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The homotopy decomposition of the suspension of a non-simply-connected $five$-manifold

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In this paper we determine the homotopy types of the reduced suspension space of certain connected orientable closed smooth $five$-manifolds. As applications, we compute the reduced K-groups of M and show that the suspension map between the third cohomotopy set $\pi^3(M)$ and the fourth cohomotopy set $\pi^4(\Sigma M)$ is a bijection.

Keywords: homotopy type; suspension; five-manifolds; cohomotopy sets

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1. Introduction

One of the goals of algebraic topology of manifolds is to determine the homotopy type of the (reduced) suspension space ΣM of a given manifold M. This problem has attracted a lot of attention since So and Theriault's work [**21**], which showed how the homotopy decompositions of the (double) suspension spaces of manifolds can be used to characterize some important invariants in geometry and mathematical physics, such as reduced K-groups and gauge groups. Several works have followed this direction, such as [**7**, **9**–**12**, **15**]. The integral homology groups $H_*(M)$ serve as the fundamental input for this topic. As shown by these papers, the 2-torsion of $H_*(M)$ and potential obstructions from certain Whitehead products usually prevent a complete homotopy classification of the (double) suspension space of a given manifold M.

The main purpose of this paper is to investigate the homotopy types of the suspension of a non-simply-connected orientable closed smooth $five$-manifold. Notice that Huang [**9**] studied the suspension homotopy of $five$-manifolds M that are S^1-principal bundles over a simply-connected oriented closed $four$-manifold. The homotopy decompositions of $\Sigma^2 M$ are successfully applied to determine the homotopy types of the pointed looped spaces of the gauge groups of a principal bundle

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over M. In this paper we greatly loosen the restriction on the homology groups $H_*(M)$ of the non-simply-connected $five$-manifold M by assuming that $H_1(M)$ has a torsion subgroup that is not divided by 6 and $H_2(M)$ contains a general torsion part.

To state our main results, we need the following notion and notations. Let $n \geqslant 2$. Denote by $\eta = \eta_n = \Sigma^{n-2}\eta$ the iterated suspension of the first Hopf map $\eta\colon S^3 \to S^2$. Recall from (cf. [25]) that $\pi_3(S^2) \cong \mathbb{Z}\langle\eta\rangle$, $\pi_{n+1}(S^n) \cong \mathbb{Z}/2\langle\eta\rangle$ for $n \geqslant 3$ and $\pi_{n+2}(S^n) \cong \mathbb{Z}/2\langle\eta^2\rangle$. For an abelian group G, denote by $P^{n+1}(G)$ the Peterson space characterized by having a unique reduced cohomology group G in dimension $n+1$; in particular, denote by $P^{n+1}(k) = P^{n+1}(\mathbb{Z}/k)$ the mod k Moore space of dimension $n+1$, where $\mathbb{Z}/k$ is the group of integers modulo k, $k \geqslant 2$. There is a canonical homotopy cofibration

$$S^n \xrightarrow{k} S^n \xrightarrow{i_n} P^{n+1}(k) \xrightarrow{q_{n+1}} S^{n+1},$$

where i_n is the inclusion of the bottom cell and q_{n+1} is the pinch map to the top cell. Recall that for each prime p and integer $r \geqslant 1$, there are higher order Bockstein operations β_r that detect the degree 2^r map on spheres S^n. For each $r \geqslant 1$, there are canonical maps $\tilde{\eta}_r\colon S^{n+2} \to P^{n+2}(2^r)$ satisfying the relation $q_{n+1}\tilde{\eta}_r = \eta$, see lemma 2.2. A finite CW-complex X is called an $\mathbf{A}_n^2$*-complex* if it is $(n-1)$-connected and has dimension at most $n+2$. In 1950, Chang [4] proved that for $n \geqslant 3$, every $\mathbf{A}_n^2$-complex X is homotopy equivalent to a wedge sum of finitely many spheres and mod p^r Moore spaces with p any primes and the following four *elementary (or indecomposable) Chang complexes*:

$$C_\eta^{n+2} = S^n \cup_\eta \boldsymbol{C}S^{n+1} = \Sigma^{n-2}\mathbb{C}P^2, \quad C_r^{n+2} = P^{n+1}(2^r) \cup_{i_n\eta} \boldsymbol{C}S^{n+1},$$
$$C^{n+2,s} = S^n \cup_{\eta q_{n+1}} \boldsymbol{C}P^{n+1}(2^s), \quad C_r^{n+2,s} = P^{n+1}(2^r) \cup_{i_n\eta q_{n+1}} \boldsymbol{C}P^{n+1}(2^s),$$

where $\boldsymbol{C}X$ denotes the reduced cone on X and r, s are positive integers. We recommend [14, 26–29] for recent work on the homotopy theory of Chang complexes.

Now it is prepared to state our main result. Let M be an orientable closed $five$-manifold whose integral homology groups are given by

i	1	2	3	4	0, 5	$\geqslant 6$
$H_i(M)$	$\mathbb{Z}^l \oplus H$	$\mathbb{Z}^d \oplus T$	$\mathbb{Z}^d \oplus H$	$\mathbb{Z}^l$	$\mathbb{Z}$	0

, (1.1)

where l, d are positive integers and H, T are finitely generated torsion abelian groups.

Theorem 1.1. *Let M be an orientable smooth closed $five$-manifold with $H_*(M)$ given by* (1.1). *Let $T_2 \cong \bigoplus_{j=1}^{t_2} \mathbb{Z}/2^{r_j}$ be the 2-primary component of T and suppose that H contains no* 2- *or* 3*-torsion. There exist integers c_1, c_2 that depend on M and satisfy*

$$0 \leqslant c_1 \leqslant \min\{l, d\}, \quad 0 \leqslant c_2 \leqslant \min\{l - c_1, t_2\}$$

and $c_1 = c_2 = 0$ if and only if the Steenrod square Sq^2 acts trivially on $H^2(M;\mathbb{Z}/2)$. Denote $T[c_2] = T/\oplus_{j=1}^{c_2}\mathbb{Z}/2^{r_j}$.

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(1) *Suppose M is spin, then there is a homotopy equivalence*

$$\Sigma M \simeq \left(\bigvee_{i=1}^{l} S^2\right) \vee \left(\bigvee_{i=1}^{d-c_1} S^3\right) \vee \left(\bigvee_{i=1}^{d} S^4\right) \vee \left(\bigvee_{i=1}^{l-c_1-c_2} S^5\right) \vee P^3(H) \vee P^5(H)$$
$$\vee \left(\bigvee_{i=1}^{c_1} C^5_\eta\right) \vee P^4(T[c_2]) \vee \left(\bigvee_{j=1}^{c_2} C^5_{r_j}\right) \vee S^6.$$

(2) *Suppose M is non-spin, then there are three possibilities for the homotopy types of ΣM.*

(a) *If for any $u, v \in H^4(\Sigma M; \mathbb{Z}/2)$ satisfying $\mathrm{Sq}^2(u) \neq 0$ and $\mathrm{Sq}^2(v) = 0$, there holds $u + v \notin \mathrm{im}(\beta_r)$ for any $r \geqslant 1$, then there is a homotopy equivalence*

$$\Sigma M \simeq \left(\bigvee_{i=1}^{l} S^2\right) \vee \left(\bigvee_{i=1}^{d-c_1} S^3\right) \vee \left(\bigvee_{i=2}^{d} S^4\right)$$
$$\vee \left(\bigvee_{i=1}^{l-c_1-c_2} S^5\right) \vee P^3(H) \vee P^5(H)$$
$$\vee \left(\bigvee_{i=1}^{c_1} C^5_\eta\right) \vee P^4(T[c_2]) \vee \left(\bigvee_{j=1}^{c_2} C^5_{r_j}\right) \vee C^6_\eta;$$

(b) *otherwise either there is a homotopy equivalence*

$$\Sigma M \simeq \left(\bigvee_{i=1}^{l} S^2\right) \vee \left(\bigvee_{i=1}^{d-c_1} S^3\right) \vee \left(\bigvee_{i=1}^{d} S^4\right)$$
$$\vee \left(\bigvee_{i=1}^{l-c_1-c_2} S^5\right) \vee P^3(H) \vee P^5(H)$$
$$\vee \left(\bigvee_{i=1}^{c_1} C^5_\eta\right) \vee \left(\bigvee_{j=1}^{c_2} C^5_{r_j}\right) \vee P^4\left(\frac{T[c_2]}{\mathbb{Z}/2^{r_{j_1}}}\right) \vee (P^4(2^{r_{j_1}}) \cup_{\tilde{\eta}_{r_{j_1}}} \mathrm{e}^6),$$

or there is a homotopy equivalence

$$\Sigma M \simeq \left(\bigvee_{i=1}^{l} S^2\right) \vee \left(\bigvee_{i=1}^{d-c_1} S^3\right) \vee \left(\bigvee_{i=1}^{d} S^4\right)$$
$$\vee \left(\bigvee_{i=1}^{l-c_1-c_2} S^5\right) \vee P^3(H) \vee P^5(H)$$
$$\vee \left(\bigvee_{i=1}^{c_1} C^5_\eta\right) \vee P^4(T[c_2]) \vee \left(\bigvee_{j_1 \neq j=1}^{c_2} C^5_{r_j}\right) \vee (C^5_{r_{j_1}} \cup_{i_P \tilde{\eta}_{r_{j_1}}} \mathrm{e}^6),$$

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where $i_P\colon P^5(2^{r_{j_1}})\to C^6_{r_{j_1}}$ *is the canonical inclusion map; in both cases,* r_{j_1} *is the minimum of* r_j *such that* $u+v\in \operatorname{im}(\beta_{r_{j_1}})$.

In Theorem 1.1 we characterize the homotopy types of ΣM by elementary complexes of dimension at most six, up to certain indeterminate $\mathbf{A}_n^2$-complexes. Note that wedge summands of the form $\bigvee_{i=u}^{v} X$ with $v<u$ are contractible and can be removed from the homotopy decompositions of ΣM. More generally, if M is a 5-dimensional Poincaré duality complex (i.e., a finite CW-complex whose integral cohomology satisfies the Poincaré duality theorem) satisfying the conditions in Theorem 1.1, then Theorem 1.1 gives the homotopy types of ΣM, except that there are two additional possibilities when the Steenrod square acts trivially on $H^3(M;\mathbb{Z}/2)$, See remark 4.5.

Due to lemma 2.3 (2), the 3-torsion of H can be well understood when studying the homotopy types of the double suspension $\Sigma^2 M$.

Theorem 1.2. *Let* M *be an orientable smooth closed five-manifold with* $H_*(M)$ *given by* (1.1)*, where* H *is a* 2*-torsion free group. Then the suspensions of the homotopy equivalences in Theorem 1.1 give the homotopy types of the double suspension* $\Sigma^2 M$.

In addition to the characterization of the homotopy types of iterated loop spaces of the gauge groups of principal bundles over M, as shown by Huang [**9**], we apply the homotopy types of ΣM (or $\Sigma^2 M$) to study the reduced K-groups and the cohomotopy sets $\pi^k(M)=[M,\,S^k]$ of the non-simply-connected manifold M.

Corollary 1.3 (See proposition 5.2). *Let* M *be a five-manifold given by Theorems 1.1 or 1.2. Then the reduced complex* K*-group and* KO*-group of* M *are given by*

$$\widetilde{K}(M)\cong \mathbb{Z}^{d+l}\oplus H\oplus H,\quad \widetilde{KO}(M)\cong \mathbb{Z}^{l}\oplus(\mathbb{Z}/2)^{l+d+t_2}.$$

The third cohomotopy set $\pi^3(M)$ possess the following property.

Corollary 1.4 (See proposition 5.6). *Let* M *be a five-manifold given by Theorems 1.1 or 1.2. Then the suspension* $\Sigma\colon \pi^3(M)\to\pi^4(\Sigma M)$ *is a bijection.*

We also apply the homotopy decompositions of ΣM to compute the group structure of $\pi^3(M)\cong\pi^4(\Sigma M)$, see proposition 5.6. The second cohomotopy set $\pi^2(M)$ always admits an action of $\pi^3(M)$ induced by the Hopf map $\eta\colon S^3\to S^2$, see lemma 5.3 or [**13**, Theorem 3]. Finally, it should be noting that when M is a 5-dimensional Poincaré duality complex with $H_1(M)$ torsion free, similar results have been proved independently and concurrently by Amelotte, Cutler and So [**1**].

This paper is organized as follows. Section 2 reviews some homotopy theory of $\mathbf{A}_n^2$-complexes and introduces the basic analysis methods to study the homotopy type of homotopy cofibres. In § 3 we study the homotopy types of the suspension of the CW-complex $\overline{M}$ of M with its top cell removed. The basic method is the homology decomposition of simply-connected spaces. Section 4 analyzes the homotopy types of ΣM and contains the proofs of Theorems 1.1 and 1.2. As applications of the homotopy decomposition of ΣM or $\Sigma^2 M$, we study the reduced K-groups and the cohomotopy sets of the five-manifolds M in § 5.

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2. Preliminaries

Throughout the paper we shall use the following global conventions and notations. All spaces are based CW-complexes, all maps are base-point-preserving and are identified with their homotopy classes in notation. A strict equality is often treated as a homotopy equality. Denote by $\mathbb{1}_X$ the identity map of a space X and simplify $\mathbb{1}_n = \mathbb{1}_{S^n}$. For different X, we use the ambiguous notations $i_k\colon S^k \to X$ and $q_k\colon X \to S^k$ to denote the possible canonical inclusion and pinch maps, respectively. For instance, there are inclusions $i_n\colon S^n \to C$ for each elementary Chang complex C and there are inclusions $i_{n+1}\colon S^{n+1} \to X$ for $X = C^{n+2,s}$ and $C^{n+2,s}_r$. Let $i_P\colon P^{n+1}(2^r) \to C^{n+2}_r$ and $i_\eta\colon C^{n+2}_\eta \to C^{n+2}_r$ be the canonical inclusions. Denote by C_f the homotopy cofibre of a map $f\colon X \to Y$. For an abelian group G generated by $x_1, \cdots, x_n$, denote $G \cong C_1\langle x_1\rangle \oplus \cdots \oplus C_n\langle x_n\rangle$ if x_i is a generator of the cyclic direct summand C_i, $i = 1, \cdots, n$.

2.1. Some homotopy theory of $\mathbf{A}^2_n$-complexes

For each prime p and integers $r,\ s \geqslant 1$, $n \geqslant 2$, there exists a map (with n omitted in notation)

$$B(\chi^r_s)\colon P^{n+1}(p^r) \to P^{n+1}(p^s)$$

satisfies $\Sigma B(\chi^r_s) = B(\chi^r_s)$ and the relation formulas (cf. [3]):

$$B(\chi^r_s) i_n = \chi^r_s \cdot i_n, \quad q_{n+1} B(\chi^r_s) = \chi^s_r \cdot q_{n+1}, \tag{2.1}$$

where χ^r_s is a self-map of spheres, $\chi^r_s = 1$ for $r \geqslant s$ and $\chi^r_s = p^{s-r}$ for $r < s$.

Lemma 2.1. *Let p be an odd prime and let $n \geqslant 3$, $r,\ s \geqslant 1$ be integers, $m = \min\{r,\ s\}$. There hold isomorphisms:*

(1) $\pi_3(P^3(p^r)) \cong \mathbb{Z}/p^r\langle i_2\eta\rangle$ *and* $\pi_{n+1}(P^{n+i}(p^r)) = 0$, $i = 0,\ 1$.

(2) $[P^n(p^r),\ P^n(p^s)] \cong \begin{cases} \mathbb{Z}/p^m\langle B(\chi^r_s)\rangle \oplus \mathbb{Z}/p^m\langle i_2\eta q_3\rangle, & n = 3; \\ \mathbb{Z}/p^m\langle B(\chi^r_s)\rangle, & n \geqslant 4. \end{cases}$

(3) $[P^{n+1}(p^r),\ P^n(p^s)] \cong \begin{cases} \mathbb{Z}/p^m\langle \hat{\eta}_s B(\chi^r_s)\rangle, & n = 3; \\ 0 & n \geqslant 4. \end{cases}$ *where* $\hat{\eta}_s\colon P^4(p^s) \to P^3(p^s)$ *satisfies* $\hat{\eta}_s i_3 = i_2\eta$.

Proof. The group $\pi_3(P^3(p^r))$ refers to [21, Lemma 2.1] and the groups $\pi_{n+1}(P^{n+i}) = 0$ was proved in [11, Lemma 6.3 and 6.4]. The groups and generators in (2) and (3) can be easily computed by applying the exact functor $[-,\ P^n(p^s)]$ to the canonical cofibrations for $P^{n+i}(p^r)$ with $i = 0,\ 1$, respectively; the details are omitted here. □

Lemma 2.2 (cf. [3]). *Let $n \geqslant 3$, $r \geqslant 1$ be integers.*

(1) $\pi_{n+1}(P^{n+1}(2^r)) \cong \mathbb{Z}/2\langle i_n\eta\rangle$.

(2) $\pi_{n+2}(P^{n+1}(2^r)) \cong \begin{cases} \mathbb{Z}/4\langle \tilde{\eta}_1\rangle, & r = 1; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2\langle \tilde{\eta}_r,\ i_n\eta^2\rangle, & r \geqslant 2. \end{cases}$

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The generator $\tilde{\eta}_r$ satisfies formulas

$$q_{n+1}\tilde{\eta}_r = \eta, \quad 2\tilde{\eta}_1 = i_n\eta^2, \quad B(\chi^r_s)\tilde{\eta}_r = \chi^s_r \cdot \tilde{\eta}_s. \tag{2.2}$$

(3) $[P^{n+1}(2^r),\, P^{n+1}(2^s)] \cong \begin{cases} \mathbb{Z}/4\langle \mathbb{1}_P\rangle, & r = s = 1; \\ \mathbb{Z}/2^m\langle B(\chi^r_s)\rangle \oplus \mathbb{Z}/2\langle i\eta q\rangle, & \textit{otherwise}, \end{cases}$
where $m = \min\{r, s\}$, $i\eta q = i_n\eta q_{n+1}$.

Lemma 2.3. *The following hold:*

(1) $\pi_5(P^3(3^r)) \cong \mathbb{Z}/3^{r+1}$, $\pi_5(P^3(p^r)) = 0$ *for primes* $p \geqslant 5$.

(2) *The suspension* $\Sigma\colon \pi_5(P^3(3^r)) \to \pi_6(P^4(3^r))$ *is trivial.*

Proof. (1) Let $F^3\{p^r\}$ be the homotopy fibre of $q_3\colon P^3(p^r) \to S^3$ and consider the induced exact sequence of p-local groups:

$$\pi_6(S^3; p) \to \pi_5(F^3\{p^r\}) \xrightarrow{(j_r)_\sharp} \pi_5(P^3(p^r)) \xrightarrow{(q_3)_\sharp} \pi_5(S^3; p) = 0.$$

By [18, Proposition 14.2] or [19, Theorem 3.1], there is a homotopy equivalence

$$\Omega F^3\{p^r\} \simeq S^1 \times \prod_{j=1}^{\infty} S^{2p^j-1}\{p^{r+1}\} \times \Omega\left(\bigvee_{\alpha} P^{n_\alpha}(p^r)\right),$$

where $S^{2n+1}\{p^r\}$ is the homotopy fibre of the mod p^r degree map on S^{2n+1}, $n_\alpha \geqslant 4$ and the equality holds for exactly one α. It follows that

$$\pi_5(F^3\{p^r\}) \cong \pi_4(S^{2p-1}\{p^{r+1}\}) \cong \begin{cases} \mathbb{Z}/3^{r+1}, & p = 3; \\ 0, & p \geqslant 5. \end{cases}$$

Thus $\pi_5(P^3(p^r)) = 0$ for $p \geqslant 5$. By [19, Theorem 2.10], $\pi_5(P^3(3^r))$ contains a direct summand $\mathbb{Z}/3^{r+1}$, therefore we have an isomorphism

$$(j_r)_\sharp\colon \pi_5(F^3\{3^r\}) \xrightarrow{\cong} \pi_5(P^3(3^r)) \cong \mathbb{Z}/3^{r+1}.$$

(2) Firstly, by [6] for any prime $p \geqslant 5$ and [19] for $p = 3$, there is a homotopy equivalence

$$\Omega P^4(p^r) \simeq S^3\{p^r\} \times \Omega\left(\bigvee_{k=0}^{\infty} P^{7+2k}(p^r)\right).$$

Second, for skeletal reasons, the suspension $E\colon P^3(p^r) \to \Omega P^4(p^r)$ factors as the composite $P^3(p^r) \xrightarrow{i} S^3\{p^r\} \xrightarrow{j} \Omega P^4(p^r)$, where i is the inclusion of the bottom

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Moore space and j is the inclusion of a factor. Third, there is a homotopy fibration diagram

$$\begin{array}{ccccc} E^3\{p^r\} & \longrightarrow & F^3\{p^r\} & \longrightarrow & \Omega S^3 \\ \| & & \downarrow & & \downarrow \\ E^3\{p^r\} & \longrightarrow & P^3(p^r) & \xrightarrow{i} & S^3\{p^r\} \\ & & \downarrow{\scriptstyle q_3} & & \downarrow \\ & & S^3 & = & S^3 \end{array}$$

that defines the space $E^3\{p^r\}$. By [**5**], for any prime $p \geqslant 5$ and [**19**] for $p = 3$, there is a homotopy equivalence

$$\Omega E^3\{p^r\} \simeq W_n \times \prod_{j=1}^{\infty} S^{2p^j-1}\{p^{r+1}\} \times \Omega\left(\bigvee_{\alpha} P^{n_\alpha}(p^r)\right),$$

where W_n is the homotopy fibre of the double suspension. This decomposition has the property that the factor $\prod_{j=1}^{\infty} S^{2p^j-1}\{p^{r+1}\}$ of $\Omega F^3\{p^r\}$ may be chosen to factor through $\Omega E^3\{p^r\}$.

Consequently, when $p = 3$, as the $\mathbb{Z}/3^{r+1}$ factor in $\pi_4(\Omega P^3(p^r))$ came from $\pi_4(\prod_{j=1}^{\infty} S^{2p^j-1}\{p^{r+1}\})$, it has the property that it composes trivially with the map $\Omega i\colon \Omega P^3(3^r) \to \Omega S^3\{3^r\}$. Hence, as ΩE factors through Ωi, the $\mathbb{Z}/3^{r+1}$ factor in $\pi_4(\Omega P^3(p^r))$ composes trivially with ΩE. Thus the $\mathbb{Z}/3^{r+1}$ factor in $\pi_5(P^3(p^r))$ suspends trivially. □

LEMMA 2.4 (cf. [**14**]). *Let $n \geqslant 3$ and $r \geqslant 1$. There hold isomorphisms*

(1) $\pi_{n+2}(C_\eta^{n+2}) \cong \mathbb{Z}\langle\tilde{\zeta}\rangle$, *where $\tilde{\zeta}$ satisfies $q_{n+2}\tilde{\zeta} = 2 \cdot \mathbb{1}_{n+2}$.*

(2) $\pi_{n+2}(C_r^{n+2}) \cong \mathbb{Z}\langle i_\eta\tilde{\zeta}\rangle \oplus \mathbb{Z}/2\langle i_P\tilde{\eta}_r\rangle$.

It follows that a map $f_C\colon S^{n+2} \to C$ with $C = C_\eta^{n+2}$ or C_r^{n+2} induces the trivial homomorphism in integral homology if and only if

$$f_C = \begin{cases} 0 & \text{for } C = C_\eta^{n+2}; \\ 0 \quad \text{or } i_P\tilde{\eta}_r & \text{for } C = C_r^{n+2}, \end{cases}$$

where $f = 0$ means f is null-homotopic.

The following Lemma can be found in [**14**, Theorem 3.1, (2)]; since it hasn't been published yet, we give a proof here.

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LEMMA 2.5. *For integers $n \geqslant 3$ and $r \geqslant 1$, there exists a map*

$$\bar{\xi}_r\colon C_r^{n+2} \to P^{n+1}(2^{r+1})$$

satisfying the homotopy commutative diagram of homotopy cofibrations

$$\begin{array}{ccccccc} S^n & \xrightarrow{i_n 2^r} & C_\eta^{n+2} & \xrightarrow{i_\eta} & C_r^{n+2} & \xrightarrow{q_{n+1}} & S^{n+1} \\ \Vert & & \downarrow{\bar{\zeta}} & & \downarrow{\bar{\xi}_r} & & \Vert \\ S^n & \xrightarrow{2^{r+1}} & S^n & \xrightarrow{i_n} & P^{n+1}(2^{r+1}) & \xrightarrow{q_{n+1}} & S^{n+1} \end{array}.$$

Moreover, there hold formulas

$$\bar{\xi}_r \circ i_P = B(\chi_{r+1}^r), \quad B(\chi_r^{s+1})\bar{\xi}_s(i_P\tilde{\eta}_s) = \tilde{\eta}_r \quad \text{for } r > s. \tag{2.3}$$

Proof. Dual to the relation in lemma 2.4 (1), there exists a map $\bar{\zeta}\colon C_\eta^{n+2} \to S^n$ satisfying $\bar{\zeta} i_n = 2 \cdot \mathbb{1}_n$. It follows that the first square in the Lemma is homotopy commutative, and hence the map $\bar{\xi}_r$ in the Lemma exists. Recall we have the composition

$$i_n = i_\eta \circ i_n\colon S^n \to C_\eta^{n+2} \to C_r^{n+2}.$$

Then $\bar{\xi}_r i_n = (\bar{\xi}_r i_\eta) i_n = (i_n \bar{\zeta}) i_n = 2 i_n$ implying that

$$\bar{\xi}_r \circ i_P = B(\chi_{r+1}^r) + \varepsilon \cdot i_n \eta q_{n+1}$$

for some $\varepsilon \in \{0, 1\}$. If $\varepsilon = 0$, we are done; otherwise we replace $\bar{\xi}_r$ by $\bar{\xi}_r + i_n \eta q_{n+1}$ to make $\varepsilon = 0$. Note that all the relations mentioned above still hold even if we make such a replacement. Thus we prove the first formula in (2.3), which implies the second one. □

2.2. Basic analysis methods

We give some auxiliary lemmas that are useful to study the homotopy types of homotopy cofibres.

LEMMA 2.6. *Let C_k^X be the homotopy cofibre of $f_k^X\colon X \to P^3(p^s)$, where $k \in \mathbb{Z}/p^{\min\{r, s\}}$ and $r = \infty$ for $X = S^3$,*

$$f_k^X = \begin{cases} k \cdot i_2 \eta, & X = S^3; \\ k \cdot i_2 \eta q_3, & X = P^3(p^r). \end{cases}$$

Then the cup squares in $H^*(C_k^X; \mathbb{Z}/p^{\min\{r, s\}})$ are given by

$$u_2 \smile u_2 = k \cdot u_4,$$

where $u_i \in H^i(C_k^X; \mathbb{Z}/p^{\min\{r, s\}})$ are generators, $i = 2, 4$. It follows that all cup squares in $H^*(C_k^X; \mathbb{Z}/p^{\min\{r, s\}})$ are trivial if and only if $k = 0$.

Proof. It is well-known that the map $k\eta$ has Hopf invariant $H(k\eta) = kH(\eta) = k$. Let $m = \min\{r, s\}$ and define $u_2 \smile u_2 = \bar{H}(f_k^X) \cdot u_4$ for some $\bar{H}(f_k^X) \in \mathbb{Z}/p^m$, which is

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called the *mod p^m Hopf invariant*. Then by naturality it is easy to deduce the formula

$$\bar{H}(f_k^X) = H(k\eta) \pmod{p^m} = k,$$

which completes the proof of the Lemma. □

LEMMA 2.7. *Let $k \in \mathbb{Z}/p^{\min\{r,\, s\}}$ and consider the homotopy cofibration*

$$P^4(p^r) \xrightarrow{g_k = k\cdot\hat{\eta}_s B(\chi_s^r)} P^3(p^s) \to C_{g_k}.$$

Let v_i be generators of $H^i(C_{g_k};\mathbb{Z}/p^s)$, $i = 2,\, 4$, then

$$v_2 \smile v_2 = k \cdot v_4 \in H^4(C_{g_k};\mathbb{Z}/p^s) \cong \mathbb{Z}/p^{\min\{r,s\}}.$$

It follows that g_k is null-homotopic if and only if $k = 0$.

Proof. By lemma 2.1 (3), there is a homotopy commutative diagram of homotopy cofibrations

$$\begin{array}{ccccc}
S^3 & \xrightarrow{k\chi_s^r\cdot i_2\eta} & P^3(p^s) & \longrightarrow & C_{k\chi_s^r} \\
\downarrow{\scriptstyle i_3} & & \| & & \downarrow{\scriptstyle \imath} \\
P^4(p^r) & \xrightarrow{k\cdot\hat{\eta}_s B(\chi_s^r)} & P^3(p^s) & \longrightarrow & C_{g_k} \\
\downarrow{\scriptstyle q_4} & & \downarrow & & \downarrow \\
S^4 & \longrightarrow & * & \longrightarrow & S^5
\end{array}.$$

It follows that $\imath$ in the right-most column induces an isomorphism

$$H^2(C_{g_k};\mathbb{Z}/p^s) \xrightarrow[\cong]{\imath^*} H^2(C_{k\chi_s^r};\mathbb{Z}/p^s) \cong \mathbb{Z}/p^s$$

and a monomorphism

$$H^4(C_{g_k};\mathbb{Z}/p^s) \cong \mathbb{Z}/p^{\min\{r,s\}} \xrightarrow{\imath^*} H^4(C_{k\chi_s^r};\mathbb{Z}/p^s) \cong \mathbb{Z}/p^s.$$

Let $v_i \in H^i(C_{g_k};\mathbb{Z}/p^s)$ be generators, $i = 2,\, 4$; let $u_2 = \imath^*(v_2)$ and u_4 be generators of $H^2(C_{k\chi_s^r};\mathbb{Z}/p^s)$ and $H^4(C_{k\chi_s^r};\mathbb{Z}/p^s)$, respectively. Let $\bar{H}(g_k)$ be the mod p^s Hopf invariant of g_k. By the naturality of cup products and lemma 2.6, we have

$$k\chi_s^r \cdot u_4 = u_2 \smile u_2 = \imath^*(v_2 \smile v_2) = \imath^*(\bar{H}(g_k)v_4) = \bar{H}(g_k) \cdot (\chi_s^r \cdot u_4).$$

Thus $\bar{H}(g_k) = k$, which completes the proof. □

The method of proof for the following lemma is due to [**7**, Lemma 2.4].

Lemma 2.8. *Let $X_1,\ X_2 \in \{S^2,\ P^3(2^r),\ C_s^4\}$ with $r,\ s \geqslant 1$. Let*

$$\iota_1\colon \Sigma X_1 \to \Sigma X_1 \vee \Sigma X_2, \quad \iota_2\colon \Sigma X_1 \to \Sigma X_2 \vee \Sigma X_2$$

be the canonical inclusion maps. Then any map u' in the composition

$$u\colon S^5 \xrightarrow{u'} \Sigma X_1 \wedge X_2 \xrightarrow{[\iota_1,\iota_2]} \Sigma X_1 \vee \Sigma X_2$$

is null-homotopic if and only if all cup products in $H^*(C_u;G)$ are trivial, where C_u is the homotopy cofibre of u and $G = H_2(X_1)\otimes H_2(X_2)$.

Proof. The 'only if' part is clear. For the 'if' part, consider the following homotopy commutative diagram of homotopy cofibrations

$$\begin{array}{ccccc}
S^5 & \xrightarrow{u'} & \Sigma X_1 \wedge X_2 & \xrightarrow{i'} & C_{u'} \\
\| & & \downarrow{\scriptstyle [\iota_1,\iota_2]} & & \downarrow \\
S^5 & \xrightarrow{u} & \Sigma X_1 \vee \Sigma X_2 & \xrightarrow{i} & C_u \\
\downarrow & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \bar{j}} \\
* & \longrightarrow & \Sigma X_1 \times \Sigma X_2 & = & \Sigma X_1 \times \Sigma X_2
\end{array},$$

which induces the commutative diagram with exact rows and columns:

$$\begin{array}{ccccc}
H^5(C_{u'};G) & \xrightarrow{(i')^*} & H^5(\Sigma X_1 \wedge X_2;G) & \xrightarrow{(u')^*} & H^5(S^5;G) \\
\downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta_2} & & \\
H^6(\Sigma X_1 \times \Sigma X_2;G) & = & H^6(\Sigma X_1 \times \Sigma X_2;G) & & \\
\downarrow{\scriptstyle \bar{j}^*} & & \downarrow & & \\
H^6(C_u;G) & \longrightarrow & H^6(\Sigma X_1 \vee \Sigma X_2;G) = 0 & &
\end{array}.$$

Note that $H^6(\Sigma X_1 \times \Sigma X_2;G)$ is generated by cup products, while all cup products in $H^6(C_u;G)$ are trivial by assumption. It follows that $\bar{j}^* = 0$ and hence δ_1 is surjective. The homomorphism δ_2 is obviously an isomorphism for $X_1,\ X_2 \in \{S^2,\ P^3(2^r)\}$ because $H^5(\Sigma X_1 \vee \Sigma X_2;G) = 0$; for $X_2 = C_s^4,\ X_1 = S^2,\ P^3(2^r)$ or C_r^4, we have $H^j(C_s^4;G) \cong G$ for $j = 2,\ 3,\ 4$, where $G = \mathbb{Z}/2^s$ or $\mathbb{Z}/2^{\min\{r,\,s\}}$. By computations,

$$H^5(\Sigma X_1 \wedge C_s^4;G) \cong \bigoplus_{i+j=5} \tilde{H}^i(\Sigma X_1;\tilde{H}^j(C_s^4;G)) \cong H^3(\Sigma X_1;H^2(C_s^4;G)),$$

$$H^6(\Sigma X_1 \times C_s^5;G) \cong \bigoplus_{i+j=6} H^i(\Sigma X_1;H^j(C_s^5;G)) \cong H^3(\Sigma X_1;H^3(C_s^5;G)).$$

Thus δ_2 is an isomorphism for all $X_1,\ X_2$. The upper commutative square then implies that $(i')^*$ is surjective and therefore $(u')^*$ is the zero map by exactness.

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Since $\Sigma X_1 \wedge X_2$ is 4-connected, the universal coefficient theorem for cohomology implies that

$$0 = (u')_* \colon H_5(S^5) \to H_5(\Sigma X_1 \wedge X_2).$$

Therefore u' is null-homotopic, by the Hurewicz theorem. □

LEMMA 2.9. *The Steenrod square* $\mathrm{Sq}^2 \colon H^n(C;\mathbb{Z}/2) \to H^{n+2}(C;\mathbb{Z}/2)$ *is an isomorphism for every* $(n+2)$*-dimensional elementary Chang complex* C.

Proof. Obvious or see [**27**]. □

For $n \geqslant 3$ and $r \geqslant 1$, we define homotopy cofibres

$$A^{n+3}(\tilde{\eta}_r) = P^{n+1}(2^r) \cup_{\tilde{\eta}_r} \mathrm{e}^{n+3}, \quad A^{n+3}(i_P\tilde{\eta}_r) = C_r^{n+2} \cup_{i_P\tilde{\eta}_r} \mathrm{e}^{n+3}. \tag{2.4}$$

LEMMA 2.10. *The Steenrod square* $\mathrm{Sq}^2 \colon H^{n+1}(X;\mathbb{Z}/2) \to H^{n+3}(X;\mathbb{Z}/2)$ *is an isomorphism for* $X = A^{n+3}(\tilde{\eta}_r)$ *and* $A^{n+3}(i_P\tilde{\eta}_r)$.

Proof. The statement for $X = A^{n+3}(\tilde{\eta}_r)$ refers to [**15**, Lemma 2.6]. For $X = A^{n+3}(i_P\tilde{\eta}_r)$, consider the homotopy commutative diagram of homotopy cofibrations

$$\begin{array}{ccccc} S^{n+2} & \xrightarrow{\tilde{\eta}_r} & P^{n+1}(2^r) & \longrightarrow & A^{n+3}(\tilde{\eta}_r) \\ \| & & \downarrow{\scriptstyle i_P} & & \downarrow{\scriptstyle \imath} \\ S^{n+2} & \xrightarrow{i_P\tilde{\eta}_r} & C_r^{n+2} & \longrightarrow & A^{n+3}(i_P\tilde{\eta}_r) \\ \downarrow & & \downarrow{\scriptstyle q_{n+2}} & & \downarrow \\ * & \longrightarrow & S^{n+2} & = & S^{n+2} \end{array}.$$

From the first two rows of the homotopy commutative diagram, it is easy to compute that

$$H^{n+i}(A^{n+3}(\tilde{\eta}_r);\mathbb{Z}/2) \cong H^{n+i}(A^{n+3}(i_P\tilde{\eta}_r);\mathbb{Z}/2) \cong \mathbb{Z}/2 \quad \text{for } i = 1, 3.$$

The third column homotopy cofibration implies that the induced homomorphisms $\imath^*$ are monomorphisms of mod 2 homology groups of dimension $n+1$ and $n+3$, hence it is an isomorphism. Then we complete the proof by the naturality of Sq^2. □

LEMMA 2.11 (Lemma 6.4 of [**12**]). *Let* $S \xrightarrow{f} (\bigvee_{i=1}^n A_i) \vee B \xrightarrow{g} \Sigma C$ *be a homotopy cofibration of simply-connected CW-complexes. For each* $j = 1, \cdots, n$, *let*

$$p_j \colon \left(\bigvee_i A_i\right) \vee B \to A_j, \quad q_B \colon \left(\bigvee_i A_i\right) \vee B \to B$$

be the obvious projections. Suppose that the composite $p_j f$ *is null-homotopic for each* $j \leqslant n$, *then there is a homotopy equivalence*

$$\Sigma C \simeq \left(\bigvee_{i=1}^n A_i\right) \vee C_{q_B f},$$

where $C_{q_B f}$ *is the homotopy cofibre of the composite* $q_B f$.

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Lemma 2.12. *Let* $(\bigvee_{i=1}^n A_i) \vee B \xrightarrow{f} C \to D$ *be a homotopy cofibration of CW-complexes. If the restriction of* f *to* A_i *is null-homotopic for each* $i = 1, \cdots, n$, *then there is a homotopy equivalence*

$$D \simeq \left(\bigvee_{i=1}^n \Sigma A_i \right) \vee E,$$

where E *is the homotopy cofibre of the restriction* $f|B \colon B \to C$.

Proof. Clear. □

Let $X = \Sigma X'$, $Y_i = \Sigma Y_i'$ be suspensions, $i = 1, 2, \cdots, n$. Let

$$i_l \colon Y_l \to \bigvee_{j=i}^n Y_i, \quad p_k \colon \bigvee_{i=1}^n Y_i \to Y_k$$

be respectively the canonical inclusions and projections, $1 \leqslant k, l \leqslant n$. By the Hilton–Milnor theorem, we may write a map $f \colon X \to \bigvee_{i=1}^n Y_i$ as

$$f = \sum_{k=1}^n i_k \circ f_k + \theta,$$

where $f_k = p_k \circ f \colon X \to Y_k$ and θ satisfies $\Sigma\theta = 0$. The first part $\sum_{k=1}^n i_k \circ f_k$ is usually represented by a vector $u_f = (f_1, f_2, \cdots, f_n)^t$. We say that f is completely determined by its components f_k if $\theta = 0$; in this case, denote $f = u_f$. Let $h = \sum_{k,l} i_l h_{lk} p_k$ be a self-map of $\bigvee_{i=1}^n Y_i$ which is completely determined by its components $h_{kl} = p_k \circ h \circ i_l \colon Y_l \to Y_k$. Denote by

$$M_h := (h_{kl})_{n\times n} = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} & \cdots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1} & h_{n1} & \cdots & h_{nn} \end{bmatrix}$$

Then the composition law $h(f+g) \simeq hf + hg$ implies that the product

$$M_h[f_1, f_2, \cdots, f_n]^t$$

given by the matrix multiplication represents the composite $h \circ f$. Two maps $f = u_f$ and $g = u_g$ are called *equivalent*, denoted by

$$[f_1, f_2, \cdots, f_n]^t \sim [g_1, g_2, \cdots, g_n]^t,$$

if there is a self-homotopy equivalence h of $\bigvee_{i=1}^n Y_i$, which can be represented by the matrix M_h, such that

$$M_h[f_1, f_2, \cdots, f_n]^t \simeq [g_1, g_2, \cdots, g_n]^t.$$

Note that the above matrix multiplication refers to elementary row operations in matrix theory; and the homotopy cofibres of the maps $f = u_f$ and $g = u_g$ are homotopy equivalent if f and g are equivalent.

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3. Homology decomposition of ΣM

Recall the homology decomposition of a simply-connected space X (cf. [8, Theorem 4H.3]). For $n \geqslant 2$, the *nth homology section* X_n of X is a CW-complex constructed from X_{n-1} by attaching a cone on a Moore space $M(H_nX, n-1)$; by definition, $X_1 = *$. Note that for each $n \geqslant 2$, there is a canonical map $j_n\colon X_n \to X$ that induces an isomorphism $j_{n*}\colon H_r(X_n) \to H_r(X)$ for $r \leqslant n$ and $H_r(X_n) = 0$ for $r > n$.

Firstly we note that similar arguments to the proof of [21, Lemma 5.1] proves the following lemma.

Lemma 3.1. *Let M be an orientable closed manifold with $H_1(M) \cong \mathbb{Z}^l \oplus H$, where $l \geqslant 1$ and H is a torsion abelian group. Then there is a homotopy equivalence*

$$\Sigma M \simeq \bigvee_{i=1}^{l} S^2 \vee \Sigma W,$$

where $W = M/\bigvee_{i=1}^{l} S^1$ is the quotient space with $H_1(W) \cong H$.

By lemma 3.1 and (1.1), the homology groups of ΣW is given by

i	2	3	4	5	0, 6	otherwise
$H_i(\Sigma W)$	H	$\mathbb{Z}^d \oplus T$	$\mathbb{Z}^d \oplus H$	$\mathbb{Z}^l$	$\mathbb{Z}$	0

(3.1)

Let W_i be the ith homology section of ΣW. There are homotopy cofibrations in which the attaching maps are *homologically trivial* (induce trivial homomorphisms in integral homology):

$$\begin{aligned}
&\left(\bigvee_{i=1}^{d} S^2\right) \vee P^3(T) \xrightarrow{f} P^3(H) \to W_3,\\
&\left(\bigvee_{i=1}^{d} S^3\right) \vee P^4(H) \xrightarrow{g} W_3 \to W_4,\\
&\bigvee_{i=1}^{l} S^4 \xrightarrow{h} W_4 \to W_5, \quad S^5 \xrightarrow{\phi} W_5 \to \Sigma W.
\end{aligned} \tag{3.2}$$

From now on we assume that $H \cong \bigoplus_{j=1}^{h} \mathbb{Z}/q_j^{s_j}$ where q_j are odd primes and $s_j \geqslant 1$.

Lemma 3.2. *There is a homotopy equivalence*

$$W_3 \simeq \left(\bigvee_{i=1}^{d} S^3\right) \vee P^3(H) \vee P^4(T).$$

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Proof. It suffices to show the map f in (3.2) is null-homotopic, or equivalently the following components of f are null-homotopic:

$$f^S\colon \bigvee_{i=1}^{d} S^2 \hookrightarrow \left(\bigvee_{i=1}^{d} S^2\right) \vee P^3(T) \xrightarrow{f} P^3(H),$$
$$f^T\colon P^3(T) \hookrightarrow \left(\bigvee_{i=1}^{d} S^2\right) \vee P^3(T) \xrightarrow{f} P^3(H),$$

where $\hookrightarrow$ denote the canonical inclusion maps. f is homologically trivial, so are f^S and f^T. Then the Hurewicz theorem and lemma 2.1 (1) imply f^S is null-homotopic.

Since $[P^3(p^r), P^3(q^s)] = 0$ for different primes p, q, it suffices to consider the case where T and H have the same prime factors. Denote by $T_H \cong \bigoplus_j \mathbb{Z}/q_j^{r_j}$ the component of T that has the same prime factors with H. The canonical inclusion $\imath_3\colon W_3 \to \Sigma W$ induces an isomorphism with $m_j = \min\{r_j, s_j\}$:

$$\imath_3^*\colon H^2(\Sigma W; \mathbb{Z}/q_j^{m_j}) \to H^2(W_3; \mathbb{Z}/q_j^{m_j}).$$

It follows that all the cup squares of cohomology classes of $H^2(W_3; \mathbb{Z}/q_j^{m_j})$, and hence of $H^2(C_{f^T}; \mathbb{Z}/q_j^{m_j})$ are trivial for any j. Let $C_{f_j^T}$ be the homotopy cofibre of the compositions

$$f_j^T\colon P^3(q_j^{r_j}) \hookrightarrow P^3(T) \xrightarrow{f^T} P^3(H) \twoheadrightarrow P^3(q_j^{s_j}),$$

where the unlabelled maps are the canonical inclusions and projections, respectively. Then [**21**, Lemma 4.2] implies that all cup squares of cohomology classes of $H^2(C_{f_j^T}; \mathbb{Z}/q_j^{m_j})$ are trivial for any j and hence f_j^T is null-homotopic, by lemma 2.6. Therefore f^T is also null-homotopic and we complete the proof. □

Lemma 3.3. *There is a homotopy equivalence*

$$W_4 \simeq \left(\bigvee_{i=1}^{d} (S^3 \vee S^4)\right) \vee P^3(H) \vee P^5(H) \vee P^4(T).$$

Proof. By (3.2) and lemma 3.2, W_4 is the homotopy cofibre of a homologically trivial map

$$\bar{g}\colon \left(\bigvee_{i=1}^{d} S^3\right) \vee P^4(H) \xrightarrow{g} W_3 \xrightarrow[\simeq]{e} \left(\bigvee_{i=1}^{d} S^3\right) \vee P^3(H) \vee P^4(T).$$

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Consider the compositions

$$S^3 \hookrightarrow \left(\bigvee_{i=1}^{d} S^3\right) \vee P^4(H) \xrightarrow{g} W_3 \to \bigvee_{i=1}^{d} S^3 \to S^3,$$

$$S^3 \hookrightarrow \left(\bigvee_{i=1}^{d} S^3\right) \vee P^4(H) \xrightarrow{g} W_3 \to P^4(T),$$

$$P^4(q_j^{s_j}) \hookrightarrow \left(\bigvee_{i=1}^{d} S^3\right) \vee P^4(H) \xrightarrow{g} W_3 \to \bigvee_{i=1}^{d} S^3 \to S^3,$$

$$P^4(q_j^{s_j}) \hookrightarrow \left(\bigvee_{i=1}^{d} S^3\right) \vee P^4(H) \xrightarrow{g} W_3 \to P^4(T) \to P^4(q_j^{r_j}),$$

where the unlabelled maps are the canonical inclusions and projections. Since $[P^4(p^r), S^3] = 0$, the Hurewicz theorem and lemma 2.1 (2) imply that all the above compositions are null-homotopic. Hence by lemma 2.11 there is a homotopy equivalence

$$W_4 \simeq \left(\bigvee_{i=1}^{d} S^3\right) \vee P^4(T) \vee C_{g'}$$

for some map $g' \colon \left(\bigvee_{i=1}^{d} S^3\right) \vee P^4(H) \to P^3(H)$.

By the homology decomposition for ΣW and the universal coefficient theorem for cohomology, the canonical map $\imath_4 \colon W_4 \to \Sigma W$ induces isomorphisms

$$\imath_4^* \colon H^i(\Sigma W) \to H^i(W_4), \quad i = 2, 4.$$

Consider the commutative diagram

$$\begin{array}{ccc} H^2(\Sigma W; \mathbb{Z}/q_j^{s_j}) & \xrightarrow{\smile^2} & H^4(\Sigma W; \mathbb{Z}/q_j^{s_j}) \\ \cong \big\downarrow \imath_4^* & & \cong \big\downarrow \imath_4^* \\ H^2(W_4; \mathbb{Z}/q_j^{s_j}) & \xrightarrow{\smile^2} & H^4(W_4; \mathbb{Z}/q_j^{s_j}) \end{array},$$

where $\smile^2$ denotes the cup squares. All cup squares in $H^*(\Sigma W; \mathbb{Z}/q_j^{s_j})$ are trivial implying that all cup squares in $H^4(W_4; \mathbb{Z}/q_j^{s_j})$ are trivial. Let $C_{g'_j}$ and $C_{g'_{ij}}$ be the homotopy cofibres of the compositions

$$g'_j \colon S^3 \hookrightarrow \left(\bigvee_{i=1}^{d} S^3\right) \vee P^4(H) \xrightarrow{g'} P^3(H) \twoheadrightarrow P^3(q_j^{s_j}),$$

$$g'_{ij} \colon P^4(q_j^{r_i}) \hookrightarrow \left(\bigvee_{i=1}^{d} S^3\right) \vee P^4(H) \xrightarrow{g'} P^3(H) \twoheadrightarrow P^3(q_j^{s_j}).$$

By [21, Lemma 4.2], we get the triviality of cup squares in $H^*(C_{g'_j};\mathbb{Z}/q_j^{s_j})$ and $H^*(C_{g'_{ij}};\mathbb{Z}/q_j^{s_j}))$. Then lemmas 2.6 and 2.7 imply that g'_j and g'_{ij} are both null-homotopic. Thus by lemma 2.12, there is a homotopy equivalence

$$C_{g'} \simeq \left(\bigvee_{i=1}^{d} S^4\right) \vee P^3(H) \vee P^5(H),$$

which completes the proof of the Lemma. □

Proposition 3.4. *There is a homotopy equivalence*

$$W_5 \simeq P^3(H) \vee P^5(H) \vee P^4(T_{\neq 2}) \vee \left(\bigvee_{i=1}^{d-c_1} S^3\right) \vee \left(\bigvee_{i=1}^{d} S^4\right) \vee \left(\bigvee_{i=1}^{l-c_1-c_2} S^5\right)$$
$$\vee \left(\bigvee_{i=1}^{c_1} C_\eta^5\right) \vee \left(\bigvee_{j=c_2+1}^{t_2} P^4(2^{r_j})\right) \vee \left(\bigvee_{j=1}^{c_2} C_{r_j}^5\right),$$

where $0 \leqslant c_1 \leqslant \min\{l, d\}$ *and* $0 \leqslant c_2 \leqslant \min\{l - c_1, t_2\}$; $c_1 = c_2 = 0$ *if and only if* $\mathrm{Sq}^2(H^2(M;\mathbb{Z}/2)) = 0$.

Proof. By (3.2) and lemma 3.3, W_5 is the homotopy cofibre of a map

$$\bigvee_{i=1}^{l} S^4 \xrightarrow{h} W_4 \simeq \left(\bigvee_{i=1}^{d}(S^3 \vee S^4)\right) \vee P^3(H) \vee P^5(H) \vee P^4(T).$$

Similar arguments to that in the proof of lemma 3.3 show that there is a homotopy equivalence

$$W_5 \simeq \left(\bigvee_{i=1}^{d} S^4\right) \vee P^3(H) \vee P^5(H) \vee P^4(T_{\neq 2}) \vee C_{h'}, \tag{3.3}$$

where $h'\colon \bigvee_{i=1}^{l} S^4 \to \left(\bigvee_{i=1}^{d} S^3\right) \vee \left(\bigvee_{i=1}^{t_2} P^4(2^{r_i})\right)$.

Since $\pi_4(P^4(2^r)) \cong \mathbb{Z}/2\langle i_3\eta\rangle$, we may represent the map h' by a $(d+t_2) \times l$-matrix $M_{h'}$ with entries 0, η or $i_3\eta$. There hold homotopy equivalences

$$\begin{bmatrix} \mathbb{1}_3 & 0 \\ i_3 & \mathbb{1}_P \end{bmatrix} \begin{bmatrix} \eta \\ i_3\eta \end{bmatrix} \simeq \begin{bmatrix} \eta \\ 0 \end{bmatrix} : S^4 \to S^3 \vee P^4(2^r),$$
$$\begin{bmatrix} \mathbb{1}_P & 0 \\ B(\chi_s^r) & \mathbb{1}_P \end{bmatrix} \begin{bmatrix} i_3\eta \\ i_3\eta \end{bmatrix} \simeq \begin{bmatrix} i_3\eta \\ 0 \end{bmatrix} : S^4 \to P^4(2^r) \vee P^4(2^s) \text{for } r \geqslant s.$$

Then by elementary matrix operations we have an equivalence

$$M_{h'} \sim \begin{bmatrix} D_{c_1} & O \\ O & O \\ O & \left[\begin{smallmatrix} E_{c_2} & O \\ O & O \end{smallmatrix}\right] \end{bmatrix},$$

where O denote suitable zero matrices, D_{c_1} is the diagonal matrix of rank c_1 whose diagonal entries are η, E_{c_2} is a $c_2 \times c_2$-matrix which has exactly one entry $i_3\eta$ in each row and column. It follows that there is a homotopy equivalence

$$C_{h'} \simeq \left(\bigvee_{i=1}^{l-c_1-c_2} S^5\right) \vee \left(\bigvee_{i=1}^{d-c_1} S^3\right) \vee \left(\bigvee_{j=c_2+1}^{t_2} P^4(2^{r_j})\right) \vee \left(\bigvee_{i=1}^{c_1} C^5_\eta\right) \vee \left(\bigvee_{j=1}^{c_2} C^5_{r_j}\right).$$

The proof of the Lemma then follows by (3.3) and lemma 2.9. □

4. Proof of Theorems 1.1 and 1.2

Let M be the given five-manifold described in Theorem 1.1. By (3.2) there is a homotopy cofibration $S^5 \xrightarrow{\phi} W_5 \to \Sigma W$ with W_5 (and integers c_1, c_2) given by proposition 3.4. Since ϕ is homologically trivial, so are the compositions

$$\phi_\eta \colon S^5 \xrightarrow{\phi} W_5 \twoheadrightarrow \bigvee_{i=1}^{c_1} C^5_\eta \twoheadrightarrow C^5_\eta,$$

$$\phi_{C_j} \colon S^5 \xrightarrow{\phi} W_5 \twoheadrightarrow \bigvee_{j=1}^{c_2} C^5_{r_j} \twoheadrightarrow C^5_{r_j},$$

$$\phi_{H,j} \colon S^5 \xrightarrow{\phi} W_5 \twoheadrightarrow P^3(H) \twoheadrightarrow P^3(q_j^{s_j}).$$

By lemma 2.4, ϕ_η is null-homotopic and $\phi_{C_j} = w_j \cdot i_P \tilde{\eta}_{r_j}$ for some $w_j \in \mathbb{Z}/2$. By lemma 2.3, $\phi_{H,j}$ is null-homotopic for primes $q_j \geqslant 5$ and $\Sigma\phi_{H,j}$ are null-homotopic for all odd primes q_j. Write $H = H_3 \oplus H_{\geqslant 5}$ with H_3 the 3-primary component of H. It follows by lemmas 2.1 (2) and 2.11 that there are homotopy equivalences

$$\Sigma W \simeq P^3(H_{\geqslant 5}) \vee P^5(H) \vee P^4(T_{\neq 2}) \vee \left(\bigvee_{i=1}^{l-c_1-c_2} S^5\right) \vee \left(\bigvee_{i=1}^{c_1} C^5_\eta\right) \vee C_{\bar{\phi}}, \tag{4.1}$$

$$\Sigma^2 W \simeq P^4(H) \vee P^6(H) \vee P^5(T_{\neq 2}) \vee \left(\bigvee_{i=1}^{l-c_1-c_2} S^6\right) \vee \left(\bigvee_{i=1}^{c_1} C^6_\eta\right) \vee C_{\Sigma\bar{\phi}}, \tag{4.2}$$

for some homologically trivial map

$$\bar{\phi} \colon S^5 \to P^3(H_3) \vee \left(\bigvee_{i=1}^{d-c_1} S^3\right) \vee \left(\bigvee_{i=1}^{d} S^4\right) \vee \left(\bigvee_{j=c_2+1}^{t_2} P^4(2^{r_j})\right) \vee \left(\bigvee_{j=1}^{c_2} C^5_{r_j}\right).$$

From now on we assume that $H_3 = 0$ to study the homotopy type of ΣW or the homotopy cofibre $C_{\bar{\phi}}$. By lemmas 2.2 and 2.4 we may put

$$\bar{\phi} = \sum_{i=1}^{d-c_1} x_i \cdot \eta^2 + \sum_{i=1}^{d} y_i \cdot \eta + \sum_{j=c_2+1}^{t_2} (z_j \cdot \tilde{\eta}_{r_j} + \epsilon_j \cdot i_3\eta^2) + \sum_{j=1}^{c_2} w_j \cdot i_P \tilde{\eta}_{r_j} + \theta, \tag{4.3}$$

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where all coefficients belong to $\mathbb{Z}/2$ and θ is a linear combination of Whitehead products. By the Hilton-Milnor theorem the domain Wh of θ is given by

$$\begin{aligned}\mathrm{Wh} = &\bigoplus_{1\leqslant i,j\leqslant d-c_1} \pi_5(\Sigma S_i^2 \wedge S_j^2) \oplus \bigoplus_{\substack{1\leqslant i\leqslant d-c_1\\ c_2+1\leqslant j\leqslant t_2}} \pi_5(\Sigma S_i^2 \wedge P^3(2^{r_j}))\\ &\oplus \bigoplus_{\substack{1\leqslant i\leqslant d-c_1\\ 1\leqslant j\leqslant c_2}} \pi_5(\Sigma S_i^2 \wedge C_{r_j}^4) \oplus \bigoplus_{c_2+1\leqslant i,j\leqslant t_2} \pi_5(\Sigma P^3(2^{r_i}) \wedge P^3(2^{r_j}))\\ &\oplus \bigoplus_{\substack{c_2+1\leqslant i\leqslant t_2\\ 1\leqslant j\leqslant c_2}} \pi_5(\Sigma P^3(2^{r_i}) \wedge C_{r_j}^4) \oplus \bigoplus_{1\leqslant i,j\leqslant c_2} \pi_5(\Sigma C_{r_i}^4 \wedge C_{r_j}^4).\end{aligned}$$

Note that all the spaces $\Sigma X_i \wedge X_j$ are 4-connected and hence there are Hurewicz isomorphisms $\pi_5(\Sigma X_i \wedge X_j) \cong H_5(\Sigma X_i \wedge X_j)$. For different X_i and X_j, we use the ambiguous notations

$$\iota_1 \colon \Sigma X_i \to \Sigma X_i \vee \Sigma X_j, \quad \iota_2 \colon \Sigma X_j \to \Sigma X_i \vee \Sigma X_j$$

to denote the natural inclusions. Then we can write

$$\theta = a_{ij} + b_{ij} + c_{ij} + e_{ij} + f_{ij}, \tag{4.4}$$

where

$$\begin{aligned}
&a_{ij} \colon S^5 \xrightarrow{a'_{ij}} \Sigma S_i^2 \wedge S_j^2 \xrightarrow{[\iota_1,\iota_2]} \Sigma S_i^2 \vee \Sigma S_j^2,\\
&b_{ij} \colon S^5 \xrightarrow{b'_{ij}} \Sigma S_i^2 \wedge P^3(2^{r_j}) \xrightarrow{[\iota_1,\iota_2]} \Sigma S_i^2 \vee \Sigma P^3(2^{r_j}),\\
&c_{ij} \colon S^5 \xrightarrow{c'_{ij}} \Sigma S_i^2 \wedge C_{r_j}^4 \xrightarrow{[\iota_1,\iota_2]} \Sigma S_i^2 \vee \Sigma C_{r_j}^4,\\
&d_{ij} \colon S^5 \xrightarrow{d'_{ij}} \Sigma P^3(2^{r_i}) \wedge P^3(2^{r_j}) \xrightarrow{[\iota_1,\iota_2]} \Sigma P^3(2^{r_i}) \vee \Sigma P^3(2^{r_j}),\\
&e_{ij} \colon S^5 \xrightarrow{e'_{ij}} \Sigma P^3(2^{r_i}) \wedge C_{r_j}^4 \xrightarrow{[\iota_1,\iota_2]} \Sigma P^3(2^{r_i}) \vee \Sigma C_{r_j}^4,\\
&f_{ij} \colon S^5 \xrightarrow{f'_{ij}} \Sigma C_{r_j}^4 \wedge C_{r_i}^4 \xrightarrow{[\iota_1,\iota_2]} \Sigma C_{r_i}^4 \vee \Sigma C_{r_j}^4.
\end{aligned}$$

Since the homotopy cofibre of ϕ is ΣW, similar arguments to the proof of [**7**, Lemma 4.2] show the following lemma.

LEMMA 4.1. *Let C_u be the homotopy cofibre of a map u with u given by (1) $u = a_{ij}$, (2) $u = b_{ij}$, (3) $u = c_{ij}$, (4) $u = d_{ij}$, (5) $u = e_{ij}$, (6) $u = f_{ij}$. Then all cup products in $H^*(C_u; R)$ are trivial for any principal ideal domain R.*

By lemmas 4.1 and 2.8 we then get

COROLLARY 4.2. *The Whitehead product component θ (4.4) of $\bar{\phi}$ is trivial.*

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For each $n \geqslant 2$, let Θ_n be secondary cohomology operation based on the null-homotopy of the composition

$$K_n \xrightarrow{\theta_n = \begin{bmatrix} \mathrm{Sq}^2\,\mathrm{Sq}^1 \\ \mathrm{Sq}^2 \end{bmatrix}} K_{n+3} \times K_{n+2} \xrightarrow{\varphi_n = [\mathrm{Sq}^1, \mathrm{Sq}^2]} K_{n+4},$$

where $K_m = K(\mathbb{Z}/2, m)$ denotes the Eilenberg–MacLane space of type $(\mathbb{Z}/2, m)$. More concretely, $\Theta_n \colon S_n(X) \to T_n(X)$ is a cohomology operation with

$$S_n(X) = \ker(\theta_n)_\sharp = \ker(\mathrm{Sq}^2) \cap \ker(\mathrm{Sq}^2\,\mathrm{Sq}^1)$$
$$T_n(X) = \operatorname{coker}(\Omega\varphi_n)_\sharp = H^{n+3}(X;\mathbb{Z}/2)/\operatorname{im}(\mathrm{Sq}^1 + \mathrm{Sq}^2).$$

Note that Θ_n detects the maps $\eta^2 \in \pi_{n+2}(S^n)$ and $i_n\eta^2 \in \pi_{n+2}(P^{n+1}(2^r))$ (cf. [**15**, Section 2.4]). By the method outlined in [**16**, page 32], the stable secondary operation $\Theta = \{\Theta_n\}_{n\geqslant 2}$ is *spin trivial* (cf. [**24**]), which means the following Lemma holds.

Lemma 4.3. *The secondary operation $\Theta \colon H^*(M;\mathbb{Z}/2) \to H^{*+3}(M;\mathbb{Z}/2)$ is trivial for any orientable closed smooth spin manifold M.*

Now we are prepared to classify the homotopy types of $C_{\bar\phi}$. Note that for a closed orientable smooth five-manifold M, the second Stiefel–Whitney class equals the second Wu class v_2, which satisfies $\mathrm{Sq}^2(x) = v_2 \smile x$ for all $x \in H^3(M;\mathbb{Z}/2)$ [**17**, page 132]. It follows that the orientable smooth five-manifold M is spin if and only if Sq^2 acts trivially on $H^3(M;\mathbb{Z}/2)$, which is equivalent to Sq^2 acting trivially on $H^4(\Sigma W;\mathbb{Z}/2)$ or $H^4(C_{\bar\phi};\mathbb{Z}/2)$, by lemma 3.1 and the homotopy decomposition (4.1).

Proposition 4.4. *If M is a closed orientable smooth spin five-manifold, then there is a homotopy equivalence*

$$C_{\bar\phi} \simeq \left(\bigvee_{i=1}^{d-c_1} S^3\right) \vee \left(\bigvee_{i=1}^{d} S^4\right) \vee \left(\bigvee_{j=c_2+1}^{t_2} P^4(2^{r_j})\right) \vee \left(\bigvee_{j=1}^{c_2} C^5_{r_j}\right) \vee S^6.$$

Proof. The smooth spin condition on M, together with lemma 4.3, implies that $x_i = \epsilon_j = 0$ for all i, j in (4.3). By the comments above proposition 4.4, M is spin implies that the Steenrod square Sq^2 acts trivially on $H^4(C_{\bar\phi};\mathbb{Z}/2)$. Then lemmas 2.9 and 2.10 imply $y_i = z_j = w_j = 0$ for all i, j. Thus the map $\bar\phi$ in (4.3) is null-homotopic and therefore we get the homotopy equivalence in the Proposition. □

Remark 4.5. If M is a general 5-dimensional connected Poincaré duality complex such that Sq^2 acts trivially on $H^3(M;\mathbb{Z}/2)$, then we have the following two additional possibilities for the homotopy types of $C_{\bar\phi}$ in terms of the secondary cohomology operation Θ:

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(1) If for any $u \in H^3(M;\mathbb{Z}/2)$ with $\Theta(u) \neq 0$ and any $v \in \ker(\Theta)$, there holds $\beta_r(u+v) = 0$ for all r, then there is a homotopy equivalence

$$C_{\bar\phi} \simeq \left(\bigvee_{i=2}^{d-c_1} S^3\right) \vee \left(\bigvee_{i=1}^{d} S^4\right) \vee \left(\bigvee_{j=c_2+1}^{t_2} P^4(2^{r_j})\right) \vee \left(\bigvee_{j=1}^{c_2} C^5_{r_j}\right) \vee (S^3 \cup_{\eta^2} \mathrm{e}^6).$$

(2) If there exist $u \in H^3(M;\mathbb{Z}/2)$ with $\Theta(u) \neq 0$ and $v \in \ker(\Theta)$ such that $\beta_r(u+v) \neq 0$, then there is a homotopy equivalence

$$C_{\bar\phi} \simeq \left(\bigvee_{i=1}^{d-c_1} S^3\right) \vee \left(\bigvee_{i=1}^{d} S^4\right) \vee \left(\bigvee_{j_0 \neq j=c_2+1}^{t_2} P^4(2^{r_j})\right) \vee \left(\bigvee_{j=1}^{c_2} C^5_{r_j}\right) \vee A^6(2^{r_{j_0}}\eta^2),$$

where $A^6(2^{r_{j_0}}\eta^2) = P^4(2^{r_{j_0}}) \cup_{i_3\eta^2} \mathrm{e}^6$, j_0 is the index such that r_{j_0} is the maximum of r_j satisfying $\beta_{r_j}(u+v) \neq 0$.

PROPOSITION 4.6. *Suppose that* Sq^2 *acts non-trivially on* $H^3(M;\mathbb{Z}/2)$*, or equivalently* Sq^2 *acts non-trivially on* $H^4(C_{\bar\phi};\mathbb{Z}/2)$*.*

(1) *If for any* $u, v \in H^4(C_{\bar\phi};\mathbb{Z}/2)$ *satisfying* $\mathrm{Sq}^2(u) \neq 0$ *and* $\mathrm{Sq}^2(v) = 0$*, there holds* $u + v \notin \mathrm{im}(\beta_r)$ *for any* $r \geqslant 1$*, then there is a homotopy equivalence*

$$C_{\bar\phi} \simeq \left(\bigvee_{i=1}^{d-c_1} S^3\right) \vee \left(\bigvee_{i=2}^{d} S^4\right) \vee \left(\bigvee_{j=c_2+1}^{t_2} P^4(2^{r_j})\right) \vee \left(\bigvee_{j=1}^{c_2} C^5_{r_j}\right) \vee C^6_\eta.$$

(2) *If there exist* $u, v \in H^4(C_{\bar\phi};\mathbb{Z}/2)$ *with* $\mathrm{Sq}^2(u) \neq 0$ *and* $v \in \ker(\mathrm{Sq}^2)$ *such that* $u + v \in \mathrm{im}(\beta_r)$ *for some* r*, then either there is a homotopy equivalence*

$$C_{\bar\phi} \simeq \left(\bigvee_{i=1}^{d-c_1} S^3\right) \vee \left(\bigvee_{i=1}^{d} S^4\right) \vee \left(\bigvee_{j_1 \neq j=c_2+1}^{t_2} P^4(2^{r_j})\right) \vee \left(\bigvee_{j=1}^{c_2} C^5_{r_j}\right) \vee A^6(\tilde\eta_{r_{j_1}}),$$

or there is a homotopy equivalence

$$C_{\bar\phi} \simeq \left(\bigvee_{i=1}^{d-c_1} S^3\right) \vee \left(\bigvee_{i=1}^{d} S^4\right) \vee \left(\bigvee_{j=c_2+1}^{t_2} P^4(2^{r_j})\right) \vee \left(\bigvee_{j_1 \neq j=1}^{c_2} C^5_{r_j}\right) \vee A^6(i_P\tilde\eta_{r_{j_1}}),$$

where the last two complexes are defined by (2.4) *and* r_{j_1} *is the minimum of* r_j *such that* $u + v \in \mathrm{im}(\beta_{r_j})$*.*

Proof. Recall the equation for $\bar\phi$ given by (4.3). Since Sq^2 acts non-trivially on $H^4(C_{\bar\phi};\mathbb{Z}/2)$, at least one of y_i, z_j, w_j equals 1.

(1) The conditions in (1) implying that $z_j = w_j = 0$ for all j and hence $y_i = 1$ for some i. Clearly we may assume that $y_1 = 1$ and $y_i = 0$ for all $2 \leqslant i \leqslant d$. By the equivalences

$$\begin{bmatrix}\eta\\ \eta^2\end{bmatrix}\sim\begin{bmatrix}\eta\\ 0\end{bmatrix}\colon S^5\to S^4\vee S^3,\quad \begin{bmatrix}\eta\\ i_3\eta^2\end{bmatrix}\sim\begin{bmatrix}\eta\\ 0\end{bmatrix}\colon S^5\to S^4\vee P^4(2^r),$$

we may further assume that $x_i=\epsilon_i=0$ for all i in (4.3). Thus we have

$$\bar{\phi}=\eta\colon S^5\to S^4,$$

which proves the homotopy equivalence in (1).

(2) The conditions in (2) implies that $z_j=1$ or $w_j=1$ for some j. For maps $\tilde{\eta}_r, i_3\eta^2\colon S^5\to P^4(2^r)$ and $i_P\tilde{\eta}_s\colon S^5\to C^5_s$, the formulas (2.1) and (2.2) indicate the following equivalences

$$\begin{bmatrix}\tilde{\eta}_r\\ \eta^a\end{bmatrix}\sim\begin{bmatrix}\tilde{\eta}_r\\ 0\end{bmatrix}\ (a=1,2),\quad \begin{bmatrix}i_P\tilde{\eta}_r\\ \eta^a\end{bmatrix}\sim\begin{bmatrix}i_P\tilde{\eta}_r\\ 0\end{bmatrix}\ (a=1,2);$$
$$\begin{bmatrix}\tilde{\eta}_r\\ \tilde{\eta}_s\end{bmatrix}\sim\begin{bmatrix}\tilde{\eta}_r\\ 0\end{bmatrix}\ (r\leqslant s),\quad \begin{bmatrix}i_P\tilde{\eta}_r\\ i_P\tilde{\eta}_s\end{bmatrix}\sim\begin{bmatrix}i_P\tilde{\eta}_r\\ 0\end{bmatrix}\ (r\leqslant s);$$
$$\begin{bmatrix}\tilde{\eta}_r\\ i_3\eta^2\end{bmatrix}\sim\begin{bmatrix}\tilde{\eta}_r\\ 0\end{bmatrix}(i_3\eta^2\in\pi_5(P^4(2^s)),r\neq s),\quad \begin{bmatrix}i_P\tilde{\eta}_r\\ i_3\eta^2\end{bmatrix}\sim\begin{bmatrix}i_P\tilde{\eta}_r\\ 0\end{bmatrix}.$$

It follows that we may assume that $x_i=y_i=0$ for all i regardless of whether $z_j=1$ or $w_j=1$.

(i) If $z_j=1$ for some j, we assume that $z_j=1$ for exactly one j, say $z_{j_1}=1$; in this case, $\epsilon_j=0$ for all $j\neq j_1$. Note that $\mathbb{1}_P+i_3\eta q_4$ is a self-homotopy equivalence of $P^4(2^r)$ and

$$(\mathbb{1}_P+i_3\eta q_4)(\tilde{\eta}_r+i_3\eta^2)=\tilde{\eta}_r+i_3\eta^2+i_3\eta^2=\tilde{\eta}_r,$$

we may assume that $\epsilon_{j_1}=1$ and $\epsilon_j=0$ for $j\neq j_1$.

(ii) If $w_j=1$ for some j, then $w_j=1$ for exactly one j, say $w_{j_2}=1$; in this case, $\epsilon_j=0$ for all j.

By (2.3) we have the equivalences for maps $S^5\to P^4(2^r)\vee C^5_s$:

$$\begin{bmatrix}\tilde{\eta}_r\\ i_P\tilde{\eta}_s\end{bmatrix}\sim\begin{bmatrix}\tilde{\eta}_r\\ 0\end{bmatrix}\quad\text{if } r\leqslant s;\quad \begin{bmatrix}\tilde{\eta}_r\\ i_P\tilde{\eta}_s\end{bmatrix}\sim\begin{bmatrix}0\\ i_P\tilde{\eta}_s\end{bmatrix}\quad\text{if } r>s.$$

Thus we may assume that $\bar{\phi}=\tilde{\eta}_{r_{j_1}}$ if $r_{j_1}\leqslant r_{j_2}$; otherwise $\bar{\phi}=i_P\tilde{\eta}_{r_{j_2}}$, which prove the homotopy equivalences in (2). □

Proof of Theorem 1.1. Combine lemma 3.1, the homotopy decomposition (4.1) and propositions 4.4 and 4.6. □

Proof of Theorem 1.2. The homotopy types of the discussion of the suspension $\Sigma C_{\bar{\phi}}$ is totally similar to that of $C_{\bar{\phi}}$. The Theorem then follows by lemma 3.1, the homotopy decomposition (4.2) and the suspended version of propositions 4.4 and 4.6. □

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5. Some applications

In this section we apply the homotopy decomposition of $\Sigma^2 M$ given by Theorem 1.1 to study the reduced K-groups and the cohomotopy sets of M.

5.1. Reduced K-groups

To prove Corollary 1.3 we recall that the reduced complex K-group $\widetilde{K}(S^n)$ is isomorphic to $\mathbb{Z}$ if n is even, otherwise $\widetilde{K}(S^n)=0$; the reduced KO-groups of spheres are given by

$i \pmod 8$	0	1	2	3	4	5	6	7
$\widetilde{KO}(S^i)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0

. (5.1)

Using the reduced complex K-groups and KO-groups of spheres one can easily get the following lemma, where the notations $A^7(\tilde{\eta}_r)$ and $A^7(i_P\tilde{\eta}_r)$ refer to (2.4).

Lemma 5.1. *Let m, r be positive integers and let p be a prime.*

(1) $\widetilde{K}(P^{2m}(p^r))\cong \mathbb{Z}/p^r$ *and* $\widetilde{K}(P^{2m+1}(p^r))=0$.

(2) $\widetilde{K}(C^{2m}_\eta)\cong \mathbb{Z}\oplus\mathbb{Z}$ *and* $\widetilde{K}(C^{2m+1}_\eta)=0$.

(3) $\widetilde{K}(C^6_r)\cong \widetilde{K}(A^7(i_P\tilde{\eta}_r))\cong\mathbb{Z}$, $\widetilde{K}(A^7(\tilde{\eta}_r))=0$.

(4) $\widetilde{KO}^2(P^{4+i}(p^r))=\widetilde{KO}^2(C^7_\eta)=0$ *for* $p\geqslant 3$ *and* $i=0,1,2$.

(5) $\widetilde{KO}^2(P^5(2^r))\cong\widetilde{KO}^2(A^7(\tilde{\eta}_r))\cong\mathbb{Z}/2$.

(6) $\widetilde{KO}^2(C^6_\eta)\cong\widetilde{KO}^2(C^6_r)\cong\widetilde{KO}^2(A^7(i_P\tilde{\eta}_r))\cong\mathbb{Z}\oplus\mathbb{Z}/2$.

Proposition 5.2. *Let M be an orientable smooth closed five-manifold given by Theorem 1.1 or 1.2. There hold isomorphisms*

$$\widetilde{K}(M)\cong\mathbb{Z}^{d+l}\oplus H\oplus H,\quad \widetilde{KO}(M)\cong\mathbb{Z}^l\oplus(\mathbb{Z}/2)^{l+d+t_2}.$$

Proof. We only give the proof of $\widetilde{KO}(M)$ here, because the proof of $\widetilde{K}(M)$ is similar but simpler. By Theorem 1.1 we can write

$$\Sigma^2 M\simeq\left(\bigvee_{i=1}^{l}S^3\right)\vee\left(\bigvee_{i=1}^{d-c_1}S^4\right)\vee\left(\bigvee_{i=2}^{d}S^5\right)\vee\left(\bigvee_{i=1}^{l-c_1-c_2}S^6\right)\vee P^4(H)\vee P^6(H)$$
$$\vee\left(\bigvee_{i=1}^{c_1}C^6_\eta\right)\vee P^5\Big(\frac{T[c_2]}{\mathbb{Z}/2^{r_{j_1}}}\Big)\vee\left(\bigvee_{j_2\neq j=1}^{c_2}C^6_{r_j}\right)\vee\Sigma^2 X,$$

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where $\Sigma^2X\simeq (S^5\vee P^5(2^{r_{j_1}})\vee C^6_{r_{j_2}})\cup e^7$. By lemma 5.1 and the table (5.1), there is a chain of isomorphisms

$$\begin{aligned}
\widetilde{KO}(M) &\cong \widetilde{KO}^2(\Sigma^2 M)\cong \bigoplus_{l}\widetilde{KO}^2(S^3)\oplus\bigoplus_{d-c_1}\widetilde{KO}^2(S^4)\oplus\bigoplus_{d}\widetilde{KO}^2(S^5)\\
&\oplus\bigoplus_{l-c_1-c_2}\widetilde{KO}^2(S^6)\oplus\widetilde{KO}(P^4(H)\vee P^6(H))\oplus\bigoplus_{c_1}\widetilde{KO}^2(C^6_\eta)\\
&\oplus\widetilde{KO}^2(P^5\left(\frac{T[c_2]}{\mathbb{Z}/2^{r_{j_1}}}\right))\oplus\bigoplus_{j_2\neq j=1}^{c_2}\widetilde{KO}^2(C^6_{r_j})\oplus\widetilde{KO}^2(\Sigma^2X)\\
&\cong(\mathbb{Z}/2)^{l+d-c_1}\oplus\mathbb{Z}^{l-c_1-c_2}\oplus(\mathbb{Z}\oplus\mathbb{Z}/2)^{\oplus c_1}\oplus(\mathbb{Z}/2)^{t_2-c_2-1}\\
&\oplus(\mathbb{Z}\oplus\mathbb{Z}/2)^{\oplus(c_2-1)}\oplus\widetilde{KO}^2(\Sigma^2X)\\
&\cong\mathbb{Z}^l\oplus(\mathbb{Z}/2)^{l+d+t_2},
\end{aligned}$$

where $\widetilde{KO}^2(\Sigma^2X)\cong\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$ in all cases of Theorem 1.1 can be easily computed by lemma 5.1. □

5.2. Cohomotopy sets

Let M be a closed five-manifold. It is clear that the *cohomotopy Hurewicz maps*

$$h^i\colon \pi^i(M)\to H^i(M),\quad \alpha\mapsto\alpha^*(\iota_i)$$

with $\iota_i\in H^i(S^i)$ a generator are isomorphisms for $i=1$ or $i\geqslant 5$. For $\pi^4(M)$, there is a short exact sequence of abelian groups (cf. [**22**])

$$0\to\frac{H^5(M;\mathbb{Z}/2)}{\mathrm{Sq}^2_{\mathbb{Z}}(H^3(M;\mathbb{Z}))}\to\pi^4(M)\xrightarrow{h^4}H^4(M)\to 0,$$

which splits if and only if there holds an equality (cf. [**23**, Section 6.1])

$$\mathrm{Sq}^2_{\mathbb{Z}}(H^3(M;\mathbb{Z}))=\mathrm{Sq}^2(H^3(M;\mathbb{Z}/2))\subseteq H^5(M;\mathbb{Z}/2).$$

The standard action of S^3 on $S^2=S^3/S^1$ by left translation induces a natural action of $\pi^3(M)$ on $\pi^2(M)$. More concretely, the Hopf fibre sequence

$$S^1\longrightarrow S^3\xrightarrow{\eta}S^2\xrightarrow{\imath_2}\mathbb{C}P^\infty\xrightarrow{\jmath}\mathbb{H}P^\infty$$

induces an exact sequence of sets

$$\pi^1(M)\xrightarrow{\kappa_u}\pi^3(M)\xrightarrow{\eta_\sharp}\pi^2(M)\xrightarrow{h}H^2(M)\xrightarrow{\jmath_\sharp}\pi^4(M),\tag{5.2}$$

where $[M,\mathbb{H}P^\infty]=\pi^4(M)$ because $\mathbb{H}P^\infty$ has the 6-skeleton S^4, $h=h^2$ is the second cohomotopy Hurewicz map. The homomorphism κ_u in (5.2) is given by the following lemma.

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Lemma 5.3 (cf. Theorem 3 of [13]). *The natural action of $\pi^3(M)$ on $\pi^2(M)$ is transitive on the fibres of h and the stabilizer of $u \in \pi^2(M)$ equals the image of the homomorphism*

$$\kappa_u\colon \pi^1(M) \to \pi^3(M), \quad \kappa_u(v) = \kappa(u \times v)\Delta_M,$$

where Δ_M is the diagonal map on M, $\kappa\colon S^2 \times S^1 \to S^3$ is the conjugation $(gS^1, t) \mapsto gtg^{-1}$ by setting $S^2 = S^3/S^1$.

Thus, in a certain sense we only need to determine the third cohomotopy group $\pi^3(M)$. Recall the EHP fibre sequence (cf. [20, Corollary 4.4.3])

$$\Omega^2 S^4 \xrightarrow{\Omega H} \Omega^2 S^7 \longrightarrow S^3 \xrightarrow{E} \Omega S^4 \xrightarrow{H} \Omega S^7,$$

which induces an exact sequence

$$[M, \Omega^2 S^4] \xrightarrow{(\Omega H)_\sharp} [M, \Omega^2 S^7] \longrightarrow [M, S^3] \xrightarrow{E_\sharp} [M, \Omega S^4] \to 0, \tag{5.3}$$

where $0 = [M, \Omega S^7] = [\Sigma M, S^7]$ by dimensional reason.

Lemma 5.4. *Let M be a 5-manifold given by Theorem 1.1. Then*

1. *$[\Sigma^2 M, S^7] \cong \mathbb{Z}\langle q_7\rangle$, where q_7 is the canonical pinch map;*
2. *$[\Sigma^2 M, S^4]$ contains a direct summand $\mathbb{Z}\langle \nu_4 q_7\rangle$, where $\nu_4\colon S^7 \to S^4$ is the Hopf map.*

Proof. By Theorem 1.1, there is a homotopy decomposition

$$\Sigma^2 M \simeq U \vee V,$$

where U is a 6-dimensional complex and V belongs to the set

$$\mathcal{S} = \{S^7,\, C^7_\eta,\, A^7(\tilde{\eta}_{r_{j_1}}) = P^5(2^{r_{j_1}}) \cup_{\tilde{\eta}_{r_{j_1}}} \mathrm{e}^7,\, A^7(i_P\tilde{\eta}_{r_{j_1}}) = C^6_{r_{j_1}} \cup_{i_P\tilde{\eta}_{r_{j_1}}} \mathrm{e}^7\}.$$

Let $q_V\colon \Sigma^2 M \to V$ be the pinch map onto V. Then it is clear that the pinch map q_7 factors as the composite $\Sigma^2 M \xrightarrow{q_V} V \xrightarrow{q_7 \text{ or } \mathbb{1}_7} S^7$. We immediately have the chain of isomorphisms

$$[\Sigma^2 M, S^7] \underset{\cong}{\xleftarrow{q_V^\sharp}} [V, S^7] \cong \mathbb{Z}\langle q_7\rangle.$$

For the group $[\Sigma^2 M, S^4]$, we show that the direct summand $[V, S^4]$ (through the homomorphism $q_V^\sharp$) is isomorphic to $\mathbb{Z}\langle \nu_4 q_7\rangle \oplus \mathbb{Z}/12$ for any $V \in \mathcal{S}$.

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If $V = S^7$, we clearly have $[S^7, S^4] \cong \mathbb{Z}\langle \nu_4 \rangle \oplus \mathbb{Z}/12$. If $V = C^7_\eta$, then from the homotopy cofibre sequence

$$S^6 \xrightarrow{\eta} S^5 \xrightarrow{i_5} C^7_\eta \xrightarrow{q_7} S^7 \xrightarrow{\eta} S^6$$

we have an exact sequence

$$0 \to \pi_7(S^4) \xrightarrow{q_7^\sharp} [C^7_\eta, S^4] \xrightarrow{i_5^\sharp} \pi_5(S^4) \xrightarrow{\eta^\sharp} \pi_6(S^4).$$

Since $\eta^\sharp$ is an isomorphism, $i_5^\sharp$ is trivial and hence $q_7^\sharp$ is an isomorphism. Thus we have

$$[C^7_\eta, S^4] \cong (q_7)^\sharp(\pi_7(S^4)) \cong \mathbb{Z}\langle \nu_4 q_7 \rangle \oplus \mathbb{Z}/12.$$

If $V = A^7(\tilde{\eta}_r) = P^5(2^{r_{j_1}}) \cup_{\tilde{\eta}_{r_{j_1}}} \mathrm{e}^7$, the homotopy cofibre sequence

$$S^6 \xrightarrow{\tilde{\eta}_{r_{j_1}}} P^5(2^{r_{j_1}}) \xrightarrow{i_P} A^7(\tilde{\eta}_r) \xrightarrow{q_7} S^7 \longrightarrow P^6(2^{r_{j_1}})$$

implying an exact sequence

$$0 \to \pi_7(S^4) \xrightarrow{q_7^\sharp} [A^7(\tilde{\eta}_r), S^4] \xrightarrow{i_P^\sharp} [P^5(2^{r_{j_1}}), S^4] \xrightarrow{\tilde{\eta}_{r_{j_1}}^\sharp} \pi_6(S^4).$$

Since $[P^5(2^{r_{j_1}}), S^4] \cong \mathbb{Z}/2\langle \eta q_5 \rangle$, the formula $q_5 \tilde{\eta}_{r_{j_1}} = \eta$ in (2.2) then implying $\tilde{\eta}_{r_{j_1}}^\sharp$ is an isomorphism. Thus

$$[A^7(\tilde{\eta}_r), S^4] \cong (q_7)^\sharp(\pi_7(S^4)) \cong \mathbb{Z}\langle \nu_4 q_7 \rangle \oplus \mathbb{Z}/12.$$

The computations for $V = A^7(i_P\tilde{\eta}_r)$ is similar. First, it is clear that

$$[C^6_{r_{j_1}}, S^4] \xleftarrow[\cong]{i_P^\sharp} [P^5(2^{r_{j_1}}), S^4] \cong \mathbb{Z}/2\langle \eta q_5 \rangle.$$

Recall we have the composite $q_5\colon P^5(2^{r_{j_1}}) \xrightarrow{i_P} C^6_{r_{j_1}} \xrightarrow{q_5} S^5$. It follows that the homomorphism $[C^6_{r_{j_1}}, S^4] \xrightarrow{(i_P\tilde{\eta}_{r_{j_1}})^\sharp} \pi_6(S^4)$ is an isomorphism, and thus there is an isomorphism

$$[A^7(i_P\tilde{\eta}_r), S^4] \cong (q_7)^\sharp(\pi_7(S^4)) \cong \mathbb{Z}\langle \nu_4 q_7 \rangle \oplus \mathbb{Z}/12. \qquad \square$$

LEMMA 5.5. *Let $r \geqslant 1$ be an integer. There hold isomorphisms*

(1) $[C^5_\eta, S^4] = 0$ *and* $[C^5_r, S^4] \cong \mathbb{Z}/2^{r+1}$.

(2) $[A^6(\tilde{\eta}_r), S^4] \cong \mathbb{Z}/2^{r-1}$, *where* $\mathbb{Z}/1 = 0$ *for* $r = 1$.

(3) $[A^6(i_P\tilde{\eta}_r), S^4] \cong \mathbb{Z}/2^r$.

Proof. (1) The groups in (1) refer to [**2**] or [**14**].

(2) The homotopy cofibre sequence for $A^6(\tilde{\eta}_r)$, as given in the proof of lemma 5.4, implying an exact sequence

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$$[P^5(2^r), S^4] \xrightarrow[\cong]{\tilde{\eta}_r^\sharp} [S^6, S^4] \to [A^6(\tilde{\eta}_r), S^4] \xrightarrow{i_P^\sharp} [P^4(2^r), S^4] \xrightarrow{\tilde{\eta}_r^\sharp} [S^5, S^4].$$

Thus $(i_P)^\sharp$ is a monomorphism and $\mathrm{im}(i_P)^\sharp = \ker(\tilde{\eta}_r^\sharp) \cong \mathbb{Z}/2^{r-1}\langle 2q_4\rangle$.

(3) The computation of the group $[A^6(i_P\tilde{\eta}_r),\, S^4]$ is similar, by noting the isomorphism $[C_r^5,\, S^4] \cong \mathbb{Z}/2^{r+1}\langle q_4\rangle$ (cf. [2]). □

Proposition 5.6. *Let M be a 5-manifold given by Theorems 1.1 or 1.2. The homomorphism $(\Omega H)_\sharp$ in (5.3) is surjective and hence there is an isomorphism*

$$\Sigma\colon \pi^3(M) \to \pi^4(\Sigma M).$$

Moreover, let M be the 5-manifold, together with the integers c_1, c_2 and r_{j_1}, given by Theorem 1.1, then we have the following concrete results:

(1) *if M is spin, then*

$$\pi^3(M) \cong \mathbb{Z}^d \oplus (\mathbb{Z}/2)^{l+1-c_1-c_2} \oplus T[c_2] \oplus \left(\bigoplus_{j=1}^{c_2} \mathbb{Z}/2^{r_j+1}\right);$$

(2) *if M is non-spin and the conditions in (a) hold, then*

$$\pi^3(M) \cong \mathbb{Z}^d \oplus (\mathbb{Z}/2)^{l-c_1-c_2} \oplus T[c_2] \oplus \left(\bigoplus_{j=1}^{c_2} \mathbb{Z}/2^{r_j+1}\right);$$

(3) *if M is non-spin and the conditions in (b) hold, then $\pi^3(M)$ is isomorphic to one of the following groups:*

$$(i) \quad \mathbb{Z}^d \oplus (\mathbb{Z}/2)^{l-c_1-c_2} \oplus \frac{T[c_2]}{\mathbb{Z}/2^{r_{j_1}}} \oplus \left(\bigoplus_{j=1}^{c_2} \mathbb{Z}/2^{r_j+1}\right) \oplus \mathbb{Z}/2^{r_{j_1}-1},$$

$$(ii) \quad \mathbb{Z}^d \oplus (\mathbb{Z}/2)^{l-c_1-c_2} \oplus T[c_2] \oplus \left(\bigoplus_{j_1\neq j=1}^{c_2} \mathbb{Z}/2^{r_j+1}\right) \oplus \mathbb{Z}/2^{r_{j_1}}.$$

Proof. We first apply the exact sequence (5.3) to show that the suspension $\pi^3(M) \xrightarrow{\Sigma} \pi^4(\Sigma M)$ is an isomorphism. By duality, it suffices to show the second James–Hopf invariant H induces a surjection $H_\sharp\colon [\Sigma^2 M,\, S^4] \to [\Sigma^2 M,\, S^7]$. By lemma 5.4, there hold isomorphisms

$$[\Sigma^2 M, S^7] \cong \mathbb{Z}\langle q_7\rangle \quad \text{and} \quad [\Sigma^2 M, S^4] \cong \mathbb{Z}\langle \nu_4 q_7\rangle \oplus G$$

for some abelian group G. Then the surjectivity of $H_\sharp$ follows by the homotopy equalities

$$H(\nu_4) = \mathbb{1}_7, \quad H(\nu_4 q_7) = H(\nu_4) q_7 = q_7.$$

Note the first statement only depends the homotopy type of the double suspension $\Sigma^2 M$, so we can also assume that M is the five-manifold satisfying conditions in Theorem 1.1.

The computations of the group $[\Sigma M, S^4]$ follows by Theorem 1.1, lemma 5.5:

(1) If M is spin, then

$$[\Sigma M, S^4] \cong \left(\bigoplus_{i=1}^{d}[S^4, S^4]\right) \oplus \left(\bigoplus_{i=1}^{l-c_1-c_2}[S^5, S^4]\right) \oplus [P^4(T[c_2]), S^4] \oplus \left(\bigoplus_{j=1}^{c_2}[C^5_{r_j}, S^4]\right) \oplus [S^6, S^4].$$

(2) If M is non-spin and ΣM is given by (a), then

$$[\Sigma M, S^4] \cong \left(\bigoplus_{i=2}^{d}[S^4, S^4]\right) \oplus \left(\bigoplus_{i=1}^{l-c_1-c_2}[S^5, S^4]\right) \oplus [P^4(T[c_2]), S^4] \oplus \left(\bigoplus_{j=1}^{c_2}[C^5_{r_j}, S^4]\right) \oplus [C^6_\eta, S^4].$$

(3) If M is non-spin and ΣM is given by (b), then

$$[\Sigma M, S^4] \cong \left(\bigoplus_{i=1}^{d}[S^4, S^4]\right) \oplus \left(\bigoplus_{i=1}^{l-c_1-c_2}[S^5, S^4]\right) \oplus \left[P^4\left(\frac{T[c_2]}{\mathbb{Z}/2^{r_{j_1}}}\right), S^4\right] \oplus \left(\bigoplus_{j=1}^{c_2}[C^5_{r_j}, S^4]\right) \oplus [A^6(\tilde{\eta}_{r_{j_1}}), S^4],$$

or

$$[\Sigma M, S^4] \cong \left(\bigoplus_{i=1}^{d}[S^4, S^4]\right) \oplus \left(\bigoplus_{i=1}^{l-c_1-c_2}[S^5, S^4]\right) \oplus [P^4(T[c_2]), S^4] \oplus \left(\bigoplus_{j_1\neq j=1}^{c_2}[C^5_{r_j}, S^4]\right) \oplus [A^6(i_P\tilde{\eta}_{r_{j_1}}), S^4].$$

□

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