On the ergodicity of geodesic flows on surfaces without focal points

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Abstract. In this paper, we study the ergodicity of the geodesic flows on surfaces with no focal points. Let M be a smooth connected and closed surface equipped with a C^∞ Riemannian metric g, whose genus $\mathfrak{g} \geq 2$. Suppose that (M, g) has no focal points. We prove that the geodesic flow on the unit tangent bundle of M is ergodic with respect to the Liouville measure, under the assumption that the set of points on M with negative curvature has at most finitely many connected components.

1. *Introduction*

Assume that (M, g) is a smooth, connected, and closed manifold equipped with a C^∞ Riemannian metric g. The geodesic flow g^t, generated by the Riemannian metric g, is defined on the unit tangent bundle SM by the formula:

$$g^t(v) = \gamma'_v(t),$$

where $\gamma'_v(t)$ is the unit vector tangent to the geodesic $\gamma_v(t)$ uniquely determined by the initial vector $v \in SM$. In this paper, we study the ergodicity of the geodesic flow with respect to the Liouville measure ν on SM, where (M, g) is assumed to be a surface of genus $\mathfrak{g} \geq 2$ having no focal points.

Our work was originally inspired by the classical results on the ergodicity of the geodesic flows on Riemannian manifolds with non-positive curvature. The geodesic flows

on Riemannian manifolds with negative or non-positive curvature have very rich dynamics and broad applications. In the last century, this class of geodesic flows have always been attracting the interest of mathematicians in dynamical systems and related areas. Many beautiful results on the dynamics of the geodesic flows have been exhibited. Among which, the ergodic properties, such as the ergodicity and the mixing properties, the measure of maximal entropy, etc., have a special importance and receive extensive attention. The statistical properties of geodesic flows on surfaces with negative curvature were first studied by Hadamard and Morse in the beginning of the twentieth century. Hopf [16, 17] proved the ergodicity of the geodesic flow with respect to the Liouville measure ν on SM for compact surfaces of variable negative curvature and for compact manifolds of constant negative sectional curvature in any dimension. The general case for compact manifolds of variable negative curvature was established by Anosov and Sinai [1, 2]. The geodesic flows on compact manifolds of negative curvature is a primary example of the *uniformly hyperbolic flows* (or Anosov flows). Its ergodicity was established based on the classical Hopf argument and results in hyperbolic geometry (see, for example, the appendix in [3]).

Geodesic flows on manifolds of non-positive curvature have also been intensively studied since the 1970's. However, even for surfaces of non-positive curvature, the geodesic flows present certain *non-uniformly hyperbolic* behaviors. The ergodicity for the geodesic flows faces a great challenge due to the existence of 'flat' geodesics. Consider a closed surface M of genus $\mathfrak{g} \geq 2$ and of non-positive curvature. Let

$$\Lambda := \{v \in SM : K(\gamma_v(t)) \equiv 0 \text{ for all } t \in \mathbb{R}\},$$

where K denotes the curvature of the point. We call γ_v a *flat geodesic* if $v \in \Lambda$, that is, γ_v is a flat geodesic if the curvature along it is constantly 0. It is still not known if Λ is small in measure ($\nu(\Lambda) = 0$ or not), in general. However, from the dynamical point of view, Λ should be a very small set. For example, in [20], Knieper showed the strict inequality for geodesic flows on rank 1 manifolds of non-positive curvature:

$$h(g^1|_\Lambda) < h(g^1),$$

where g^1 is the time-one map of the geodesic flow g^t, h denotes the topological entropy, and Λ denotes the irregular set of the geodesic flow, which is a counterpart of the above defined set in arbitrary dimensions. This means that the geodesic flow restricted on Λ has less complexity than the whole geodesic flow. Knieper [20] (see also [7]) proved that on rank-1 surfaces of non-positive curvature, the geodesic flow on Λ has zero topological entropy. In higher dimensions, it is possible to have positive entropy on Λ; an example was given by Gromov [15].

For geodesic flows on rank-1 surfaces of non-positive curvature, the orbits inside Λ are also believed to have a simple behavior. In all the known examples, all the orbits in Λ are closed. In a recent survey, Burns asks the question: Does there exist a non-closed flat geodesic? [8, Question 6.2.1]. In this paper, we will show that all flat geodesics are closed on surfaces without focal points, under our assumption. Nevertheless, the most important topic on the set Λ is still its Liouville measure (that is, how small it is). One expects that on surfaces with non-positive curvature, Λ should have 0 Liouville measure (this leads to the ergodicity of the geodesic flows, see [5]). This is the following well-known conjecture on

the ergodicity for geodesic flows on surfaces with non-positive curvature. (Some experts in the area expect a negative answer to the conjecture. Our results in the paper support the conjecture under an additional assumption.)

Conjecture 1.1. (Cf. [22]) Let (M, g) be a smooth, connected, and closed surface of genus $\mathfrak{g} \geq 2$, which has non-positive curvature. Then all flat geodesics are closed and there are only finitely many homotopy classes of such geodesics. In particular, $\nu(\Lambda) = 0$, and hence the geodesic flow on SM is ergodic.

We declare that the terminology 'ergodicity' in this paper means the ergodicity with respect to the Liouville measure ν on SM. The problem we are considering in this paper is the ergodicity of the geodesic flows on surfaces without focal points. First of all, we give the definition of the focal points.

Definition 1.2. Let (M, g) be a Riemannian manifold and γ a geodesic on M. Points $q = \gamma(t_0)$ and $p = \gamma(t_1)$ are called *focal* if there exists a Jacobi field J along γ such that $J(t_0) = 0$, $J'(t_0) \neq 0$, and $d/dt \|J(t)\|^2 |_{t=t_1} = 0$. The Riemannian manifold (M, g) is said to be *without focal points* if there are no focal points on any geodesic of M.

It is not hard to see that the manifolds with non-positive curvature have no focal points. If M is a surface of genus 1 and has no focal points, then it must be a flat torus [6, 18]. Therefore, the geodesic flow on M is obviously not ergodic. However, if M has higher genus, the curvature is allowed to vary. In this paper, we always assume that the surface M we are considering has genus greater than 1.

In the 1970's, by using his theory of non-uniform hyperbolicity, Pesin obtained a celebrated result on the ergodicity of the geodesic flows on manifolds without focal points, which satisfy the uniform visibility axiom [5, Theorem 12.2.12]. We are not going to give the explicit definition of the uniform visibility axiom here, but remark that it is satisfied by every closed surface of genus $\mathfrak{g} \geq 1$. To state Pesin's result for surfaces without focal points, we define the sets:

$$\Delta^+ := \{v \in SM : \chi(v, \xi) < 0 \text{ for any } \xi \in E^+(v)\},$$
$$\Delta^- := \{v \in SM : \chi(v, \xi) > 0 \text{ for any } \xi \in E^-(v)\},$$
$$\Delta := \Delta^+ \cap \Delta^-,$$

where χ denotes the Lyapunov exponents and $E^\pm$ denotes the stable and unstable distributions on SM with respect to the geodesic flow, respectively. Here, Δ is called the *regular set* with respect to the geodesic flow. For details, see §2. Pesin proved the following theorem.

THEOREM 1.3. (Cf. [5]) *For the geodesic flow on a surface without focal points, we have that $\nu(\Delta) > 0$, and $g^t|_\Delta$ is ergodic.*

Our first result in this paper is the following relation between the regular set Δ and the set Λ of unit vectors tangent to flat geodesics. We remark that all our results are established under *the assumption of no focal points* so, sometimes, we omit the statement of this assumption in the following theorems.

THEOREM 1.4. $\nu(\Lambda^c \setminus \Delta) = 0$.

By Theorems 1.3 and 1.4, if $\nu(\Lambda) = 0$, the regular set $\Delta \subset SM$ is a full measure set and then the geodesic flow is ergodic on SM. The condition $\nu(\Lambda) = 0$ holds in all the known examples so far. However, it is still not proved, even for the surfaces of non-positive curvature. Recent progress on this problem was made by the first author in [25]. We conclude the main result of [25] in the following theorem.

THEOREM 1.5. (Cf. [25]) *Let (M, g) be a smooth, connected, and closed surface of genus $\mathfrak{g} \geq 2$, which has non-positive curvature. Suppose that the set $\{p \in M : K(p) < 0\}$ has finitely many connected components, then $\nu(\Lambda) = 0$. In particular, the geodesic flow is ergodic.*

In this paper, we generalize Theorem 1.5 from the setting of surfaces with non-positive curvature to surfaces without focal points. This means that we are going to prove the ergodicity of the geodesic flows on surfaces which can have positive curvature in a subset. To achieve this goal, we explore the properties of flat geodesics, which are also of independent interest. Among them is the following result. Here, we let $\text{Per}(g^t)$ denote the set of periodic points of the geodesic flow and $\mathcal{O}(z)$ denote the orbit of z under the geodesic flow. The following theorem says that non-closed flat orbits can accumulate only on non-closed flat orbits.

THEOREM 1.6. $\Lambda \cap (Per (g^t))^c$ *is a closed subset of SM.*

According to the dichotomy: (1) $\Lambda \subset \text{Per}(g^t)$; (2) $\Lambda \cap (\text{Per} (g^t))^c \neq \emptyset$, and we prove the following two results.

THEOREM 1.7. *If $\Lambda \subset Per(g^t)$, then there is a finite decomposition of Λ:*

$$\Lambda = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \mathcal{O}_k \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_l,$$

where each $\mathcal{O}_i$, $1 \leq i \leq k$, is an isolated periodic orbit and each $\mathcal{F}_j$, $1 \leq j \leq l$, consists of vectors tangent to a flat strip. Here, k or l is allowed to be 0 if there is no isolated closed flat geodesic or no flat strip.

We remark that if $\Lambda \subset \text{Per}(g^t)$, then Theorem 1.7 immediately implies $\nu(\Lambda) = 0$, and therefore the geodesic flow is ergodic.

THEOREM 1.8. *If $\Lambda \cap (Per(g^t))^c \neq \emptyset$, then there exist $y, z \in \Lambda$, $y \notin \mathcal{O}(z)$, such that*

$$d(g^t(y), g^t(z)) \to 0 \quad \text{as } t \to +\infty.$$

Our main result is the following theorem, which means that under certain conditions, the scenario in Theorem 1.8 cannot happen.

THEOREM 1.9. *If the set $\{p \in M : K(p) < 0\}$ has at most finitely many connected components, then $\Lambda \subset Per(g^t)$. In particular, the geodesic flow is ergodic.*

Theorem 1.9 gives a negative answer to Question 6.2.1 asked by Burns in [8] for surfaces without focal points when $\{p \in M : K(p) < 0\}$ has at most finitely many connected

components. Furthermore, Theorem 1.7 exhibits that in fact there are at most finitely many flat strips and isolated closed flat geodesics in this case.

So far, it is still unknown whether Conjecture 1.1 is true or not in general. In §4, we discover several properties of the flat geodesics on surfaces without focal points, which include:

- all flat strips are closed;
- a unit vector not tangent to a flat strip has the expansivity property;
- an ideal triangle with a flat geodesic which is asymptotic to a closed geodesic as an edge has infinite area.

All these results together with our Theorem 1.6 are believed to be important toward Conjecture 1.1 in future research.

The paper is organized as follows. In §2, we will present some preliminaries on the geodesic flows on surfaces without focal points. The proof of Theorem 1.4 is shown in §3. In §4, we prove Theorem 1.6 and the above properties of the flat geodesics. Our main Theorems 1.7, 1.8, and 1.9 are proved in the last section. Throughout the remainder of the paper, we always let M be a smooth, connected, and closed surface with genus $\mathfrak{g} \geq 2$, and equipped with a C^{∞} Riemannian metric g without focal points.

2. *Preliminaries on surfaces without focal points*

2.1. *Jacobi fields, and stable and unstable distributions.*

To study the dynamics of geodesic flows, we should investigate the geometry of the second tangent bundle TTM. Let $\pi : TM \to M$ be the natural projection, that is, $\pi(v) = p$, where $v \in T_p M$. The connection map $K_v : T_v TM \to T_{\pi(v)} M$ is defined as follows. For any $\xi \in T_v TM$, $K_v \xi := (\nabla X)(t)|_{t=0}$, where $X : (-\epsilon, \epsilon) \to TM$ is a smooth curve satisfying $X(0) = v$ and $X'(0) = \xi$, and ∇ is the covariant derivative along the curve $\pi(X(t)) \subset M$. Then the standard Sasaki metric on TTM is given by

$$\langle \xi, \eta \rangle_v = \langle d\pi_v \xi, d\pi_v \eta \rangle + \langle K_v \xi, K_v \eta \rangle, \quad \xi, \eta \in T_v TM.$$

Recall that the *Jacobi equation* along a geodesic $\gamma_v(t)$ is

$$J''(t) + R(\gamma_v'(t), J(t))\gamma_v'(t) = 0, \tag{1}$$

where R is the curvature tensor, and $J(t)$ is a Jacobi field along $\gamma_v(t)$ and perpendicular to $\gamma_v'(t)$. Suppose $J_\xi(t)$ is the solution of equation (1) which satisfies the initial conditions

$$J_\xi(0) = d\pi_v \xi, \quad \frac{d}{dt}\bigg|_{t=0} J_\xi(t) = K_v \xi.$$

Then, it follows that (cf. [5, p. 386])

$$J_\xi(t) = d\pi_{g^t v} dg_v^t \xi, \quad \frac{d}{dt} J_\xi(t) = K_{g^t v} dg_v^t \xi.$$

On the surface M, we have the Fermi coordinates $\{e_1(t), e_2(t)\}$ along the geodesic $\gamma_v(t)$, obtained by the time t-parallel translations along $\gamma_v(t)$ of an orthonormal basis $\{e_1(0), e_2(0)\}$ where $e_1(0) = \gamma_v'(0)$. Thus, $e_1(t) = \gamma_v'(t)$ and $e_2(t) \perp \gamma_v'(t)$. Suppose that

$J(t) = j(t)e_2(t)$. Then the Jacobi equation (1) becomes

$$j''(t) + K(t)j(t) = 0, \tag{2}$$

where $K(t) = K(\gamma_v(t))$ is the curvature at point $\gamma_v(t)$. Let $u(t) = j'(t)/j(t)$. Then the Jacobi equation (2) can be written in an equivalent form

$$u'(t) + u^2(t) + K(t) = 0, \tag{3}$$

which is called *the Riccati equation*.

Using the Fermi coordinates, we can write equation (1) in the matrix form

$$\frac{d^2}{dt^2}A(t) + K(t)A(t) = 0. \tag{4}$$

The following result is a standard fact.

PROPOSITION 2.1. (Cf. [12]) *Given $s \in \mathbb{R}$, let $A_s(t)$ be the unique solution of equation (4) satisfying $A_s(0) = \mathrm{Id}$ and $A_s(s) = 0$, then there exists a limit*

$$A^+ = \lim_{s \to +\infty} \left. \frac{d}{dt} \right|_{t=0} A_s(t).$$

Now we can define the *positive limit solution* $A^+(t)$ as the solution of equation (4) satisfying the initial conditions

$$A^+(0) = \mathrm{Id}, \quad \left. \frac{d}{dt} \right|_{t=0} A^+(t) = A^+.$$

It is easy to see that $A^+(t)$ is non-degenerate for all $t \in \mathbb{R}$. Similarly, letting $s \to -\infty$, one can define the *negative limit solution* $A^-(t)$ of equation (4).

For each $v \in SM$, define

$$E^+(v) := \{\xi \in T_v SM : \langle \xi, V(v) \rangle = 0 \text{ and } J_\xi(t) = A^+(t)\, d\pi_v \xi\},$$
$$E^-(v) := \{\xi \in T_v SM : \langle \xi, V(v) \rangle = 0 \text{ and } J_\xi(t) = A^-(t)\, d\pi_v \xi\},$$

where V is the vector field generated by the geodesic flow and J_ξ is the solution of equation (1) satisfying

$$J_\xi(0) = d\pi_v \xi, \quad \left. \frac{d}{dt} \right|_{t=0} J_\xi(t) = K_v \xi.$$

One can check the following properties of $E^+(v)$ and $E^-(v)$ (see [5] for more details).

PROPOSITION 2.2. (Cf. [5, Proposition 12.1.1]) *$E^+(v)$ and $E^-(v)$ have the following properties.*
(1) *$E^+(v)$ and $E^-(v)$ are one-dimensional subspaces of $T_v SM$.*
(2) *$d\pi_v E^+(v) = d\pi_v E^-(v) = \{w \in T_{\pi(v)}M : w \text{ is orthogonal to } v\}$.*

(3)   The subspaces $E^+(v)$ and $E^-(v)$ are continuous and invariant under the geodesic flow.

(4)   Let $\tau : SM \to SM$ be the involution defined by $\tau v = -v$, then

$$E^+(-v) = d\tau E^-(v) \quad \text{and} \quad E^-(-v) = d\tau E^+(v).$$

(5)   If the curvature satisfies $K(p) \geq -a^2$ for some $a > 0$, then $\|K_v \xi\| \leq a\|d\pi_v \xi\|$ for any $\xi \in E^+(v)$ or $\xi \in E^-(v)$.

(6)   If $\xi \in E^+(v)$ or $\xi \in E^-(v)$, then $J_\xi(t) \neq 0$ for each $t \in \mathbb{R}$.

(7) $\xi \in E^+(v)$ (respectively, $\xi \in E^-(v)$) if and only if

$$\langle \xi, V(v) \rangle = 0 \quad \text{and} \quad \|d\pi_{g^t v} dg_v^t \xi\| \leq c$$

for each $t > 0$ (respectively, $t < 0$) and some $c > 0$.

(8)   For $\xi \in E^+(v)$ (respectively, $\xi \in E^-(v)$), the function $t \mapsto \|J_\xi(t)\|$ is non-increasing (respectively, non-decreasing).

When $\gamma_v(t)$ is a flat geodesic, there exists a non-trivial element $\xi \in E^+(v) \cap E^-(v)$, and J_ξ is a parallel Jacobi field along $\gamma_v(t)$, that is, $J_\xi'(t) = 0$ for all $t \in \mathbb{R}$. In this case, $E^+(v)$ and $E^-(v)$ do not span the whole second tangent space $T_v SM$. The distributions E^s and E^u on SM are integrable and their integral manifolds form foliations W^s and W^u of SM, respectively. These two foliations are both invariant under g^t, known as the stable and unstable horocycle foliations.

2.2. *Universal cover.* Let $\widetilde{M}$ be the universal Riemannian cover of M, that is, a simply connected complete Riemannian manifold for which $M = \widetilde{M}/\Gamma$, where Γ is a discrete subgroup of the group of isometries of $\widetilde{M}$, isomorphic to $\pi_1(M)$. Recall that we assume M has no focal points. According to the Hadmard–Cartan theorem, for each two points on $\widetilde{M}$, there is a unique geodesic segment joining them. Therefore, $\widetilde{M}$ can be identified with the open unit disk in the plane. The lifting of a geodesic γ from M to $\widetilde{M}$ is denoted by $\widetilde{\gamma}$. Two geodesics $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ are said to be *asymptotes* if $d(\widetilde{\gamma}_1(t), \widetilde{\gamma}_2(t)) \leq C$ for some $C > 0$ and for all $t > 0$. It is easy to check that the asymptotes relation is an equivalence relation. Let $\widetilde{M}(\infty)$ be the set of all the equivalence classes, which can be identified with the boundary of the unit disk. Then the set

$$\overline{M} := \widetilde{M} \cup \widetilde{M}(\infty)$$

can be identified with the closed unit disk in the plane. Denote by $\widetilde{\gamma}(+\infty)$ the asymptote class of the geodesic $\widetilde{\gamma}$, and by $\widetilde{\gamma}(-\infty)$ the one of the reversed geodesic of $\widetilde{\gamma}$. We use $\widetilde{W}^s$ and $\widetilde{W}^u$ to denote the lifting of W^s and W^u to $S\widetilde{M}$, respectively. It is obvious that if $w \in \widetilde{W}^s(v)$, then geodesics $\widetilde{\gamma}_v(t)$ and $\widetilde{\gamma}_w(t)$ are asymptotic.

An isometry α of $\widetilde{M}$ is called *axial* if there exist a geodesic $\widetilde{\gamma}$ on $\widetilde{M}$ and a $t_1 > 0$ such that for all $t \in \mathbb{R}$, $\alpha(\widetilde{\gamma}(t)) = \widetilde{\gamma}(t + t_1)$. The corresponding geodesic $\widetilde{\gamma}$ is called an *axis* of α. The following result is due to Watkins [24], which is proved for rank-1 manifolds without focal points. Here we only need it for surfaces without focal points.

LEMMA 2.3. (Cf. [24, Theorem 6.11]) *Let $\widetilde{\gamma}$ be an axis of an isometry α of $\widetilde{M}$. Suppose that $\widetilde{\gamma}$ is not the boundary of a flat half-plane. Then for all neighborhoods $U \subseteq \overline{M}$ of*

$\widetilde{\gamma}(-\infty)$ *and* $V \subseteq \overline{M}$ *of* $\widetilde{\gamma}(+\infty)$, *there is an integer* $N \in \mathbb{N}$ *such that*

$$\alpha^n(\overline{M} - U) \subseteq V, \quad \alpha^{-n}(\overline{M} - V) \subseteq U,$$

for all $n \geq N$.

Obviously, every closed geodesic γ in M can be lifted to a geodesic $\widetilde{\gamma}$ on $\widetilde{M}$, such that

$$\widetilde{\gamma}(t + t_0) = \phi(\widetilde{\gamma}(t)) \quad \text{for all } t \in \mathbb{R},$$

for some $t_0 > 0$ and $\phi \in \pi_1(M)$. Therefore, $\widetilde{\gamma}$ is an axis of ϕ. In this case, we also say that ϕ fixes $\widetilde{\gamma}$, written as $\phi(\widetilde{\gamma}) = \widetilde{\gamma}$. Here, ϕ acts on $\widetilde{M}(\infty)$ in a natural way and fixes exactly the two points $\widetilde{\gamma}(\pm\infty)$. Moreover, by Lemma 2.3, for any $x \in \widetilde{M}(\infty)$, $x \neq \widetilde{\gamma}(\pm\infty)$, we have

$$\lim_{n \to +\infty} \phi^n(x) = \widetilde{\gamma}(+\infty) \quad \text{and} \quad \lim_{n \to -\infty} \phi^n(x) = \widetilde{\gamma}(-\infty).$$

3. *The regular set*

This section is devoted to proving Theorem 1.4. The proof of Theorem 1.4 in the non-positive curvature case is given in [25, Lemma 1.1]. In fact, it is already well known in folklore that, after Pesin's Theorem 1.3, all that remains for ergodicity of the geodesic flow is to show that Λ has zero Liouville measure. Nevertheless, to prove Theorem 1.4 in the no focal points case, we need to use some geometric properties of the geodesic flow which we will present below.

For a given $\xi \in T_v SM$, we always let $J_\xi(t)$ be the unique Jacobi field satisfying the Jacobi equation (1) under initial conditions

$$J_\xi(0) = d\pi_v \xi, \quad \frac{d}{dt}\bigg|_{t=0} J_\xi(t) = K_v \xi.$$

Suppose that $J_\xi(t)$ is perpendicular to $\gamma_v'(t)$, then $J_\xi(t) = j_\xi(t)e_2(t)$ and $j_\xi(t) = \|J_\xi(t)\|$. Denote $u_\xi(t) = j_\xi'(t)/j_\xi(t)$. Recall that u_ξ is a solution of the Riccati equation (3).

Given $\xi \in T_v SM$, the *Lyapunov exponent* $\chi(v, \xi)$ is defined as

$$\chi(v, \xi) := \limsup_{T \to \infty} \frac{1}{T} \log \|dg^T \xi\|.$$

The following proposition shows the connection between the Lyapunov exponent $\chi(v, \xi)$ and the function u_ξ.

PROPOSITION 3.1. *For any* $v \in SM$ *and* $\xi \in E^+(v)$, *one has*

$$\chi(v, \xi) = \limsup_{T \to \infty} \frac{1}{T} \int_0^T u_\xi(t) \, dt.$$

Proof. By the definition of Lyapunov exponents and Proposition 2.2(5), we have

$$\chi(v, \xi) = \limsup_{T \to \infty} \frac{1}{T} \log \|dg^T \xi\| = \limsup_{T \to \infty} \frac{1}{T} \log \sqrt{\|J_\xi(T)\|^2 + \|J_\xi'(T)\|^2}$$

$$= \limsup_{T \to \infty} \frac{1}{T} \log \|J_\xi(T)\| = \limsup_{T \to \infty} \frac{1}{T} \int_0^T (\log j_\xi(t))' \, dt$$

$$= \limsup_{T \to \infty} \frac{1}{T} \int_0^T \frac{j_\xi'(t)}{j_\xi(t)} dt = \limsup_{T \to \infty} \frac{1}{T} \int_0^T u_\xi(t) \, dt. \qquad \square$$

Throughout this section, if $\xi \in E^+(v)$, we write $j(t) := j_\xi(t)$ and $u(t) := u_\xi(t)$ for simplicity. By Proposition 2.2(8) and the definition of $u(t)$, we know that $u(t) \le 0$ for all $t \in \mathbb{R}$.

The following notion of uniformly recurrent vectors appeared in [4].

Definition 3.2. (Cf. [4]) A vector $x \in SM$ is said to be uniformly recurrent if for any neighborhood U of x in SM,

$$\liminf_{t \to \infty} \frac{1}{T} \int_0^T \mathbf{I}_U(g^t(x)) \, dt > 0,$$

where $\mathbf{I}_U$ is the characteristic function of U.

The next lemma about the set of uniformly recurrent vectors was stated in [4] without a proof, so we provide a proof here. It will be used later in our proof of Theorem 1.4.

LEMMA 3.3. *Let Γ be the set of all the uniformly recurrent vectors. Then Γ has full Liouville measure.*

Proof. Let $\{U_n\}_{n \in \mathbb{N}}$ be a countable base consisting of open sets on SM. By the Birkhoff ergodic theorem, there exists a set $X \subset SM$ of full measure such that for all $x \in X$ and all $n \in \mathbb{N}$, the limit

$$f_n(x) := \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbf{I}_{U_n}(g^t(x)) \, dt$$

exists and

$$\int_{SM} f_n(x) dv(x) = v(U_n).$$

Assume the contrary that $v(\Gamma^c) > 0$. Then, $\Gamma^c \cap X$ is non-empty. For each $y \in \Gamma^c \cap X$, which is not uniformly recurrent, there exists a neighborhood U of y in SM such that

$$\liminf_{T \to \infty} \frac{1}{T} \int_0^T \mathbf{I}_U(g^t(y)) \, dt = 0.$$

Then there exists an $n(y)$ such that $U_{n(y)} \subset U$ and

$$f_{n(y)}(y) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbf{I}_{U_n}(g^t(y)) \, dt \le \liminf_{T \to \infty} \frac{1}{T} \int_0^T \mathbf{I}_U(g^t(y)) \, dt = 0. \qquad (5)$$

Since there are only countably many U_n, we can find some N such that $v(U_N \cap \Gamma^c \cap X) > 0$. By equation (5), $f_N(y) = 0$ for any $y \in U_N \cap \Gamma^c \cap X$.

However, the Birkhoff ergodic theorem implies that for almost every $y \in U_N \cap \Gamma^c \cap X$, one has

$$g(y) := \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbf{I}_{(U_N \cap \Gamma^c \cap X)}(g^t(y)) \, dt$$

exists with

$$\int_{SM} g(y) \, d\nu(y) = \nu(U_N \cap \Gamma^c \cap X) > 0. \tag{6}$$

However, by equation (5), we have $g(y) \le f_N(y) = 0$ for all $y \in U_N \cap \Gamma^c \cap X$, which contradicts equation (6). This proves the lemma. $\qquad\square$

Given an open set $U \subset SM$ and a unit vector $w \in SM$, we say that the orbit $g^t w$ has positive frequency of return to U if $\liminf_{t \to \infty}(T_U(t)/t) > 0$, where $T_U(t)$ denotes the total length of the set

$$\mathfrak{T}_U(t) := \{\tau : 0 \le \tau \le t \text{ and } g^\tau w \in U\}.$$

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Choose an arbitrary $v \in \Lambda^c \cap \Delta^c \cap \Gamma$, where Γ denotes the set of uniformly recurrent vectors. Recall that $\nu(\Gamma) = 1$ by Lemma 3.3. Without loss of generality, we assume $v \in (\Delta^+)^c$. We claim that $K(\gamma_v(t)) \ge 0$ for all $t \in \mathbb{R}$.

Assume the contrary that $K(\gamma_v(t_0)) < 0$ for some $t_0 > 0$. Since $K(\gamma_v(t_0)) < 0$, we can choose two open neighborhoods $W_1 \supset W_2$ of $g^{t_0}v$, such that $-\delta_2 < K|_{\pi(W_1)} < -\delta_1 < 0$ and $\text{dist}(\partial W_1, \partial W_2) > \sigma$ for some $\delta_2 > \delta_1 > 0$ and $\sigma > 0$.

Choose an open neighborhood U of v which is small enough, such that for any $w \in U$, one has $g^{t_0}(w) \in W_2$. Since $\liminf_{T \to \infty}(1/T) \int_0^T \mathbf{I}_U(g^t v) \, dt > 0$, we have

$$\liminf_{T \to \infty} \frac{1}{T} \int_{t_0}^{T+t_0} \mathbf{I}_{W_2}(g^t v) \, dt > 0.$$

Then the orbit of v has positive frequency of return to W_2, that is,

$$\liminf_{T \to \infty} \frac{T_{W_2}(T)}{T} = \liminf_{T \to \infty} \frac{1}{T+t_0} \int_0^{T+t_0} \mathbf{I}_{W_2}(g^t v) \, dt$$

$$= \liminf_{T \to \infty} \frac{T}{T+t_0} \cdot \frac{1}{T} \int_{t_0}^{T+t_0} \mathbf{I}_{W_2}(g^t v) \, dt > 0. \tag{7}$$

LEMMA 3.4. *There exists a constant $c > 0$ such that if $g^t v \in W_2$, then $u(t) \le -c$ for all $t \ge 0$.*

Proof of Lemma 3.4. We prove the lemma by contradiction. Assume this is not true. Then there exists a sequence of $t_i \ge 0$ with $g^{t_i} v \in W_2$, but $u(t_i) \to 0$ as $i \to \infty$. There exist $s_{i,1} < t_i < s_{i,2}$ such that $g^{s_{i,1}} v, g^{s_{i,2}} v \in \partial W_1$ and $g^t v \in W_1$ for any $s_{i,1} < t < s_{i,2}$. In fact, $s_{i,1} + \sigma < t_i < s_{i,2} - \sigma$.

Recall the Riccati equation $u'(t) + u^2(t) + K(t) = 0$. If i is large enough, then $u'(t_i) = -u^2(t_i) - K(t_i) > \delta_1 > 0$. We claim that $u(t)$ is strictly increasing in the interval $(t_i, s_{i,2})$. Indeed, if not, there is a smallest number $s_i \in (t_i, s_{i,2})$ such that $u'(s_i) = 0$. Then

$u(s_i) > u(t_i)$, since $u'(t) > 0$ for all $t \in (t_i, s_i)$. Therefore, $u'(s_i) = -u^2(s_i) - K(s_i) > \delta_1 > 0$, which is a contradiction.

It follows that $u'(t) = -u^2(t) - K(t) > \delta_1 > 0$ for all $t \in (t_i, s_{i,2})$. Thus,

$$
\begin{aligned}
u(s_{i,2}) &= u(t_i) + \int_{t_i}^{s_{i,2}} u'(t) \, dt \\
&> u(t_i) + \delta_1(s_{i,2} - t_i) > u(t_i) + \delta_1\sigma.
\end{aligned}
$$

If i is large enough, then $u(t_i)$ is close enough to 0, and hence $u(s_{i,2}) > 0$. This contradicts the fact that $u(t) \leq 0$ for all $t \in \mathbb{R}$. □

Let us go on with the proof of Theorem 1.4. By Proposition 3.1, Lemma 3.4, and equation (7), one has

$$
\chi(v, \xi) = \limsup_{T \to \infty} \frac{1}{T} \int_0^T u(t) \, dt \leq \limsup_{T \to \infty} \frac{1}{T} \cdot T_{W_2}(v) \cdot (-c) < 0,
$$

where $\xi \in E^+(v)$. This contradicts $v \in (\Delta^+)^c$. Thus, $K(\gamma_v(t)) \geq 0$ for all $t \geq 0$. Analogously, we can prove that $K(\gamma_v(t)) \geq 0$ for all $t \leq 0$. Thus,

$$
K(\gamma_v(t)) \geq 0 \quad \text{for all } t \in \mathbb{R}.
$$

Now recall the Riccati equation $u'(t) + u^2(t) + K(t) = 0$ again. Since $K(t) \geq 0$ along $\gamma_v(t)$, we have $u'(t) \leq 0$ for all $t \in \mathbb{R}$. We have the following three possibilities.

(1) $\lim_{t \to \infty} u(t) = 0$. Since $u(t) \leq 0$ and $u'(t) \leq 0$ for all $t \in \mathbb{R}$, we must have $u(t) \equiv 0$ for all $t \in \mathbb{R}$. Then $u'(t) \equiv 0$ for all $t \in \mathbb{R}$. It follows from the Riccati equation that $K(t) \equiv 0$ for all $t \in \mathbb{R}$.

(2) $\lim_{t \to \infty} u(t) = -d < 0$ for some $d > 0$. Then,

$$
u'(t) = -u^2(t) - K(t) \leq -u^2(t) < 0 \quad \text{for all } t \in \mathbb{R}.
$$

This contradicts the fact that $\lim_{t \to \infty} u'(t) = 0$.

(3) $\lim_{t \to \infty} u(t) = -\infty$. Since $J(t)$ is a stable Jacobi field along $\gamma_v(t)$, we have $|u(t)| = |j'(t)/j(t)| \leq a$. We also arrive at a contradiction.

In summary, we must have $K(\gamma_v(t)) \equiv 0$ for all $t \in \mathbb{R}$. This contradicts our assumption $v \in \Lambda^c$. Therefore, $\Lambda^c \cap (\Delta^+)^c \cap \Gamma = \emptyset$. The case $v \in (\Delta^-)^c$ can be dealt with similarly and leads to the same result. Based on the discussion above, we can conclude that $\Lambda^c \cap \Delta^c \cap \Gamma = \emptyset$. Since Γ is a full measure set, we immediately know that

$$
\nu(\Lambda^c \cap \Delta^c) = 0.
$$

We are done with the proof of Theorem 1.4. □

4. Flat geodesics

4.1. *Flat strips are closed.* A flat strip means a totally geodesic isometric imbedding $r : \mathbb{R} \times [0, c] \to \widetilde{M}$, where $\mathbb{R} \times [0, c]$ is a strip in a Euclidean plane. The projection of a flat strip from $\widetilde{M}$ to M is also called a flat strip. We have the following flat strip lemma.

LEMMA 4.1. (Cf. [21]) *If two distinct geodesics $\widetilde{\alpha}$ and $\widetilde{\beta}$ satisfy $d(\widetilde{\alpha}(t), \widetilde{\beta}(t)) \leq C$ for some $C > 0$ and for all $t \in \mathbb{R}$, then they are the boundary curves of a flat strip in $\widetilde{M}$.*

The flat strip lemma for non-positively curved manifolds was established by Eberlein and O'Neill in [13]. The above flat strip lemma for manifolds without focal points is due to Green in dimension two [14], and O'Sullivan in arbitrary dimensions [21]. The following lemma is also useful in our work.

LEMMA 4.2. (Cf. [25, Lemma 3.6]) *If $w' \in W^s(w) \subset SM$ and*

$$\lim_{t \to +\infty} d(\gamma_w(t), \gamma_{w'}(t)) = \delta > 0,$$

then $\gamma_w(t)$ and $\gamma_{w'}(t)$ converge to the boundaries of a flat strip of width δ.

In view of Conjecture 1.1, our aim is to show that all flat geodesics are closed. An important progress was made by Cao and Xavier on the flat geodesics inside flat strips on manifolds of non-positive curvature, in an unpublished preprint [9] (see also [11]). We state it in the following theorem.

THEOREM 4.3. (Cf. [9]) *Let M be a smooth, connected, and closed surface with genus $\mathfrak{g} \geq 2$. Suppose that M has non-positive curvature. Then any flat strip on M consists of closed geodesics in the same homotopy type.*

Based on the flat strip Lemma 4.1, we generalize the above result to the manifolds without focal points. We adapt the argument of Cao and Xavier to surfaces without focal points.

THEOREM 4.4. *Let M be a smooth, connected, and closed surface with genus $\mathfrak{g} \geq 2$. Suppose that M has no focal points. Then any flat strip on M consists of closed geodesics in the same homotopy type.*

Proof. Observe that in the universal cover $\widetilde{M}$, there exists an upper bound for the width of all the flat strips. Indeed, let $D > \mathrm{diam}(M)$. Then a flat strip of width greater than $2D$ contains a fundamental domain in $\widetilde{M}$. Hence, M must be a flat torus. This contradicts the fact that M has genus $\mathfrak{g} \geq 2$.

Let $\widetilde{G} : (-\infty, \infty) \times [0, \epsilon_0] \to \widetilde{M}$ be a flat strip in $\widetilde{M}$ and $G = p(\widetilde{G})$, where $p : \widetilde{M} \to M$ is the universal covering map. Consider a sequence of unit vectors $v_i \in SM$ where $v_i = \partial G / \partial t(i, \epsilon_0)$, $i = 1, 2, \dots$. Since SM is compact, there exists a subsequence of $\{v_i\}_{i=1}^{\infty}$ which converges to a unit vector $v_0 \in SM$. For simplicity of notation, we still let $\{v_i\}_{i=1}^{\infty}$ denote the subsequence. Recall that $\pi : SM \to M$ is the canonical projection. Let x_i denote the foot point of v_i, that is, $x_i = \pi(v_i)$, $i = 0, 1, 2, \dots$.

Let δ be the injectivity radius of M. For sufficiently large j, we may assume $d(v_j, v_0) < \delta/2$. We choose a preimage $\widetilde{x}_0 \in p^{-1}(x_0)$ such that $\widetilde{x}_0$ is the nearest point to the flat strip $\widetilde{G}$ in $p^{-1}(x_0)$. Then $p|_{B(\widetilde{x}_0, \delta)} : B(\widetilde{x}_0, \delta) \to B(x_0, \delta)$ and $\Psi := dp|_{SB(\widetilde{x}_0, \delta)} : SB(\widetilde{x}_0, \delta) \to SB(x_0, \delta)$ are both isometries.

Denote $w_i = \Psi^{-1}(v_i) \in SB(\widetilde{x}_0, \delta)$, $i = 0, 1, 2, \ldots$. Let $F_j : (-\infty, \infty) \times [0, \epsilon_0] \to \widetilde{M}$ denote the lifted flat strip tangent to w_j, $j = 1, 2, \ldots$. Then the limit of F_j is a flat strip $\widetilde{G}_0 : (-\infty, \infty) \times [0, \epsilon_0] \to \widetilde{M}$ tangent to w_0. There are two distinct cases.

(1) $p \circ F_{j_0}$ is periodic for some $j_0 \in \mathbb{N}$. As $p \circ F_{j_0}$ is the flat strip tangent to v_{j_0}, it coincides with G. Hence, G is periodic.

(2) $p \circ F_j$ is not periodic for any $j \in \mathbb{N}$. Then F_j and $\widetilde{G}_0$ are a pair of transversal flat strips of the same width ϵ_0. Suppose that they intersect at q_j with angle α_j, where $q_j \in \partial F_j \cap \widetilde{G}_0((-\infty, \infty) \times \{\epsilon_0\})$, $j = 1, 2, \ldots$. Because $F_j \cup \widetilde{G}_0$ has curvature 0 everywhere, we can construct a rectangle $R_j = [0, L_j] \times (0, \epsilon_0/8)$ contained in the closure of $F_j - \widetilde{G}_0$ such that:

- one side of R_j, $[0, L_j] \times \{0\}$ is contained in the line $\widetilde{G}_0((-\infty, \infty) \times \{\epsilon_0\})$);
- $L_j \geq \epsilon_0/16 \sin \alpha_j$.

Attaching R_j to $\widetilde{G}_0$, we obtain an isometric embedding $\widetilde{G}_0 : [c_j, c_j + L_j] \times [0, 9\epsilon_0/8]$ for some $c_j \in \mathbb{R}$. Let $\widetilde{u}_j$ be the unit vector tangent to $\widetilde{G}_0([c_j, c_j + L_j] \times \{0\})$ at the point $\widetilde{G}_0(c_j + L_j/2, 0)$. Write $u_j = dp(\widetilde{u}_j)$ and suppose that a subsequence of $\{u_j\}$ converges to u_0. As $L_j \to \infty$ as $j \to \infty$, we know that there exists a flat strip tangent to u_0 of width $9\epsilon_0/8$. Hence, there exits a flat strip $\widetilde{G}_1$ of width $9\epsilon_0/8$ in $\widetilde{M}$.

We are done if we arrive at the first case above. If we have the second case, we then repeat the argument for the new flat strip $\widetilde{G}_1$. However, we cannot enlarge our flat strips by a factor $9/8$ again and again, as the width of the flat strips in $\widetilde{M}$ have an upper bound $2D$. Thus, we must arrive at the first case at some step. It follows that G is periodic. $\square$

4.2. *Expansivity.* The proof of Theorems 1.7 and 1.8 uses an argument based on the following *expansivity* property of a vector $x \in SM$ not tangent to a flat strip. This argument will be used later several times, in the proof of Theorems 1.7, 1.8, and Lemma 4.11.

Definition 4.5. (Cf. [19, Definition 3.2.11]) We say $x \in SM$ has the expansivity property if there exists a small $\delta_0 > 0$, such that whenever $d(g^t(x), g^t(y)) < \delta_0$ for all $t \in \mathbb{R}$, then $y = g^{t_0}(x)$ for some t_0 with $|t_0| < \delta_0$.

The flat strip lemma (Lemma 4.1) for surfaces without focal points guarantees the expansivity property for a unit vector which is not tangent to a flat strip.

LEMMA 4.6. *If $x \in SM$ is not tangent to a flat strip, then it has the expansivity property.*

Proof. We prove this lemma by contradiction. Assume the lemma does not hold. Then for an arbitrarily small $\epsilon > 0$ less than the injectivity radius of M, there exists a point $y \in SM$ such that $y \notin \mathcal{O}(x)$ and $d(\gamma_x(t), \gamma_y(t)) < \epsilon$ for all $t \in \mathbb{R}$. By the choice of ϵ, we can lift $\gamma_x(t)$ and $\gamma_y(t)$ to the universal cover $\widetilde{M}$ such that

$$d(\widetilde{\gamma}_x(t), \widetilde{\gamma}_y(t)) < \epsilon \quad \text{for all } t \in \mathbb{R}.$$

Thus, by the flat strip Lemma 4.1, $\widetilde{\gamma}_x(t)$ and $\widetilde{\gamma}_y(t)$ bound a flat strip. Hence, x is tangent to a flat strip, which is a contradiction. $\square$

4.3. *Area of ideal triangles.* Given $x, y, z \in \widetilde{M}(\infty)$, an ideal triangle with vertices x, y, z means the region in $\widetilde{M}$ bounded by the three geodesics joining the vertices. In the case when at least one of x, y, z is on $\widetilde{M}(\infty)$ (the other points can be inside $\widetilde{M}$), we also call the region bounded by the three geodesics an ideal triangle. The following theorem about the ideal triangle is proved in [23].

The following theorem is a version of [23, Theorem 1] for surfaces without focal points.

THEOREM 4.7. *Let M be a smooth, connected, and closed surface with genus $\mathfrak{g} \geq 2$ with no focal points. Suppose that γ is a flat geodesic which is asymptotic to a closed geodesic β. Then every ideal triangle having $\widetilde{\gamma}(t)$ as an edge has infinite area.*

Proof. Suppose that $v = \widetilde{\gamma}'(0)$, and $c(s) \in S\widetilde{M}, s \in [0, a]$ is the curve in the stable horosphere $H^s(v)$ with $c(0) = v$. We want to show that the area of $\pi g^{[0,\infty]} c[0, a]$ is infinite. Assume the contrary. Then obviously,

$$\lim_{t \to \infty} l(\pi g^t c[0, a]) = 0, \tag{8}$$

where l denotes the length of the curve.

As in the remark after the proof of Proposition 3.1, we denote by $u(w), w \in S\widetilde{M}$ the function satisfying Riccati equation (3) with respect to stable Jacobi field $J^s(w)$. Then, $u(w) = \|J_w^s\|'/\|J_w^s\|$ and $u(w) \leq 0$. Given $T > 0$, define

$$u_T(w) := \int_0^T u(g^t w) \, dt.$$

Note that u and hence u_T are continuous functions. Since $\tilde{\gamma}$ is flat, $u(g^t v) = 0$ for all $t \in \mathbb{R}$. By continuity of u, for any $s \in [0, a]$,

$$\lim_{T \to \infty} \frac{1}{T} u_T(c(s)) = 0.$$

Then there exists large enough $T > 0$ which depends on the unit vector $c(s)$ such that

$$-\frac{1}{T} u_T(c(s)) \leq l(\pi c[0, s]).$$

By assumption, γ is asymptotic to a closed geodesic β. Without loss of generality, assume that $\tilde{\gamma}(0)$ and $\tilde{\beta}(0)$ are close enough and $v = \tilde{\gamma}'(0)$ is in the stable manifold of $\tilde{\beta}'(0)$. On compact manifold M, $\pi g^{[0,\infty]}(c[0, a])$ is contained in a small compact neighborhood of the closed geodesic β. By the compactness and continuity, we can find a uniform constant $T > 0$ independent of $t \geq 0$ and $s \in [0, a]$ such that

$$-\frac{1}{T} u_T(g^t c(s)) \leq l(\pi g^t c[0, s]) \quad \text{for all } t \geq 0, \text{ for all } s \in [0, a]. \tag{9}$$

We introduce the following useful lemma. The proof is similar to that of [10, Lemma 3.13].

LEMMA 4.8. *Let ψ be a non-negative function and $\psi_T(t) := \int_0^T \psi(t + \tau)\, d\tau$. For $a, b \in \mathbb{R}$ with $b - a \geq T$, we have*

$$\frac{1}{T}\int_a^b \psi_T(t)\, dt \geq \int_{a+T}^b \psi(t)\, dt.$$

Note that $u \leq 0$ in our case. Choose $L \geq T$. Then by the above lemma, we have

$$\frac{1}{T}\int_{-T}^L -u_T(g^t w)\, dt \geq \int_0^L -u(g^t w)\, dt \tag{10}$$

for any $w \in S\widetilde{M}$. We have

$$l(\pi g^{L+T}(c[a, b])) = \int_0^a \|dg^{L+T}c(s)\|\, ds$$

$$= \int_0^a \|J_{c(s)}^s(L + T)\|\, ds$$

$$= \int_0^a (\pi g^T c(s))' e^{\int_T^{L+T} u(g^t c(s))\, dt}\, ds.$$

Given $s < t$, denote by $A_{s,t}$ the area of the region bounded by $\pi g^s(c[0, a])$, $\pi g^t(c[0, a])$, γ and the geodesic tangent to $c(a)$. Then by (10) and (9), we have

$$l(\pi g^{L+T}(c[a, b])) \geq \int_0^a (\pi g^T c(s))' e^{1/T \int_0^{L+T} u_T(g^t c(s))\, dt}\, ds$$

$$\geq \int_0^a (\pi g^T c(s))' e^{-\int_0^{L+T} l(g^t c(s))\, dt}\, ds$$

$$= l(\pi g^T(c[a, b])) e^{-A_{0,L+T}}.$$

Similarly, we can prove

$$l(\pi g^{2L+T}(c[a, b])) \geq l(\pi g^{L+T}(c[a, b])) e^{-A_{L,2L+T}} \geq l(\pi g^T(c[a, b])) e^{-2A_{0,2L+T}}.$$

By induction, we have

$$l(\pi g^{nL+T}(c[a, b])) \geq l(\pi g^T(c[a, b])) e^{-2A_{0,nL+T}}.$$

Taking limit, since we are assuming $\lim_{n\to\infty} A_{0,nL+T} < \infty$, we get

$$\lim_{n\to\infty} l(\pi g^{nL+T}(c[a, b])) \geq l(\pi g^T(c[a, b])) e^{-2\lim_{n\to\infty} A_{0,nL+T}} > 0.$$

This contradicts equation (8) and the theorem follows.   □

Remark 4.9. There is significant difference between the above proof and the proof in the non-positive curvature case by Ruggiero [23]. In non-positive curvature, in the proof of Lemma 4.1 in [23], we can have by Taylor's formula
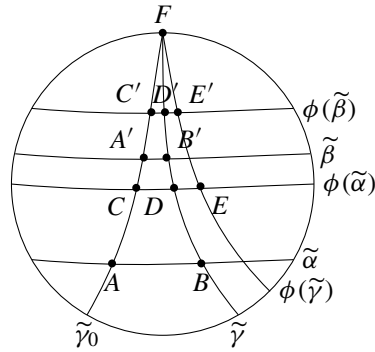
$$K_g(p) = -y(p)^2 f_s(y(p)),$$

FIGURE 1. Proof of Lemma 4.10.

since the first derivative of the curvature function is zero at p. In the no focal points case, there could be positive curvature, and thus we may not have this formula. So the method using comparison theorem in [23] breaks down here.

The idea using $(1/T)u_T$ instead of u in our method has appeared in [10]. The assumption that the flat geodesic is asymptotic to a closed geodesic is used to get the uniform T in equation (9). However, we do not know a proof without this assumption.

4.4. *Non-closed flat geodesics.*   In this subsection, we discuss some important properties of the flat geodesics. Our Theorem 1.6 is a straightforward corollary of these properties. In fact, it is closely related to the following two lemmas (Lemmas 4.10 and 4.11). The first lemma shows that if a flat geodesic converges to a closed one (no matter flat or not), then the former geodesic must also be closed and coincide with the latter.

LEMMA 4.10. *Suppose that $y \in \Lambda$ and the ω-limit set $\omega(y) = \mathcal{O}(z)$, where $\mathcal{O}(z)$ is periodic. Then $\mathcal{O}(y) = \mathcal{O}(z)$. In particular, $\mathcal{O}(y)$ is periodic.*

Proof. Since $\omega(y) = \mathcal{O}(z)$, we can lift geodesics $\gamma_z(t)$, $\gamma_y(t)$ to the universal cover $\widetilde{M}$, denoted by $\widetilde{\gamma}_0(t)$ and $\widetilde{\gamma}(t)$, respectively, such that $\lim_{t \to +\infty} d(\widetilde{\gamma}_0(t), \widetilde{\gamma}(t)) = 0$. In particular, $\widetilde{\gamma}_0(+\infty) = \widetilde{\gamma}(+\infty)$.

Since $\gamma_z(t)$ is a closed geodesic, there exists an isometry ϕ of $\widetilde{M}$ such that $\phi(\widetilde{\gamma}_0(t)) = \widetilde{\gamma}_0(t + t_0)$. Moreover, on the boundary of the disk $\widetilde{M}(\infty)$, ϕ fixes exactly two points $\widetilde{\gamma}_0(\pm\infty)$, and for any other point $a \in \widetilde{M}(\infty)$, $\lim_{n \to +\infty} \phi^n(a) = \widetilde{\gamma}_0(+\infty)$.

Assume $\widetilde{\gamma}$ is not fixed by ϕ. Then $\widetilde{\gamma}$ and $\phi(\widetilde{\gamma})$ do not intersect, since $\phi(\widetilde{\gamma})(+\infty) = \widetilde{\gamma}(+\infty)$. By Lemma 2.3, replacing ϕ by ϕ^N for a large enough $N \in \mathbb{N}$ if necessary, we know that the position of $\phi(\widetilde{\gamma})$ must be as shown in Figure 1. We then pick another two geodesics $\widetilde{\alpha}$ and $\widetilde{\beta}$ as in Figure 1. The image of $ABB'A'$ under ϕ is $CEE'C'$. Since ϕ is an isometry, it preserves area. So the area of $ABCD$ is equal to the region $A'B'DEE'C'$, and thus greater than the area of the region $DEE'D'$. We can let A' and B' approach F. In this process, the area of the region $DEE'D'$ approaches the area of the ideal triangle DEF. However, since γ_y is a flat geodesic asymptotic to a closed geodesic γ_z, the area of

DEF is infinite by Theorem 4.7. Then $ABCD$ has infinite area which is absurd. So $\phi(\widetilde{\gamma})$ and $\widetilde{\gamma}$ must coincide.

Therefore, $\widetilde{\gamma}(\pm\infty) = \widetilde{\gamma}_0(\pm\infty)$. Then either $\widetilde{\gamma}(t)$ and $\widetilde{\gamma}_0(t)$ bound a flat strip by the flat strip Lemma 4.1 or $\widetilde{\gamma}(t) = \widetilde{\gamma}_0(t)$. Recall that $\lim_{t\to+\infty} d(\widetilde{\gamma}(t), \widetilde{\gamma}_0(t)) = 0$, we must have $\widetilde{\gamma}(t) = \widetilde{\gamma}_0(t)$. Hence, $\mathcal{O}(y) = \mathcal{O}(z)$. ☐

Lemma 4.10 can be strengthened to the following.

LEMMA 4.11. *Suppose that $y \in \Lambda$ and $z \in \omega(y)$, where z is periodic. Then $\mathcal{O}(y) = \mathcal{O}(z)$. In particular, y is periodic.*

The proof of Lemma 4.11 follows from an argument similar to the one in the proof of [25, Lemma 3.8]. The argument relies on the expansivity property of a unit vector not tangent to a flat strip, as stated in Lemma 4.6. Then Theorem 1.6 follows from an almost identical argument. For this reason, we omit the proof here. Readers can check the proof of [25, Lemma 3.8, Theorem 1.5] for details of the argument.

5. *Proof of main theorems*

5.1. *Proof of Theorem 1.8.* Now we assume that $\Lambda \cap (\mathrm{Per}\,(g^t))^c \neq \emptyset$, in other words, there exists an aperiodic orbit $\mathcal{O}(x)$ in Λ. We will construct the points $y, z \in \Lambda$ as stated in Theorem 1.8 starting from $\mathcal{O}(x)$ based on the expansivity property of x. A first observation is that we can always find two arbitrarily nearby points on the orbit $\mathcal{O}(x)$. We state this result in the following lemma (see [25, Lemma 3.3] for the proof).

LEMMA 5.1. *For any $k \in \mathbb{N}$, there exist two sequences $t_k \to +\infty$ and $t'_k \to +\infty$, such that $t'_k - t_k \to +\infty$ and*

$$d(x_k, x'_k) < \frac{1}{k} \quad where\ x_k = g^{t_k}(x),\ x'_k = g^{t'_k}(x).$$

For each pair x_k, x'_k with large enough k, we can check the expansivity in the positive direction of the flow by using the idea in the proof of [25, Proposition 3.4]. In fact, the expansivity in one direction (either positive or negative) of the flow is sufficient for our purpose.

PROPOSITION 5.2. *Fix an arbitrarily small $\epsilon_0 > 0$. Then there exists $s_k \to +\infty$ or $s_k \to -\infty$, such that*

$$d(g^{s_k}(x_k), g^{s_k}(x'_k)) = \epsilon_0,$$

and $d(g^s(x_k), g^s(x'_k)) < \epsilon_0$ for all $0 \leq s < s_k$ or for all $s_k < s \leq 0$, respectively.

Proof. Assume the contrary. Then x does not have expansivity property. By Lemma 4.6, x is tangent to a flat strip. Then by Lemma 4.4, x must be periodic. This contradicts the assumption $x \in (\mathrm{Per}\,(g^t))^c$. ☐

Without loss of generality, we suppose that $s_k \to +\infty$ in the remainder of the paper. For the case $s_k \to -\infty$, everything remains true by a slight modification.

PROPOSITION 5.3. *There exist $\epsilon_0 > 0$, $a, b \in \Lambda \cap (Per\,(g^t))^c$ such that*

$$d(a, b) = \epsilon_0, \tag{11}$$

$$d(g^t(a), g^t(b)) \leq \epsilon_0 \quad \text{for all } t < 0, \tag{12}$$

$$a \notin \mathcal{O}(b), \tag{13}$$

$$a \in W^u(b). \tag{14}$$

Proof. We apply Proposition 5.2. Pick a subsequence $k_i \to +\infty$ such that both of the sequences $\{g^{s_{k_i}}(x_{k_i})\}$ and $\{g^{s_{k_i}}(x'_{k_i})\}$ converge. Let

$$a := \lim_{k_i \to +\infty} g^{s_{k_i}}(x_{k_i}) \quad \text{and} \quad b := \lim_{k_i \to +\infty} g^{s_{k_i}}(x'_{k_i}).$$

Then $d(a, b) = \lim_{k_i \to +\infty} d(g^{s_{k_i}}(x_{k_i}), g^{s_{k_i}}(x'_{k_i})) = \epsilon_0$. We get equation (11).

For any $t < 0$, since $0 < s_{k_i} + t < s_{k_i}$ for some large k_i, one has

$$d(g^t(a), g^t(b)) = \lim_{k_i \to +\infty} (d(g^{s_{k_i}+t}(x_{k_i}), g^{s_{k_i}+t}(x'_{k_i}))) \leq \epsilon_0.$$

Hence, we get equation (12).

Next assume that a is periodic. Since

$$\lim_{k_i \to +\infty} g^{t_{k_i}+s_{k_i}}(x) = \lim_{k_i \to +\infty} g^{s_{k_i}}(x_{k_i}) = a,$$

then x is periodic by Lemma 4.11. This is a contradiction. So $a \in (Per\,(g^t))^c$. Similarly, $b \in (Per\,(g^t))^c$. Thus, $a, b \in \Lambda \cap (Per\,(g^t))^c$.

Now we prove equation (13), that is, $a \notin \mathcal{O}(b)$. For a simpler notation, we write

$$\lim_{k \to +\infty} g^{s_k}(x_k) = a \quad \text{and} \quad \lim_{k \to +\infty} g^{s_k}(x'_k) = b.$$

The geodesics $\gamma_{x_k}(t), \gamma_{x'_k}(t)$ on M can be lifted to $\widetilde{\gamma}_k, \widetilde{\gamma}'_k$ on $\widetilde{M}$ in the way such that $d(x_k, x'_k) < 1/k$, $d(y_k, y'_k) = \epsilon_0$, where $y_k = g^{s_k}(x_k)$, $y'_k = g^{s_k}(x'_k)$, and moreover $y_k \to a$, $y'_k \to b$. Here we use a same notation for the lift of a point since no confusion is caused. Then $\widetilde{\gamma}_k$ converges to $\widetilde{\gamma} = \widetilde{\gamma}_a$, $\widetilde{\gamma}'_k$ converges to $\widetilde{\gamma}' = \widetilde{\gamma}_b$, and $d(a, b) = \epsilon_0$. See Figure 2 (we use the same notation for a vector and its footpoint).

First we show that $d(y_k, \widetilde{\gamma}'_k)$ is bounded away from 0. Write $d_k := d(y_k, \widetilde{\gamma}'_k) = d(y_k, z_k)$, $l_k := d(y_k, x'_k)$, $b_k := d(x'_k, z_k)$, and $b'_k := d(z_k, y'_k)$. And we already know that $d(x'_k, y'_k) = s_k$. Suppose that $d_k \to 0$ as $k \to +\infty$. By the triangle inequality, $\lim_{k \to +\infty}(l_k - b_k) = 0$. Since $\lim_{k \to +\infty}(l_k - s_k) \leq \lim_{k \to +\infty} d(x_k, x'_k) = 0$, we have that $\lim_{k \to +\infty} b'_k = \lim_{k \to +\infty} |(l_k - b_k) - (l_k - s_k)| = 0$. However, the triangle inequality implies $\epsilon_0 \leq d_k + b'_k \to 0$, which is a contradiction. Now $\widetilde{\gamma} \neq \widetilde{\gamma}'$ follows from $d(a, \widetilde{\gamma}') = \lim_{k \to +\infty} d(y_k, \widetilde{\gamma}'_k) \geq d_0$ for some $d_0 > 0$.

Next we suppose there exists a deck transformation ϕ such that $\phi(\widetilde{\gamma}) = \widetilde{\gamma}'$. See Figure 3. Observe that $\widetilde{\gamma}(-\infty) = \widetilde{\gamma}'(-\infty)$ since $d(g^t(a), g^t(b)) \leq \epsilon_0$ for all $t < 0$. Let $\widetilde{\gamma}_0$ be the closed geodesic such that $\phi(\widetilde{\gamma}_0) = \widetilde{\gamma}_0$. Then $\widetilde{\gamma}(-\infty) = \widetilde{\gamma}_0(-\infty)$. By Lemma 4.11, $\widetilde{\gamma}$ is a closed geodesic, that is, a is a periodic point. We arrive at a contradiction. Hence, for any deck transformation ϕ, $\phi(\widetilde{\gamma}) \neq \widetilde{\gamma}'$. So $a \notin \mathcal{O}(b)$ and we get equation (13).
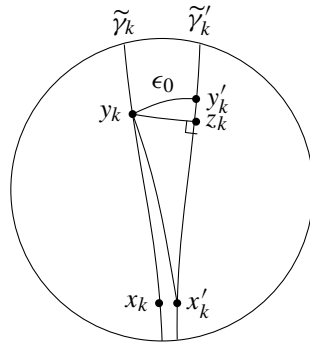
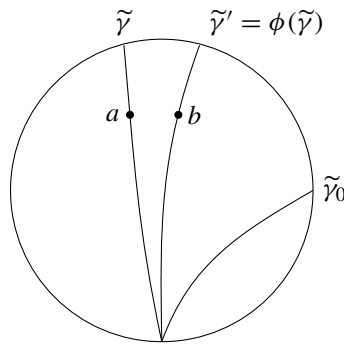FIGURE 2. Proof of $\widetilde{\gamma} \neq \widetilde{\gamma}'$.

FIGURE 3. Proof of $\phi(\widetilde{\gamma}) \neq \widetilde{\gamma}'$.

At last, if $a \notin W^u(b)$, we can replace a by some $a' \in \mathcal{O}(a)$, b by some $b' \in \mathcal{O}(b)$ such that $a' \in W^u(b')$, and the above three properties still hold for a different ϵ_0. We get equation (14). □

Proof of Theorem 1.8. We apply Proposition 5.3. Let $y = -a$, $z = -b$. Then $y, z \in \Lambda \cap$ $(\text{Per}\,(g^t))^c$, $d(g^t(y), g^t(z)) \leq \epsilon_0$ for all $t > 0$, $z \notin \mathcal{O}(y)$ and $y \in W^s(z)$.

If ϵ_0 is small enough, we can lift geodesics $\gamma_y(t)$ and $\gamma_z(t)$ to $\widetilde{\gamma}_y(t)$ and $\widetilde{\gamma}_z(t)$, respectively, on $\widetilde{M}$, such that $d(\widetilde{\gamma}_y(t), \widetilde{\gamma}_z(t)) \leq \epsilon_0$ for any $t > 0$ and $y \in \widetilde{W}^s(z)$. Suppose $\lim_{t \to +\infty} d(\widetilde{\gamma}_y(t), \widetilde{\gamma}_z(t)) = \delta > 0$. Then by Lemma 4.2, $\widetilde{\gamma}_y(t)$ and $\widetilde{\gamma}_z(t)$ converge to the boundary of a flat strip. Hence, y and z are periodic by Lemma 4.11, which is a contradiction. So we have $\lim_{t \to +\infty} d(\widetilde{\gamma}_y(t), \widetilde{\gamma}_z(t)) = 0$. Hence,

$$d(g^t(y), g^t(z)) \to 0 \quad \text{as } t \to +\infty. \qquad \square$$

5.2. *Proof of Theorem 1.7.* In the proof of Theorem 1.7, an argument similar to the one in Proposition 5.3 will be used.

Proof of Theorem 1.7. Suppose that $\Lambda \subset \text{Per}\,(g^t)$. We will prove that if $x \in \Lambda$, then x is tangent to an isolated closed flat geodesic or to a flat strip.

Assume the contrary to Theorem 1.7. Then there exists a sequence of different vectors $x'_k \in \Lambda$ such that $\lim_{k \to +\infty} x'_k = x$ for some $x \in \Lambda$. Here, different x'_k are tangent to different closed geodesics or to different flat strips, and x is tangent to a closed geodesic or to a flat strip. For large enough k, we suppose that $d(x'_k, x) < 1/k$. Fix a small number $\epsilon_0 > 0$. It is impossible that $d(g^t(x'_k), g^t(x)) \leq \epsilon_0$ for all $t > 0$. For otherwise, $\widetilde{\gamma}_{x'_k}(t)$ and $\widetilde{\gamma}_x(t)$ are positively asymptotic closed geodesics. They must be tangent to a common flat strip by Lemmas 4.2 and 4.10. This is impossible since different x'_k are tangent to different closed geodesics or to different flat strips. Hence, there exists a sequence $s_k \to +\infty$ such that

$$d(g^{s_k}(x'_k), g^{s_k}(x)) = \epsilon_0,$$

and

$$d(g^s(x'_k), g^s(x)) \leq \epsilon_0 \quad \text{for all } 0 \leq s < s_k.$$

Let $y_k := g^{s_k}(x)$ and $y'_k := g^{s_k}(x'_k)$. Without loss of generality, we suppose that $y_k \to a$ and $y'_k \to b$. A similar proof as in Proposition 5.3 shows that $d(a, b) = \epsilon_0$ and $d(g^t(a), g^t(b)) \leq \epsilon_0$ for all $t \leq 0$. Replacing x, x'_k by $-x, -x'_k$, respectively, and applying the same argument, we can obtain two points a^-, b^- such that $d(a^-, b^-) = \epsilon_0$ and $d(g^t(a^-), g^t(b^-)) \leq \epsilon_0$ for all $t \leq 0$. Then we have the following three cases.

(1) $\lim_{t \to \infty} d(g^t(-a), g^t(-b)) = 0$. By Lemma 4.10, $-a = -b$. This contradicts $d(a, b) = \epsilon_0$.

(2) $\lim_{t \to \infty} d(g^t(-a^-), g^t(-b^-)) = 0$. Also by Lemma 4.10, $-a^- = -b^-$. This contradicts $d(a^-, b^-) = \epsilon_0$.

(3) $\lim_{t \to \infty} d(g^t(-a), g^t(-b)) > 0$ and $\lim_{t \to \infty} d(g^t(-a^-), g^t(-b^-)) > 0$.

For case (3), by Lemmas 4.2 and 4.11, γ_a and γ_x coincide. And moreover, γ_x and γ_b are boundaries of a flat strip of width $\delta_1 > 0$. Similarly, γ_x and γ_{b^-} are boundaries of a flat strip of width $\delta_2 > 0$. We claim that these two flat strips lie on different sides of γ_x. Indeed, we choose ϵ_0 small enough and consider the ϵ_0-neighborhood of the closed geodesic γ_x which contains two regions lying on different sides of γ_x. By the definition of b and b^-, they must lie in different regions as above. This implies the claim.

In this way, we get a flat strip of width $\delta_1 + \delta_2$ and x is tangent to the interior of this flat strip. Since $g^{s_k}(x'_k) \to b$, we can repeat all the arguments above to b, $g^{s_k}(x'_k)$ instead of x, x'_k. Then either we are arriving at a contradiction as in case (1) or case (2) and we are done, or we get a flat strip of width greater than $\delta_1 + \delta_2$ and b is tangent to the interior of the flat strip. However, we can not enlarge a flat strip repeatedly in this way on a compact surface M. So we are done with the proof. $\square$

5.3. *Proof of Theorem 1.9.* We shall prove Theorem 1.9 by showing that the second one of the dichotomy cannot happen if $\{p \in M : K(p) < 0\}$ has at most finitely many connected components. The proof is an adaption of the one of [25, Theorem 1.6] to surfaces without focal points. Moreover, we fix a gap in that proof, which was pointed out by Keith Burns.
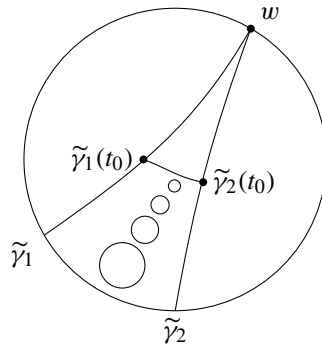
FIGURE 4. Proof of Theorem 1.9.

Proof of Theorem 1.9. Suppose $\Lambda \cap (\text{Per}\,(g^t))^c \neq \emptyset$. Consider the two points y and z given by Theorem 1.8. We lift the geodesics $\gamma_y(t)$ and $\gamma_z(t)$ to the geodesics in the universal cover $\widetilde{M}$, which are denoted by $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$, respectively. See Figure 4.

Consider the connected components of $\{p \in \widetilde{M} : K(p) < 0\}$ on $\widetilde{M}$ and we want to see how they distribute inside the ideal triangle bounded by $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$. Since $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ are flat geodesics, no connected component intersects $\widetilde{\gamma}_1$ or $\widetilde{\gamma}_2$. Note that such connected component may be not simply one lifting of (hence, not isometric to) one connected component of $\{p \in M : K(p) < 0\}$ on the base space M. However, each of them projects onto a connected component on M.

We claim that the maximal radius of inscribed disks inside each connected component is bounded away from 0. Indeed, if this is not true, then there exists an isometry between the inscribed disk with very small radius inside a connected component and an inscribed disk inside a connected component of $\{p \in M : K(p) < 0\}$. This is impossible because the number of connected components of $\{p \in M : K(p) < 0\}$ is finite, and therefore the maximal radius of their inscribed disks is bounded away from 0. The claim follows.

Let T denote the ideal triangle $\widetilde{\gamma}_1(0)\widetilde{\gamma}_2(0)w$. We claim that a connected component of $\{p \in \widetilde{M} : K(p) < 0\}$ is bounded inside T, that is, it cannot approach w. Assume the contrary. Let D be a fundamental domain inside the ideal triangle bounded by $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$. Then there exist x_i in D and $\alpha_i \in \Gamma$, $i = 1, 2, \ldots$, such that $\alpha_i x_i$ are all in one connected component of $\{p \in \widetilde{M} : K(p) < 0\}$ and $\alpha_i x_i \to w$ in T as $i \to \infty$. For i large enough, consider $\alpha_i \widetilde{\gamma}_1$ and $\alpha_i \widetilde{\gamma}_2$, which are two asymptotic non-closed flat geodesics. If $\alpha_i \widetilde{\gamma}_1$ and $\alpha_i \widetilde{\gamma}_2$ approach w, then w is a fixed point for $\alpha_i \in \Gamma$. Then $\widetilde{\gamma}_1$ is asymptotic to an axis of α_i, and hence itself is closed by Lemma 4.10. This contradicts the assumption. So $\alpha_i \widetilde{\gamma}_1$ and $\alpha_i \widetilde{\gamma}_2$ approach some $w_i \neq w \in \widetilde{M}(\infty)$ and $w_i \to w$ as $i \to \infty$. Since $\alpha_i T$ contains $\alpha_i x_i \in T$, at least one of $\alpha_i \widetilde{\gamma}_1$ and $\alpha_i \widetilde{\gamma}_2$ must intersect T. Since the considered connected component cannot intersect flat geodesics, it must be bounded. This proves the claim.

Since the radii of the inscribed disks are bounded away from zero, there exists a $t_0 > 0$ such that the infinite triangle $\widetilde{\gamma}_1(t_0)\widetilde{\gamma}_2(t_0)w$ does not contain any such inscribed disk, see Figure 4. Note that if two connected components project to the same connected component on M, they must be isometric. Thus, by the second claim above, we can find

$t_1 > t_0$ such that the infinite triangle $\widetilde{\gamma}_1(t_1)\widetilde{\gamma}_2(t_1)w$ is a flat region. Then $d(\widetilde{\gamma}_1(t), \widetilde{\gamma}_2(t)) \equiv d(\widetilde{\gamma}_1(t_1), \widetilde{\gamma}_2(t_1))$ for all $t \geq t_1$. Indeed, if we construct a geodesic variation between $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$, then the Jacobi fields are constant for $t \geq t_1$ since $K \equiv 0$. Thus, $d(\widetilde{\gamma}_1(t), \widetilde{\gamma}_2(t))$ is constant when $t \geq t_1$. We get a contradiction since $d(\widetilde{\gamma}_1(t), \widetilde{\gamma}_2(t)) \to 0$ as $t \to +\infty$ by Theorem 1.8.

Finally we can conclude that $\Lambda \subset \mathrm{Per}\,(g^t)$. In particular, the geodesic flow is ergodic by Theorem 1.7 and Pesin's theorem (Theorem 1.3). □

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