{ or } (k = n + 2 + j \ \& \ j \in g). \end{cases}$$

(Note that the join of an ω -sequence of reals is defined, as in Definition 5.11, in such a way that $g(n)$ depends only on the \mathbb{S}_k -generics for $k < n$.)

If $M \models V = L$, and G is $\mathbb{S}C_n$ -generic over M , then by Proposition 2.4, in $M[G]$ the degrees of constructibility form a lattice of height ω and width 2 (a tower of lines and diamonds) that (uniformly) codes (n, g) , where g is the generic real. In particular, in $M[G]$ there is a unique $\mathbb{S}C_n$ generic over M (and no $\mathbb{S}C_m$ -generic over M for $m \neq n$). Furthermore, the lattice of degrees of constructibility in $M[G]$ contains a unique minimal nonzero degree.

DEFINITION 4.2. A self-coding real with base n is any real x such that the degrees of constructibility below x form a lattice coding x in the same way that the generic real g for $\mathbb{S}C_n$ is coded by the degrees of constructibility below g .

It is not hard to see that if x is a self-coding real, the base n is uniquely determined by x . Also, if x is a self-coding real, then $\omega_1^{L[x]} = \omega_1^L$.

CLAIM 4.3. *There is a formula $\varphi_{sc}(n)$ such that*

1. *If $X \subset \omega_1^L$, $(L_{\omega_1^L}, \in, X) \models \varphi_{sc}(n)$, $x = X \cap \omega$, and $\omega_1^{L[x]} = \omega_1^L$, then x is a self-coding real with base n ;*
2. *If x is a self-coding real with base n , then there is a unique $X \subset \omega_1^L$ such that $X \cap \omega = x$ and $(L_{\omega_1^L}, \in, X) \models \varphi_{sc}(n)$.*

PROOF. Choose a canonical way of coding the structure of $L_{\omega_1^L}[x]$, for any real x , into $Y \subseteq \omega_1^L$, coding the truth predicate of the model $L_{\omega_1^L}[x]$ coded by Y into $Z \subseteq \omega_1^L$, and coding x , Y , and Z into $X \subseteq \omega_1^L$ in such a way that $X \cap \omega = x$.

Then $\varphi_{sc}(n)$ asserts that X is the canonical code for $L_{\omega_1^L}[X \cap \omega]$, and that the model coded by X satisfies a sentence asserting that the universe is $L_{\omega_1^L}[r]$ for a self-coding real r with base n . ⊢

DEFINITION 4.4. If x is a self-coding real with base n , then the unique $X \subset \omega_1^L$ such that $X \cap \omega = x$ and $(L_{\omega_1^L}, \in, X) \models \varphi_{sc}(n)$ is denoted $C(x)$, the canonical code for $L_{\omega_1^L}[x]$.

This implies the following fact.

CLAIM 4.5. *Let N be a model of ZF in which x is the unique self-coding real with base n . Then in N we have $\text{Imp}_{\omega_1^L} = (\text{Imp}_{\omega_1^L})^L$ (by Proposition 1.3), $C(x) \in \text{Imp}_{\omega_1^L+1}$, and $x \in \text{Imp}_{\omega_1^L+2}$.*

DEFINITION 4.6. The forcing \mathbb{P} is a countable-support product

$$\mathbb{S} \times \prod_{i \in (\omega_1 \times \omega)} \mathbb{P}_i$$

where

$$\mathbb{P}_{(\alpha, n)} = \mathbb{S}C_n.$$

If G is \mathbb{P} -generic, then G is equivalent to the sequence of generics

$$\langle G_{\mathbb{S}}, G_i \mid i \in (\omega_1 \times \omega) \rangle$$

or to the sequence of generic reals

$$\langle g_{\mathbb{S}}, g_i \mid i \in (\omega_1 \times \omega) \rangle.$$

CLAIM 4.7. *If $M \models V = L$, and G is \mathbb{P} -generic over M , then in $M[G]$ the only self-coding reals with base n are the generic reals $g_{(\alpha, n)}$.*

PROOF. By Proposition 2.4, the only self-coding real in $M[G_{(\alpha,n)}]$ is the generic real $g_{(\alpha,n)}$. By the technical lemma (Lemma 2.5), any real not in any $M[G_{(\alpha,n)}]$ lies above at least two different minimal (nonzero) L -degrees of reals, and therefore is not a self-coding real with base n . \dashv

DEFINITION 4.8. $N = M[H]$ is the submodel of $M[G]$ defined by setting h to be the join of g_S and the reals $g_{0,n}$, and setting

$$H = \langle g_S, g_{(\alpha,n)} \mid \alpha < \omega_1 \ \& \ (\alpha = 0 \text{ or } n \notin h) \rangle.$$

CLAIM 4.9. *In N , there is a unique self-coding real with base n iff $n \in h$.*

PROOF. By Claim 4.7, the only self-coding reals in $M[G]$ are the $g_{(\alpha,n)}$. By definition of H , if $n \notin h$, every $g_{(\alpha,n)}$ is in $N = M[H]$, so there are many self-coding reals with base n .

If $n \in h$, by an argument like the proof of Claim 3.2, the only $g_{(\alpha,n)}$ in N is $g_{(0,n)}$, so there is a unique self-coding real with base n in N . \dashv

CLAIM 4.10. *Let $H' = \langle g_S, g_{(0,n)} \rangle_{n < \omega}$. Then:*

1. $(Imp)^N = M[H']$. In particular, $g_S \in (Imp)^N$.
2. $g_S \notin (Imp)^{M[H']}$.

PROOF. We know by Proposition 1.3 that in both N and $M[H']$, we have $Imp_{\omega_1^t} = L_{\omega_1^t}$.

For part (1), by Claim 4.9, if $n \in h$, then in N there is a unique self-coding real with base n , and so there is a unique $X \subseteq \omega_1^t$ such that $(L_{\omega_1^t}, \in, X) \models \varphi_{sc}(n)$, namely $C(g_{0,n})$. Hence, $C(g_{0,n}) \in Imp_{\omega_1^t+1}$. On the other hand, if $n \notin h$, then we will show that $C(g_{\alpha,n}) \notin Imp_{\omega_1^t+1}$, hence there is no $X \in Imp_{\omega_1^t+1}$ such that $(Imp_{\omega_1^t}, \in, X) \models \varphi_{sc}(n)$

To see this, suppose that $p \in \mathbb{P}$ and p forces $(L_{\omega_1^t}, \in, C(g_{\alpha,n})) \models \psi(a_1, \dots, a_k)$. Since \mathbb{P} is a product forcing, and $C(g_{\alpha,n})$ is defined from the $\mathbb{P}_{(\alpha,n)}$ generic, it must be the case that $p(\alpha, n)$, as a condition in $\mathbb{P}_{(\alpha,n)} = \mathbb{S}C_n$, forces $(L_{\omega_1^t}, \in, C(g)) \models \psi(a_1, \dots, a_k)$. We can extend p to p' such that, for some $(\beta, n) \notin dom(p)$, we have $p'(\beta, n) = p(\alpha, n)$. Thus, p' forces $(L_{\omega_1^t}, \in, C(g_{\beta,n})) \models \psi(a_1, \dots, a_k)$. This shows that $C(g_{\alpha,n})$ cannot be the unique X such that $(L_{\omega_1^t}, \in, X) \models \psi(a_1, \dots, a_k)$, and therefore $C(g_{(\alpha,n)}) \notin Imp_{\omega_1^t+1}$.

This shows that in N , the real h is definable over $Imp_{\omega_1^t+1}$, and therefore $h \in (Imp)^N$. It follows, since H' is constructible from h , that $M[H'] \subseteq (Imp)^N$.

To see the reverse inclusion, by Proposition 2.2 it suffices to note that N is obtained from $M[H']$ as a generic extension for an almost-homogeneous notion of forcing, namely

$$\overline{\mathbb{P}} = \{ p \upharpoonright \{(\alpha, n) \mid 0 < \alpha < \omega_1 \ \& \ n \notin h\} \mid p \in \mathbb{P} \}$$

(where \mathbb{P} is defined in M).

To see that $\overline{\mathbb{P}}$ is almost-homogeneous, let the restrictions $p = \langle p_s, \overline{p} \rangle$ and $q = \langle q_s, \overline{q} \rangle$ be any conditions in $\overline{\mathbb{P}}$, and let $\gamma < \omega_1$ be such that the supports of \overline{p} and \overline{q}

are contained in $(\gamma \times \omega)$. Then the permutation φ of $\omega_1 \times \omega$ defined by

$$\varphi(\beta, n) = \begin{cases} (\beta, n) & \text{if } \beta = 0 \text{ or } \beta \geq 2\gamma; \\ (\gamma + m - 1, n) & \text{if } \beta = m \text{ and } 0 < m < \omega; \\ (\gamma + \delta, n) & \text{if } \beta = \delta \text{ and } \omega \leq \delta < \gamma; \\ (m + 1, n) & \text{if } \beta = \gamma + m \text{ and } m < \omega; \\ (\delta, n) & \text{if } \beta = \gamma + \delta \text{ and } \omega \leq \delta < \gamma; \end{cases}$$

induces an automorphism $\overline{\varphi}$ of \mathbb{P} that fixes h , H' , and $\overline{\mathbb{P}}$. In $M[H']$, $\overline{\varphi}$ gives an automorphism of $\overline{\mathbb{P}}$ that sends $p \upharpoonright \{(\alpha, n) \mid 0 < \alpha < \omega_1 \ \& \ n \notin h\}$ to a condition compatible with $q \upharpoonright \{(\alpha, n) \mid 0 < \alpha < \omega_1 \ \& \ n \notin h\}$.

To see that N is a $\overline{\mathbb{P}}$ -generic extension of $M[H']$, note that $N = M[H'][(H'')]$ where

$$H'' = \langle g_{\alpha, n} \mid 0 < \alpha < \omega_1 \ \& \ n \notin h \rangle.$$

Suppose that D is a dense subset of $\overline{\mathbb{P}}$ in $M[H']$, and $p = \langle p_s, \underline{p} \rangle \in \mathbb{P}$. Choose $q = \langle q_s, \underline{q} \rangle \leq \langle p_s, \underline{p} \upharpoonright (\{0\} \times \omega) \rangle$ (so q is a condition for adding H') and $r \in \mathbb{P}$ such that q forces

$$\begin{aligned} \overline{r} \in D \text{ and } \overline{r} \leq p \upharpoonright \{(\alpha, n) \mid \\ 0 < \alpha < \omega_1 \ \& \ n \notin h\} \text{ where } \overline{r} = r \upharpoonright \{(\alpha, n) \mid 0 < \alpha < \omega_1 \ \& \ n \notin h\}. \end{aligned}$$

Note that if $\alpha > 0$ and $r(\alpha, n) \not\leq p(\alpha, n)$, then $q \Vdash n \in h$. Define $p' \leq p$ by $p' = \langle q_s, \underline{p}' \rangle$ where

$$\underline{p}'(\alpha, n) = \begin{cases} q(\alpha, n) & \text{if } \alpha = 0; \\ r(\alpha, n) & \text{if } \alpha > 0 \text{ and } r(\alpha, n) \leq p(\alpha, n); \\ p(\alpha, n) & \text{otherwise.} \end{cases}$$

Then p' forces the generic filter adding H'' to meet D .

For part (2), H' is generic over M for the product forcing $\mathbb{S} \times \prod_{n \in \omega} \mathbb{S}\mathbb{C}_n$. Since this is a product forcing, $M[H']$ is a generic extension of $M[\langle g_{0, n} \mid n \in \omega \rangle]$ by the almost-homogeneous forcing $(\mathbb{S})^M$, which adds the generic real $g_{\mathbb{S}}$. Therefore in $M[H']$, by Proposition 2.2, we have that $g_{\mathbb{S}} \notin \text{Imp}$. \dashv

This shows that in N we have $g_{\mathbb{S}} \in \text{Imp}$ and $g_{\mathbb{S}} \notin (\text{Imp})^{\text{Imp}}$, proving the following theorem.

THEOREM 4.11. *If ZF is consistent, so is ZFC + $((\text{Imp})^{\text{Imp}} \neq \text{Imp})$.*

§5. Proof of technical lemma. In this section we prove the technical lemma.

LEMMA 5.1. *Suppose $M \models V = L$, the poset $\mathbb{P} \in M$ is as described in Section 2, and G is \mathbb{P} -generic over M , where $G = \langle G_i \mid i \in I \rangle$. If x is a real in $M[G]$ and for all $i \in I$ we have $x \notin M[G_i]$, then the L -degree of x lies above at least two minimal (nonzero) L -degrees of reals.*

To establish notation and intuition, we begin by reviewing Sacks forcing.

Sacks forcing conditions are perfect trees, binary trees in which branching nodes are dense. A subtree is a stronger condition. The generic G is equivalent to the generic real g , the unique real that is a branch through all the trees in G .

DEFINITION 5.2. If $T \subseteq 2^{<\omega}$ is downward closed, and $\sigma \in T$, we say σ splits in T if $\sigma \smallfrown 0 \in T$ and $\sigma \smallfrown 1 \in T$. We may call σ a *splitting node* of T .

A *perfect tree* is a downward closed $T \subseteq 2^{<\omega}$ such that for every $\sigma \in T$ there is some $\tau \supseteq \sigma$ that splits in T .

A branch of T is $b \in 2^\omega$ such that, for all $n < \omega$, the restriction $b \upharpoonright n$ is an element of T . The set of all branches of T is denoted $[T]$.

Sacks forcing \mathbb{S} has as conditions perfect trees, ordered by $T' \leq T$ (T' is stronger than T) iff $T' \subseteq T$.

If $G \subseteq \mathbb{S}$ is \mathbb{S} -generic over M , then the \mathbb{S} -generic real g is defined in $M[G]$ by $g = \bigcap \{ [T] \mid T \in G \}$.

If $G \subseteq \mathbb{S}$ is \mathbb{S} -generic over M , then $G = \{ T \in (\mathbb{S})^M \mid g \in [T] \}$. Hence, $M[G] = M[g]$.

DEFINITION 5.3. Let T be a perfect tree. The root, or stem, of T is the shortest $\tau \in T$ that splits in T .

For $\sigma \in 2^n$, we define $rt_\sigma(T)$ by induction on n :

If $\sigma = \langle \rangle$ is the empty sequence, then $rt_\sigma(T)$ is the root of T .

Inductively, $rt_{\sigma \smallfrown i}(T)$ is the shortest $\tau \supseteq (rt_\sigma(T)) \smallfrown i$ that splits in T .

REMARK 5.4. The collection of splitting nodes $\{rt_\sigma(T) \mid \sigma \in 2^{<\omega}\}$ comprises an isomorphic copy of the complete binary tree $2^{<\omega}$ inside T .

Any branch b through T is determined by $\{\sigma \mid rt_\sigma(T) \subset b\}$, and any branch b' through 2^ω determines a branch through T , given by the downward closure of $\{rt_\sigma(T) \mid \sigma \subset b'\}$.

We use the $rt_\sigma(T)$ to construct fusion sequences (defined below), an essential tool for Sacks forcing arguments.

DEFINITION 5.5. For $T \in \mathbb{S}$ and $n < \omega$, the n^{th} splitting level of T is $S_n(T) = \{rt_\sigma(T) \mid \sigma \in 2^n\}$.

If $T' \leq T$, and $S_m(T') = S_m(T)$ for all $m < n$, we say $T' \leq_n T$.

A *fusion sequence* for \mathbb{S} is a decreasing (with respect to the partial ordering) sequence of conditions $\langle T_m \mid m \in \omega \rangle$ such that

$$(\forall n)(\exists k_n)(\forall m, m') (k_n \leq m \leq m' \implies T_{m'} \leq_n T_m).$$

The *fusion* of the sequence is $\bigcap \{ T_m \mid m < \omega \}$.

REMARK 5.6. The set $S_n(T)$ is a maximal antichain of nodes of T ; any branch through T extends exactly one element of $S_n(T)$.

$T' \leq_0 T$ iff $T' \leq T$.

The fusion T of a fusion sequence $\langle T_m \mid m \in \omega \rangle$ is a condition, $T = \bigwedge_{m < \omega} T_m$.

Furthermore, for all m we have $T \leq T_m$, and for all $m \geq k_n$ we have $T \leq_n T_m \leq_n T_{k_n}$.

Although \mathbb{S} is not countably closed, we can use closure under fusions of fusion sequences in place of countable closure to prove, for example, that \mathbb{S} does not collapse ω_1 , and that any function $f : \omega \rightarrow \omega$ in a generic extension by \mathbb{S} is dominated by a function in the ground model.

Fusion sequences are also used to prove the property for which Sacks forcing was developed: If G is Sacks generic over M , then every element of $M[G]$ is either in M or equivalent (equidefinable using parameters from M) to G . In particular, if $M \models V = L$, then the generic real g is of minimal (nonzero) degree of constructibility.

Typically, constructing a fusion sequence uses the notion of restriction:

DEFINITION 5.7. If $\tau \in T$, then T_τ , sometimes called the restriction of T to τ , is

$$\{\rho \in T \mid \rho \subseteq \tau \text{ or } \tau \subseteq \rho\}.$$

If $\sigma \in 2^{<\omega}$, then $T_{(\sigma)} = T_{rt_\sigma(T)}$.

REMARK 5.8. If $\sigma \in 2^n$, then $rt_\sigma(T) \in S_n(T)$ is the root of $T_{(\sigma)}$.

For each $n < \omega$, the collection $\{[T_{(\sigma)}] \mid \sigma \in 2^n\}$ is a partition of the branches of T . The collection $\{T_{(\sigma)} \mid \sigma \in 2^n\}$ is a maximal antichain of conditions extending T .

The condition T forces that exactly one element of $S_n(T) = \{rt_\sigma(T) \mid \sigma \in 2^n\}$ is an initial segment of g , and g is a branch through exactly one $T_{(\sigma)}$ for $\sigma \in 2^n$.

For all T, T' in \mathbb{S} ,

$$T' \leq_n T \iff (\forall \sigma \in 2^n) (T'_{(\sigma)} \leq T_{(\sigma)}).$$

It will be convenient to use this alternative characterization of \leq_n when we generalize the definition to products and iterations.

REMARK 5.9. Suppose $D \subseteq \mathbb{S}$ is an open dense set. Given T and n , we can extend every $T_{(\sigma)}$ for $\sigma \in 2^n$ to a condition $S(\sigma) \in D$, and put those extensions together to form $S \leq_n T$ such that $S_{(\sigma)} = S(\sigma) \in D$ for each $\sigma \in 2^n$. (Formally, $S = \bigcup_{\sigma \in 2^n} S(\sigma)$.) Thus, S forces the generic to contain one of finitely many elements $S_{(\sigma)}$ of D .

If D_n is an open dense set for each $n \in \omega$, we can begin with any condition T and build a fusion sequence, $\langle T_n \mid n < \omega \rangle$ with $T_0 = T$, $T_{n+1} \leq_n T_n$, and, for all $\sigma \in 2^n$, $(T_{n+1})_{(\sigma)} \in D_n$. The fusion S of this fusion sequence forces that, for each n , the generic contains one of finitely many elements $(T_{n+1})_{(\sigma)}$ of D_n .

If x is a term for a function from ω to M , and D_n is the set of conditions forcing a value for $x(n)$, then S forces each $x(n)$ to lie within a finite set of possible values. Hence, if $x : \omega \rightarrow \omega_1$, the range of x is contained in some countable set in M ; this shows \mathbb{S} does not collapse ω_1 . If $x : \omega \rightarrow \omega$, then S determines a ground model function $f : \omega \rightarrow \omega$ such that S forces x to be dominated by f ; $f(n)$ is an upper bound for the possible values of $x(n)$.

Suppose that x is a term for an element of 2^ω that is not in the ground model M . To show x is equivalent to the generic real g , construct a fusion sequence below any condition T , so that the fusion S provides a method for using x to determine $\{\sigma \in 2^{<\omega} \mid S_{(\sigma)} \in G\}$ (and hence, to determine g). To do this, when extending $R = T_n$ to $T_{n+1} \leq_n R$, instead of extending each individual $R_{(\sigma)}$ to lie in some dense set, extend each pair $R_{(\sigma)}$ and $R_{(\tau)}$ (for $\sigma \neq \tau$) to force contradictory facts about x (that is, for some k , one forces $x(k) = 0$ and the other forces $x(k) = 1$). Then the resulting \overline{R} will force that, for $\sigma \in 2^n$, we have $\overline{R}_{(\sigma)} \in G$ iff $\overline{R}_{(\sigma)}$ forces only correct facts about x ; since $S \leq_n T_{n+1} \leq_n \overline{R}$, we have $S_{(\sigma)} \in G$ iff $S_{(\sigma)}$ forces only correct facts about x . In this way x recovers $\{\sigma \mid rt_\sigma(S) \subset g\}$, and therefore x recovers g .

We can always \leq_n -extend R in this way: Since x is forced not to be in M , there must be some k such that $R_{(\sigma)}$ does not decide the value of $x(k)$. Then we can extend $R_{(\tau)}$ to decide the value of $x(k)$, and $R_{(\sigma)}$ to decide the opposite value. Repeating this for all pairs $\sigma \neq \tau$ from 2^n , we produce the desired T_{n+1} .

To extend the fusion technique to products and iterations of \mathbb{S} , we use coordinatewise definitions of restrictions, fusion sequences, and fusions.

In particular, suppose $p = \langle p(i) \mid i \in I \rangle$ is a condition. To define an analogue of $p_{(\sigma)}$, we break up σ into finitely many subsequences σ_k , for each k choose a coordinate i , and replace $p(i)$ with the restriction $(p(i))_{(\sigma_k)}$.

To organize this, we introduce notation for finite and infinite joins.

DEFINITION 5.10. For $x, y \in 2^\omega$, the *join* of x and y is $x \oplus y$, defined by

$$(x \oplus y)(n) = \begin{cases} x(k) & \text{if } n = 2k; \\ y(k) & \text{if } n = 2k + 1. \end{cases}$$

We make a similar definition for $\sigma \in 2^m$ and $\tau \in 2^m$ or $\tau \in 2^{m-1}$: $\sigma \oplus \tau$ has domain $2m$ in the first case, and $2m - 1$ in the second, and

$$(\sigma \oplus \tau)(n) = \begin{cases} \sigma(k) & \text{if } n = 2k; \\ \tau(k) & \text{if } n = 2k + 1. \end{cases}$$

If $z \in 2^{\leq \omega}$, we can view z as a join, and define its left and right parts: If $z = x \oplus y$, then $\ell(z) = x$ and $r(z) = y$.

DEFINITION 5.11. Let $[\cdot, \cdot] : \omega \times \omega \rightarrow \omega$ be a computable bijection, increasing in each coordinate, such that $[0, 0] = 0$, $[0, 1] = 1$, and otherwise $[m, n] > \max(m, n)$.

If σ is a sequence of length at most ω , for each $n < \omega$ define the sequence $c(\sigma, n)$ by $c(\sigma, n)(m) = \sigma([n, m])$. If $[n, m]$ is not in the domain of σ , then m is not in the domain of $c(\sigma, n)$.

The least n such that, for σ of length k and all $m \geq n$, the domain of $c(\sigma, m)$ is empty, is denoted $W(k)$.

If, for each n , x_n is a sequence of length ω , the join $\bigoplus_{n < \omega} x_n$ is the sequence x defined by $x([n, m]) = x_n(m)$.

The bijection $[\cdot, \cdot]$ allows us to view a (possibly partial) function σ on ω as a function on $\omega \times \omega$. If we view $\omega \times \omega$ as a two-dimensional grid, then $c(\sigma, n)$ is the restriction of σ to the n^{th} column of the grid.

If σ is a finite sequence, then $c(\sigma, n)$ is a finite sequence for all n , and for all but finitely many n we have $c(\sigma, n) = \langle \rangle$. If we view σ of length k as a partial function on the $\omega \times \omega$ grid, then $W(k)$ is the width of the domain of σ , that is, the number of columns having nonempty intersection with the domain of σ .

Taking the join of $\langle x_n \mid n < \omega \rangle$ is the reverse process, viewing each x_n as a function on the n^{th} column of the grid.

REMARK 5.12. If $\sigma = \bigoplus_{n < \omega} x_n$, then $c(\sigma, n) = x_n$.

For combinations of iterations and products of Sacks forcing over a model of $V = L$, we want to employ the method of fusion sequences to analyze the degrees of constructibility in the generic extension.

We define fusion sequences and fusions coordinatewise, with an inductive component to the definition in the case of iteration.

DEFINITION 5.13. 1. A decreasing sequence $\langle p_m \mid m < \omega \rangle$ in $\mathbb{S} \times \mathbb{S}$ is a fusion sequence if it is a fusion sequence in each coordinate; that is, if $p_m = (S_m, T_m)$, then both $\langle S_m \mid m < \omega \rangle$ and $\langle T_m \mid m < \omega \rangle$ are fusion sequences in \mathbb{S} .

Its fusion is the coordinatewise fusion, $\bigwedge_{m < \omega} p_m = \left(\bigcap_{m < \omega} S_m, \bigcap_{m < \omega} T_m \right)$.

2. A fusion sequence for an iteration \mathbb{Q} of length α , and its fusion, are defined coordinatewise. Inductively:

A decreasing sequence $\langle p_m \mid m < \omega \rangle$ is a fusion sequence if for all $\beta < \alpha$, the sequence $\langle p_m \upharpoonright \beta \mid m < \omega \rangle$ is a fusion sequence in $\langle \mathbb{Q}_\gamma \mid \gamma < \beta \rangle$, and its fusion forces $\langle p_m(\beta) \mid m < \omega \rangle$ to be a fusion sequence in \mathbb{Q}_β .

The fusion of the sequence is the condition $p = \bigwedge_{m < \omega} p_m$ such that, for all

$\beta < \alpha$, we have that $p \upharpoonright \beta = \left(\bigwedge_{m < \omega} p_m \upharpoonright \beta \right)$ and $p(\beta)$ denotes the fusion $\bigwedge_{m < \omega} (p_m(\beta))$.

3. A fusion sequence for \mathbb{P} , and its fusion, are defined coordinatewise:

A decreasing sequence $\langle p_m \mid m < \omega \rangle$ is a fusion sequence if for all i the sequence $\langle p_m(i) \mid m < \omega \rangle$ is a fusion sequence for \mathbb{P}_i .

Its fusion is defined by $\left(\bigwedge_{m < \omega} p_m \right)(i) = \bigwedge_{m < \omega} (p_m(i))$.

As with \mathbb{S} , the fusion, or infimum, of a fusion sequence is a condition.

For constructing fusion sequences in this setting, we want to generalize the definitions of $T_{(\sigma)}$ and \leq_n .

For countable products and iterations, we can decompose σ into subsequences $c(\sigma, n)$, and use a fixed enumeration $\{i(n) \mid n < \omega\}$ of the support of the product or iteration to make a coordinatewise definition of $p_{(\sigma)}$.

For an uncountable product or iteration, we define a notion $p_{(\sigma, \vec{s})}$, where \vec{s} identifies the coordinates to which we associate those subsequences $c(\sigma, n)$ that are nonempty.

DEFINITION 5.14. 1. Suppose $p = (T_0, T_1) \in \mathbb{S} \times \mathbb{S}$.

For $\sigma \in 2^{<\omega}$, we define $p_{(\sigma)} = ((T_0)_{(\ell(\sigma))}, (T_1)_{(r(\sigma))})$.

We define $p \leq_n q$ iff $(\forall \sigma \in 2^n) (p_{(\sigma)} \leq q_{(\sigma)})$. (As in Definition 5.10, $\ell(q_\sigma)$ and $r(\sigma)$ denote the left and right parts of σ .)

2. For \mathbb{Q} of countable length α , fix an enumeration $\{\beta_m \mid m < \omega\}$ of α , such that $\beta_0 = 0$.

For $p \in \mathbb{Q}$ and $\sigma \in 2^{<\omega}$ we define $p_{(\sigma)}$ coordinatewise: $p_{(\sigma)}(\beta_m)$ is a term for $(p(\beta_m))_{c(\sigma, m)}$.

We define $p \leq_n q$ iff $(\forall \sigma \in 2^n) (p_{(\sigma)} \leq q_{(\sigma)})$.

3. For $p \in \mathbb{P}$, $\sigma \in 2^n$, and $\vec{s} = \langle i(k) \mid k < \delta \rangle$, where $W(n) \leq \delta \leq \omega$, we define $p_{(\sigma, \vec{s})}$ by $p_{(\sigma, \vec{s})}(i(k)) = p(i(k))_{(c(\sigma, k))}$, and for $i \notin \{i(k) \mid k < \delta\}$, we set $p_{(\sigma, \vec{s})}(i) = p(i)$.

We define $p \leq_{n, \vec{s}} q$ iff $(\forall \sigma \in 2^n) (p_{(\sigma, \vec{s})} \leq q_{(\sigma, \vec{s})})$.

REMARK 5.15. For $\mathbb{S} \times \mathbb{S}$ and \mathbb{Q} , if $p \in G$, then the sequence $\langle \sigma \mid p_{(\sigma)} \in G \rangle$ is equivalent to the generic G .

For \mathbb{P} , if $\vec{s} = \langle i(k) \mid k < \omega \rangle$ and $p \in G$, the sequence $\langle \sigma \mid p_{(\sigma, \vec{s})} \in G \rangle$ is equivalent to the portion of the generic $\langle G_{i(k)} \mid k < \omega \rangle$.

This is also true coordinatewise: In $\mathbb{S} \times \mathbb{S}$, from $\{\ell(\sigma) \mid p_{(\sigma)} \in G\}$ and p , we can recover g_0 , and similarly for g_1 . In \mathbb{Q} , from $\{c(\sigma, k) \mid p_{(\sigma)} \in G\}$, $G \upharpoonright \beta_k$, and p , we can recover g_{β_k} . In \mathbb{P} , from $\{c(\sigma, k) \mid p_{(\sigma)} \in G\}$ and p , we can recover $g_{i(k)}$.

For $\mathbb{S} \times \mathbb{S}$ and \mathbb{Q} , if $\vec{p} = \langle p_m \mid m < \omega \rangle$ is a sequence of conditions such that

$$(\forall n)(\exists k_n)(\forall m, m') (k_n \leq m \leq m' \implies p'_m \leq_n p_m),$$

then \vec{p} is a fusion sequence. Furthermore, if p is its fusion, then for all $m \geq k_n$ we have $p \leq_n p_m$.

For \mathbb{P} , if $\vec{p} = \langle p_m \mid m < \omega \rangle$ is a sequence of conditions and $\vec{s} = \langle i(k) \mid k < \omega \rangle$ is an enumeration of $\bigcup \{supp(p_m) \mid m < \omega\}$ such that

$$(\forall n)(\exists k_n)(\forall m, m') (k_n \leq m \leq m' \implies p'_m \leq_{(n, \vec{s})} p_m),$$

then \vec{p} is a fusion sequence. Furthermore, if p is its fusion, then for all $m \geq k_n$ we have $p \leq_{(n, \vec{s})} p_m$.

Note, in this case, that $p \leq_{(n, \vec{s})} q$ is equivalent to $p \leq_{(n, \vec{s} \upharpoonright m)} q$, for any $m \geq W(n)$. We will use this in constructing fusion sequences, when we may need to find $p \leq_{(n, \vec{s})} q$ although only some initial segment of \vec{s} has been defined.

To produce $r \leq_n p$ such that the restrictions $r_{(\sigma)}$ (or $r_{(\sigma, \vec{s})}$) for $\sigma \in 2^n$ all have some given property, we wish, as in the case of Sacks forcing, to extend each $p_{(\sigma)}$ individually, and then put the results together to form r . However, we can no longer extend the $p_{(\sigma)}$ independently; extending $p_{(\sigma)}$ generally changes $p_{(\tau)}$ for $\tau \neq \sigma$. To facilitate extending the $p_{(\sigma)}$ sequentially, for $q \leq p_{(\sigma)}$ we define the amalgamation of q into p above σ , essentially the result of extending $p_{(\sigma)}$ and then plugging the extension q back into p .

The amalgamation of q into p above σ will be the maximal (weakest) $r \leq_n p$ such that $r_{(\sigma)} = q$.

DEFINITION 5.16. 0. For $\sigma \in 2^n$, $T \in \mathbb{S}$, and $S \leq T_{(\sigma)}$, the amalgamation of S into T above σ is $Am_{\sigma}(T, S) = S \cup \bigcup \{T_{(\tau)} \mid \tau \in 2^n \ \& \ \tau \neq \sigma\}$.

1. For $\sigma \in 2^n$, and $\mathbb{S} \times \mathbb{S}$ conditions $p = (T_0, T_1)$ and $q = (S_0, S_1) \leq p_{(\sigma)}$, we define $Am_{\sigma}(p, q)$, the amalgamation of q into p above level n , to be the coordinatewise amalgamation $(Am_{\ell(\sigma)}(T_0, S_0), Am_{r(\sigma)}(T_1, S_1))$.
2. For $\sigma \in 2^n$, $p \in \mathbb{Q}$, and $q \leq p_{(\sigma)}$, we define $Am_{\sigma}(p, q)$, the amalgamation of q into p above σ , inductively: Letting r denote $Am_{\sigma}(p, q)$,

$$r(\beta_m) = \begin{cases} Am_{c(\sigma, m)}(p(\beta_m), q(\beta_m)) & \text{if } (\forall \beta_k < \beta_m) (p(\beta_k)_{(c(\sigma, k))} \in G_{\beta_k}); \\ p(\beta_m) & \text{otherwise.} \end{cases}$$

3. For $p \in \mathbb{P}$, $\sigma \in 2^n$, $\vec{s} = \langle i(k) \mid k < \delta \rangle$ (with $\delta \geq W(n)$), and $q \leq p_{(\sigma, \vec{s})}$, the amalgamation of q into p above (σ, \vec{s}) is defined to be the coordinatewise

amalgamation: Letting r denote the amalgamation $Am_{(\sigma, \bar{s})}(p, q)$, we define $r(i(k)) = Am_{(c(\sigma, k))}(p(i(k)), q(i(k)))$, and for $i \notin \{i(k) \mid k < \delta\}$, we set $r(i) = q(i)$.

REMARK 5.17. To somewhat clarify clause (2) above, note that p is a condition for the iterated forcing $\mathbb{Q} = \langle \mathbb{Q}_\beta \mid \beta \leq \alpha \rangle$, so $p(\beta_m)$ is a name. If \mathbb{Q}_{β_m} is forced to be \mathbb{S} , then $p(\beta_m)$ is forced to be a perfect tree in $M[G \upharpoonright \beta_m]$. Then $r(\beta_m)$ is a name for a perfect tree in $M[G \upharpoonright \beta_m]$. It is constructed so that $r(\beta_m)$ is forced by $p \upharpoonright \beta_m$ to satisfy the given definition. In somewhat more detail, $\langle p(\beta_k)_{(c(\sigma, k))} \mid \beta_k < \beta_m \rangle$ forces $r(\beta_m)$ to equal $Am_{c(\sigma, m)}(p(\beta_m), q(\beta_m))$, and conditions in $\mathbb{Q} \upharpoonright \beta_m$ incompatible with $\langle p(\beta_k)_{(c(\sigma, k))} \mid \beta_k < \beta_m \rangle$ force $r(\beta_m)$ to equal $p(\beta_m)$.

REMARK 5.18. In the case of \mathbb{S} , for $\sigma \in 2^n$, and $R = Am_\sigma(T, S)$, we have $R_{(\sigma)} = S$, and for $\tau \in 2^n$ with $\tau \neq \sigma$, we have $R_{(\tau)} = T_{(\tau)}$.

In the case of $\mathbb{S} \times \mathbb{S}$, for $\sigma \in 2^n$, $q \leq p_{(\sigma)}$, and $r = Am_\sigma(p, q)$, we have $r_{(\sigma)} = q$. For $\tau \in 2^n$ with $\tau \neq \sigma$, we have $r_{(\tau)} \leq p_{(\tau)}$, but we do not in general have $r_{(\tau)} = p_{(\tau)}$. (However, if $\ell(\tau) \neq \ell(\sigma)$, we have equality in the first coordinate, and if $r(\tau) \neq r(\sigma)$, we have equality in the second coordinate.)

In the case of \mathbb{Q} as in clause (2) above, for $\sigma \in 2^n$, $q \leq p_{(\sigma)}$, and $r = Am_\sigma(p, q)$, we have $r_{(\sigma)} = q$. For $\tau \in 2^n$ with $\tau \neq \sigma$, we have $r_{(\tau)} \leq p_{(\tau)}$, but we do not in general have $r_{(\tau)} = p_{(\tau)}$. However, we do have the following (which will be used later): If $c(\sigma, 0) \neq c(\tau, 0)$, then $r_{(\tau)} = p_{(\tau)}$.

An illustrative case is the two-step iteration of Sacks forcing $\mathbb{Q} = \langle \mathbb{Q}_0, \mathbb{Q}_1 \rangle$, where \mathbb{Q}_0 and \mathbb{Q}_1 are both Sacks forcing. A condition in \mathbb{Q} is a pair $p = \langle T, T' \rangle$, where T is a perfect tree in M , and T' is a term for a perfect tree in $M[G_0]$. Suppose that $\sigma = \langle 0, 0 \rangle = \langle 0 \rangle \oplus \langle 0 \rangle$, and $q = \langle S, S' \rangle \leq p_{(\sigma)} = \langle T_{(\langle 0 \rangle)}, T'_{(\langle 0 \rangle)} \rangle$. (For purposes of illustration we are using the pairwise join, rather than the infinite join, to decompose σ .) If r is the amalgamation of q into p above σ , then

$$\begin{aligned} r_{(\langle 0 \rangle \oplus \langle 0 \rangle)} &= r_{(\sigma)} = \langle S, S' \rangle = q, \\ r_{(\langle 0 \rangle \oplus \langle 1 \rangle)} &= \langle S, T'_{(\langle 1 \rangle)} \rangle, \\ r_{(\langle 1 \rangle \oplus \langle 0 \rangle)} &= \langle T_{(\langle 1 \rangle)}, T'_{(\langle 0 \rangle)} \rangle = p_{(\langle 1 \rangle \oplus \langle 0 \rangle)}, \text{ and} \\ r_{(\langle 1 \rangle \oplus \langle 1 \rangle)} &= \langle T_{(\langle 1 \rangle)}, T'_{(\langle 1 \rangle)} \rangle = p_{(\langle 1 \rangle \oplus \langle 1 \rangle)}. \end{aligned}$$

This is precisely what is needed for $r \leq_2 p$ with $r_{(\sigma)} = q$ to be maximal (as weak as possible). Here $\ell(\tau)$ is playing the role of $c(\tau, 0)$, and where $\ell(\tau) = \langle 1 \rangle \neq \ell(\sigma)$, we have $r_{(\tau)} = p_{(\tau)}$, as claimed.

In the case of \mathbb{P} as in clause (3) above, for $q \leq p_{(\sigma, \bar{s})}$, and $r = Am_{(\sigma, \bar{s})}(p, q)$, we have $r_{(\sigma, \bar{s})} = q$. For $\tau \in 2^n$ with $\tau \neq \sigma$, we have $r_{(\tau, \bar{s})} \leq p_{(\tau, \bar{s})}$, but we do not in general have $r_{(\tau, \bar{s})} = p_{(\tau, \bar{s})}$. However, derived from the above, we do have the following (which will be used later):

Suppose $i = i(k)$ for some $k < \delta$, and for $\tau \in 2^n$ we set $b(\tau) = c(c(\tau, k), 0)$. (That is, $b(\tau)$ is the subsequence of τ that determines the initial path of the generic real for $(\mathbb{P}_i)_{0.}$) If $b(\tau) \neq b(\sigma)$, then $r_{(\tau, \bar{s})}(i) = p_{(\tau, \bar{s})}(i)$.

LEMMA 5.19. *Let $M \models V = L$.*

In M , let $\mathbb{P} = \prod_{i \in I} \mathbb{P}_i$ be a countable-support product where each \mathbb{P}_i is a countable-support iteration $\langle \mathbb{Q}_\beta \mid \beta < \alpha_i \rangle$ of countable length α_i , such that each \mathbb{Q}_β is either \mathbb{S} or $\mathbb{S} \times \mathbb{S}$ (which one may depend on the generic sequence below β). Let x be a \mathbb{P} -term for a function from ω to the ordinals.

Let $i \in I$, and $p \in \mathbb{P}$, such that $p \Vdash x \notin M[\langle G_j \mid j \neq i \rangle]$. Let $n < \omega$, and $\vec{s} = \langle i(k) \mid k < \delta \rangle$ be a finite sequence from I of length at least $W(n)$, with $i(0) = i$.

For $\tau \in 2^n$, set $b(\tau) = c(c(\tau, 0), 0)$. Suppose $\sigma, \tau \in 2^n$ and $b(\tau) \neq b(\sigma)$. Then there are a condition $r \leq_{n, \vec{s}} p$, a number $d < \omega$, and ordinals $\gamma \neq \gamma'$ such that $r_{(\sigma, \vec{s})} \Vdash x(d) = \gamma$ and $r_{(\tau, \vec{s})} \Vdash x(d) = \gamma'$.

PROOF. Suppose by way of contradiction that, for all $d < \omega$, all $q \leq p_{\sigma, \vec{s}}$, and all γ such that $q \Vdash x(d) = \gamma$, if $r = Am_{\sigma, \vec{s}}(p, q)$, we have $r_{(\tau, \vec{s})} \Vdash x(d) = \gamma$. By Remark 5.18, since $r_{(\tau, \vec{s})}(i) = p_{(\tau, \vec{s})}(i)$ (and since amalgamation is defined coordinatewise), the condition $r_{(\tau, \vec{s})}$ is determined by p and $q \upharpoonright \{j \mid j \neq i\}$. Specifically, $r_{(\tau, \vec{s})} = (Am_{(\sigma, \vec{s})}(p, \bar{q}))_{(\tau, \vec{s})}$, where $\bar{q}(i) = p_{(\tau, \vec{s})}(i)$ and $\bar{q}(j) = q(j)$ for $j \neq i$.

Then $p_{\sigma, \vec{s}}$ forces that $x(d) = \gamma$ if and only if there is $q \in G \upharpoonright \{j \mid j \neq i\}$ such that $(Am_{\sigma, \vec{s}}(p, \bar{q}))_{(\tau, \vec{s})} \Vdash x(d) = \gamma$, where $\bar{q}(i) = p_{(\tau, \vec{s})}(i)$ and $\bar{q}(j) = q(j)$ for $j \neq i$. But this contradicts $p \Vdash x \notin M[\langle G_j \mid j \neq i \rangle]$.

Hence, we can find $d < \omega$, γ , and $q \leq p_{(\sigma, \vec{s})}$, such that $q \Vdash x(d) = \gamma$, but $(Am_{\sigma, \vec{s}}(q, p))_{(\tau, \vec{s})} \not\Vdash x(d) = \gamma$. Choose $q' \leq (Am_{\sigma, \vec{s}}(q, p))_{(\tau, \vec{s})}$ such that $q' \Vdash x(d) = \gamma'$ for $\gamma' \neq \gamma$, and let $r = Am_{\tau, \vec{s}}(Am_{\sigma, \vec{s}}(p, q), q')$. Then $r_{(\tau, \vec{s})} = q' \Vdash x(d) = \gamma'$, and $r_{(\sigma, \vec{s})} \leq (Am_{\sigma, \vec{s}}(p, q))_{\sigma, \vec{s}} = q \Vdash x(d) = \gamma$, as desired. \dashv

PROPOSITION 5.20. Let $M \models V = L$.

In M , let $\mathbb{P} = \prod_{i \in I} \mathbb{P}_i$ be a countable-support product where each \mathbb{P}_i is a countable-support iteration $\langle \mathbb{Q}_\beta \mid \beta < \alpha_i \rangle$ of countable length α_i , such that each \mathbb{Q}_β is either \mathbb{S} or $\mathbb{S} \times \mathbb{S}$ (which one may depend on the generic sequence below β). Let x be a term for a function from ω to the ordinals.

Let $i \in I$, and $p \in \mathbb{P}$, such that $p \Vdash x \notin M[\langle G_j \mid j \neq i \rangle]$. Then $p \Vdash (G_i)_0 \leq_L x$.

PROOF. We define a fusion sequence $\langle p_m \mid m < \omega \rangle$ and an enumeration $\vec{s} = \langle i(k) \mid k < \omega \rangle$ of $\bigcup_{m < \omega} \text{supp}(p_m)$, such that from the fusion $\bigwedge p_m$, the sequence \vec{s} , and x , we can recover the generic real $(g_i)_0$ for $(\mathbb{Q}_i)_0$.

At step m we define p_m and $i(m)$, using a diagonalization strategy to insure the range of \vec{s} is $\bigcup_{m < \omega} \text{supp}(p_m)$.

We will guarantee we have a fusion sequence by making $p_{m+1} \leq_{m+1, \vec{s} \upharpoonright (m+1)} p_m$. Let $p_0 = p$, and $i(0) = i$.

Inductively, suppose p_m and $\vec{s}_m = \langle i(0), \dots, i(m) \rangle$ have been defined, with the following property: Say that conditions q and r are separated by x if there are a number $d < \omega$ and ordinals $\gamma \neq \gamma'$ such that $p \Vdash x(d) = \gamma$ and $q \Vdash x(d) = \gamma'$. For $\sigma \in 2^m$, as in Lemma 5.19, define $b(\sigma) = c(c(\sigma, 0), 0)$. Then, for all $\sigma, \tau \in 2^m$, if $b(\sigma) \neq b(\tau)$, then $(p_m)_{(\sigma, \vec{s}_m)}$ and $(p_m)_{(\tau, \vec{s}_m)}$ are separated by x .

Choose $i(m+1)$ according to our diagonalization strategy.

By Lemma 5.19, for any $\sigma, \tau \in 2^{m+1}$ with $b(\sigma) \neq b(\tau)$, there is $r \leq_{m+1, \vec{s} \upharpoonright (m+1)} p_m$ such that $r_{(\sigma, \vec{s}_{m+1})}$ and $r_{(\tau, \vec{s}_{m+1})}$ are separated by x . Therefore, by a finite iteration of choosing \leq_{m+1} extensions, we may choose $p_{m+1} \leq_{m+1, \vec{s} \upharpoonright (m+1)} p_m$ such that, for

all $\sigma, \tau \in 2^{m+1}$ with $b(\sigma) \neq b(\tau)$, we have that $(p_{m+1})_{(\sigma, \vec{s}_{m+1})}$ and $(p_{m+1})_{(\tau, \vec{s}_{m+1})}$ are separated by x .

Let q be the fusion $\bigwedge p_m$. Then, q forces x to recover the generic real $(g_i)_0$ for $(\mathbb{Q}_i)_0$ as follows: Let $\rho \in 2^n$, and m such that if $\tau \in 2^m$, then $b(\tau) \in 2^n$. Then p_m forces that $rt_\rho((p_m(i))(0)) \subset (g_i)_0$ iff there is $\tau \in 2^m$ with $b(\tau) = \rho$ such that $(p_m)_{(\tau, \vec{s}\upharpoonright m)}$ forces no incorrect facts about x ; that is, whenever $(p_m)_{(\tau, \vec{s}\upharpoonright m)} \Vdash x(d) = \gamma$, then in fact $x(d) = \gamma$. ⊣

PROPOSITION 5.21. *In M , let $\mathbb{P} = \prod_{i \in I} \mathbb{P}_i$ be a countable-support product where each \mathbb{P}_i is a countable-support iteration $\langle \mathbb{Q}_\beta \mid \beta < \alpha_i \rangle$ of countable length α_i , such that each \mathbb{Q}_β is either \mathbb{S} or $\mathbb{S} \times \mathbb{S}$ (which one may depend on the generic sequence below β). Let x be a term for a function from ω to ordinals.*

Then, in $M[G]$, one of:

1. $x \in M$;
2. $x \in M[G_i]$, where G_i is \mathbb{P}_i -generic for some $i \in I$; or
3. x lies above at least two minimal (nonzero) L -degrees of reals.

This proves the technical lemma.

REMARK 5.22. Unlike the previous proposition, this depends on the fact that the domain of x is countable.

PROOF. Let p be any condition, and extend p so one of

1. $p \Vdash (\forall i \in I) (x \in M[\langle G_j \mid j \neq i \rangle])$;
2. For some $k \in I$, we have $p \Vdash (x \notin M[\langle G_j \mid j \neq k \rangle])$, and $p \Vdash (\forall i \in I) (i \neq k \implies x \in M[\langle G_j \mid j \neq i \rangle])$;
3. For some $i_0, i_1 \in I$, we have $p \Vdash (x \notin M[\langle G_j \mid j \neq i_0 \rangle])$ and $p \Vdash (x \notin M[\langle G_j \mid j \neq i_1 \rangle])$.

In Case (3), by Proposition 5.20, p forces x to lie above the (nonzero) minimal L -degrees added by G_{i_0} and G_{i_1} . We show that in Case (1), we can extend p to force $x \in M$, and in Case (2), to force $x \in M[G_{i_0}]$.

In each case, we build a decreasing sequence of conditions $\langle p_n \mid n < \omega \rangle$, and simultaneously build an enumeration $\vec{s} = \langle i(n) \mid n < \omega \rangle$ of $\bigcup_{n < \omega} \text{supp}(p_n)$. For $q \in \mathbb{P}$, we let $q[-n]$ denote $q \upharpoonright \{i \mid i \notin \{i(0), i(1), \dots, i(n)\}\}$. We let $\mathbb{P}[-n]$ denote $\prod_{i \notin \{i(0), \dots, i(n)\}} \mathbb{P}_i$.

For Case (1), choose $i(0)$. Since $p \Vdash (x \in M[\langle G_j \mid j \neq i_0 \rangle])$, we can choose $p_0 \leq p$ and a term x_0 for forcing with $\mathbb{P}[-0]$ such that $p_0 \Vdash x = x_0$ and $p_0 \Vdash x_0(0) = \gamma_0$.

Since x_0 is a term for forcing with $\mathbb{P}[-0]$ and $p_0 \Vdash (x_0 \in M[\langle G_j \mid j \neq i(1) \rangle])$, we also have $p_0 \Vdash (x_0 \in M[\langle G_j \mid j \notin \{i(0), i(1)\} \rangle])$, and $p_0[-0] \Vdash (x_0 \in M[\langle G_j \mid j \notin \{i(0), i(1)\} \rangle])$. Therefore we can choose $p_1[-0] \leq p_0[-0]$ and a term x_1 for forcing with $\mathbb{P}[-1]$ such that $p_1[-0] \Vdash x_0 = x_1$ and $p_1[-0] \Vdash x_1(1) = \gamma_1$. Expand $p_1[-0]$ to $p_1 \leq p_0$ by setting $p_1(i_0) = p_0(i_0)$.

Then $p_1 \leq p_0 \leq p$, $p_1(i(0)) = p_0(i(0))$, and $p_1 \Vdash (x = x_1 \ \& \ x(0) = \gamma_0 \ \& \ x(1) = \gamma_1)$.

Inductively, assume that we have $p \geq p_0 \geq \dots \geq p_{n-1}$, x_0, x_1, \dots, x_{n-1} , and $\gamma_0, \gamma_1, \dots, \gamma_{n-1}$ such that for all $m' < m < n$:

1. $p_m(i(m')) = p_{m'}(i(m'))$;
2. x_m is a term for forcing with $\mathbb{P}[-m]$ and $p_m \Vdash x = x_m$;
3. $p_m \Vdash x(m) = \gamma_m$.

Then, as before, we can choose $p_n[-(n - 1)] \leq p_{n-1}[-(n - 1)]$ and a term x_n for forcing with $\mathbb{P}[-n]$ such that $p_n[-(n - 1)] \Vdash x_{n-1} = x_n$, and $p_n[-(n - 1)] \Vdash x_n(n) = \gamma_n$. Expand $p_n[-(n - 1)]$ to p_n by, for $m < n$, setting $p_n(i(m)) = p_m(i(m))$. This preserves the inductive hypothesis.

Finally, let q be the limit of the p_n : $q(i(n)) = p_n(i(n))$. Then, for all n , we have $q \Vdash x(n) = \gamma_n$. Therefore, $q \Vdash x \in M$.

For Case (2), we combine the construction of Case (1) with the construction of a fusion sequence for \mathbb{P}_k . During the course of the construction, we construct $\vec{s} = \langle i(n) \mid n < \omega \rangle$ enumerating $\left(\bigcup_{n < \omega} \text{supp}(p_n) \right) - \{k\}$. For $q \in \prod_{i \neq k} \mathbb{P}_i$ and $r \in \mathbb{P}_k$, we let $q \hat{\wedge} r$ denote the condition defined by setting $(q \hat{\wedge} r)(i) = q(i)$ for $i \neq k$ and $(q \hat{\wedge} r)(k) = r$.

Choose $p_0 \leq p$ such that $p_0 \Vdash x = x_{\langle \rangle}$, where $x_{\langle \rangle}$ is a term for forcing with $\mathbb{P}[-0]$, and $p_0 \Vdash x_{\langle \rangle}(0) = \gamma_{\langle \rangle}$.

Inductively, assume we have constructed a decreasing sequence of conditions $\langle p_m \mid m < n \rangle$, a collection of terms $\langle x_\sigma \mid \sigma \in 2^m, m < n \rangle$, and a collection of ordinals $\langle \gamma_\sigma \mid \sigma \in 2^m, m < n \rangle$ such that, for all $m' < m < n$,

1. $p_m(k) \leq_{m'} p_{m'}(k)$;
2. $p_m(i(m')) = p_{m'}(i(m'))$;
3. For $\sigma \in 2^m$, x_σ is a term for forcing with $\mathbb{P}[-m]$;
4. For $\sigma \in 2^m$, $((p_m(k))_{(\sigma)}) \hat{\wedge} (p_m \upharpoonright I - \{k, i(0), \dots, i(m - 1)\}) \Vdash x_\sigma(m) = \gamma_\sigma$;
5. For $\tau \in 2^{m-1}$, $((p_m(k))_{(\tau \hat{\wedge} \ell)}) \hat{\wedge} (p_m \upharpoonright I - \{k, i(0), \dots, i(m - 1)\}) \Vdash x_\tau = x_{\tau \hat{\wedge} \ell}$.

We can extend this to n as follows:

Let q_0 denote $p_{n-1}(k)$ and r_0 denote $p_{n-1} \upharpoonright (I - \{k, i(0), i(1), \dots, i(n - 1)\})$, and enumerate 2^n as $\{\sigma_j \mid j = 1, \dots, b\}$.

Inductively, for $j = 1, \dots, b$, if $\sigma_j = \tau \hat{\wedge} \ell$, choose a condition $(r \leq (q_{j-1})_{(\sigma_j)}) \hat{\wedge} r_{j-1}$, a term x_{σ_j} for forcing with $\mathbb{P}[-n]$, and an ordinal γ_{σ_j} such that $r \Vdash x_\tau = x_{\sigma_j}$ and $r \Vdash x_{\sigma_j}(n) = \gamma_{\sigma_j}$. Let $q_j = Am_{\sigma_j}(q_{j-1}, r(k))$ and $r_j = r \upharpoonright (I - \{k, i(0), i(1), \dots, i(n - 1)\})$.

Define p_n by $p_n(k) = q_k$, $p_n(i(j)) = p_{n-1}(i(j))$ for $j = 0, \dots, n - 1$, and $p_n \upharpoonright (I - \{k, i(0), \dots, i(n - 1)\}) = r_b$.

Now, define $q \leq p$ by setting $q(k) = \bigwedge_{n \in \omega} p_n(k)$ and $q(i(n)) = p_n(i(n))$. Then q forces that, for all n , we have $(x(n) = \gamma_\sigma) \iff ((q(k))_{(\sigma)} \in G_k)$; that is, q forces $x \in M[G_k]$. -1

The technical lemma proved here (Lemma 2.5) is a special case of the general analysis of degrees in generic extensions by forcing notions built from Sacks forcing. A reasonably general result, proved using the ideas in Case 2 of the proof of Proposition 5.21, is stated in the following proposition.

This proposition concerns generalized iterations as defined in Groszek and Jech [4]. For a well-founded partial ordering I , a generalized iteration $\langle \mathbb{P}_i \mid i \in I \rangle$ is defined in the natural way, so that \mathbb{P}_i is a term for a partial ordering in the generic

extension by $\langle \mathbb{P}_j \mid j < i \rangle$. Generalized iterations encompass products, iterations, and various combinations of products and iterations.

An ordering is ω_2 -like if every point has fewer than ω_2 -many predecessors. In the proposition below, using an ω_2 -like partial ordering I guarantees cardinal preservation, as shown in [4].

PROPOSITION 5.23. *Let $M \models V = L$. In M , let I be any well-founded, ω_2 -like partial ordering, and let \mathbb{P} be a countable-support generalized iteration along I such that every \mathbb{P}_i is forced to be either \mathbb{S} or the trivial forcing (which one may depend on the generic below i). Suppose G is \mathbb{P} -generic over M .*

If x is any real in $M[G]$, then $x \equiv_L \langle X_n \mid n < \omega \rangle$ where, for some $\{i(n) \mid n < \omega\} \subseteq I$ in M , and some $\{F(n) \mid n < \omega\}$ in M such that each $F(n)$ is a finite set of finite binary sequences,

$$X_n = \begin{cases} G_{i(n)} & \text{if } (\exists \sigma \in F(n)) (\sigma \subset \bigoplus_{m < n} X_m); \\ \emptyset & \text{otherwise.} \end{cases}$$

Furthermore, for all $i \in I$,

$$(G_i \leq_L x) \iff ((\exists n) (G_i \leq_L X_n)) \iff ((\exists n) (X_n = G_{i(n)} \ \& \ i \leq i(n))).$$

REMARK 5.24. The general idea of this proposition is that every real in the generic extension is equivalent to a join $\langle G_i \mid i \in J \rangle$ of countably many of the explicitly added generic reals. However, this is complicated by the fact that the set J may not be in the ground model. For example, suppose that \mathbb{P} is a product of countably many copies of Sacks forcing, adding a generic sequence $\langle G_n \mid n < \omega \rangle$, and x is $\langle G_{i+1} \mid G_0(i) = 0 \rangle$. Then, in the notation of this proposition, x is equivalent to $\langle X_n \mid n < \omega \rangle$ where X_0 is G_0 , and X_{i+1} is G_{i+1} if $G_0(i) = 0$, and \emptyset otherwise. The sequence $\{F(n) \mid n < \omega\}$ captures the way in which the value of X_0 at i determines X_{i+1} .

The fusion argument to prove the proposition begins with a variant of Lemma 5.19. Intuitively, this variant assumes that x is forced not to lie in $M[\langle G_j \mid j \leq i \rangle]$, and says that p can be $\leq_{n,\vec{s}}$ -extended to r so that if the stems of $r_{(\sigma,\vec{s})}(i)$ and $r_{(\tau,\vec{s})}(i)$ are different, then $r_{(\sigma,\vec{s})}(i)$ and $r_{(\tau,\vec{s})}(i)$ force different things about x . By choosing each $r_{(\sigma,\vec{s})}$ to also determine the stem of $r_{(\sigma,\vec{s})}(i)$, this allows x to compute which stem is associated with $r_{(\sigma,\vec{s})}(i)$ for $r \in G$, and thus ultimately to recover G_i . In this case, if i is the n^{th} element of the support of the eventual fusion condition, we have $X_n = G_i$. If, on the other hand, x is forced to lie in $M[\langle G_j \mid j \leq i \rangle]$, we have $X_n = \emptyset$.

The picture is complicated because we cannot initially force whether $x \in M[\langle G_j \mid j \leq i \rangle]$ for all i in the support; instead, we consider a new i at each stage of the fusion construction. When we are committed to $\leq_{n,\vec{s}}$ -extending our current condition p to a new r , we may not be able to decide whether $x \in M[\langle G_j \mid j \leq i \rangle]$; the best we can do is to say that $r_{(\sigma,\vec{s})}$ decides whether $x \in M[\langle G_j \mid j \leq i \rangle]$ for each σ of appropriate length. We can determine σ by knowing the stems of $r(j)$ for the j already considered. These G_j will appear earlier in the X sequence than the newly considered G_i ; the F sequence will capture the way in which this initial part of the X -sequence determines the stems of the conditions and therefore whether G_j or \emptyset appears next in the sequence.

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