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ON THE NONNEGATIVITY OF THE DIRICHLET ENERGY OF A WEIGHTED GRAPH

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Abstract

Motivated by considerations of the quadratic orthogonal bisectional curvature, we address the question of when a weighted graph (with possibly negative weights) has nonnegative Dirichlet energy.

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To motivate a question about the nonnegativity of the Dirichlet energy of a weighted graph, we first discuss some background on curvatures and flows on complex manifolds.

DEFINITION 1. Let (M^n, ω) be a Hermitian manifold. The *quadratic orthogonal bisectional curvature* (from now on, QOBC) is the function

$$\mathrm{QOBC}_\omega : \mathcal{F}_M \times \mathbb{R}^n \backslash \{0\} \to \mathbb{R}, \quad \mathrm{QOBC}_\omega : (\vartheta, v) \mapsto \frac{1}{|v|_\omega^2} \sum_{\alpha,\gamma=1}^{n} R_{\alpha\bar{\alpha}\gamma\bar{\gamma}}(v_\alpha - v_\gamma)^2,$$

where the $R_{\alpha\bar{\alpha}\gamma\bar{\gamma}}$ denote the components of the Chern connection of ω with respect to the unitary frame ϑ (a section of the unitary frame bundle $\mathcal{F}_M$).

This curvature first appeared implicitly in [1] and is the Weitzenböck curvature operator (in the sense of [12, 13, 14]) acting on real $(1, 1)$-forms. (See [3] for alternative descriptions of the QOBC.) From [9], the QOBC is strictly weaker than the orthogonal bisectional curvature $\mathrm{HBC}_\omega^\perp$ (the restriction of the holomorphic bisectional curvature HBC_ω to pairs of orthogonal $(1,0)$-tangent vectors). From [8], the Kähler–Ricci flow on a compact Kähler manifold with $\mathrm{HBC}_\omega^\perp \geq 0$ converges to a Kähler metric $\mathrm{HBC}_\omega \geq 0$. Hence, Mok's extension [10] of the solution of the Frankel conjecture [11, 16] shows that all compact Kähler manifolds with $\mathrm{HBC}_\omega^\perp \geq 0$ are biholomorphic to a product of Hermitian symmetric spaces (of rank ≥ 2) and projective spaces.

In particular, although $\mathrm{HBC}_\omega^\perp$ is an algebraically weaker curvature notion than HBC_ω, the positivity of $\mathrm{HBC}_\omega^\perp$ does not generate new examples.

Wu *et al.* [17] showed that if (M, ω) is a compact Kähler manifold with $\mathrm{QOBC}_\omega \geq 0$, then every non-Kähler class on the boundary of the Kähler cone affords a semi-positive representative (which is certainly not true in general; see [7]). Further, Chau and Tam [4] showed that all harmonic $(1,1)$-forms are parallel on compact Kähler manifolds with $\mathrm{QOBC}_\omega \geq 0$.

In [2], I showed that the real bisectional curvature of Yang and Zheng [18] was best understood as a Rayleigh quotient. Continuing this program, we find that the QOBC is best understood as the Dirichlet energy of a certain weighted graph. In particular, we realise the difference of the Hodge and metric Laplacians, acting on $(1,1)$-forms, as the Dirichlet energy of a weighted graph.

To analyse the QOBC, we recall the following definition. Let G be a finite weighted graph, with vertices $V(G) = \{x_1, \ldots, x_n\}$, and weighting specified by its adjacency matrix $A \in \mathbb{R}^{n \times n}$. The Dirichlet energy for a weighted graph is defined by

$$\mathcal{E}(f) := \sum_{i,j=1}^{n} A_{ij}(f(x_i) - f(x_j))^2,$$

where $f : V(G) \to \mathbb{R}$ is a function defined on the vertices of G.

To understand when the QOBC is nonnegative, we can ask the following (equally natural) question.

QUESTION 2. Given a finite weighted graph (G, A), where $A \in \mathbb{R}^{n \times n}$ is a real matrix, what conditions on A are necessary or sufficient for the inequality $\mathcal{E}(f) \geq 0$ to hold for all $f : V(G) \to \mathbb{R}$?

The main theorem of this note gives an answer to this problem. To this end, let us recall some terminology arising from distance geometry.

DEFINITION 3. Let $A = (A_{ij}) \in \mathbb{R}^{n \times n}$ be a real symmetric matrix. We say that A is a Euclidean distance matrix if there is a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $A_{ij} = (x_i - x_j)^2$ for each $i, j = 1, \ldots, n$.

The set of all $n \times n$ Euclidean distance matrices (EDMs) forms a convex cone that we denote by $\mathbb{EDM}^n$. Recall that the Frobenius inner product of two matrices $A, B \in \mathbb{R}^{n \times n}$ is defined by

$$(A, B)_{\mathrm{F}} := \mathrm{tr}(AB^t).$$

This dual pairing allows us to define the dual EDM cone $(\mathbb{EDM}^n)^*$.

DEFINITION 4. The dual EDM cone $(\mathbb{EDM}^n)^*$ is given by

$$(\mathbb{EDM}^n)^* := \{A \in \mathbb{R}^{n \times n} : (A, B)_{\mathrm{F}} \geq 0 \text{ for all } B \in \mathbb{EDM}^n\}.$$

THEOREM 5. *Let (G, A) be a weighted finite graph. Then the Dirichlet energy $\mathcal{E}$ is nonnegative if and only if A lies in the dual EDM cone.*

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PROOF. If $V(G) = \{x_1, \ldots, x_n\}$ is the vertex set of some graph, then we may construct a Euclidean distance matrix $B(f)$ from a graph function $f : V(G) \to \mathbb{R}$ by setting $B(f)_{ij} = (f(x_i) - f(x_j))^2$. In particular, since

$$\mathrm{tr}(AB(f)) = \sum_{i,j=1}^{n} A_{ij} B(f)_{ij} = \sum_{i,j=1}^{n} A_{ij} (f(x_i) - f(x_j))^2,$$

we see that the Dirichlet energy $\mathcal{E}$ of a weighted graph (G, A) is nonnegative if and only if $\mathrm{tr}(AB) \geq 0$ for all Euclidean distance matrices $B \in \mathbb{EDM}^n$. □

It is natural to ask what is the relation (if any) between the EDM cone (and its dual) and the $\mathbb{PSD}^n$ cone, that is, the cone of (symmetric) positive semi-definite matrices. Dattorro [5] has shown that

$$\mathbb{EDM}^n = \mathbb{S}^n_{\mathrm{H}} \cap ((\mathbb{S}^n_{\mathrm{C}})^{\perp} - \mathbb{PSD}^n) \subset \mathbb{R}^{n \times n}_{\geq 0}.$$

Here, $\mathbb{S}^n_{\mathrm{H}}$ denotes the space of symmetric $n \times n$ hollow matrices, that is, symmetric matrices with no nonzero entries on the diagonal, and $\mathbb{S}^n_{\mathrm{C}}$ denotes the geometric centring subspace

$$\mathbb{S}^n_{\mathrm{C}} := \{A \in \mathbb{S}^n : A\mathbf{e} = 0\},$$

where $\mathbf{e} = (1, \ldots, 1)^t$. It is more natural to refer to $\mathbb{S}^n_{\mathrm{C}}$ as the annihilator of $\mathbf{e} = (1, \ldots, 1)^t \in \mathbb{R}^n$. The orthogonal complement of $\mathbb{S}^n_{\mathrm{C}}$ is then

$$(\mathbb{S}^n_{\mathrm{C}})^{\perp} = \{u\mathbf{e}^t + \mathbf{e}u^t : u \in \mathbb{R}^n\}.$$

In particular, standard properties of cones (see [6, page 434]) give the next proposition.

PROPOSITION 6. *Let (G, A) be a weighted finite graph. The Dirichlet energy $\mathcal{E}$ is nonnegative if and only if A lies in*

$$(\mathbb{EDM}^n)^* = \mathbb{D}^n - \mathbb{S}^n_C \cap \mathbb{PSD}^n,$$

where $\mathbb{D}^n$ is the cone of diagonal matrices.

REMARK 7. As discussed in [3], let us introduce the nonstandard terminology of *Perron weights*. The well-known Schoenberg criterion [15] states that a symmetric hollow matrix Σ is a Euclidean distance matrix if and only if it is negative semi-definite on the hyperplane $H = \{x \in \mathbb{R} : x^t\mathbf{e} = 0\}$, where $\mathbf{e} = (1, \ldots, 1)^t$. The Perron–Frobenius theorem asserts that the largest eigenvalue (the Perron root) of the EDM Σ is positive and occurs with eigenvector in the nonnegative orthant $\mathbb{R}^n_{\geq 0}$. Therefore, if $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$ denote the eigenvalues of a nontrivial Euclidean distance matrix Σ (that is, a Euclidean distance matrix with $\delta_1 > 0$), then $\delta_1 > 0$ and $\delta_2, \ldots, \delta_n \leq 0$.

DEFINITION 8. The Perron weights $r_2, \ldots, r_n$ of an $n \times n$ Euclidean distance matrix Σ, with eigenvalues $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$, are the ratios $r_k := -\delta_k/\delta_1$.

With this terminology in mind, appealing to the eigenvalue characterisation of the dual EDM cone given in [3] yields the following corollary.

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COROLLARY 9. *Let $A \in \mathbb{R}^{n\times n}$ be a real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then $A \in (\mathbb{EDM}^n)^*$ if and only if, for every Euclidean distance matrix Σ, the Perron weights $r_2, \ldots, r_k$ of Σ satisfy*

$$\lambda_1 \geq \sum_{k=1}^{n} r_k \lambda_k.$$

REMARK 10. Let us note that the Perron weights of a Euclidean distance matrix always satisfy $0 \leq r_2 \leq r_3 \leq \cdots \leq r_n \leq 1$.

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References

[1] R. L. Bishop and S. I. Goldberg, 'On the second cohomology group of a Kaehler manifold of positive curvature', *Proc. Amer. Math. Soc.* **16** (1965), 119–122.
[2] K. Broder, 'The Schwarz lemma in Kähler and non-Kähler geometry', Preprint, 2021, arxiv:2109.06331.
[3] K. Broder, 'An eigenvalue characterisation of the dual EDM cone', *Bull. Aust. Math. Soc.* to appear, doi:10.1017/S0004972721000915.
[4] A. Chau and L.-F. Tam, 'On quadratic orthogonal bisectional curvature', *J. Differential Geom.* **92**(2) (2012), 187–200.
[5] J. Dattorro, 'Equality relating Euclidean distance cone to positive semidefinite cone', *Linear Algebra Appl.* **428**(11–12) (2008), 2597–2600.
[6] J. Dattorro, *Convex Optimization & Euclidean Distance Geometry* 2ε (MeBoo, Palo Alto, CA, 2019), available at the author's webpage at https://convexoptimization.com/TOOLS/0976401304.pdf.
[7] J.-P. Demailly, Y. Peternell and M. Schneider, 'Compact complex manifolds with numerically effective tangent bundles', *J. Algebraic Geom.* **3** (1994), 295–345.
[8] H. Gu and Z. Zhang, 'An extension of Mok's theorem on the generalized Frankel conjecture', *Sci. China Math.* **53** (2010), 1–12.
[9] Q. Li, D. Wu and F. Zheng, 'An example of compact Kähler manifold with non-negative quadratic bisectional curvature', *Proc. Amer. Math. Soc.* **141**(6) (2013), 2117–2126.
[10] N. Mok, 'The uniformization theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature', *J. Differential Geom.* **27** (1988), 179–214.
[11] S. Mori, 'Projective manifolds with ample tangent bundles', *Ann. of Math. (2)* **110**(3) (1979), 593–606.
[12] P. Petersen, *Riemannian Geometry*, 3rd edn, Graduate Texts in Mathematics, 171 (Springer, Cham, 2016).
[13] P. Petersen and M. Wink, 'New curvature conditions for the Bochner technique', *Invent. Math.* **224** (2021), 33–54.
[14] P. Petersen and M. Wink, 'Vanishing and estimation results for Hodge numbers', *J. reine angew. Math.* **2021** (2021), 197–219.
[15] I. J. Schoenberg, 'Remarks to Maurice Fréchet's article "Sur la définition axiomatique d'une classe déspace distanciés vectoriellement applicable sur l'espace de Hilbert"', *Ann. of Math. (2)* **36**(3) (1935), 724–732.
[16] Y.-T. Siu and S.-T. Yau, 'Compact Kähler manifolds of positive bisectional curvature', *Invent. Math.* **59**(2) (1980), 189–204.

[17] D. Wu, S.-T. Yau and F. Zheng, 'A degenerate Monge–Ampère equation and the boundary classes of Kähler cones', *Math. Res. Lett.* **16**(2) (2009), 365–374.
[18] X. Yang and F. Zheng, 'On real bisectional curvature for Hermitian manifolds', *Trans. Amer. Math. Soc.* **371**(4) (2019), 2703–2718.

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