Proceedings of the Edinburgh Mathematical Society (2015) **58**, 807–808

BOOK REVIEWS

DOI:10.1017/S0013091515000607

David B. Ellis and Robert Ellis *Automorphisms and equivalence relations in topological dynamics* (Cambridge University Press, 2014), 281 pp, 978-1-107-63322-3 (paperback), £45/$70.

Topological dynamics—formally the study of the properties of an action of a group (or semigroup) by homeomorphisms (or continuous maps) of a topological space—has its origins in questions about the asymptotic behaviour of solutions of ordinary differential equations. In particular, for differential equations describing the evolution in time of some physical system, questions about the existence of asymptotically repetitive solutions, stationary points, stability, existence of dense orbits, and so on arise naturally. Important contributions were made by Poincaré in settings arising from celestial mechanics, but the movement towards extracting and developing the essential mathematical features of the problems was given great impetus by work of George Birkhoff in the first third of the twentieth century. While specific categories of topological dynamical systems (e.g. smooth diffeomorphisms of manifolds, continuous maps on low-dimensional spaces or even on intervals, symbolic systems, systems of algebraic origin) remain of great interest, the abstract theory of continuous maps on compact topological spaces has its own interesting history. Despite formally subsuming any specific class of system, abstract topological dynamics has its own distinctive character and significant themes. The distinctive rather abstract character was expressed well in an influential systematic treatment of the field by Gottschalk and Hedlund [**7**].

One of the major themes had a structural flavour but concerned algebraic objects associated with topological dynamical systems rather than the structure of the dynamical system itself. The second author introduced [**2**] the *enveloping semigroup* of a topological dynamical system (X, G), where X is a compact Hausdorff space and G is a topological group acting continuously on X by homeomorphisms. Writing $i\colon G \to \mathrm{Homeo}(X)$ for the action, the enveloping semigroup $E(X, G)$ is then defined to be the closure of the subset $i(G)$ of maps induced by the action in the compact space X^X of all maps $X \to X$ with the product topology. The binary operation of composition of maps makes $E(X, G)$ into a right topological compact semigroup, and the map sending $g \in G$ to $i(g)$ exhibits $E(X, G)$ as a right topological compactification of G. Any topological dynamical system therefore has an associated semigroup, which is an object of interest in itself but also, as it turns out, a source of powerful and novel ideas in understanding the original dynamical system. For example, a different major theme was initiated by a structure theorem of Furstenberg for minimal distal flows [**4**], leading to a great deal of significant work on the structure of minimal and other categories of flows. The proof of this structure theorem made essential use of the enveloping semigroup, and in particular the result of the second author that the enveloping semigroup is a group for distal systems [**1**].

However, there is necessarily a price to pay for this level of abstraction in several different senses. Topologically, in most situations the enveloping semigroup will not be a metrizable space, nor is it in any other topological sense small or tractable. Algebraically, as the nomenclature suggests, the enveloping semigroup is usually not a group. Viewed through the prism of the Borel measurable structure carried by (X, G), most of the maps in the enveloping semigroup when viewed as maps on X are not measurable with respect to the Borel σ-algebra. Nonetheless, the topological and algebraic structure of $E(X, G)$ has become a powerful tool in topological dynamics, with many subtle results concerned with dynamical properties of (X, G) finding their most natural proof via an excursion through the seemingly exotic space $E(X, G)$. This algebraic approach to topological dynamics was strongly influenced by the second author's monograph [**3**], which gave an influential presentation of the theory at the time. Half a century later, the original idea of the enveloping semigroup continues to prove useful in topological dynamics, as reflected, for example, in a recent survey by Glasner [**5**] and the monograph on proximal flows by the same author [**6**].

The present volume is, as the authors make clear, in some sense an updated version of the lectures from 1969, but the changes are more substantial than the word update suggests. It is a fundamental conceptual revisiting of the theory developed in the original lectures by the second author, together with an updating in the usual sense of presenting more recent results. In particular, great emphasis is placed on the construction of a universal object in the category of minimal topological systems, which permits each minimal system (that is, a system without non-trivial closed invariant subsets under the action) to be presented as a quotient of the universal minimal model by some invariant closed equivalence relation. The system may then also be analysed using properties of this defining equivalence relation. In particular, this equivalence relation may be used in contrast to a more familiar approach of using the algebra of continuous functions on the underlying compact space. An additional technical change is to make greater use of the group of automorphisms of a system, which permits some of the constructions used in the theory to be presented in a more canonical manner.

The book comprises five parts: universal constructions; equivalence relations and automorphisms; the τ topology on the group of automorphisms; relations between subgroups and the dynamics of minimal flows; and extensions of minimal flows. A final chapter demonstrates the various approaches developed here by discussing versions of Furstenberg's fundamental structure theorem for distal flows from the original point of view, from the viewpoint of the invariant closed equivalence relation construction presented here, from the viewpoint of the notion of regional proximality, and group theoretically.

The style of presentation used in this book is a little unconventional and is described by the authors as being in 'theorem–proof format'. While it would be reasonable to expect any mathematical monograph to adopt such a format, the text goes beyond this, presenting many of the proofs in a highly structured and almost algorithmic format, with a large number of carefully justified steps. This perhaps loses something in fluency and an overall view of the next destination but certainly gains something in clarity and rigour. There are other structural choices that the authors have made to facilitate use of the book by non-experts, which is to be welcomed as parts of the theory can seem arcane on first exposure. The book is aimed at postgraduate students in topological dynamics as well as researchers, and it is certain to prove useful as a carefully thought through and precise presentation of the material.

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