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GLOBAL ACTIONS AND VECTOR K-THEORY

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Abstract

Purely algebraic objects like abstract groups, coset spaces, and G-modules do not have a notion of hole as do analytical and topological objects. However, equipping an algebraic object with a global action reveals holes in it and thanks to the homotopy theory of global actions, the holes can be described and quantified much as they are in the homotopy theory of topological spaces. Part I of this article, due to the first author, starts by recalling the notion of a global action and describes in detail the global actions attached to the general linear, elementary, and Steinberg groups. With these examples in mind, we describe the elementary homotopy theory of arbitrary global actions, construct their homotopy groups, and revisit their covering theory. We then equip the set $Um_n(R)$ of all unimodular row vectors of length n over a ring R with a global action. Its homotopy groups $\pi_i(Um_n(R)), i \geqslant 0$ are christened the vector K-theory groups $K_{i+1}(Um_n(R)), i \geqslant 0$ of $Um_n(R)$. It is known that the homotopy groups $\pi_i(\mathrm{GL}_n(R))$ of the general linear group $\mathrm{GL}_n(R)$ viewed as a global action are the Volodin K-theory groups $K_{i+1,n}(R)$. The main result of Part I is an algebraic construction of the simply connected covering map $StUm_n(R) \to EUm_n(R)$ where $EUm_n(R)$ is the path connected component of the vector $(1, 0, \dots, 0) \in Um_n(R)$. The result constructs the map as a specific quotient of the simply connected covering map $St_n(R) \to E_n(R)$ of the elementary global action $E_n(R)$ by the Steinberg global action $St_n(R)$. As expected, $K_2(Um_n(R))$ is identified with $\mathrm{Ker}(StUm_n(R) \to EUm_n(R))$. Part II of the paper provides an exact sequence relating stability for the Volodin K-theory groups $K_{1,n}(R)$ and $K_{2,n}(R)$ to vector K-theory groups.

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1. Introduction

The first goal of this article is to use global actions to construct the vector K-theory functors $K_i Um_n$, $i > 0$, of rings and to algebraically describe the functors $K_i Um_n$, $i = 1, 2$. The second goal is to relate vector K-theory to stability in

Volodin K-theory. The first goal is handled in Part I of the article and the results are due to the first author. The second goal is handled in Part II.

We describe now the results in Part I.

Part I: Global actions and vector K-theory

The concept global action is a rigorous formulation of the idea that group actions express motion. Formally, a global action is a set X equipped with a family of group actions $X_\alpha \curvearrowleft G_\alpha$ on subsets X_α of X, subject to a compatibility condition. The group actions tell us how we can move and deform subsets of X and thereby construct homotopy groups of X and a homotopy theory of global actions.

The current paper focuses on two specific global actions. The first is defined by letting the standard unipotent subgroups of the general linear group $\mathrm{GL}_n(R)$ of a ring R act by right multiplication on (all of) $\mathrm{GL}_n(R)$. The homotopy groups $\pi_i(\mathrm{GL}_n(R))$ of this global action are the Volodin K-theory groups $K_{i+1,n}(R)$. The second global action is defined by letting the standard unipotent subgroups above act by right multiplication on the set $Um_n(R)$ of unimodular row vectors of length n with entries in R. The vector K-theory groups $K_{i+1,n}(R)$ are by definition the homotopy groups $\pi_i(Um_n(R))$. Of course π_0 is generally just a pointed set.

The main results of Part I are an algebraic description of the vector K-theory objects $K_{i,n}(R), i = 1, 2, n \geqslant 3$ and of the simply connected covering of the path-connected component of the vector $(1, 0, \ldots, 0) \in Um_n(R)$. We state these results next.

Let R be associative ring with identity. A unimodular row vector of length n over R is by definition a row vector $v = (v_1, \ldots, v_n)$, having entries $v_i \in R$ with the property there is a row vector $w = (w_1, \ldots, w_n)$, $w_i \in R$, such that $v \cdot w^t = \langle v, w\rangle := \sum_i v_i w_i = 1$. Let $Um_n(R)$ denote the set of all unimodular row vectors of length n over R, equipped with the global action above.

A path in $Um_n(R)$ is by definition a sequence $v, vg_1, \ldots, vg_k$ of vectors such that each g_i belongs to a standard unipotent subgroup of $\mathrm{GL}_n(R)$. Since the standard unipotent subgroups generate the elementary subgroup $E_n(R)$ of $\mathrm{GL}_n(R)$, it follows that the path connected components of $Um_n(R)$ are the orbits $Um_n(R)/E_n(R)$ of the action of $E_n(R)$ on $Um_n(R)$. Thus, by definition, $\pi_0(Um_n(R)) = Um_n(R)/E_n(R)$.

Let $e = (1, 0, \ldots, 0)$ be the base point of $Um_n(R)$. Let $EUm_n(R)$ be the path connected component of e. Clearly $EUm_n(R) = eE_n(R)$. $eE_n(R)$ has a global action defined by the action of the standard unipotent subgroups of $E_n(R)$ on $eE_n(R)$. Let $StUm_n(R) \to eE_n(R)$ be the simply connected covering map of $eE_n(R)$. We algebraically describe this map and $K_2(Um_n(R))$ where by definition $K_2(Um_n(R)) = \pi_1(Um_n(R))$. Let:

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(1) $P_n(R) = \{\sigma \in \mathrm{GL}_n(R) | e_1\sigma = e_1\}$;

(2) $EP_n(R) = P_n(R) \cap E_n(R)$.

There is a canonical map $E_n(R) \to eE_n(R)$, $\varepsilon \mapsto e\varepsilon$, which induces a bijection $EP_n(R) \setminus E_n(R) \to eE_n(R)$ of $E_n(R)$-sets where $EP_n(R) \setminus E_n(R)$ is the set of right cosets of $EP_n(R)$ in $E_n(R)$. This map becomes an isomorphism of global actions as soon as we give $EP_n(R) \setminus E_n(R)$ the global action defined by the canonical right action of the standard unipotent subgroups of $E_n(R)$ on $EP_n(R) \setminus E_n(R)$. We identify $eE_n(R)$ with $EP_n(R) \setminus E_n(R)$, using this isomorphism. Let $St_n(R)$ denote the Steinberg group (recalled in Section 2.2) and $\theta_n : St_n(R) \to E_n(R)$ the canonical surjective homomorphism. Let Φ_n be an index set for the set of all standard unipotent subgroups $E_n(R)_\alpha$ of $E_n(R)$, $\alpha \in \Phi_n$. In Section 2.2, it is shown that there is a canonical lifting $St_n(R)_\alpha$ of each $E_n(R)_\alpha$ to the Steinberg group. Let:

(3) $\widetilde{P}_n(R) = \theta_n^{-1}(EP_n(R))$;

(4) $B_n(R) = \langle x^{-1}abx \in \widetilde{P}_n(R) \mid x \in St_n(R), a \in St_n(R)_\alpha, b \in St_n(R)_\beta, \alpha, \beta \in \Phi_n \rangle$.

Let $B_n(R) \setminus St_n(R)$ denote the set of all right cosets $B_n(R)x$, $x \in St_n(R)$, of $B_n(R)$ in $St_n(R)$. Let $B_n(R) \setminus \widetilde{P}_n(R)$ denote the set of all right cosets of $B_n(R)$ in $\widetilde{P}_n(R)$. Since $B_n(R)$ is clearly normal in $\widetilde{P}_n(R)$, $B_n(R) \setminus \widetilde{P}_n(R)$ is a group.

The main theorem of Part I is the following.

THEOREM 1.1 (Main theorem). *Let $B_n(R) \setminus St_n(R)$ have the global action defined by letting the subgroups $St_n(R)_\alpha, \alpha \in \Phi_n$, of $St_n(R)$ act by right multiplication on $B_n(R) \setminus St_n(R)$. Then the canonical map*

$$B_n(R) \setminus St_n(R) \to EP_n(R) \setminus E_n(R),$$
$$B_n(R)x \mapsto EP_n(R)\theta_n(x)$$

is a simply connected covering morphism of global actions. Furthermore the sequence

$$B_n(R) \setminus \widetilde{P}_n(R) \rightarrowtail B_n(R) \setminus St_n(R) \twoheadrightarrow EP_n(R) \setminus E_n(R)$$

is short exact and

$$K_2(Um_n(R)) \cong B_n(R) \setminus \widetilde{P}_n(R).$$

Of course, the isomorphism above can be used as an algebraic definition of $K_2(Um_n(R)$. This is analogous to defining $K_{2,n}(R) = \mathrm{Ker}(St_n(R) \to E_n(R))$.

To conclude Part I, we mention a very general exact sequence which motivates results in Part II.

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THEOREM 1.2. *Let $f : (B, b) \to (A, a)$ be a morphism of pointed global actions. Then there is an infinite homotopy exact sequence*

$$\cdots \to \pi_{i+1}(\mathrm{rel}(f)) \to \pi_{i+1}(B) \to \pi_{i+1}(A) \to \pi_i(\mathrm{rel}(f)) \to \cdots$$

for all $i \geqslant 0$, terminating at $\pi_0(A)$.

Proof. The proof is carried out mimicking any classical proof of the analogous result in algebraic topology. Details are left to the reader. □

Part II: Stability in algebraic K-theory and vector K-theory

The goal of this part is to relate stability in Volodin K-theory to vector K-theory by an exact sequence.

Give $\mathrm{GL}_n(R)$ the global action obtained by letting the standard unipotent subgroups of $\mathrm{GL}_n(R)$ act on it by right multiplication. Define $K_{i,n}(R) = \pi_{i-1}(\mathrm{GL}_n(R))$ for all $i > 0$. There is a canonical morphism $f : \mathrm{GL}_n(R) \to Um_n(R)$, $g \mapsto eg = \mathrm{firstrow}(g)$ of global actions where $e = (1, 0, \ldots, 0)$. By Theorem 1.2, there is an exact sequence

$$\begin{aligned} K_2(\mathrm{rel}(f)) \to K_{2,n}(R) \to K_2(Um_n(R)) \to \\ K_1(\mathrm{rel}(f)) \to K_{1,n}(R) \to K_1(Um_n(R)). \end{aligned}$$

Volodin K-theory computes $K_{2,n}(R)$ and $K_{1,n}(R)$ and the results in Part I algebraically compute $K_2(Um_n(R))$ and $K_1(Um_n(R))$. So to understand the exact sequence above, it suffices to compute $K_2(\mathrm{rel}(f))$ and $K_1(\mathrm{rel}(f))$ and the missing maps, namely those going in and out of $K_i(\mathrm{rel}(f))$. Exactness follows for free.

But this is not done in the current paper. Instead, an exact sequence is constructed *ad hoc*, based on the fact that there is one to find. So we guess at objects to replace $K_2(\mathrm{rel}(f))$ and $K_1(\mathrm{rel}(f))$. They are respectively the group $K_{2,n}(R)_2$ and the pointed set $(K_{1,n-1}(R))_2 \setminus K_{1,n-1}(R)$. Then maps in and out of these objects are defined and exactness is proved. The main result is the following.

THEOREM 1.3. *There is an 8-term exact sequence*

$$\begin{aligned} (K_{2,n}(R))_2 \to K_{2,n}(R) \to \pi_1(EUm_n(R)) \to (K_{1,n-1}(R))_2 \setminus K_{1,n-1}(R) \to \\ K_{1,n}(R) \to \pi_0(Um_n(R)) \to K^s_{0,n-1}(R) \to 0 \end{aligned}$$

of pointed sets, whose first two maps on the left are group homomorphisms. By definition,

$$(K_{2,n}(R))_2 = K_{2,n}(R) \cap B_n(R)$$

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and contains the

$$\mathrm{image}(K_{2,n-1}(R) \to K_{2,n}(R))$$

(see Section 4);

$$(K_{1,n-1}(R))_2 = E_{n-1}(R) \setminus (BEP_n(R)) \cap \mathrm{GL}_{n-1}(R)),$$

where

$$BEP_n(\mathrm{R}) = \langle \varepsilon^{-1}ab\varepsilon \in EP_n(\mathrm{R}) \mid \varepsilon \in E_n(R), \\ a \in E_n(R)_\alpha, b \in E_n(R)_\beta, \alpha, \beta \in \Phi_n\rangle.$$

It is clearly a normal subgroup of $EP_n(R)$, which contains $E_{n-1}(R)$ (see Section 4).

$K^s_{0,m}(R)$ = *set of all isomorphism classes of finitely generated, projective, left R-modules P such that $P \oplus R \cong {}^{m+1}R$. The base point of $K^s_{0,m}(R)$ is the isomorphism class of ${}^mR$.*

The maps in the exact sequence are defined in Section 4.1.

The first three terms of the exact sequence above, starting from the left, come equipped with group structures and the maps between them are group homomorphisms. So this much of the sequence is an exact sequence of groups. Suppose that $E_{n-1}(R)$ and $E_n(R)$ are normal in $\mathrm{GL}_{n-1}(R)$ and $\mathrm{GL}_n(R)$, respectively. Then $K_{1,n-1}(R)$ and $K_{1,n}(R)$ are groups and it turns out that $(K_{1,n-1}(R))_2$ is a normal subgroup of $\mathrm{Ker}(K_{1,n-1}(R) \to K_{1,n}(R))$ and that the map $\pi_1(Um_n(R)) \to (K_{1,n-1}(R))_2 \setminus K_{1,n-1}(R)$ has as image the group $[\mathrm{Ker}(K_{1,n-1}(R) \to (K_{1,n-1}(R))_2 \setminus K_{1,n}(R))]$ and is a group homomorphism to this group. So in this case, the first five terms is an exact sequence of groups. It is an interesting problem to find group structures on the remaining objects so that the entire sequence is an exact sequence of groups.

Assuming the ring R is commutative and Noetherian of finite Krull dimension d and n is sufficiently large relative to d, van der Kallen [**10**, **11**] found a group structure on $\pi_0(Um_n(R))$, but showed that the map $\mathrm{GL}_n(R) \to \pi_0(Um_n(R))$ is not always a group homomorphism. On the other hand, Ravi Rao and van der Kallen [**7**] found (nontrivial) examples where it is a group homomorphism. In these examples, we get a 6-term exact sequence of groups. An interesting problem is to find a group structure on $K^s_{0,n-1}(R)$ such that the map involving this group is a group homomorphism.

The rest of the paper is organized as follows. In Section 2.1, the definition of a global action and of a single domain global action are recalled. Section 2.2

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carefully describes important examples of global actions including the single domain global actions equipping the general linear, elementary, and Steinberg groups. It concludes with the notion of geometric group which includes as special cases the single domain global actions above and with the notions of geometric set and geometric coset space which include as special cases all of the remaining single domain global actions of the paper. Section 3.1 recalls the notion of homotopy for arbitrary morphisms of global actions. Section 3.2 recalls the definition of end-point- preserving homotopy of paths (also called path-homotopy) and defines the fundamental group π_1 of a pointed global action. Section 3.3 defines loop- homotopy of n-loops and the higher homotopy groups π_n of a pointed global action. Section 3.4 equips the set $Um_n(R)$ of all unimodular row vectors of length n with a single domain global action. The notion of covering and covering morphism is recalled and the Covering Classification Theorem 3.23 is proved. Theorem 1.1 is then deduced as Theorem 3.22, from Theorem 3.23. Section 4.1 proves Theorem 1.3.

Part I: Global actions and vector K-theory

2. Preliminaries

The results of this section are due to the first author.

2.1. Global actions. In this section, we recall from [**1**, **2**] the definitions of global action and single domain global action and describe important examples which are used in the rest of the paper. The section concludes with a concise conceptualization of all single domain global actions occuring in the paper.

DEFINITION 2.1. Let G be a group and X a set. Then a (right) *group action* of G on X is a function $X \times G \to X$, denoted by $(x, g) \mapsto x \cdot g$, such that:

(1) $x \cdot 1 = x$, for all $x \in X$, where 1 is the identity of the group G; and

(2) $x \cdot (g_1 g_2) = (x \cdot g_1) \cdot g_2$, for all $x \in X$ and $g_1, g_2 \in G$.

A group action will be denoted by $X \curvearrowleft G$.

DEFINITION 2.2. Let $X \curvearrowleft G$ and $Y \curvearrowleft H$ be group actions. A *morphism of group actions* $(\psi, \varphi) : X \curvearrowleft G \to Y \curvearrowleft H$ consists of a function $\psi : X \to Y$ and a group homomorphism $\varphi : G \to H$ such that $\psi(x \cdot g) = \psi(x) \cdot \varphi(g)$. Two morphisms $(\psi, \phi), (\psi', \phi') : X \curvearrowleft G \to Y \curvearrowleft H$ are considered the *same*, if $\psi = \psi'$. In this case, we write $(\psi, \phi) = (\psi', \phi')$ (The paper [**1**] defines notions of global actions which are stronger than the one defined in the current paper and

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require morphisms of group actions whose notion of sameness is stronger than that defined above, for example two morphisms as above are the same, if $\psi = \psi'$ and $\phi = \phi'$.)

DEFINITION 2.3. A *global action* $A = (X, \Phi, G, \theta)$ consists of a set X called the *underlying set* of A, together with:

(1) A set Φ equipped with a reflexive relation $\leqslant$. Φ is called the *index system* of A. (Equivalently, Φ is a directed graph such that there is a directed loop at each vertex.)

(2) A set $\{X_\alpha \curvearrowleft G_\alpha \mid \alpha \in \Phi\}$ of group actions on subsets X_α of X. The groups G_α are called the *local groups* of A, the subsets X_α the *local sets* of A, and the group actions $X_\alpha \curvearrowleft G_\alpha$ the *local actions* of A.

(3) *Compatibility condition.* For each relation $\alpha \leqslant \beta$ of Φ, the local group G_α leaves $X_\alpha \cap X_\beta$ invariant and there is stipulated a group homomorphism

$$\theta_{\alpha\leqslant\beta} : G_\alpha \to G_\beta,$$

called a *structure homomorphism* such that the map

$$(\iota_{\alpha\leqslant\beta}, \theta_{\alpha\leqslant\beta}) : (X_\alpha \cap X_\beta) \curvearrowleft G_\alpha \to X_\beta \curvearrowleft G_\beta$$

is a morphism of group actions where

$$\iota_{\alpha\leqslant\beta} : (X_\alpha \cap X_\beta) \to X_\beta$$

is the canonical inclusion of sets.

A global action $A = (X, \Phi, G, \theta)$ is called a *single domain global action*, if $X_\alpha = X$ for all $\alpha \in \Phi$.

It is useful to think of the map $G : \Phi \to ((\text{group actions}))$, $\alpha \mapsto X_\alpha \curvearrowleft G_\alpha$, together with the structure maps $(\iota_{\alpha\leqslant\beta}, \theta_{\alpha\leqslant\beta}) : (X_\alpha \cap X_\beta) \curvearrowleft G_\alpha \to X_\beta \curvearrowleft G_\beta$ as a notion of prefunctor on the directed graph Φ with values in group actions.

LEMMA 2.4. *Let $A = (X, \Phi, G, \theta)$ be a single domain global action such that the relation $\leqslant$ of Φ is reflexive and transitive. Then the prefunctor above is a functor if and only $\theta_{\alpha\leqslant\alpha} : G_\alpha \to G_\alpha$ is the identity homomorphism for all $\alpha \in \Phi$ and $\theta_{\beta\leqslant\gamma}\theta_{\alpha\leqslant\beta} = \theta_{\alpha\leqslant\gamma}$ for all $\alpha \leqslant \beta \leqslant \gamma \in \Phi$.*

Proof. Straightforward. □

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We recall the definition of a morphism of global actions. To do this we need the notion of a local frame.

DEFINITION 2.5. Let $A = (X, \Phi, G, \theta)$ be a global action. A *local frame* of A is a finite set F of elements of X such that for some $\alpha \in \Phi$, $F \subseteq X_\alpha$ and $F \subseteq$ an orbit of G_α.

DEFINITION 2.6. Let A and B be global actions with underlying sets X and Y, respectively. A *morphism* $f : A \to B$ of global actions is a function $f : X \to Y$ which preserves local frames.

EXAMPLE 2.7. Let A be a global action. Then the identity function on the underlying set of A is a morphism of global actions.

2.2. Important examples of global actions. We give below examples of global actions by describing their underlying set, index system, local sets, local groups, local actions, and structure homomorphisms. It is easy to check that the compatibility condition holds.

- **The line action.** The *line action* denoted by L is a global action with underlying set $X = \mathbb{Z}$ and index system $\Phi = \mathbb{Z} \cup \{*\}$. The relations of Φ are $* \leqslant *$, $* \leqslant n$, and $n \leqslant n$ for all $n \in \mathbb{Z}$. The local sets are $X_* = \mathbb{Z}$ and $X_n = \{n, n + 1\}$ for all $n \in \mathbb{Z}$. The local groups are $G_* = 1$ and $G_n = \mathbb{Z}/2\mathbb{Z}$ for all $n \in \mathbb{Z}$. The local action $L \curvearrowleft G_*$ is the unique one. The local action $\{n, n+1\} \curvearrowleft G_n$ is the unique nontrivial action. The structure homomorphisms $\theta_{*\leqslant *} : G_* \to G_*$ and $\theta_{*\leqslant n} : G_* \to G_n$ are the unique ones for all $n \in \mathbb{Z}$. The structure homomorphisms $\theta_{n\leqslant n} : G_n \to G_n$ are the identity for all $n \in \mathbb{Z}$.

- **The graph global action.** The graph global action is a generalization of the line action. Recall that a graph $G = (V, E)$ is a pair consisting of a set V of elements called *vertices* and a reflexive, antisymmetric relation $E \subseteq V \times V$ called *edges*. If $e = (v, w) \in E$, then one should think of e as an edge between v and w. Let $\text{vert}(e) =$ the set $\{v, w\}$. The *graph global action* of G is the global action with underlying set V and index system $\Phi = E \cup \{*\}$. Let the relations of Φ be $* \leqslant *$, $* \leqslant \alpha$, and $\alpha \leqslant \alpha$ for all $\alpha \in E$. The local sets are $V_* = V$ and $V_\alpha = \{\text{vert}(\alpha)\}$ for all $\alpha \in E$. The local groups are $G_* = 1$ and $G_\alpha = \mathbb{Z}/2\mathbb{Z}$ for all $\alpha \in E$. The local action $V \curvearrowleft G_*$ is the unique one. The local action $V_\alpha \curvearrowleft G_\alpha$ is the unique nontrivial one for all $\alpha \in E$. The structure homomorphisms $\theta_{*\leqslant *} : G_* \to G_*$ and $\theta_{*\leqslant\alpha} : G_* \to G_\alpha$ are the unique ones for all $\alpha \in E$. The structure homomorphisms $\theta_{\alpha\leqslant\alpha} : G_\alpha \to G_\alpha, \alpha \in \Phi(G)$, are the

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identity for all $\alpha \in E$. Abusing notation, we also let G denote the global action just constructed.

- **The general linear global action.** If $n \geqslant 3$, let $J_n = [1, n] \times [1, n]) \setminus \{(i, i) \mid 1 \leqslant i \leqslant n\}$, that is the Cartesian product of the set $\{1, 2, \ldots, n\}$ with itself with the diagonal removed. It is worth mentioning that J_n is the root system of the Chevalley group A_{n-1} over commutative rings.

  A subset $\alpha \subseteq J_n$ is called *nilpotent* if the following conditions hold:

  - if $(i, j) \in \alpha$, then $(j, i) \notin \alpha$;
  - if $(i, j), (j, k) \in \alpha$, then $(i, k) \in \alpha$.

  Note that the empty set is a nilpotent subset and that the intersection of nilpotent subsets is nilpotent. Let Φ_n denote the index system

$$\Phi_n = \{\alpha \mid \alpha \text{ nilpotent subset of } J_n\}$$

  equipped with the reflexive relation $\leqslant$ defined by $\alpha \leqslant \beta \Leftrightarrow \alpha \subseteq \beta$. There is a canonical embedding

$$J_n \rightarrowtail \Phi_n, (i, j) \mapsto \{(i, j)\}$$

  whose image consists of the smallest nonempty nilpotent subsets of J_n.

  Let R denote an associative ring with identity. The *general linear global action* is a single domain global action with underlying set the general linear group $\mathrm{GL}_n(R)$, index system Φ_n, and local groups $\mathrm{GL}_n(R)_\alpha$, $\alpha \in \Phi_n$, consisting of all matrices whose diagonal entries are 1 and whose nondiagonal entry at position (i, j) is 0 if $(i, j) \notin \alpha$ and arbitrary if $(i, j) \in \alpha$. This means that the empty subset of Φ_n is assigned the trivial subgroup of $\mathrm{GL}_n(R)$. The local groups $\mathrm{GL}_n(R)_\alpha$ act on $\mathrm{GL}_n(R)$ by right multiplication. The structure homomorphisms $\theta_{\alpha\leqslant\beta} : \mathrm{GL}_n(R)_\alpha \to \mathrm{GL}_n(R)_\beta$ are by definition the natural inclusions. Clearly $\mathrm{GL}_n(R)_\alpha \cap \mathrm{GL}_n(R)_\beta = \mathrm{GL}_n(R)_{\alpha\cap\beta}$. It follows that the assignment

$$\begin{aligned} \Phi_n &\to \text{ subgroups of } \mathrm{GL_n(R)}, \\ \alpha &\mapsto \mathrm{GL}_n(R)_\alpha \end{aligned}$$

  preserves not only partial orderings, that is $\alpha \leqslant \beta \implies \mathrm{GL}_n(R)_\alpha \subseteq \mathrm{GL}_n(R)_\beta$ (and is therefore a functor), but also intersections, that is $\mathrm{GL}_n(R)_{\alpha\cap\beta} = \mathrm{GL}_n(R)_\alpha \cap \mathrm{GL}_n(R)_\beta$.

  The subgroups $\mathrm{GL}_n(R)_\alpha$, $\alpha \in \Phi_n$ are known in the literature as the *standard unipotent subgroups* of $\mathrm{GL}_n(R)$.

  Abusing notion, we also let $\mathrm{GL}_n(R)$ denote the single domain global action just constructed.

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We recall the definition of an elementary matrix. If $(i, j) \in J_n$, let e_{ij} denote the $n \times n$ matrix whose (i, j)th entry is 1 and all other entries are 0. For $r \in R$, let $E_{ij}(r) = I_n + re_{ij}$, where I_n denotes the $n \times n$ identity matrix. $E_{ij}(r)$ is called an *elementary matrix*. The subgroup of $\mathrm{GL}_n(R)$ generated by all elementary matrices is called the *elementary group* and is denoted by $E_n(R)$.

The map $J_n \rightarrowtail \Phi_n$, $(i, j) \mapsto \{(i, j)\}$, tells us that any elementary matrix is contained in some local group $\mathrm{GL}_n(R)_\alpha$. Conversely it follows from [6, Lemma 9.14], that any local group $\mathrm{GL}_n(R)_\alpha$ is generated by all elementary matrices $E_{ij}(r)$ such that $(i, j) \in \alpha$. For this reason, we introduce the notation

$$E_n(R)_\alpha = \mathrm{GL}_n(R)_\alpha$$

and note that $E_n(R)$ is generated by the groups $E_n(R)_\alpha$, as α ranges over Φ_n.

- **The elementary global action.** The *elementary global action* is the single domain global action with underlying set the elementary group $E_n(R)$. Its index system Φ_n, local groups $E_n(R)_\alpha$, and structure homomorphisms $\theta_{\alpha \leqslant \beta}$ are the same as those of the general linear global action. The action of $E_n(R)_\alpha$ on $E_n(R)$ is by right multiplication. Abusing notation, we also let $E_n(R)$ denote the single domain global action just constructed.

- **The special linear global action.** Suppose R is commutative. The *special linear global action* is the single domain global action with underlying set the special linear group $\mathrm{SL}_n(R)$. Its index system Φ_n, local groups $E_n(R)_\alpha$, and structure homomorphisms $\theta_{\alpha \leqslant \beta}$ are the same as those of the elementary or general linear global action. The action of $E_n(R)_\alpha$ on $SL_n(R)$ is by right multiplication. Abusing notation as usual, we also let $SL_n(R)$ denote the single domain global action just constructed.

Clearly, the canonical inclusion $E_n(R) \to \mathrm{GL}_n(R)$ and when R is commutative, the canonical maps $E_n(R) \to \mathrm{SL}_n(R) \to \mathrm{GL}_n(R)$ are morphisms of global actions.

Before we describe the Steinberg global action, we recall the definition of the Steinberg group, following [6, Section 5].

Recall that elementary matrices satisfy the following identities:

- $E_{ij}(r)\, E_{ij}(s) = E_{ij}(r + s)$, for all $r, s \in R$;
- $[E_{ij}(r)\, E_{kl}(s)] = 1$, if $j \neq k, i \neq l, r, s \in R$;
- $[E_{ij}(r)\, E_{jl}(s)] = E_{il}(rs)$, if $i \neq l, r, s \in R$.

The *Steinberg group* $St_n(R)$ of a ring R is the free group on the generators $X_{ij}(r)$, $(i, j) \in J_n$, $r \in R$ modulo the normal subgroup generated by substituting

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$X_{ij}(r)$ for $E_{ij}(r)$ in the relations above. The assignment $X_{ij}(r) \mapsto E_{ij}(r)$ sends the relations among the generators of $St_n(R)$ into valid identities between elementary matrices and defines a canonical surjective group homomorphism $St_n(R) \to E_n(R)$.

- **The Steinberg global action.** The Steinberg global action is the single domain global action with underlying set the Steinberg group $St_n(R)$, index system Φ_n, and local groups $St_n(R)_\alpha$, $\alpha \in \Phi_n$ where $St_n(R)_\alpha$ is defined as the subgroup of $St_n(R)$ generated by all $X_{ij}(r)$ such that $(i, j) \in \alpha$ and $r \in R$. The action of $St_n(R)_\alpha$ on $St_n(R)$ is by right multiplication. The structure homomorphisms $\theta_{\alpha\leqslant\beta}$ are by definition the natural inclusions $St_n(R)_\alpha \subseteq St_n(R)_\beta$. Abusing notation, we also let $St_n(R)$ denote the single domain global action just constructed.

Clearly the canonical surjective group homomorphism $\theta_n : St_n(R) \to E_n(R)$ described above is a morphism of global actions and maps each local group $St_n(R)_\alpha$ onto the corresponding local group $E_n(R)_\alpha$. The next proposition shows that the maps of local groups are group isomorphisms and establishes a canonical isomorphism $St_n(R) \to \operatorname{colim}_{\alpha\in\Phi_n} E_n(R)_\alpha$ of global actions.

PROPOSITION 2.8. *Let $\theta_n : St_n(R) \to E_n(R)$ denote the canonical homomorphism. Let $E : \Phi_n \to \mathrm{Subgr}(E_n(R))$, $\alpha \mapsto E(\alpha) = E_n(R)_\alpha$, and let $St : \Phi_n \to \mathrm{Subgr}(St_n(R))$, $\alpha \mapsto St(\alpha) = St_n(R)_\alpha$. Clearly, E and St preserve partial orderings and are functors with values in groups. Obviously the homomorphism θ_n defines an inclusion preserving map $\theta : \mathrm{Subgr}(St_n(R)) \to \mathrm{Subgr}(E_n(R))$, $H \mapsto \theta_n(H)$, such that the diagram*

$$\begin{array}{ccc} \Phi_n & \xrightarrow{\quad St \quad} & \mathrm{Subgr}(St_n(R)) \\ & \searrow^{E} & \downarrow \theta \\ & & \mathrm{Subgr}(E_n(R)) \end{array}$$

commutes and induces a natural transformation $\tau : St \to E$ of functors defined by $\tau_\alpha : St_n(R)_\alpha \to E_n(R)_\alpha$, $x \mapsto \theta_n(x)$. The following holds:

(1) *There is an action of the symmetric group S_n on Φ_n, $St_n(R)$, and $E_n(R)$ such that the diagram above is S_n-equivariant.*

(2) *τ is a natural isomorphism $St \xrightarrow{\sim} E$ of functors.*

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(3) *St and E preserve intersections.*

(4) *Give* $\operatorname{colim}_{\alpha\in\Phi_n} E_n(R)_\alpha$ *its canonical, single domain global action defined by the right action of each* $E_n(R)_\alpha$ *on the colimit. Similarly give* $\operatorname{colim}_{\alpha\in\Phi_n} St_n(R)_\alpha$ *its canonical, single domain global action. Then the canonical group homomorphisms*

$$\operatorname*{colim}_{\alpha\in\Phi_n} E_n(R)_\alpha \longleftarrow \operatorname*{colim}_{\alpha\in\Phi_n} St_n(R)_\alpha \longrightarrow St_n(R)$$

are group isomorphisms and isomorphisms of global actions.

(5) *the canonical map*

$$\bigcup_{\alpha\in\Phi_n} St_n(R)_\alpha \to \bigcup_{\alpha\in\Phi_n} E_n(R)_\alpha$$

is bijective. (From this result, it follows by Definition 3.20 that $\theta_n : St_n(R) \to E_n(R)$ *is a covering morphism of global actions. From here, it is not difficult to show, for example along the lines of the proof of Corollary 3.24, that* θ_n *is a simply connected covering morphism.)*

Proof. (1) Let $\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ (1)\pi & (2)\pi & \cdots & (n)\pi \end{pmatrix} \in S_n$. To each element π, we associate the permutation matrix M_π, whose $(i)\pi$th column has zeros in all positions except the ith position where it has 1.

We let the group S_n act on $\mathrm{GL}_n(R)$ via conjugation by permutation matrices. This action leaves $E_n(R)$ invariant, because $E_{ij}(r)^\pi = M_{\pi^{-1}} E_{ij}(r) M_\pi = E_{(i\pi)(j\pi)}(r)$. This also shows that the action of S_n on $E_n(R)$ preserves the three relations above for elementary matrices. Thus the action of S_n on $E_n(R)$ uniquely lifts to an action of S_n on $St_n(R)$ such that the homomorphism θ_n is S_n-equivariant. We let S_n act on J_n in the obvious way, namely $(i, j)\pi = (i\pi, j\pi)$. There is clearly an induced action of S_n on Φ_n. It is obvious that the maps E and St are S_n-equivariant. This completes the proof of (1).

(2) Let δ denote the nilpotent set

$$\{(i, j) \mid i < j, 1 \leqslant i, j \leqslant n\} \subset \Phi_n.$$

The set δ is a maximal nilpotent subset. It is easy to check that any nilpotent subset is contained in a maximal nilpotent subset and that any maximal nilpotent subset is conjugate under the action of S_n to δ.

To prove that τ is a natural isomorphism $St \xrightarrow{\sim} E$ of functors, we must show that for any $\alpha \in \Phi_n$, the surjective canonical homomorphism $St_n(R)_\alpha \to E_n(R)_\alpha$ is injective as well. By the previous paragraph, we can assume that

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$\alpha \subseteq \delta$. By [6, Lemma 9.14], the map $St_n(R)_\delta \to E_n(R)_\delta$ is bijective. It follows that the map $St_n(R)_\alpha \to E_n(R)_\alpha$ is bijective.

(3) Since the map $\Phi_n \to \mathrm{Subgr}(\mathrm{GL}_n(R)), \alpha \mapsto \mathrm{GL}_n(R)_\alpha$, preserves intersections and $E_n(R)_\alpha = \mathrm{GL}_n(R)_\alpha$, it follows that E preserves intersections. Thus St preserves intersections, by (2).

(4) It follows immediately from (2) that the group homomorphism $\mathrm{colim}_{\alpha\in\Phi_n} E_n(R)_\alpha \longleftarrow \mathrm{colim}_{\alpha\in\Phi_n} St_n(R)_\alpha$ is an isomorphism. By the definition of colimit, there is a canonical group homomorphism $\mathrm{colim}_{\alpha\in\Phi_n} St_n(R)_\alpha \longrightarrow St_n(R)$ and it is obviously surjective. Using the definition of the Steinberg group by generators and relations, we can construct straightforward an inverse to this homomorphism, since any defining relation of $St_n(R)$ is contained in some local subgroup $St_n(R)_\alpha$. It is obvious that the group isomorphisms just establishes are isomorphisms of global actions.

(5) Let $x \in St_n(R)_\alpha$ and $y \in St_n(R)_\beta$. Let $\gamma = \alpha \cap \beta$. Suppose $\theta_n(x) = \theta_n(y)$. We must show $x = y$. Clearly, $\theta_n(x) = \theta_n(y)$ in $E_n(R)_\gamma$. Let $z \in St_n(R)_\gamma$ be such that $\theta_n(z) = \theta_n(x)$. Since $St_n(R)_\gamma \subseteq St_n(R)_\alpha$, it follows that $x = z$, because the homomorphism $St_n(R)_\alpha \to E_n(R)_\alpha$ is bijective. Similarly, $y = z$. □

The 4 single domain global actions above are examples of a geometric group. It is defined next along with the related concepts of geometric set and geometric coset space. Together they account for all of the single domain global actions occurring in the current paper, as well as in the vector K-theory of quadratic and Hermitian forms and the vector K-theory of Chevalley groups. The new topics will be covered in separate papers.

DEFINITION 2.9. Let G be a group. Let Φ be an index set for a set $\Phi(G) = \{G_\alpha \mid \alpha \in \Phi\}$ of subgroups G_α of G. We give $\Phi(G)$ the reflexive, transitive relation defined by the natural inclusion of one subgroup in another and Φ the relation defined by its bijective correspondence with $\Phi(G)$. From this data we construct a single domain global action as follows. Its underlying set is G, its index system is Φ, its local groups are $\{G_\alpha \mid \alpha \in \Phi\}$ acting on G by right multiplication, and its structure homomorphisms $\theta_{\alpha\leqslant\beta}$ are the natural inclusions $G_\alpha \subseteq G_\beta$. We call this global action a *geometric group*. If G is acting on a set X on the right, then we equip X with the single domain global action whose index system Φ, local groups $\Phi(G)$, and structure homomorphisms $\theta_{\alpha\leqslant\beta}$ are the same as those above and the action of each local group G_α on X is the canonical one induced by the action of G. We call this global action a *geometric set* of G. Suppose H is a subgroup of

G. Let $H \setminus G$ be the set of right cosets $\{Hg \mid g \in G\}$ of H in G. Then G acts on $H \setminus G$ on the right by multiplication and the resulting geometric set is called a *geometric coset space* of G.

3. Elementary homotopy theory and covering theory

The results of this section are due to the first author.

The section summarizes in a convenient form constructions and results from the homotopy theory and covering theory of global actions. Emphasis will be on single domain global actions. The main result on covering theory is Theorem 3.23. It classifies in a constructive way all connected coverings of a connected single domain global action. It is used to deduce Theorem 1.1 of the Introduction.

3.1. The notion of homotopy. The most natural notion of homotopy is the following.

Suppose A and B are global actions with underlying sets X and Y, respectively, and index systems Φ_A and Φ_B, respectively. If $\alpha \in \Phi_A$ and $\beta \in \Phi_B$, let $X_\alpha \curvearrowleft G_\alpha$, and $X_\beta \curvearrowleft G_\beta$ respectively denote the corresponding local group actions. Define the *product global action* $A \times B$ as follows: its underlying set is the Cartesian product $X \times Y$ of sets and its index system the Cartesian product $\Phi_A \times \Phi_B$ with relation defined by $(\alpha, \beta) \leqslant (\alpha', \beta')$ if and only if $\alpha \leqslant \alpha'$ and $\beta \leqslant \beta'$. The local set $(X \times Y)_{(\alpha,\beta)}$ is the Cartesian product $X_\alpha \times Y_\beta$ and the local group $G_{(\alpha,\beta)}$ is the product group $G_\alpha \times G_\beta$. Its action on $(X \times Y)_{(\alpha,\beta)}$ is the obvious one, namely coordinatewise.

Let L denote the line action, see Section 2, with underlying set $\mathbb{Z}$. For $n \in \mathbb{Z}$, let $\iota_n : Y \to Y \times L$, $y \mapsto (y, n)$. This assignment clearly defines a morphism $\iota_n : B \to B \times L$ of global actions.

DEFINITION 3.1. Morphisms $f, f' \in \mathrm{Mor}(B, A)$ are called *homotopic* if there is a morphism $H : B \times L \to A$ of global actions and integers $n_- \leqslant n_+$ such that for all $n \leqslant n_-$, $f = H\iota_n = H\iota_{n_-}$ and for all $n_+ \leqslant n$, $f' = H\iota_n = H\iota_{n_+}$. The morphism H is called, as in topology, a *homotopy* from f to f'. If f is homotopic to f', we write $f \sim f'$.

In situations such as that of paths, a concept of homotopy which leaves certain points fixed is needed. We define this for paths next.

3.2. End-point-preserving homotopy of paths and the fundamental group. The goal of this section is to define end-point-preserving homotopy of paths and

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to define the fundamental group functor π_1. We also define the path-connected component functor π_0.

Throughout this section, A and B denote global actions with underlying sets X and Y, respectively, and L the line action.

The easiest way of defining a *path* in A is as a finite sequence $x_0, \ldots, x_n$ of points $x_i \in X$ such that for each $i < n$ there is an element g_i in a local group G_α of A such that $x_i \in X_\alpha$ and $x_i g_i = x_{i+1}$. However, this definition of path is far too rigid for our purposes. The following definition has the pliability we need.

DEFINITION 3.2. Let $\omega : L \to A$ be a morphism. We say that it *stabilizes below* or *negatively* if there is an integer n_- such that for all $n \leqslant n_-$, $\omega(n) = \omega(n_-)$. In this case, we say that ω *stabilizes below* or *negatively* to $x = \omega(n_-)$. Similarly we say that ω *stabilizes above* or *positively* if there is an integer n_+ such that for all $n \geqslant n_+$, $\omega(n) = \omega(n_+)$. In this case, we say that ω *stabilizes above* or *positively* to $x = \omega(n_+)$. A *path* is a morphism $\omega : L \to A$ which stabilizes below and above. A *loop* is a path which stabilizes below and above to the same element of X.

A path $\omega : L \to A$ is called *constant* if $\omega(m) = \omega(n)$ for all $m, n \in \mathbb{Z}$. If ω is not constant, then it is always the case that $n_- < n_+$. On the other hand, if ω is constant then n_- and n_+ can be any integer. For this reason, we exclude constant paths from the following definition.

DEFINITION 3.3. Let ω denote a nonconstant path. The *lower degree* of ω is defined by

$$\ell\mathrm{d}(\omega) = \sup\{n_- \in \mathbb{Z} \mid \omega(n) = \omega(n_-) \text{ for all } n \leqslant n_-\}.$$

The *upper* degree of ω is defined by

$$\mathrm{ud}(\omega) = \inf\{n_+ \in \mathbb{Z} \mid \omega(n) = \omega(n_+) \text{ for all } n \geqslant n_+\}.$$

Next we define the notion of composition for paths.

DEFINITION 3.4. Let ω and ω' be paths. Define the *initial point* (in X) of a nonconstant path ω by

$$\mathrm{in}(\omega) = \omega(\ell\mathrm{d}(\omega)).$$

Define the *terminal point* of a nonconstant path ω by

$$\mathrm{ter}(\omega) = \omega(\mathrm{ud}(\omega)).$$

Define the *initial* and *terminal* points of a constant path ω taking the constant value $x \in X$ by

$$\mathrm{in}(\omega) = \mathrm{ter}(\omega) = x.$$

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The *composition* $\omega \cdot \omega'$ of paths ω and ω' exists if $\text{ter}(\omega') = \text{in}(\omega)$ and is defined as follows:

$$(\omega \cdot \omega') = \begin{cases} \omega & \text{if } \omega' \text{ is constant} \\ \omega' & \text{if } \omega \text{ is constant.} \end{cases}$$

If ω and ω' are nonconstant then

$$(\omega \cdot \omega')(n) = \begin{cases} \omega'(n) & \text{for all } n \leqslant \text{ ud}(\omega') \\ \omega(n - \text{ud}(\omega') + \ell\text{d}(\omega)) & \text{for all } n \geqslant \text{ ud}(\omega'). \end{cases}$$

DEFINITION 3.5. Let

$$P(A) = \{\omega : L \to A \mid \omega \text{ a path}\}.$$

Under composition of paths, $P(A)$ is an associative, partial magma with local identities. That is: (i) Let $\omega, \mu, \nu \in P(A)$. If either $(\omega \cdot \mu) \cdot \nu$ or $\omega \cdot (\mu \cdot \nu)$ is defined then both are defined and $(\omega \cdot \mu) \cdot \nu = \omega \cdot (\mu \cdot \nu)$. (ii) The constant paths are the local identities.

DEFINITION 3.6. Points $p, p' \in A$ are called *path-connected*, if there is a path $\omega \in P(A)$ such that $\text{in}(\omega) = p$ and $\text{ter}(\omega) = p'$. In this case, we write $p \sim p'$. The relation path-connected for ordered pairs of points $p, p' \in A$ is obviously reflexive and transitive. If $\omega \in P(A)$, define its *inverse* (or *reverse*) *path* ω^{-1} by

$$\omega^{-1}(n) = \omega(-n).$$

Clearly $\text{in}(\omega^{-1}) = \text{ter}(\omega)$ and $\text{ter}(\omega^{-1}) = \text{in}(\omega)$. Thus if $p, p' \in A$ and ω tells us that $p \sim p'$, then ω^{-1} tells us that $p' \sim p$. Thus the relation path-connected is symmetric and therefore an equivalence relation on ordered pairs of points of A. Let

$$\pi_0(A)$$

be the set of equivalence classes of $\sim$. Each equivalence class is called a *path-connected component* of $\sim$. If A is equipped with a base point a, let

$$\pi_0(A, a) = \pi_0(A)$$

with base point the path connected component of a.

We turn now to the notion of end-point-preserving homotopy for paths.

DEFINITION 3.7. A homotopy $H : L \times L \to A$ of paths is called *end-point-preserving* or a *path-homotopy*, if for any $n \in \mathbb{Z}$, $H\iota_n$ is a path and if for all $m, n \in \mathbb{Z}$,

$$\text{in}(H\iota_m) = \text{in}(H\iota_n) \quad \text{and} \quad \text{ter}(H\iota_m) = \text{ter}(H\iota_n).$$

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Let $H : L \times L \to A$ be a path-homotopy. Let $n_- \leqslant n_+$ be integers and let ω and ω' be paths with the same initial and terminal points such that $\omega = H\iota_n$ for all $n \leqslant n_-$ and $\omega' = H\iota_n$ for all $n \geqslant n_+$. Then we say that ω is *end-point-preserving homotopic* or *path-homotopic* to ω' and write $\omega \sim \omega'$.

The notion of homotopy for morphisms and of end-point-preserving homotopy for paths are generalizations of the notion of path and both notions of homotopy above have a notion of composition extending that for paths. Whereas a path is a formal notion for moving a point around in a global action A, the notions of homotopy and path-homotopy are formal notions for moving around as well as deforming larger objects inside A.

A homotopy $H : B \times L \to A$ is called *constant*, if $H\iota_m = H\iota_n$ for all m, n.

DEFINITION 3.8. Let H be a nonconstant homotopy. The *lower degree* of H is defined by

$$\ell\mathrm{d}(H) = \sup\{n_- \in \mathbb{Z} \mid H\iota_n = H\iota_{n_-} \text{for all } n \leqslant n_-\}.$$

The *upper degree* of H is defined by

$$\mathrm{ud}(H) = \inf\{n_+ \in \mathbb{Z} \mid H\iota_n = H\iota_{n_+} \text{for all } n \geqslant n_+\}.$$

Next we define the notion of composition of homotopies.

DEFINITION 3.9. Let $H : B \times L \to A$ be a homotopy. Define the *initial morphism* (in $\mathrm{Mor}(B, A)$) of a nonconstant homotopy H by

$$\mathrm{in}(H) = H\iota_{\ell\mathrm{d}(H)}.$$

Define the *terminal morphism* (in $\mathrm{Mor}(B, A)$) of a nonconstant homotopy H by

$$\mathrm{ter}(H) = H\iota_{\mathrm{ud}(H)}.$$

Define the *initial* and *terminal* morphisms of a constant homotopy H taking the constant value $f \in \mathrm{Mor}(B, A)$ by

$$\mathrm{in}(H) = \mathrm{ter}(H) = f.$$

If H is an end-point-preserving homotopy of paths then $\mathrm{in}(H)$ is called the *initial path* and $\mathrm{ter}(H)$ the *terminal path*.

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The *composition* $H \cdot H'$ of homotopies $H, H' : B \times L \to A$ exists if $\mathrm{ter}(H') = \mathrm{in}(H)$ and is defined as follows:

$$(H \cdot H') = \begin{cases} H & \text{if } H' \text{ is constant} \\ H' & \text{if } H \text{ is constant.} \end{cases}$$

If H and H' are nonconstant then

$$(H \cdot H')(b, n) = \begin{cases} H'(b, n) & \text{for all } n \leqslant \mathrm{ud}(H') \\ H(b, n - \mathrm{ud}(H') + \ell\mathrm{d}(H) & \text{for all } n \geqslant \mathrm{ud}(H'). \end{cases}$$

The *composition* $H \cdot H'$ of end-point-preserving homotopies $H, H' : L \times L \to A$ of paths exists if $\mathrm{ter}(H') = \mathrm{in}(H)$ (this implies that for all m, n, $\mathrm{in}(H\iota_m) = \mathrm{in}(H\iota_n) = \mathrm{in}(H'i_m) = \mathrm{in}(H'\iota_n)$ and $\mathrm{ter}(H\iota_m) = \mathrm{ter}(H\iota_n) = \mathrm{ter}(H'\iota_m) = \mathrm{ter}(H'\iota_n)$) and is defined as follows:

$$(H \cdot H') = \begin{cases} H & \text{if } H' \text{ is constant} \\ H' & \text{if } H \text{ is constant.} \end{cases}$$

If H and H' are nonconstant then

$$(H \cdot H')(b, n) = \begin{cases} H'(b, n) & \text{for all } n \leqslant \mathrm{ud}(H') \\ H(b, n - \mathrm{ud}(H') + \ell\mathrm{d}(H) & \text{for all } n \geqslant \mathrm{ud}(H'). \end{cases}$$

DEFINITION 3.10. If $H : B \times L \to A$ is a homotopy or end-point preserving homotopy of paths (in which case $B = L$), define the *inverse* or *reverse homotopy* H^{-1} by

$$H^{-1}(b, n) = H(b, -n).$$

Clearly $\mathrm{in}H^{-1} = \mathrm{ter}H$ and $\mathrm{ter}H^{-1} = \mathrm{in}H$. Thus if $f, f' \in \mathrm{Mor}(B, A)$ and H tells us that $f \sim f'$, then H^{-1} tells us that $f' \sim f$. Thus the homotopy relation on $\mathrm{Mor}(B, A)$ is symmetric and therefore an equivalence relation. Similarly the relation of end-point-preserving homotopy on $P(A)$ is an equivalence relation.

REMARK 3.11. It is obvious that composition of homotopies is a generalization of that of paths. In fact, we can use the exponential map $\mathrm{Hom}(B \times L, A) \to \mathrm{Hom}(L, \mathrm{Hom}(B, A))$ of global actions in [1], Section 3, to interpret each homotopy $H : B \times L \to A$ as a path $\omega_H \in P(\mathrm{Mor}(B, A))$ and then use composition of paths to define composition of homotopies.

DEFINITION 3.12. Let $a \in A$ be a base point for A. A *loop at a* is a path $\omega \in P(A)$ such that $\mathrm{in}(\omega) = \mathrm{ter}(\omega) = a$. Two loops at a are called *loop-homotopic*, if they are end-point preserving homotopic. Define

$$L(A, a) = \{\omega \in P(A) \mid \omega \text{ a loop at } a\}.$$

Clearly two loops at a are composable. By Definition 3.5, $L(A, a)$ is a monoid with identity the constant loop at a. Define the *fundamental monoid $\Pi_1(A, a)$ at a*, by

$$\Pi_1(A, a) = L(A, a).$$

Mimicking standard arguments in topology, we can show that the equivalence relation of loop-homotopy for loops at a commutes with composition of loops. Moreover, for any loop ω at a, its inverse loop ω^{-1} has the property that $\omega \cdot \omega^{-1} \sim \omega^{-1} \cdot \omega \sim$ the constant loop at a. Thus the loop-homotopy classes of loops at a form a group

$$\pi_1(A, a)$$

under composition called the *fundamental group* of (A, a). Furthermore the canonical map $\Pi_1(A, a) \to \pi_1(A, a)$ is a surjective monoid homomorphism. (A, a) is called *simply connected*, if A is path connected and $\pi_1(A, a) = 1$.

The following corollary is worth taking note of, although it is not required in the rest of the paper. Its proof is routine and left to the reader.

COROLLARY 3.13. *The path-homotopy classes of paths of $P(A)$ form a groupoid*

$$FG(A)$$

under composition of paths called the fundamental groupoid of A. It is an associative partial magma and the canonical surjective map $P(A) \to FG(A)$ is a morphism of associative partial magmas.

3.3. Loop-homotopy of higher loops and higher homotopy groups. Throughout this subsection, A denotes a global action with base point a.

Let L denote the line action. For $1 \leqslant i \leqslant n$, let $L_i = L$. Set

$$L^n = L_1 \times \cdots \times L_n.$$

DEFINITION 3.14. An *n-dimensional loop* or simply *n-loop* at a is a morphism $\omega : L^n \to A$ such that $\omega(z) = a$ for almost all $z \in L^n$. Let

$$L^n(A, a) = \{\omega \mid \omega \text{ an } n\text{-loop at } a\}.$$

Clearly $L^1(A, a) = L(A, a)$ as defined in 3.12.

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We show next how to compose n-loops.

DEFINITION 3.15. Let ω be a nonconstant n-loop at a. The *lower degree* of ω is defined by

$$\ell\mathrm{d}(\omega) = \sup\{(z_n)_- \in \mathbb{Z} \mid \omega(z_1, \ldots, z_n) = a \text{ for all } z_n \leqslant (z_n)_- \text{ and all } z_1, \ldots, z_{n-1}\}.$$

The *upper degree* of ω is defined by

$$\mathrm{ud}(\omega) = \inf\{(z_n)_+ \in \mathbb{Z} \mid \omega(z_1, \ldots, z_n) = a \text{ for all } z_n \geqslant (z_n)_+ \text{ and all } z_1, \ldots, z_{n-1}\}.$$

DEFINITION 3.16. The *composition* $\omega \cdot \omega'$ of n-loops ω, ω' at a is defined as follows:

$$(\omega \cdot \omega') = \begin{cases} \omega & \text{if } \omega' \text{ is constant} \\ \omega' & \text{if } \omega \text{ is constant.} \end{cases}$$

If ω and ω' are nonconstant, then define

$$(\omega \cdot \omega')(z_1, \ldots, z_n) = \begin{cases} \omega'(z_1, \ldots, z_n) & \text{for all } n \leqslant \mathrm{ud}(\omega') \\ \omega(z_1, \ldots, z_{n-\mathrm{ud}(\omega')+\ell\mathrm{d}(\omega)}) & \text{for all } n \geqslant \mathrm{ud}(\omega'). \end{cases}$$

DEFINITION 3.17. We extend the notion of loop-homotopy of 1-loops to n-loops. A *loop-homotopy* $H : L^n \times L \to A$ is a homotopy such that $H\iota_n$ is an n-loop for all n. It follows from Definition 3.10 that loop-homotopy is an equivalence relation on n-loops at a.

DEFINITION 3.18. By Definition 3.16, n-loops at a are composable. Composition is trivially associative and has an identity, namely the constant n-loop at a. Thus $L^n(A, a)$ is a monoid. Define the *nth fundamental monoid*

$$\Pi_n(A, a) = L^n(A, a).$$

Mimicking standard arguments in topology, we can show that the equivalence relation of loop-homotopy of n-loops at a commutes with composition of n-loops at a. Moreover, for any n-loop ω, its inverse loop ω^{-1} defined by

$$\omega^{-1}(z_1, \ldots, z_{n-1}, z_n) = \omega(z_1, \ldots, z_{n-1}, -z_n)$$

has the property that $\omega \cdot \omega^{-1} \sim \omega^{-1} \cdot \omega \sim$ the constant loop at a. Thus the loop-homotopy classes of n-loops at a form a group

$$\pi_n(A, a)$$

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under composition called the *nth homotopy group* of (A, a). Furthermore the canonical map $\Pi_n(A, a) \to \pi_n(A, a)$ is a surjective monoid homomorphism. (A, a) is called *n-simply connected*, if A is path-connected and $\pi_i(A, a) = 1$ for all $1 \leqslant i \leqslant n$.

3.4. The unimodular row global action and its simply connected covering. This subsection draws heavily on [2, Section 11]. We begin by defining the unimodular row global action and computing its path-connected components.

The unimodular row global action. The *unimodular row global action* has as underlying set $Um_n(R)$, the set of all R-unimodular row vectors $v = (v_1, \ldots, v_n)$ of length n, with coefficients $v_i \in R$. Recall that v is unimodular means there is a row vector $w = (w_1, \ldots, w_n)$ such that $v \cdot w^t = \sum_i v_i w_i = 1$, where t denotes the transpose operator on (not necessarily square) matrices. The general linear group $\mathrm{GL}_n(R)$ acts on $Um_n(R)$ on the right, in the usual way. The index system Φ_n and local groups $E_n(R)_\alpha, \alpha \in \Phi_n$, of the unimodular row global action are the same as those of the global action $\mathrm{GL}_n(R)$. Each local set is the whole of $Um_n(R)$ and the right action of each local group $E_n(R)_\alpha$ on $Um_n(R)$ is via that of $\mathrm{GL}_n(R)$ on $Um_n(R)$. Abusing notation, we also let $Um_n(R)$ denote the single domain global action we just described. We give $Um_n(R)$ the base point $e = (1, 0, \ldots, 0)$. In the language of Definition 2.9, $Um_n(R)$ is a geometric set of the geometric group $GL_n(R)$.

PROPOSITION 3.19. *The path-connected component of e in $Um_n(R)$ is the orbit $eE_n(R)$ of the right action of $E_n(R)$ at e. $\pi_0(Um_n(R)) = Um_n(R)/E_n(R) =$ the orbit space of the right action of $E_n(R)$ on $Um_n(R)$. $\pi_0(Um_n(R), e) = Um_n(R)/E_n(R)$ with base point $eE_n(R)$.*

Proof. We prove that v, w belong to the same path-component of $Um_n(R)$ if and only if there exists $\varepsilon \in E_n(R)$ such that $v\varepsilon = w$, i.e. if and only if $vE_n(R) = wE_n(R)$.

Let $v, w \in Um_n(R)$ belong to the same path-component and let ω be a path from v to w. As ω is a morphism of global actions, there exist $\varepsilon_i \in E_n(R)_{\alpha_i}$, $1 \leqslant i \leqslant N$, such that $v\varepsilon_1 \cdots \varepsilon_N = w$. Then $\varepsilon := \prod_i \varepsilon_i \in E_n(R)$ has the property that $v\varepsilon = w$.

Conversely suppose $w = v\varepsilon$, for some $\varepsilon \in E_n(R)$. By definition, there exist elementary matrices $E_1, \ldots, E_k$ such that $\varepsilon = \prod_i E_i$. As each E_i lies in some local group, we can trivially define a path from v to w. Thus, $\pi_0(Um_n(R)) = Um_n(R)/E_n(R)$.

From the above it follows that the path-connected component of the base point e is $eE_n(R)$. □

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We introduce next a global action structure on the path-connected component $eE_n(R)$ of e.

- **The elementary unimodular row global action.** Let $EUm_n(R) = eE_n(R)$. Clearly $EUm_n(R) \subseteq Um_n(R)$. The action of each local group $E_n(R)_\alpha, \alpha \in \Phi_n$, on $Um_n(R)$ leaves $EUm_n(R)$ invariant and induces a single domain action on $EUm_n(R)$, whose index system and local groups are the same as those of $Um_n(R)$. Abusing notation, as usual, we also let $EUm_n(R)$ denote this global action. It is called the *elementary unimodular row global action*. We give it the base point e. In the language of Definition 2.9, $EUm_n(R)$ is a geometric set of $E_n(R)$.

- **The Steinberg unimodular row global action.** Let $P_n(R)$ denote the subgroup of $\mathrm{GL}_n(R)$ which leaves e fixed. Clearly each matrix in $P_n(R)$ takes the form $\left(\begin{smallmatrix} 1 & 0 \\ v & \tau \end{smallmatrix}\right)$, for some $v \in M_{n-1,1}(R)$, $\tau \in \mathrm{GL}_{n-1}(R)$.

  Let $EP_n(R) = P_n(R) \cap E_n(R)$. Give the right coset space $EP_n(R) \setminus E_n(R)$ a single domain action structure defined by letting each local group $E_n(R)_\alpha$, $\alpha \in \Phi_n$, act naturally by right multiplication on $EP_n(R) \setminus E_n(R)$. Let $EP_n(R) \setminus E_n(R)$ also denote the resulting single domain global action. In the language of Definition 2.9, it is a geometric coset space of $E_n(R)$. Let the trivial coset $EP_n(R)$ be the base point of $EP_n(R) \setminus E_n(R)$. Clearly the map $E_n(R) \to EUm_n(R), \varepsilon \mapsto e\varepsilon$, defines a base point preserving isomorphism $EP_n(R) \setminus E_n(R) \cong EUm_n(R)$ of global actions. We shall construct the Steinberg unimodular row global action as a global action with a canonical morphism onto $EP_n(R) \setminus E_n(R)$.

  Let $\theta_n : St_n(R) \to E_n(R)$ denote the canonical morphism. Let

$$B_n(R) = \langle x^{-1}abx \in \theta_n^{-1}(EP_n(R)) \mid x \in St_n(R), \\ a \in St_n(R)_\alpha, b \in St_n(R)_\beta, \alpha, \beta \in \Phi_n \rangle.$$

  Give the right coset space $B_n(R) \setminus St_n(R)$ a single domain action structure by letting each local group $St_n(R)_\alpha, \alpha \in \Phi_n$, act naturally by right multiplication on $B_n(R) \setminus St_n(R)$. Let the trivial coset $B_n(R)$ be the base point of $B_n(R) \setminus St_n(R)$. Clearly θ_n induces a canonical base point preserving morphism $B_n(R) \setminus St_n(R) \to EP_n(R) \setminus E_n(R)$ of global actions. Define the *Steinberg unimodular row global action* $StUm_n(R)$ to be the single domain global action $B_n(R) \setminus St_n(R)$ with base point the trivial coset $B_n(R)$. In the language of Definition 2.9, $StUm_n(R)$ is a geometric coset space of $St_n(R)$.

The main result of this subsection is that the canonical map $StUm_n(R) \to EP_n(R) \setminus E_n(R)$ is a simply connected covering morphism. To state and prove

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this result, we need the notion of star global action $\mathrm{star}(x)$ at a point x of a global action and the notion of covering morphism.

DEFINITION 3.20. Let A be a global action with underlying set X, index system Φ, local actions $X_\alpha \curvearrowleft G_\alpha$, $\alpha \in \Phi$, and structure homomorphisms $\theta_{\alpha\leqslant\beta}$. If $x \in X$, define the *star global action* $\mathrm{star}(x)$ at x as follows: Its index system $\Phi_{\mathrm{star}(x)} = \{\alpha \in \Phi \mid x \in X_\alpha\}$ equipped with the reflexive relation inherited as a subset of Φ and its underlying set

$$X_{\mathrm{star}(x)} = \bigcup_{\alpha\in\Phi_{\mathrm{star}(x)}} x \cdot G_\alpha.$$

If $\alpha \in \Phi_{\mathrm{star}(x)}$, let $(X_{\mathrm{star}(x)})_\alpha = xG_\alpha$ and $(G_{\mathrm{star}(x)})_\alpha = G_\alpha$. If $\alpha \leqslant \beta \in \Phi_{\mathrm{star}(x)}$, let the structure homomorphism $(\theta_{\mathrm{star}(x)})_{\alpha\leqslant\beta} = \theta_{\alpha\leqslant\beta}$. (Even if A is a single domain global action, $\mathrm{star}(x)$ usually is not.)

DEFINITION 3.21. A morphism $f : A \to B$ of global actions is *surjective*, if it is a surjective map on the underlying sets. A base point preserving surjective morphism $f : (A, a) \to (B, b)$ of path connected global actions is called a *covering morphism*, if for every $x \in X_A$, the induced map $f : \mathrm{star}(x) \to \mathrm{star}(f(x))$ is an isomorphism of global actions. A covering morphism $(A, a) \to (B, b)$ is called *simply connected*, if (A, a) is simply connected.

Theorem 1.1 is a trivial consequence of the following result.

THEOREM 3.22. *The canonical morphism*

$$B_n(R) \setminus St_n(R) \to EP_n(R) \setminus E_n(R)$$

is a simply connected covering morphism. Let $\theta_n : St_n(R) \to E_n(R)$ denote the canonical morphism. Let

$$\widetilde{P}_n(R) = \theta_n^{-1}(EP_n(R)).$$

Then $B_n(R) \subseteq \widetilde{P}_n(R)$ as a normal subgroup and there is a short exact sequence

$$B_n(R) \setminus \widetilde{P}_n(R) \rightarrowtail B_n(R) \setminus St_n(R) \twoheadrightarrow EP_n(R) \setminus E_n(R)$$

of canonical morphisms and a canonical group isomorphism

$$K_2(Um_n(R)) \cong B_n(R) \setminus \widetilde{P}_n(R).$$

We deduce the theorem above from a very useful and very general result, essentially proved in [2], having applications far beyond the scope of the current paper. The result classifies in a constructive way all connected coverings of a connected single domain global action.

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THEOREM 3.23. **Classification of coverings of connected single domain global actions** *(following* [2, 11.2]*). Let A be a pointed, path-connected single domain global action. Following the notation of* [2]*, let*

$$G_A = \operatorname*{colim}_{\alpha \in \Phi} G_\alpha.$$

Let X be the underlying set of A and let $a \in X$ be the base point of A. Thanks to the compatibility condition of A, there is a canonical action of G_A on X. Let

$$H_A$$

be the subgroup of G_A of all elements which leave the base point a fixed. Give the right coset space $H_A \setminus G_A$ the single domain global action obtained by letting each local group G_α, $\alpha \in \Phi$, of A act by right multiplication on $H_A \setminus G_A$. Let the trivial coset H_A be the base point of $H_A \setminus G_A$. The group G_A itself is a path connected, single domain global action, with the same index system Φ as A and the same local groups G_α, $\alpha \in \Phi$, as A, acting by right multiplication on G_A. The canonical morphism $G_A \to X$, $g \mapsto ag$, of global actions is surjective because A is path connected and induces a base point preserving isomorphism

$$H_A \setminus G_A \cong A$$

of pointed, path connected, global actions. Let

$$B_A = \langle g g_\alpha g_\beta g^{-1} \mid g g_\alpha g_\beta g^{-1} \in H_A, g \in G_A, g_\alpha \in G_\alpha, g_\beta \in G_\beta, \alpha, \beta \in \Phi \rangle$$

and

$$\mathcal{H}_A = \{H \in \mathrm{Subgr}(G_A) \mid B_A \subseteq H \subseteq H_A\}.$$

Clearly B_A is a normal subgroup of H_A. The assertion of [2, *Proposition 11.2*] *is that each morphism of $\{H \setminus G_A \to H_A \setminus G_A \mid H \in \mathcal{H}_A\}$ is a covering morphism and there is a 1-1 correspondence between the morphisms in the set above and the isomorphism classes of covering morphisms of (A, a).*

Proof. [2, 11.2] defines $\mathcal{H}_A$ as $\{H \in Subgr(G_A) \mid H \subseteq H_A, gHg^{-1} \cap G_\alpha G_\beta = gH_A g^{-1} \cap G_\alpha G_\beta, \forall g \in G_A, \forall \alpha, \beta \in \Phi\}$. We shall show that the definition of $\mathcal{H}_A$ in [2, 11.2], agrees with that given in the theorem above. The conclusion of the theorem above will then follow trivially from that of [2, Proposition 11.3]. Clearly the condition $gHg^{-1} \cap G_\alpha G_\beta = gH_A g^{-1} \cap G_\alpha G_\beta$ holds $\Leftrightarrow$ the condition $H \cap g^{-1} \alpha G_\beta g = H_A \cap g^{-1} G_\alpha G_\beta g$ holds. But the latter condition holds $\forall g \in G_A$ and $\forall \alpha, \beta \in \Phi \Leftrightarrow B_A \subseteq H$. Thus the definitions of $\mathcal{H}_A$ in [2, 11.2] and the current paper coincide. □

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The following corollary is alluded to in [2, 11.2].

COROLLARY 3.24. *Maintain the hypothesis of the theorem above. Then the canonical morphism*

$$B_A \backslash G_A \to H_A \backslash G_A \cong A$$

is a simply connected covering morphism and there is a canonical short exact sequence

$$\pi_1(A, a) \rightarrowtail B_A \backslash G_A \twoheadrightarrow H_A \backslash G_A$$

which identifies the group $\pi_1(A, a)$ *with the group* $B_A \backslash H_A$.

Proof. By [2, 10.17] and the theorem above, there is an $H \in \mathcal{H}_A$ such that the canonical morphism $H \backslash G_A \to H_A \backslash G_A$ is a simply connected covering. We shall show $B_A = H$. Since the composite morphism $B_A \backslash G_A \to H \backslash G_A \to H_A \backslash G_A$ is a covering morphism, it follows from [2, Proposition 10.10] that $f : B_A \backslash G_A \to H \backslash G_A$ is also a covering morphism. Since $H \backslash G_A$ is simply connected, there is by [2, Proposition 10.11 and Lemma 10.4] a unique surjective morphism $\tilde{f} : H \backslash G_A \to B_A \backslash G_A$ such that $f\tilde{f}$ = identity map on $H \backslash G_A$. Suppose $B_A \subsetneq H$. Then f has a nontrivial kernel and so $f\tilde{f}$ also has a nontrivial kernel, because $\tilde{f}$ is surjective. This contradicts that $f\tilde{f}$ = identity. So $B_A = H$.

The short exact sequence follows now from the covering theory of [2, Section 10]. □

Proof of Theorem 3.22. By Proposition 2.8.4, the canonical group homomorphism $\text{colim}_{\alpha\in\Phi} St_n(R)_\alpha \to St_n(R)$ is a group isomorphism. It follows trivially that it is a global action isomorphism. This makes the theorem a special case of Corollary 3.24. □

Part II: Stability in algebraic K-theory and vector K-theory

4. Stability in K-theory and the fundamental group of the unimodular row global action

In this section we construct certain K-theoretic (or homotopy theoretic) exact sequences of pointed sets and prove Theorem 1.3. Under suitable conditions on the stable rank or dimension of a ring R, some exact sequences of pointed sets are exact sequences of groups. The sandwiching of $\pi_1(EUm_n(R))$ in an exact sequence of groups helps us to make conclusions (in certain situations) about the vanishing of $\pi_1(EUm_n(R))$.

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4.1. A $K_2, \pi_1, K_1, \pi_0, K_0$ pointed exact sequence. Notation is as in the Introduction. Let R be a ring. Let $K^s_{0,m}(R)$ be the set of all isomorphism classes of finitely generated projective left R-modules P such that $P \oplus R \simeq {}^{m+1}R$. The base point of $K^s_{0,m}(R)$ is the isomorphism class of ${}^m R$.

Groups and coset spaces will have their standard base points, namely the identity element and the trivial coset, respectively.

Recall that by definition $K_i(Um_n(R)) = \pi_{i-1}(Um_n(R)$. However, in this section we shall occasionally write $\pi_{i-1}(Um_n(R))$ instead of $K_i(Um_n(R))$.

PROPOSITION 4.1. *Let R be a ring and $n \geqslant 3$ be an integer. Then, there is an exact sequence of pointed sets*

$$(K_{2,n}(R))_2 \xrightarrow{\delta} K_{2,n}(R) \xrightarrow{\eta} K_2(Um_n(R)) \xrightarrow{\mu} (K_{1,n-1}(R))_2 \setminus K_{1,n-1}(R) \xrightarrow{\lambda}$$
$$K_{1,n}(R) \xrightarrow{\alpha} K_1(Um_n(R)) \xrightarrow{\beta} K^s_{0,n-1}(R) \to 0$$

where the base point of $K_1(Um_n(R))$ $(= \pi_0(Um_n(R)))$ is $[e]$ and by definition,

$$K_2(Um_n(R)) = \pi_1(EUm_n(R))$$
$$(K_{2,n}(R))_2 = K_{2,n}(R) \cap B_n(R)$$
$$(K_{1,n-1}(R))_2 = E_{n-1}(R) \setminus (\mathrm{GL}_{n-1}(R) \cap (BEP_n(R)))$$

and $B_n(R)$ and $BEP_n(R)$ are defined as in the Introduction. The maps in the exact sequence are defined as exactness is being proved.

Proof. Exactness at $K_{2,n}(R)$. The map δ is the natural inclusion map. Define $\eta : K_{2,n}(R) \to \pi_1(EUm_n(R))$ by $\eta(Y) = B_n(R)Y$. Exactness at $K_2(R)$ is clearly trivial.

Exactness at $K_2(Um_n(R))$ and $(K_{1,n-1}(R))_2 \setminus K_{1,n-1}(R)$. We define the maps λ and μ, in that order. $\lambda : K_{1,n-1}(R))_2 \setminus K_{1,n-1}(R) \to K_{1,n}(R)$ is easy. It is induced by the diagonal inclusion of $\mathrm{GL}_{n-1}(R) \to \mathrm{GL}_n(R)$, $\sigma \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & \sigma \end{smallmatrix}\right)$. We define the map $\mu : \pi_1(EUm_n(R)) \to (K_{1,n-1}(R))_2 \setminus K_{1,n-1}(R)$. Recall that by Theorem 3.22, $\pi_1(EUm_n(R)) \simeq B_n(R) \setminus \widetilde{P}_n(R)$. Obviously the canonical homomorphism $\theta_n : St_n(R) \to E_n(R)$ defines a surjective morphism $B_n(R) \setminus St_n(R) \to BEP_n(R) \setminus E_n(R)$, $B_n(R)x \mapsto BEP_n(R)\theta_n(x)$, of global actions, whose kernel is $B_n(R) \setminus B_n(R)K_{2,n}(R)$. Given $\sigma \in EP_n(R)$, there exists a unique $\tau \in \mathrm{GL}_{n-1}(R)$ such that $\sigma = \left(\begin{smallmatrix} 1 & 0 \\ v & \tau \end{smallmatrix}\right) \in E_n(R)$. Clearly $\tau \in \mathrm{GL}_{n-1}(R) \cap E_n(R)$ and defines an element $E_{n-1}(R)\tau$ of $K_{1,n-1}(R)$, which we denote by $[\sigma_{rd}]$, the class of the right diagonal of σ. If $x \in \widetilde{P}_n(R)$, let $\Delta(x) = [\theta_n(x)_{rd}]$. It is straightforward to check that $\Delta(B_n(R)) = K_{1,n-1}(R)_2$ and that Δ induces a morphism, denoted by $\mu : B_n(R) \setminus \widetilde{P}_n(R) \to (K_{1,n-1}(R))_2 \setminus K_{1,n-1}(R)$ of global actions. Examining μ, it is clear that $\mathrm{Ker}(\mu) = K_{2,n}(R) \cap B_n(R))$. This proves *exactness at* $K_2(Um_n(R))$. Moreover, $\mathrm{image}(\mu) = K_{1,n-1}(R)_2 \setminus (E_{n-1} \setminus \mathrm{GL}_{n-1}(R) \cap E_n(R)) = \mathrm{Ker}(\lambda)$. This proves *exactness at* $K_{1,n-1}(R))_2 \setminus K_{1,n-1}(R)$.

Exactness at $K_{1,n}(R)$. Let $\sigma \in \mathrm{GL}_n(R)$ represent the element $[\sigma] \in K_1(R)$. By definition, $\alpha([\sigma]) = [e\sigma]$. Clearly $[e\sigma] = [e] \Leftrightarrow \sigma \in P_n(R)E_n(R) \Leftrightarrow [\sigma] \in \mathrm{image}(\lambda)$.

Exactness at $K_1(Um_n(R))$. Define $\alpha : K_{1,n}(R) \to K_1(Um_n(R))$ by $[\sigma] \mapsto e\sigma$. This map is well defined because, if $\varepsilon \in E_n(R)$ then $e\sigma\varepsilon$ and $e\sigma$ are in the same path-connected component. Clearly this map preserves base points. Define $\beta : K_1(Um_n(R)) \to K^s_{0,n-1}(R)$ as follows. If $v \in Um_n(R)$, define the R-linear map $\beta_v : {}^nR \to R, w \mapsto w \cdot v^t = \sum_i v_i w_i$. Since v is unimodular, there is a $w \in {}^nR$ such that $\beta_v(w) = 1$. Thus β_v is surjective and $\mathrm{Ker}(\beta_v) \in K^s_{0,n-1}(R)$. Define $\beta(v) = [\mathrm{Ker}(\beta_v)]$. If $\sigma \in \mathrm{GL}_n(R)$ then $\mathrm{Ker}(\beta_{v\sigma}) \cong \mathrm{Ker}(\beta_v)$. Taking $\sigma \in E_n(R)$, we obtain that β is well defined. Clearly $\beta(e) = [{}^{n-1}R]$, so β is base point preserving. Furthermore, $\beta\alpha(\sigma) = \beta(e\sigma) = \beta(e) = [{}^{n-1}R]$. Conversely, if $\beta(v) \cong [{}^{n-1}R]$, then there is an element $\sigma \in \mathrm{GL}_n(R)$ such that $v\sigma = e$. This proves exactness at $K_1(Um_n(R))$.

Exactness at $K^s_{0,n-1}(R)$. This is trivial. □

We recall some results regarding injective and surjective stability of Volodin algebraic K-groups of commutative rings.

THEOREM 4.2. *Let R be a commutative ring such that the maximal ideal space of R is a Noetherian space of dimension $\leqslant d$ (for example R is a commutative, Noetherian ring of Krull dimension at most d). Then:*

(1) *The canonical map $K_{1,n}(R) \to K_{1,n+1}(R)$ is a group homomorphism for all $n \geqslant 3$ and is surjective (respectively bijective), if $n \geqslant d+1$ (respectively $n \geqslant d+2$). ([3], [4] and [9])*

(2) *The canonical map $K_{2,n}(R) \to K_{2,n+1}(R)$ is surjective (respectively bijective), if $n \geqslant d+2$ (respectively $n \geqslant d+3$). ([4]) If R is a Dedekind ring of arithmetic type with infinitely many units (as in Bass–Milnor–Serre [5]), then the map $K_{2,2}(R) \to K_{2,3}(R)$ is surjective and the map $K_{2,3}(R) \to K_{2,4}(R)$ bijective.*

We deduce an interesting corollary of the result above.

COROLLARY 4.3. *Let R be a commutative ring such that the maximal ideal space of R is a Noetherian space of dimension $\leqslant d$. Then, $K_2(Um_n(R)) = 1$, for all $n \geqslant d+3$.*

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Proof. Note that $n \geqslant d+3$ implies that $K_1(Um_n(R)) = 1$. This gives the following exact sequence of groups:

$$\begin{aligned}(K_{2,n}(R))_2 &\to K_{2,n}(R) \to K_2(Um_n(R)) \\ &\to (K_{1,n-1}(R))_2 \setminus K_{1,n-1}(R) \to K_{1,n}(R) \to 1.\end{aligned}$$

Noting that $(K_{2,n}(R))_2$ contains the image$(K_{2,n-1}(R) \to K_{2,n}(R))$, we have

$$\frac{K_{2,n}(R)}{(K_{2,n}(R))_2} \simeq \frac{K_{2,n}(R)/\text{image}(K_{2,n-1}(R) \to K_{2,n}(R))}{(K_{2,n}(R))_2/\text{image}(K_{2,n-1}(R) \to K_{2,n}(R))}.$$

Thus,

$$\frac{K_{2,n}(R)}{(K_{2,n}(R))_2} \simeq \frac{\text{cokernel}(K_{2,n-1}(R) \to K_{2,n}(R))}{(K_{2,n}(R))_2/\text{image}(K_{2,n-1}(R) \to K_{2,n}(R))}.$$

This gives the following exact sequence of groups:

$$\begin{aligned}1 &\to \frac{\text{cokernel}(K_{2,n-1}(R) \to K_{2,n}(R))}{(K_{2,n}(R))_2/\text{image}(K_{2,n-1}(R) \to K_{2,n}(R))} \to K_2(Um_n(R)) \\ &\to \frac{\ker(K_{1,n-1}(R) \to K_{1,n}(R))}{(K_{1,n-1}(R))_2} \to 1.\end{aligned}$$

That $K_2(Um_n(R))(= \pi_1(EUm_n(R))) = 1$ for all $n \geqslant d+3$ now follows from Theorem 4.2. □

COROLLARY 4.4. *Let R be a Dedekind ring of arithmetic type with infinitely many units. Then, $\pi_1(EUm_n(R)) = 1$ for all $n \geqslant 3$.*

Proof. Follows from the corresponding stability results for Dedekind rings of arithmetic type with infinitely many units. □

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