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RINGS WITH A SPECIAL KIND OF AUTOMORPHISM

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ABSTRACT. In this paper we examine the nature of rings R with unit having an automorphism $\phi \neq 1$ such that $x - \phi(x)$ is 0 or invertible for every $x \in R$. We show that the only examples of such rings are $R = D$, $R = D_2$, and $R = D \oplus D$, where D is a division ring. Furthermore, for the case D_2, we describe the division rings that are possible.

In a recent paper [1] we considered the structure of a ring R with 1 having non-zero derivation, d, such that $d(x) = 0$ or is invertible for every $x \in R$. If R has no 2-torsion we showed that R is either a division ring, D, or D_2, the ring of 2×2 matrices over D. Moreover we completely characterized those D for which D_2 has such a derivation. Even if R has 2-torsion similar results were obtained, when R is semi-prime or when d is inner.

If ϕ is an automorphism of a ring R then the map δ defined by $\delta(x) = x - \phi(x)$ behaves almost like a derivation in that $\delta(xy) = \delta(x)y + \phi(x)\delta(y)$. We shall consider here the nature of rings R with 1 having an automorphism $\phi \neq 1$ such that $x - \phi(x)$ is 0 or invertible for every $x \in R$. Note that unlike the situation for derivations described above, outside of D and D_2 there is another obvious candidate, namely $R = D \oplus D$, where D is any division ring. The automorphism ϕ defined on R by $\phi(a, b) = (b, a)$ has the property that $\phi(x) - x$ is 0 or invertible for every $x \in R$. We shall show that these 3 examples, D, D_2, and $D \oplus D$ are the only possible rings with such an automorphism. Furthermore, for the case D_2, we describe the division rings that are possible.

In what follows, R will always be a ring with 1 and $\phi \neq 1$ will be an automorphism of R such that $x - \phi(x)$ is either 0 or invertible, for every $x \in R$.

We begin with

LEMMA 1. *If $\phi(x) = x$ then $x = 0$ or x is invertible.*

Proof. Since $\phi = 1$ there is an $r \in R$ such that $a = \phi(r) - r \neq 0$, hence a is invertible. Suppose that $0 \neq \phi(x) = x$; then $\phi(rx) - rx = ax \neq 0$ since a is invertible and $x \neq 0$, hence ax is invertible. Thus x is invertible.

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COROLLARY. *If $L \neq R$ is a left ideal of R then $L \cap \phi(L) = 0$.*

Proof. We may suppose that $L \neq 0$; let $0 \neq x \in L \cap \phi(L)$, then $x = \phi(y)$ for some $y \in L$ and $y - \phi(y) \in L$. Since $L \neq R$, $y - \phi(y)$ cannot be invertible, hence $y = \phi(y)$. Since $y \neq 0$, by the Lemma we have that y is invertible, implying that $L = R$.

We continue the study of R by giving a closer look at its left ideals.

LEMMA 2. *Every non-trivial left ideal of R is a minimal left ideal.*

Proof. If $L \neq 0$, R is a left ideal of R then $I = L + \phi(L)$ is also a left ideal of R. Since $I \cap \phi(I) \supset \phi(L) \neq 0$, by the Corollary to Lemma 1 we get that $R = I = L + \phi(L)$. Also by the Corollary to Lemma 1, $L \cap \phi(L) = 0$, so R is the direct sum of L and $\phi(L)$. If $0 \neq L_1 \subset L$ is a left ideal of R then, by the same token, $R = L_1 + \phi(L_1)$, so if $0 \neq t \in L$ then $t = u + \phi(v)$ where $u, v \in L_1$. Thus $\phi(v) = t - u \in L \cap \phi(L)$ so $v = 0$ by the Corollary to Lemma 1. This gives us $t = u \in L_1$, hence $L \subset L_1$, and so $L = L_1$. This proves the minimality of L.

Lemma 2 tells us that every left ideal of R is both minimal and maximal, hence R is certainly artinian of Goldie rank at most 2. From Lemma 2 we now obtain

LEMMA 3. *If R is not simple, then $R = I_1 \oplus I_2$ where I_1 is an ideal of R, $I_2 = \phi(I_1)$, and I_1, I_2 are isomorphic division rings.*

Proof. Let $I \neq 0$, R be an ideal of R. By Lemmas 1 and 2, $R = I \oplus \phi(I)$, and I is a minimal left ideal of R. Moreover, I has no non-trivial left ideals (of itself) for, if $0 \neq J$ is a left ideal of I then, since $1 \in R$, $0 \neq RJ = (I \oplus \phi(I))J = IJ \subset J$, so J is a left ideal of R, whence $J = I$ by Lemma 2. Since I has no non-trivial left ideals, I is a division ring.

We also have

LEMMA 4. *If R is simple then, for some division ring D, $R = D$ or $R = D_2$.*

Proof. If R is not a division ring then, by Lemma 2, all non-trivial left ideals of R are minimal and maximal. Since R is then simple artinian we immediately have that $R = D_2$ for some division ring D.

In view of Lemma 4 the question naturally arises for what D does D_2 possess an automorphism ϕ of the required kind? If ϕ is inner, say $\phi(x) = txt^{-1}$ for all $x \in D_2$ then the condition $\phi(x) - x = 0$ or invertible becomes $tx - xt = 0$ or invertible for all $x \in D_2$. This situation was completely described by Lemma 9 of [1]; the answer is that D does not contain all quadratic extensions of its center Z. Thus our interest here is mainly in the case in which ϕ is not inner. A complete answer to the question is furnished us in

LEMMA 5. *D_2 has a non-inner automorphism ϕ such that, for all $x \in D_2$,*

$\phi(x)=x$ or $x-\phi(x)$ is invertible if and only if D has a non-inner automorphism ψ such that $\psi^2(x)=u^{-1}xu$ for all $x\in D$, where $\psi(u)=u$ and $u\neq y\psi(y)$ for all $y\in D$.

Proof. If D has such an automorphism ψ define ϕ on D_2 by

$$\phi\begin{pmatrix} x & y \\ z & w \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}\begin{pmatrix} \psi(x) & \psi(y) \\ \psi(z) & \psi(w) \end{pmatrix}\begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}^{-1}.$$

Clearly ϕ is an automorphism of D_2 and is not inner, for if ϕ is inner we get that

$$\begin{pmatrix} \psi(x) & \psi(y) \\ \psi(z) & \psi(w) \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}.$$

This implies that $\psi(x)a=ax$, $\psi(x)b=bx$, and not both $a=0$ and $b=0$ (since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible), contradicting that ψ is non-inner on D.

We verify that $\phi(x)-x$ is invertible or 0 for all $x\in D_2$. Clearly

$$\phi\begin{pmatrix} x & y \\ z & w \end{pmatrix}-\begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

is 0 or invertible according as

$$A=\begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}\begin{pmatrix} \psi(x) & \psi(y) \\ \psi(z) & \psi(w) \end{pmatrix}-\begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}=\begin{pmatrix} \psi(z)-yu & \psi(w)-x \\ u\psi(x)-wu & u\psi(y)-z \end{pmatrix}$$

is 0 or invertible. Since $x=\psi(t)$, $y=\psi(s)$ for some $s,t\in D$ and ψ^2 is inner by u, $\psi(u)=u$, we have that

$$A=\begin{pmatrix} -\psi(b) & -\psi(a) \\ au & b \end{pmatrix}\quad \text{where}\quad a=t-w,\, b=su-z.$$

If either $a=0$ or $b=0$ it is immediate to see that $A=0$ or is invertible; suppose then that $a\neq 0$, $b\neq 0$. Then A is invertible if and only if

$$B=\begin{pmatrix} \psi(b)-\psi(a)b^{-1}au & \psi(a) \\ 0 & b \end{pmatrix}$$

is invertible, that is, if and only if $\psi(b)\neq\psi(a)b^{-1}au$. This latter is certainly the case, for if $\psi(b)=\psi(a)b^{-1}au$ then $u=(a^{-1}b)\psi(a^{-1}b)$, contradicting our hypothesis on u.

Suppose, on the other hand, that D_2 has a non-inner automorphism ϕ such that for all $x\in D_2$, $x-\phi(x)$ is 0 or invertible. By the nature of automorphisms on matrix rings,

$$\phi\begin{pmatrix} x & y \\ z & w \end{pmatrix}=t\begin{pmatrix} \psi(x) & \psi(y) \\ \psi(z) & \psi(w) \end{pmatrix}t^{-1},$$

where ψ is an automorphism of D and $t\in D_2$ is invertible. Our condition on ϕ

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immediately implies that ψ is not inner on D and that:

(1) $t\begin{pmatrix}\psi(x) & \psi(y)\\ \psi(z) & \psi(w)\end{pmatrix}-\begin{pmatrix}x & y\\ z & w\end{pmatrix}t=0$ or is invertible for all $\begin{pmatrix}x & y\\ z & w\end{pmatrix}\in D_2$.

Let $t=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$; then, by Lemma 1,

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}-\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\neq 0,$$

so is invertible, that is $\begin{pmatrix}0 & -b\\ c & 0\end{pmatrix}$ is invertible. Hence $b\neq 0$, $c\neq 0$. Let

$$\Psi\begin{pmatrix}x & y\\ z & w\end{pmatrix}=\begin{pmatrix}\psi(x) & \psi(y)\\ \psi(z) & \psi(w)\end{pmatrix}.$$

If $A\in D_2$ is invertible and $Y\in D_2$ then $t\Psi(AYA^{-1})-AYA^{-1}t=A(A^{-1}t\Psi(A)\Psi(Y)-YA^{-1}t\Psi(A))\Psi(A)^{-1}$ is 0 or invertible, hence the same holds for $A^{-1}t\psi(A)\psi(Y)-YA^{-1}t\psi(A)$ for all $Y\in D_2$. Thus we can change the t in condition (1) into $A^{-1}t\psi(A)$ for any invertible $A\in D_2$.

We claim that we may assume that $t=\begin{pmatrix}0 & 1\\ u & 0\end{pmatrix}$. We see this in a few steps. Firstly we assert that we may assume that $a=0$, for, if not, then using

$$A=\begin{pmatrix}1 & ac^{-1}\\ 0 & 1\end{pmatrix},\ t_1=A^{-1}t\psi(A)=\begin{pmatrix}0 & b_1\\ c_1 & d_1\end{pmatrix};$$

as we saw above, $b_1\neq 0$, $c_1\neq 0$. Thus

$$B^{-1}t_1\psi(B)=\begin{pmatrix}0 & 1\\ u & e\end{pmatrix}\quad\text{where}\quad B=\begin{pmatrix}b_1 & 0\\ 0 & 1\end{pmatrix}.$$

We claim that $e=0$. For, if $x\in D$ then

$$\begin{aligned}W&=\begin{pmatrix}0 & 1\\ u & e\end{pmatrix}\begin{pmatrix}\psi(x) & \psi(x)u^{-1}e-u^{-1}e\psi(u)\psi^2(x)\psi(u)^{-1}\\ 0 & \psi(u\psi(x)u^{-1})\end{pmatrix}\\ &\quad-\begin{pmatrix}x & x\psi^{-1}(u^{-1}e)-\psi^{-1}(u^{-1}e)u\psi(x)u^{-1}\\ 0 & u\psi(x)u^{-1}\end{pmatrix}\begin{pmatrix}0 & 1\\ u & e\end{pmatrix}\\ &=\begin{pmatrix}x\psi^{-1}(u^{-1}e)u-\psi^{-1}(u^{-1}e)u\psi(x) & *\\ 0 & 0\end{pmatrix}.\end{aligned}$$

Therefore, by (1), $W=0$, and so, for all $x\in D$, $\psi^{-1}(u^{-1}e)u\psi(x)=x\psi^{-1}(u^{-1}e)u$. But ψ is not inner, hence $\psi^{-1}(u^{-1}e)u=0$, from which we get that $e=0$.

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Therefore we may assume that $t=\begin{pmatrix}0 & 1\\ u & 0\end{pmatrix}$. Computing

$$\begin{pmatrix}0 & 1\\ u & 0\end{pmatrix}\begin{pmatrix}\psi(x) & 0\\ 0 & \psi(u\psi(x)u^{-1})\end{pmatrix}-\begin{pmatrix}x & 0\\ 0 & u\psi(x)u^{-1}\end{pmatrix}\begin{pmatrix}0 & 1\\ u & 0\end{pmatrix}$$
$$=\begin{pmatrix}0 & \psi(u)\psi^2(x)\psi(u)^{-1}-x\\ 0 & 0\end{pmatrix}$$

we obtain from (1) that $\psi^2(x)=\psi(u)^{-1}x\psi(u)$. Also

$$\begin{pmatrix}0 & 1\\ u & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ \psi(u) & 0\end{pmatrix}-\begin{pmatrix}0 & 1\\ u & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ u & 0\end{pmatrix}=\begin{pmatrix}\psi(u)-u & 0\\ 0 & 0\end{pmatrix},$$

so by (1), $\psi(u)=u$; thus $\psi^2(x)=u^{-1}xu$. Finally, for $x\in D$,

$$\begin{pmatrix}0 & 1\\ u & 0\end{pmatrix}\begin{pmatrix}\psi(x) & 0\\ 1 & 0\end{pmatrix}-\begin{pmatrix}x & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ u & 0\end{pmatrix}=\begin{pmatrix}1 & -x\\ u\psi(x) & -1\end{pmatrix}$$

is invertible by Lemma 1, hence

$$\begin{pmatrix}1-xu\psi(x) & -x\\ 0 & -1\end{pmatrix}$$

is invertible. Thus $1-xu\psi(x)\neq 0$ for all $x\in D$, hence $u\neq y\psi(y)$ for all $y\in D$.

This proves the lemma.

Putting together all the pieces, we summarize what we have obtained in the

THEOREM. *Let R be a ring with 1 and $\phi\neq 1$ an automorphism of R such that for every $x\in R$, $x=\phi(x)$ or $x-\phi(x)$ is invertible in R. Then R is either*

1. *a division ring D, or*
2. *$D\oplus D$, or*
3. *D_2.*

Furthermore, D_2 is possible, with ϕ non-inner, if and only if D has a non-inner automorphism ψ such that $\psi^2(x)=u^{-1}xu$ for all $x\in D$, where $\psi(u)=u$ and $u\neq y\psi(y)$ for all $y\in D$, or with ϕ inner if and only if D does not contain all quadratic extensions of its center Z.

Note that if char $D\neq 2$ then D does not contain all quadratic extensions of Z if and only if some $\alpha\in Z$ fails to be a square in D. In that case, using $\psi=1$ as the automorphism of D, clearly $\psi^2(x)=x=\alpha^{-1}x\alpha$ and $\alpha\neq y\psi(y)$ for all $y\in D$. Thus, in this case, the division between inner and non-inner disappears, and the theorem reads: if $2R\neq 0$ and R has an automorphism $\phi\neq 1$ such that $x-\phi(x)$ is 0 or invertible for all $x\in R$ then $R=D, D\oplus D$, or D_2, for some division ring D; furthermore, D_2 is possible if and only if D has an automorphism ψ such that $\psi^2(x)=u^{-1}xu$ and $\psi(u)=u$ and $u\neq y\psi(y)$ for all $y\in D$.

In line with what we did in [1] we also consider the situation in which we merely suppose that the automorphism ϕ behaves in a given pattern, not on R itself, but merely on a left ideal of R.

Let R be a ring, $L \neq 0$, R a left ideal of R and ϕ an automorphism of R such that is not the identity on L, and such that $x - \phi(x)$ is 0 or invertible for each $x \in L$. We shall show, as before, that R is either D, D_2, or $D \oplus D$ for some division ring D (although the condition we obtained previously on D does not necessarily carry over).

If $0 \neq r \in R$ and $\phi(r) = r$ then, since $rL \subset L$, we get that $r(\phi(x) - x) = 0$ or invertible for all $x \in L$; however, since $\phi(x) - x \neq 0$ for some $x \in L$, allows us to conclude that r is invertible. As before, this immediately implies that L is a minimal left ideal of R and $R = L \oplus \phi(L)$; this latter clearly implies that R is artinian.

R is semi-simple artinian, for if $A^2 = 0$ for some ideal A of R then $A = AR = AL \oplus A\phi(L)$; because $L, \phi(L)$ are minimal left ideals of R and $AL = L$ would force $0 = A^2L = AL = L$, we conclude that $AL = A\phi(L) = 0$. Thus $A = 0$.

Since R is semi-simple artinian and $R = L \oplus \phi(L)$ with L, $\phi(L)$ minimal left ideals of R, if L is not a 2-sided ideal of R, $R = D_2$ follows by Wedderburn's theorem. If L is a 2-sided ideal of R, then L must be a division ring, D, and $R \approx D \oplus D$. This proves the assertion made above.

References

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