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Actions of $\mathbb{Z}^k$ associated to higher rank graphs

ALEX KUMJIAN† and DAVID PASK‡

† *Department of Mathematics, University of Nevada, Reno NV 89557, USA*
(e-mail: alex@unr.edu)
‡ *Mathematics, SMPS, The University of Newcastle, NSW 2308, Australia*
(e-mail: davidp@maths.newcastle.edu.au)

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Abstract. An action of $\mathbb{Z}^k$ is associated to a higher rank graph Λ satisfying a mild assumption. This generalizes the construction of a topological Markov shift arising from a non-negative integer matrix. We show that the stable Ruelle algebra of Λ is strongly Morita equivalent to $C^*(\Lambda)$. Hence, if Λ satisfies the aperiodicity condition, the stable Ruelle algebra is simple, stable and purely infinite.

1. *Introduction*

The shift map defines a homeomorphism on the space of two-sided infinite paths in a finite directed graph, a compact zero-dimensional space when endowed with the natural topology. Such dynamical systems, called topological Markov shifts or shifts of finite type, form a key class of examples in symbolic dynamics. Higher-dimensional analogues which exhibit many of the same dynamical properties include axiom A diffeomorphisms studied by Smale [**Sm**]. The local hyperbolic nature of the homeomorphisms in many of the examples has led to the axiomatization of Smale spaces (see [**Ru1**]). In [**Ru2, Pt1**] (see also [**KPS**] for a short survey and [**Pt2**] for extended notes) certain C^*-algebras were associated to a Smale space making use of the asymptotic, stable and unstable equivalence relations engendered by the homeomorphism. The Ruelle algebras, crossed products of the stable and unstable algebras by the canonical automorphism, may be regarded as higher-dimensional generalizations of Cuntz–Krieger algebras (see [**CK, Pt2, PtS**]). If the graph is irreducible, the stable Ruelle algebra associated to the Markov shift is strongly Morita equivalent to the Cuntz–Krieger algebra associated to the incidence matrix of the graph (cf. [**CK**, Theorem 3.8] and [**KPS**, Proposition 3.7] for similar results).

Following [**KP**] a k-graph is defined to be a higher rank analogue of a directed graph. The definition of a k-graph is motivated by the geometrical examples of Robertson and Steger arising from group actions on buildings (see [**RSt1, RSt2**]). Given a k-graph Λ, we define a universal C^*-algebra, $C^*(\Lambda)$, the Cuntz–Krieger algebra of Λ. Under a mild assumption we form the ‘two-sided path space’ of Λ, a natural zero-dimensional space associated with a k-graph on which there is a $\mathbb{Z}^k$ action by an analogue of the shift. We establish that the key dynamical properties identified by Ruelle (see [**Ru1**]), when properly interpreted, hold for this action. Our program then follows the one set out by Putnam. If Λ is irreducible and has finitely many vertices, then, as in [**Pt1, Pt2**], we construct C^*-algebras from the stable and unstable equivalence relations on which there are natural $\mathbb{Z}^k$ actions. We then form the resulting crossed products, the Ruelle algebras, R_s and R_u. Furthermore, we show that the Ruelle algebra R_s is strongly Morita equivalent to $C^*(\Lambda)$. Then, if Λ satisfies the aperiodicity condition, the Ruelle algebra R_s is a Kirchberg algebra; that is, R_s is simple, nuclear and purely infinite (a similar result holds for R_u). See [**PtS**] for general results on the Ruelle algebras of Smale spaces.

The paper is organized as follows. In §2 we establish our notation and collect facts for later use. We define a k-graph (Λ, d) to be a small category Λ equipped with a degree map d satisfying a certain factorization property. When (Λ, d) satisfies the standing assumption, every vertex of Λ receives and emits a finite but non-zero number of edges of any given degree; we form Λ^Ω the one-sided infinite path space of Λ. Pairs of shift-tail equivalent paths in Λ^Ω give rise to elements in the path groupoid $\mathcal{G}_\Lambda$. The groupoid C^*-algebra $C^*(\mathcal{G}_\Lambda)$ is naturally isomorphic to $C^*(\Lambda)$ (see [**KP**, Corollary 3.5]). There is a canonical gauge action α of $\mathbb{T}^k$ on $C^*(\Lambda)$ whose fixed point algebra $C^*(\Lambda)^\alpha$ is an AF-algebra (approximately finite dimensional C^*-algebra) which coincides with the C^*-algebra of a subgroupoid Γ_Λ of $\mathcal{G}_\Lambda$ under this identification. We conclude the section by stating some facts about principal proper groupoids.

In §3 we build a topological dynamical system from a k-graph which is generated by k commuting homeomorphisms. We show that it satisfies analogues of the two conditions (SS1) and (SS2) for a Smale space defined in [**Ru1**, §7.1]. The two-sided path space Λ^Δ of Λ has a zero-dimensional topology generated by cylinder sets which is also given by a metric ρ; it is compact if Λ^0 is finite. For $n \in \mathbb{Z}^k$ the shift $\sigma^n : \Lambda^\Delta \to \Lambda^\Delta$ gives rise to an expansive $\mathbb{Z}^k$-action which is topologically mixing if Λ is primitive. We show that condition (SS1) is satisfied, in particular there is a map $(x, y) \mapsto [x, y]$, defined for $x, y \in \Lambda^\Delta$ with $\rho(x, y) < 1$ taking values in Λ^Δ, which endows the space Λ^Δ with a local product structure. For $x \in \Lambda^\Delta$ there are subsets E_x and F_x of Λ^Δ such that $E_x \times F_x$ is homeomorphic to a neighbourhood of x (under this bracket map). Moreover, if $e = (1, \dots, 1) \in \mathbb{Z}^k$ then the shift σ^e contracts the distance between points in E_x and expands them on F_x. This is our analogue of condition (SS2) for a single homeomorphism. As in [**Pt1**] we define the stable and unstable relations which may be characterized in terms of tail equivalences on Λ^Δ, since the topology of Λ^Δ is generated by cylinder sets. The stable and unstable relations give rise to the stable and unstable groupoids, G_s and G_u. Since the unstable relation for Λ^Δ is exactly the stable relation for the opposite k-graph Λ^{op} (the k-graph formed by reversing all the arrows of Λ), we focus our attention on the stable case. Finally, we examine the internal structure of the stable groupoid G_s; it is the inductive limit of a sequence of mutually isomorphic principal proper groupoids $G_{s,m}$, for $m \in \mathbb{Z}^k$.

In §4 we associate certain C^*-algebras to an irreducible k-graph Λ with Λ^0 finite. First we state a suitable version of the Perron–Frobenius Theorem, which gives rise to a shift-invariant measure μ on Λ^Δ. The measure μ decomposes in a manner which

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respects the local product structure; this in turn gives rise to Haar systems for G_s and G_u. The stable and unstable C^*-algebras may then be defined: $S = C^*(G_s)$ and $U = C^*(G_u)$. The $\mathbb{Z}^k$-action on Λ^Δ induces actions β_s on S and β_u on U which scale the canonical densely-defined traces. The Ruelle algebras are defined to be the corresponding crossed products, $R_s = S \times_{\beta_s} \mathbb{Z}^k$ and $R_u = U \times_{\beta_u} \mathbb{Z}^k$.

In the last section we prove our main results. Suppose that Λ is an irreducible k-graph which has finitely many vertices. Then:

(i) S is strongly Morita equivalent to $C^*(\Lambda)^\alpha$ (see Theorem 5.3);

(ii) R_s is strongly Morita equivalent to $C^*(\Lambda)$ (see Theorem 5.6).

Similar assertions hold for U and R_u when Λ is replaced by Λ^{op}. We establish our main results using the notion of equivalence of groupoids (in the sense of [**MRW**]). From the established properties of $C^*(\Lambda)$ flow many important consequences. The stable algebra S is an AF-algebra and if Λ is primitive then S is simple. The Ruelle algebra R_s is nuclear and in the bootstrap class $\mathcal{N}$ for which the UCT (Universal Coefficient Theorem of [**RSc**]) holds. Further, if Λ satisfies the aperiodicity condition then R_s is simple, stable and purely infinite. The Kirchberg–Phillips Theorem therefore applies, so the isomorphism class of R_s is completely determined by its K-theory (see [**Ki, Ph**]).

2. *Preliminaries*

In this section we first give a little background, then we establish our notation and conventions about a k-graph Λ and its path groupoid $\mathcal{G}_\Lambda$ which are taken from [**KP**]. We define the C^*-algebra of a k-graph, $C^*(\Lambda)$, which may be realized as $C^*(\mathcal{G}_\Lambda)$. Finally, we state some results concerning principal proper groupoids and their Haar systems which are taken from [**Rn2, MW1, KMRW**].

We use $\mathbb{N}$ to denote the set of natural numbers $\{0, 1, 2, \dots\}$; $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{T}$ denote the sets of integers, real numbers and complex numbers with unit modulus, respectively. For $k > 0$ we endow $\mathbb{N}^k$ and $\mathbb{Z}^k$ with the coordinatewise ordering.

A category is said to be small if its morphisms form a set; the objects are often identified with a subset of morphisms ($x \mapsto 1_x$). A *groupoid* is a small category Γ in which every morphism is invertible. For $\gamma \in G$ we have $r(\gamma) = \gamma\gamma^{-1}$ and $s(\gamma) = \gamma^{-1}\gamma$, then $r, s : \Gamma \to \Gamma^0$ where Γ^0 is the *unit space* (or space of objects) of Γ. If the groupoid Γ is furnished with a topology for which the groupoid operations are continuous then Γ is called a *topological groupoid*. We assume that our groupoids are equipped with a locally compact, Hausdorff, second countable topology. If the groupoid Γ has a *left Haar system* $\mu = \{\mu^x : x \in \Gamma^0\}$, an equivariant system of measures on the fibres $r^{-1}(x)$, then we may form the full and reduced C^*-algebras, $C^*(\Gamma)$ and $C^*_r(\Gamma)$. Since we only deal with left Haar systems we henceforth omit the qualifier left. If Γ is amenable then its full and reduced C^*-algebras coincide. The groupoid Γ is called r-*discrete* if r is a local homeomorphism; in this case the counting measures form a Haar system. For more definitions and properties of groupoids and their C^*-algebras, consult [**Rn1, M**]. For the most part we have followed the conventions of [**Rn1**], with the exception that s replaces d for the source map. A good reference for amenable groupoids may be found in [**AR**]. We shall frequently invoke the notion of equivalence of groupoids [**MRW**, Definition 2.1] which (in the presence of Haar systems) gives rise to the strong Morita equivalence of their

C^*-algebras [**MRW**, Theorem 2.8]. A good reference for C^*-algebras and their crossed products is [**Pd**].

Let k be a positive integer. Recall the notion of k-graph (see [**KP**]).

Definition 2.1. A k-graph is a pair (Λ, d), where Λ is a countable small category and $d : \Lambda \to \mathbb{N}^k$ is a morphism, called the degree map, such that the factorization property holds: for every $n_1, n_2 \in \mathbb{N}^k$ and $\lambda \in \Lambda$ with $d(\lambda) = n_1 + n_2$, there exist unique elements $\nu_1, \nu_2 \in \Lambda$ with

$$\lambda = \nu_1\nu_2, \quad n_1 = d(\nu_1), \quad n_2 = d(\nu_2).$$

For $n \in \mathbb{N}^k$ write $\Lambda^n = \{\lambda \in \Lambda : d(\lambda) = n\}$. It will be convenient to identify Λ^0 with the objects of Λ. Let $r, s : \Lambda \to \Lambda^0$ denote the range and source maps.

Let $E = (E^0, E^1)$ be a (countable) directed graph. Then the set of finite paths E^* together with the length map defines a 1-graph (the roles of r and s must be switched).

If Λ is a k-graph then the opposite category Λ^{op} can also be made into a k-graph by setting $d(\lambda^{\mathrm{op}}) = d(\lambda)$.

The k-graph which gives us the prototype for a (one-sided) infinite path is

$$\Omega = \Omega_k = \{(m, n) : m, n \in \mathbb{N}^k : m \le n\}.$$

The structure maps are given by

$$r(m, n) = m, \quad s(m, n) = n, \quad (\ell, n) = (\ell, m)(m, n), \quad d(m, n) = n - m \tag{1}$$

where the object space is identified with $\mathbb{N}^k$ (see [**KP**, Example 1.7ii]). For other examples of k-graphs consult [**KP**].

Definition 2.2. A k-graph Λ is said to be irreducible (or strongly connected) if, for every $u, v \in \Lambda^0$, there is $\lambda \in \Lambda$ with $d(\lambda) \neq 0$ such that $u = r(\lambda)$ and $v = s(\lambda)$. We say that Λ is primitive if there is a non-zero $p \in \mathbb{N}^k$ so that for every $u, v \in \Lambda^0$ there is $\lambda \in \Lambda^p$ with $r(\lambda) = u$ and $s(\lambda) = v$.

Suppose that Λ is primitive, then there is an N such that, for all $p \ge N$ and every $u, v \in \Lambda^0$, there is $\lambda \in \Lambda^p$ with $r(\lambda) = u$ and $s(\lambda) = v$. Moreover, under the following Standing Assumption 2.3, Λ^0 must be finite.

To ensure that the analogue of the two-sided infinite path space (to be discussed in the next section) is non-empty and locally compact we shall need the following standing hypothesis.

STANDING ASSUMPTION 2.3. *For each $p \in \mathbb{N}^k$ the restrictions of r and s to Λ^p are surjective and finite to one.*

The standing hypothesis used here is equivalent to the requirement that both Λ and Λ^{op} satisfy the condition of [**KP**, §1]. Recall from [**KP**] the definition of the universal C^*-algebra of a k-graph.

Definition 2.4. Let Λ be a k-graph. Then $C^*(\Lambda)$ is defined to be the universal C^*-algebra generated by a family $\{s_\lambda : \lambda \in \Lambda\}$ of partial isometries satisfying:

(i) $\{s_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections;

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(ii) $s_{\lambda\mu} = s_\lambda s_\mu$ for all $\lambda, \mu \in \Lambda$ such that $s(\lambda) = r(\mu)$;
(iii) $s_\lambda^* s_\lambda = s_{s(\lambda)}$ for all $\lambda \in \Lambda$;
(iv) for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, we have $s_v = \sum_{\lambda \in \Lambda^n(v)} s_\lambda s_\lambda^*$.

Let Λ be a k-graph and set

$$\Lambda^\Omega = \{x : \Omega \to \Lambda : x \text{ is a } k\text{-graph homomorphism}\}; \tag{2}$$

note that a k-graph morphism must preserve degree, so that for $x \in \Lambda^\Omega$, we have $d(x(m,n)) = n - m$. In [**KP**] the set Λ^Ω was denoted Λ^∞. By Standing Assumption 2.3, $\Lambda^\Omega \neq \varnothing$. For each $\lambda \in \Lambda$ we put

$$Z(\lambda) = \{x \in \Lambda^\Omega : x(0, d(\lambda)) = \lambda\} \tag{3}$$

then again by Standing Assumption 2.3, $Z(\lambda) \neq \varnothing$. The collection of all such cylinder sets forms a basis for a topology on Λ^Ω under which each such subset is compact. For $p \in \mathbb{N}^k$ define a map $\sigma^p : \Lambda^\Omega \to \Lambda^\Omega$ by

$$(\sigma^p x)(m,n) = x(m+p, n+p) \tag{4}$$

note that σ^p is a local homeomorphism. Now we form the *path groupoid* (for more details see [**KP**])

$$\mathcal{G}_\Lambda = \{(x,n,y) : x, y \in \Lambda^\Omega, n \in \mathbb{Z}^k, \sigma^\ell x = \sigma^m y, n = \ell - m \text{ for some } \ell, m \in \mathbb{N}^k\},$$

with structure maps

$$r(x,n,y) = x, \quad s(x,n,y) = y \quad \text{and} \quad (x,m,y)(y,n,z) = (x, m+n, z),$$

where we have identified Λ^Ω with the unit space by $x \mapsto (x, 0, x)$.

By [**KP**, Corollary 3.5(i)], $C^*(\mathcal{G}_\Lambda) = C^*(\Lambda)$. There is a canonical gauge action $\alpha : \mathbb{T}^k \to \operatorname{Aut}(C^*(\Lambda))$ which is realized on the dense subalgebra $C_c(\mathcal{G}_\Lambda)$ by

$$\alpha_t(f)(x,n,y) = t^n f(x,n,y)$$

where $t^n = \prod_i t_i^{n_i}$. The fixed point algebra for this action $C^*(\Lambda)^\alpha$ is the closure of the subalgebra of $C_c(\mathcal{G}_\Lambda)$ consisting of functions which vanish at points of the form (x,n,y) with $n \neq 0$. Hence $C^*(\Lambda)^\alpha$ is isomorphic to $C^*(\Gamma_\Lambda)$ where Γ_Λ is the open subgroupoid of $\mathcal{G}_\Lambda$ given by

$$\Gamma_\Lambda = \{(x,0,y) : x, y \in \Lambda^\Omega, \sigma^m x = \sigma^m y \text{ for some } m \in \mathbb{N}^k\}.$$

Recall that

$$C^*(\Lambda)^\alpha = \mathcal{F}_\Lambda = \lim_{m \to \infty} \mathcal{F}_m$$

where

$$\mathcal{F}_m \cong \bigoplus_{v \in \Lambda^0} \mathcal{K}(\ell^2(\{\lambda \in \Lambda^m : s(\lambda) = v\})),$$

hence, $C^*(\Gamma_\Lambda) = C^*(\Lambda)^\alpha$ is an AF-algebra (see [**KP**, Lemmas 3.2 and 3.3]).

Let Λ_i be a k_i graph for $i = 1, 2$, then $\Lambda_1 \times \Lambda_2$ is a $(k_1 + k_2)$-graph in a natural way (see [**KP**, Proposition 1.8]). By [**KP**, Corollary 3.5iv] we have

$$C^*(\Lambda_1 \times \Lambda_2) \cong C^*(\Lambda_1) \otimes C^*(\Lambda_2).$$

If $k_1 = k_2 = k$ then we may form a k-graph

$$\Lambda_1 \diamond \Lambda_2 = \{(\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2 : d(\lambda_1) = d(\lambda_2)\}$$

with $d(\lambda_1, \lambda_2) = d(\lambda_1)$ and the other structure maps inherited from $\Lambda_1 \times \Lambda_2$. Note that $\Lambda_1 \diamond \Lambda_2 = f^*(\Lambda_1 \times \Lambda_2)$ where $f : \mathbb{N}^k \to \mathbb{N}^k \times \mathbb{N}^k$ is given by $f(m) = (m, m)$ (cf. [**KP**, Example 1.10iii]).

Let G be a compact Abelian group and for $i = 1, 2$ let $\alpha^i : G \to \mathrm{Aut}(A_i)$ be a strongly continuous action of G on the C^*-algebra A_i. Let $A_1 \otimes_G A_2$ denote the fixed-point algebra $(A_1 \otimes A_2)^\eta$ where $\eta : G \to \mathrm{Aut}(A_1 \otimes A_2)$ is given by $\eta_g(a \otimes b) = \alpha^1_g(a) \otimes \alpha^2_{g^{-1}}(b)$. This is the natural notion of tensor product in the category of C^*-algebras with a given G-action, see [**OPT**, §2]. Now with Λ_i as above and taking α^i to be the gauge action on $C^*(\Lambda_i)$ for $i = 1, 2$ we have

$$C^*(\Lambda_1 \diamond \Lambda_2) \cong C^*(\Lambda_1) \otimes_{\mathbb{T}^k} C^*(\Lambda_2). \tag{5}$$

This follows by an argument similar to the proof of [**Ku2**, Proposition 2.7].

In the remainder of this section we state some standard facts concerning principal proper groupoids in a convenient form. Recall that a groupoid G is said to be *principal*, if it is isomorphic to an equivalence relation; that is, if $r \times s : G \to G^0 \times G^0$ is an embedding. If, in addition, the image is a closed subset of $G^0 \times G^0$, it is said to be *proper* (see [**MW1**]).

LEMMA 2.5. *Let $\pi : X \to Y$ be a continuous open surjection between two locally compact Hausdorff spaces. Then*

$$X \star_\pi X = \{(x, y) \in X \times X : \pi(x) = \pi(y)\}$$

is a principal proper groupoid, with structure maps $r(x, y) = x$, $s(x, y) = y$ and $(x, y)(y, z) = (x, z)$. Moreover, X is an $(X \star_\pi X, Y)$-equivalence.

Proof. Evidently $X \star_\pi X$ is a principal groupoid; since $X \star_\pi X$ is a closed subset of $X \times X$, it is proper. By [**MRW**, Example 2.5] X is a $(X \star_\pi X, Y)$-equivalence. □

The following definition is taken from [**Rn2**, §1] (see also [**M**, Definition 5.42]).

Definition 2.6. With π as above, a π-system consists of a family

$$\mu = \{\mu^y : y \in Y\}$$

of positive Radon measures on X such that the support of μ^y is contained in $\pi^{-1}(y)$ for each $y \in Y$ and the function

$$\mu(f)(y) = \int f(x)\, d\mu^y(x)$$

lies in $C_c(Y)$ for each $f \in C_c(X)$. If the support of each μ^y is all of $\pi^{-1}(y)$ for all $y \in Y$, then the π-system is said to be full.

Note that a Haar system on a groupoid is an equivariant r-system. A full π-system gives rise to a Haar system for $X \star_\pi X$.

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PROPOSITION 2.7. *Let π be as above and $\mu = \{\mu^y : y \in Y\}$ be a full π-system. Then, for $x \in X$,*

$$\tilde{\mu}^x = \delta_x \times \mu^{\pi(x)}$$

defines a Haar system $\tilde{\mu} = \{\tilde{\mu}^x : x \in X\}$ for $X \star_\pi X$. Moreover, $C^*(X \star_\pi X)$ is strongly Morita equivalent to $C_0(Y)$. There is a densely defined $C_0(Y)$-valued trace on $C^*(X \star_\pi X)$ given by

$$\tau_\mu(f)(y) = \int_X f(x, x)\, d\mu^y(x) \tag{6}$$

for $f \in C_c(X \star_\pi X)$.

Proof. The first assertion follows from [**KMRW**, Proposition 5.2] (see also [**M**, Theorem 5.51]). The Morita equivalence now follows from [**MRW**, Theorem 2.8] and Lemma 2.5 (see also [**MW1**, Proposition 2.2]). A routine computation shows that $\tau_\mu(fg) = \tau_\mu(gf)$ for $f, g \in C_c(X \star_\pi X)$. □

3. $\mathbb{Z}^k$ *actions*

In this section we adapt the methods of [**Pt1**] for the $\mathbb{Z}$-action associated to shifts of finite type to analyze an analogous $\mathbb{Z}^k$ action on a topological space associated to a k-graph. Many of the constructions of [**Ru1, Pt1**] can be generalized to this setting. Following [**Pt1, Pt2**], we provide a description of the stable, unstable and asymptotic relations for our $\mathbb{Z}^k$ action and the topology of the associated groupoids.

There is a natural $\mathbb{Z}^k$ action on the analogue of the two-sided path space of a k-graph satisfying Standing Assumption 2.3. First, we form a k-graph which gives us the prototype of a two-sided infinite path. Set

$$\Delta = \Delta_k = \{(m, n) : m, n \in \mathbb{Z}^k, m \le n\},$$

with structure maps given as in (1), it is straightforward to check that (Δ, d) is a k-graph.

Next we use Δ to form the two-sided infinite path space (cf. (2)). Set

$$\Lambda^\Delta = \{x : \Delta \to \Lambda : x \text{ is a } k\text{-graph morphism}\},$$

then by Standing Assumption 2.3, $\Lambda^\Delta \neq \varnothing$. We endow Λ^Δ with a topology as follows (cf. (3)): for each $n \in \mathbb{Z}^k$ and $\lambda \in \Lambda$ set

$$Z(\lambda, n) = \{x \in \Lambda^\Delta : x(n, n + d(\lambda)) = \lambda\}.$$

Again by Standing Assumption 2.3, $Z(\lambda, n) \neq \varnothing$. The collection of all such cylinder sets forms a basis for a topology on Λ^Δ for which each such subset is compact. It follows that Λ^Δ is a zero-dimensional space and if Λ^0 is finite, then Λ^Δ is itself compact (since $\Lambda^\Delta = \bigcup_{v \in \Lambda^0} Z(v, 0)$). Now for each $n \in \mathbb{Z}^k$ we define a map $\sigma^n : \Lambda^\Delta \to \Lambda^\Delta$ by

$$\sigma^n(x)(\ell, m) = x(\ell + n, m + n).$$

Note that σ^n is a homeomorphism for every $n \in \mathbb{Z}^k$, $\sigma^{n+m} = \sigma^n \sigma^m$ for $n, m \in \mathbb{Z}^k$ and σ^0 is the identity map.

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We define a metric on Λ^Δ as follows. We set $e = (1, \dots, 1) \in \mathbb{Z}^k$ and for $j \in \mathbb{N}$, let $\theta_j \in \Delta$ denote the element $(-je, je)$; note that $\theta_0 = 0$. Given $x, y \in \Lambda^\Delta$, set

$$h(x, y) = \begin{cases} 0 & x(0) \neq y(0) \\ 1 + \sup\{j : x(\theta_j) = y(\theta_j)\} & \text{otherwise.} \end{cases}$$

Fix $0 < r < 1$; we may define a metric ρ on Λ^Δ by the formula $\rho(x, y) = r^{h(x,y)}$ for $x, y \in \Lambda^\Delta$ (note that $\rho(x, x) = r^\infty = 0$). The topology induced by this metric is the same as the one above.

PROPOSITION 3.1. *The $\mathbb{Z}^k$-action $n \mapsto \sigma^n$ on Λ^Δ is expansive in the sense that there is an $\varepsilon > 0$ such that for all $x, y \in \Lambda^\Delta$ if $\rho(\sigma^n(x), \sigma^n(y)) < \varepsilon$ for all n then $x = y$. Moreover, if Λ is primitive then σ is topologically mixing in the sense that for any two non-empty open sets U and V in Λ^Δ there is a $Q \in \mathbb{Z}^k$ so that $U \cap \sigma^q(V) \neq \varnothing$ for all $q \geq Q$.*

Proof. To show that the action is expansive, observe that $\varepsilon = r$ will suffice (if $x(n - e, n + e) = y(n - e, n + e)$ for all $n \in \mathbb{Z}^k$, then $x = y$). If Λ is primitive, there is an $M \in \mathbb{N}^k$ such that for all $m \geq M$ and every $u, v \in \Lambda^0$ there is $\lambda \in \Lambda^m$ with $r(\lambda) = u$ and $s(\lambda) = v$. To show that for any two non-empty open sets U and V in Λ^Δ there is a $Q \in \mathbb{Z}^k$ so that $U \cap \sigma^q(V) \neq \varnothing$ for all $q \geq Q$, it suffices to demonstrate this for cylinder sets. So, let $U = Z(\lambda, \ell)$ and $V = Z(\nu, n)$. Set $Q = M + d(\nu) + n - \ell$; then given $q \geq Q$, there is $\lambda' \in \Lambda$ with $d(\lambda') = M + q - Q$ such that $r(\lambda') = s(\nu)$ and $s(\lambda') = r(\lambda)$. Observe that

$$Z(\nu\lambda'\lambda, n - q) \subset Z(\lambda, \ell) \cap \sigma^q(Z(\nu, n)).$$

Therefore, $U \cap \sigma^q(V) \neq \varnothing$ for all $q \geq Q$, as required. □

Remark 3.2. Consider the 1-graph obtained by restricting consideration to powers of Λ^e; note that this 1-graph may be regarded as the k-graph $f^*(\Lambda)$ where $f : \mathbb{N} \to \mathbb{N}^k$ is given by $f(j) = je$ (see [**KP**, Definition 1.9]). By arguing as in [**KP**, Proposition 2.9] it follows that the restriction map $\Lambda^\Delta \to f^*(\Lambda)^\Delta$ is a homeomorphism. Under this identification the generator of the action of $\mathbb{Z}$ on $f^*(\Lambda)^\Delta$ is identified with σ^e. Many attributes of this restricted dynamical system are reflected in the action of $\mathbb{Z}^k$, as we shall see below.

The space Λ^Δ decomposes locally into contracting and expanding directions for the shift. For $x \in \Lambda^\Delta$ set

$$E_x = \{y \in \Lambda^\Delta : x(m, n) = y(m, n), \text{ for all } 0 \leq m \leq n\}$$
$$F_x = \{y \in \Lambda^\Delta : x(m, n) = y(m, n), \text{ for all } m \leq n \leq 0\}.$$

Observe that for $j \in \mathbb{N}$ we have (see [**Ru1**, §7.1], also [**Pt1**])

$$\rho(\sigma^{je}(y), \sigma^{je}(z)) \leq r^j \rho(y, z) \quad \text{for } y, z \in E_x$$
$$\rho(\sigma^{-je}(y), \sigma^{-je}(z)) \leq r^j \rho(y, z) \quad \text{for } y, z \in F_x.$$

For $p \geq 0$ a simple calculation shows that $\sigma^p E_x \subseteq E_{\sigma^p x}$ and $\sigma^{-p} F_x \subseteq F_{\sigma^{-p} x}$.

PROPOSITION 3.3. (cf. [**Ru1, Pt1**]) *There exists a unique map*

$$[\cdot,\cdot] : \{(x, y) \in \Lambda^{\Delta} \times \Lambda^{\Delta} : \rho(x, y) < 1\} \to \Lambda^{\Delta}$$

satisfying

$$\begin{aligned} [x, y](m, n) &= x(m, n) \quad \textit{if } m \le n \le 0 \\ [x, y](m, n) &= y(m, n) \quad \textit{if } 0 \le m \le n. \end{aligned} \tag{7}$$

Moreover, $[\cdot,\cdot]$ *is continuous,* $F_x \cap E_y = \{[x, y]\}$ *if* $\rho(x, y) < 1$ *and the following hold:*

$$[x, x] = x, \quad [[x, y], z] = [x, z], \quad [x, [y, z]] = [x, z], \quad [\sigma^n x, \sigma^n y] = \sigma^n [x, y], \tag{8}$$

wherever both sides of each equation are defined. Furthermore, for $x \in \Lambda^{\Delta}$ *the restriction of* $[\cdot,\cdot]$ *to* $E_x \times F_x$ *induces a homeomorphism* $E_x \times F_x \cong Z(x(0), 0)$.

Proof. By the factorization property, a consistent family of elements

$$\{x(-p, p) \in \Lambda^{2p} : p \ge 0\} \quad \text{with } x(-q, q) = \lambda x(-p, p)\mu,$$

for some λ, μ when $q \ge p$, will determine a unique element $x \in \Lambda^{\Delta}$ (cf. [**KP**, Remarks 2.2]). For $p \ge 0$ and x, y with $\rho(x, y) < 1$ (so that $x(0) = y(0)$) set

$$[x, y](-p, p) = x(-p, 0)y(0, p).$$

It is straightforward to check that this results in the unique map satisfying (7); moreover it is continuous. If $z \in F_x \cap E_y$, then $z(m, n) = x(m, n)$ for $m \le n \le 0$ and $z(m, n) = y(m, n)$ for $0 \le m \le n$; hence, $z = [x, y]$ by (7).

The properties (8) are straightforward to verify. For the last assertion, it is clear that the restriction of $[\cdot,\cdot]$ to $E_x \times F_x$ is one-to-one. To see that the image is $Z(x(0), 0)$, let $z \in Z(x(0), 0)$; then $[z, x] \in E_x$, $[x, z] \in F_x$ and $z = [[z, x], [x, z]]$. The restriction is clearly continuous as is its inverse $z \mapsto ([z, x], [x, z])$. □

Note that if $\rho(x, y) < 1$, then $y \in E_x$ if and only if $[x, y] = x$ and similarly $y \in F_x$ if and only if $[y, x] = x$.

As in [**Pt1**] we define the stable and unstable equivalence relations on Λ^{Δ} as follows. Given $x, y \in \Lambda^{\Delta}$ define

$$\begin{aligned} x \sim_{\mathrm{s}} y \quad &\text{if } \lim_{j\to\infty} \rho(\sigma^{je}(x), \sigma^{je}(y)) = 0 \\ x \sim_{\mathrm{u}} y \quad &\text{if } \lim_{j\to-\infty} \rho(\sigma^{je}(x), \sigma^{je}(y)) = 0. \end{aligned}$$

Note that $x \sim_{\mathrm{s}} y$ if and only if there is $m \in \mathbb{Z}^k$ such that for all $n \in \mathbb{Z}^k$ with $m \le n$ we have $x(m, n) = y(m, n)$. Similarly $x \sim_{\mathrm{u}} y$ if and only if there is $n \in \mathbb{Z}^k$ such that for all $m \in \mathbb{Z}^k$ with $m \le n$ we have $x(m, n) = y(m, n)$.

These equivalence relations give rise to two locally compact groupoids: the stable groupoid,

$$G_{\mathrm{s}} = G_{\mathrm{s}}(\Lambda) = \{(x, y) \in \Lambda^{\Delta} \times \Lambda^{\Delta} : x \sim_{\mathrm{s}} y\}$$

and the unstable groupoid,

$$G_{\mathrm{u}} = G_{\mathrm{u}}(\Lambda) = \{(x, y) \in \Lambda^{\Delta} \times \Lambda^{\Delta} : x \sim_{\mathrm{u}} y\};$$

the unit space of each is identified with Λ^Δ and the structure maps are the natural ones. The topology on G_s is given as follows. For $m \in \mathbb{Z}^k$, set

$$G_{s,m} = \{(x, y) \in \Lambda^\Delta \times \Lambda^\Delta : x(m, n) = y(m, n) \text{ for all } n \geq m\}.$$

Note that $G_{s,m}$ is a subgroupoid of G_s. We endow $G_{s,m}$ with the relative topology and $G_s = \bigcup_m G_{s,m}$ with the inductive limit topology. The topology on G_u is defined similarly. None of these groupoids are r-discrete in general.

There is an natural inclusion map $\Omega \hookrightarrow \Delta$ which gives rise to a surjective map $\pi : \Lambda^\Delta \to \Lambda^\Omega$ given by restriction: $\pi(x)(m, n) = x(m, n)$ for $x \in \Lambda^\Delta$ and $(m, n) \in \Omega$. It is straightforward to verify that π is continuous and open. Observe that for $x \in \Lambda^\Delta$ and $p \in \mathbb{N}^k$ we have

$$\pi \circ \sigma^p(x) = \sigma^p \circ \pi(x). \tag{9}$$

We now collect some facts about the topology of G_s for future use.

PROPOSITION 3.4. *Let Λ be a k-graph and G_s the groupoid defined above. Then, for all $m \in \mathbb{Z}^k$, $G_{s,m}$ is a closed subset of $\Lambda^\Delta \times \Lambda^\Delta$; indeed*

$$G_{s,m} = \Lambda^\Delta \star_{\pi\circ\sigma^m} \Lambda^\Delta = \{(x, y) \in \Lambda^\Delta \times \Lambda^\Delta : \pi(\sigma^m x) = \pi(\sigma^m y)\}$$

and hence $G_{s,m}$ is a principal proper groupoid. For all $m, n \in \mathbb{Z}^k$ we have that $G_{s,m+n} = (\sigma^{-m} \times \sigma^{-m})G_{s,n}$; in particular, $G_{s,m}$ are all isomorphic to $\Lambda^\Delta \star_\pi \Lambda^\Delta$. Moreover, for $m \leq n$, $G_{s,m}$ is an open subset of $G_{s,n}$.

Proof. For the first part, observe that $\pi(\sigma^m x) = \pi(\sigma^m y)$ if and only if $x(m, n) = y(m, n)$ for all $n \geq m$; so $G_{s,m} = \Lambda^\Delta \star_{\pi\circ\sigma^m} \Lambda^\Delta$ which is a principal proper groupoid by Lemma 2.5. For the second assertion, note that $x(n + m, \ell) = y(n + m, \ell)$ for all $\ell \geq n + m$ if and only if $\sigma^m x(n, \ell') = \sigma^m y(n, \ell')$ for all $\ell' \geq n$ and that $(\sigma^{-m} \times \sigma^{-m})$ is a homeomorphism of $\Lambda^\Delta \times \Lambda^\Delta$. To show that $G_{s,m}$ is an open subset of $G_{s,n}$ for $m \leq n$, it suffices to consider the case when $m = 0$. Suppose that $(x, y) \in G_{s,0}$, then we have $x, y \in Z(\lambda, 0)$ where $\lambda = x(0, n) = y(0, n)$. Put $U = Z(\lambda, 0) \times Z(\lambda, 0)$, then $(x, y) \in G_{s,0} \cap U$. If $(x', y') \in G_{s,n} \cap U$ then $x'(0, n) = \lambda = y'(0, n)$ and since $(x', y') \in G_{s,n}$, we have $x'(n, \ell) = y'(n, \ell)$ for $\ell \geq n$. Hence $x'(0, \ell) = y'(0, \ell)$ for all $\ell \geq 0$ and so $(x', y') \in G_{s,0} \cap U$, which shows that $G_{s,0}$ is open in $G_{s,n}$ as required. □

Remark 3.5. There is a homeomorphism $\Lambda^\Delta \to (\Lambda^{\mathrm{op}})^\Delta$ given by $x \mapsto x^{\mathrm{op}}$ where

$$x^{\mathrm{op}}(m, n) = x(-n, -m)^{\mathrm{op}}.$$

Note that for $n \in \mathbb{Z}^k$ and $x \in \Lambda^\Delta$ we have $(\sigma^{\mathrm{op}})^n(x^{\mathrm{op}}) = \sigma^{-n}(x)^{\mathrm{op}}$, where σ^{op} is the shift action of $\mathbb{Z}^k$ on $(\Lambda^{\mathrm{op}})^\Delta$. For every $x, y \in \Lambda^\Delta$ we have $x \sim_s y$ if and only if $x^{\mathrm{op}} \sim_u y^{\mathrm{op}}$ and $x \sim_u y$ if and only if $x^{\mathrm{op}} \sim_s y^{\mathrm{op}}$. Hence $G_u(\Lambda) = G_s(\Lambda^{\mathrm{op}})$ and $G_u(\Lambda^{\mathrm{op}}) = G_s(\Lambda)$.

Remark 3.6. As in [**Pt1**] (cf. [**Ru2**]) we define the asymptotic relation on Λ^Δ as follows. For $x, y \in \Lambda$, we put $x \sim_a y$ if $x \sim_s y$ and $x \sim_u y$. Observe that $x \sim_a y$ if and only if there is $m \in \mathbb{N}^k$ so that for all $n \geq m$ we have

$$x(m, n) = y(m, n) \quad \text{and} \quad x(-n, -m) = y(-n, -m). \tag{10}$$

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Let G_a denote the groupoid derived from this equivalence relation. We endow it with a topology that makes it an r-discrete groupoid. Given $(x, y) \in G_a$, there is an $m \in \mathbb{N}^k$ so that (10) holds; set $\lambda = x(-m, m)$ and $\nu = y(-m, m)$. There is a unique map $\varphi_{\nu,\lambda} : Z(\lambda, -m) \to Z(\nu, -m)$ such that $\varphi_{\nu,\lambda}(x) = y$ and

$$\varphi_{\nu,\lambda}(z)(m, n) = z(m, n) \quad \text{and} \quad \varphi_{\nu,\lambda}(z)(-n, -m) = z(-n, -m)$$

for all $z \in Z(\lambda, -m)$ and $n \geq m$. Note that the map $\varphi_{\nu,\lambda}$ is a homeomorphism and $z \sim_a \varphi_{\nu,\lambda}(z)$ for all $z \in Z(\lambda, -m)$. We let $U_{\nu,\lambda} \subset G_a$ be the graph of $\varphi_{\nu,\lambda}$

$$U_{\nu,\lambda} = \{(\varphi_{\nu,\lambda}(z), z) : z \in Z(\lambda, -m)\}.$$

The collection $\{U_{\nu,\lambda}\}_{\nu,\lambda}$ forms a basis for the topology of G_a in which $U_{\nu,\lambda}$ are compact open sets. Evidently the restriction of the range map to each $U_{\nu,\lambda}$ is a homeomorphism onto $Z(\nu, -m)$; hence G_a is r-discrete.

4. *Ruelle algebras*

As in [**Pt1**] the stable and unstable C^*-algebras are given by $S := C^*(G_s)$ and $U := C^*(G_u)$. The Ruelle algebras, R_s and R_u, are defined as the crossed products of S and U by the natural $\mathbb{Z}^k$ actions (see [**PtS**]). We need to show that G_s and G_u have Haar systems; for this it will be necessary to invoke a suitable version of the Perron–Frobenius Theorem, for an irreducible k-graph Λ with Λ^0 finite (cf. [**Pt2**]). As in [**Pt1**] we show that there is a densely-defined trace on S and U which is scaled by the $\mathbb{Z}^k$ action. Finally, we discuss the corresponding facts in the asymptotic case.

Let Λ be a k-graph. For $u, v \in \Lambda^0$, $p \in \mathbb{N}^k$ set

$$\Lambda^p(u, v) = \{\lambda \in \Lambda^p : u = r(\lambda) \text{ and } v = s(\lambda)\},$$

then for each $p \in \mathbb{N}^k$, we obtain a non-negative integer valued matrix $|\Lambda^p|$ indexed by Λ^0 given by $|\Lambda^p|(u, v) = |\Lambda^p(u, v)|$ for $u, v \in \Lambda^0$. For $p, q \in \mathbb{N}^k$, we have $|\Lambda^{p+q}| = |\Lambda^p||\Lambda^q|$. Let $\mathbb{R}_+$ denote the collection of positive real numbers.

LEMMA 4.1. (cf. [**Pt2**]) *Suppose that Λ is irreducible and Λ^0 is finite. Then there exist $t \in \mathbb{R}_+^k$, $a : \Lambda^0 \to \mathbb{R}_+$ and $b : \Lambda^0 \to \mathbb{R}_+$ with $\sum_{v\in\Lambda^0} a(v)b(v) = 1$ such that for all $p \in \mathbb{N}^k$ we have*

$$\sum_{u\in\Lambda^0} a(u)|\Lambda^p|(u, v) = t^p a(v) \quad \textit{for all } v \in \Lambda^0 \tag{11}$$

$$\sum_{v\in\Lambda^0} |\Lambda^p|(u, v)b(v) = t^p b(u) \quad \textit{for all } u \in \Lambda^0. \tag{12}$$

Proof. Since Λ is irreducible, there is an integer matrix A with all positive entries which may be written as a sum of matrices of the form $|\Lambda^p|$ for various $p \in \mathbb{N}^k$. By the Perron–Frobenius Theorem (see [**Se**, Theorem 1.5] for example) there are functions $a, b : \Lambda^0 \to \mathbb{R}_+$ satisfying $\sum_{v\in\Lambda^0} a(v)b(v) = 1$ and a number $T \in \mathbb{R}_+$ such that

$$\sum_{v\in\Lambda^0} a(u)A(u, v) = Ta(v) \quad \text{for all } v \in \Lambda^0,$$

$$\sum_{u\in\Lambda^0} A(u, v)b(v) = Tb(u) \quad \text{for all } u \in \Lambda^0.$$

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For $i = 1, \ldots, k$ let e_i denote the canonical generators of $\mathbb{N}^k$, then since A commutes with $|\Lambda^{e_i}|$ for each i there exist non-negative t_i such that the same formulas hold with A replaced by $|\Lambda^{e_i}|$ and T replaced by t_i; formulae (12) and (11) now follow with $t = (t_1, \ldots, t_k)$.

It remains to show that $t_i > 0$ for each i. Let $u \in \Lambda^0$, then by Standing Assumption 2.3 $|\Lambda^e|(u, v) > 0$ for some $v \in \Lambda^0$; applying (12) we have

$$\sum_{v \in \Lambda^0} |\Lambda^e|(u, v) b(v) = t_1 \ldots t_k b(u).$$

Since the left-hand side is evidently positive, $t_1 \ldots t_k > 0$; hence $t_i > 0$ for all i as required. □

We construct the analogue of the Parry measure μ on Λ^Δ as follows (cf. [**Pt2**]).

PROPOSITION 4.2. *Suppose that Λ is irreducible and Λ^0 is finite. Then there is a shift invariant probability measure μ on Λ^Δ such that*

$$\mu(Z(\lambda, n)) = t^{-d(\lambda)} a(r(\lambda)) b(s(\lambda)),$$

for all $\lambda \in \Lambda$ and $n \in \mathbb{Z}^k$.

Proof. We must show that μ is well defined on cylinder sets. Given $\lambda \in \Lambda$ and $n \in \mathbb{Z}^k$, observe that for $m \geq 0$ we may write $Z(\lambda, n)$ as a disjoint union by expanding on the right:

$$Z(\lambda, n) = \coprod_{\substack{\nu \in \Lambda^m \\ r(\nu) = s(\lambda)}} Z(\lambda\nu, n).$$

Then we compute μ of the right-hand side (using (12))

$$\begin{aligned}
\sum_{\substack{\nu \in \Lambda^m \\ r(\nu) = s(\lambda)}} \mu(Z(\lambda\nu, n)) &= \sum_{\substack{\nu \in \Lambda^m \\ r(\nu) = s(\lambda)}} t^{-d(\lambda\nu)} a(r(\lambda)) b(s(\nu)) \\
&= t^{-d(\lambda)} a(r(\lambda)) \sum_{v \in \Lambda^0} t^{-m} \sum_{\nu \in \Lambda^m(s(\lambda), v)} b(s(\nu)) \\
&= t^{-d(\lambda)} a(r(\lambda)) \sum_{v \in \Lambda^0} t^{-m} |\Lambda^m|(s(\lambda), v) b(v) \\
&= t^{-d(\lambda)} a(r(\lambda)) b(s(\lambda)) = \mu(Z(\lambda, n)).
\end{aligned}$$

If we write $Z(\lambda, n)$ as a disjoint union by expanding on the left,

$$Z(\lambda, n) = \coprod_{\substack{\nu \in \Lambda^m \\ s(\nu) = r(\lambda)}} Z(\nu\lambda, n - m),$$

then a similar calculation (using (11)) shows that μ of each side is the same and completes the demonstration that μ is well defined. Thus μ extends to a probability measure which is invariant under the action of $\mathbb{Z}^k$. □

Our next task is to decompose μ locally into measures which give rise to Haar systems for G_{s} and G_{u}.

Fix $x \in \Lambda^\Delta$; then for $\lambda \in \Lambda$ with $s(\lambda) = x(0)$, we define

$$Z^-(\lambda, x) = E_x \cap Z(\lambda, -d(\lambda)).$$

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Likewise, for $\lambda \in \Lambda$ with $r(\lambda) = x(0)$, we define

$$Z^+(\lambda, x) = F_x \cap Z(\lambda, 0).$$

PROPOSITION 4.3. (cf. [**Pt2**]) *Suppose that Λ is irreducible and Λ^0 is finite. Then for each $x \in \Lambda^\Delta$ there is a measure μ_s^x on E_x and a measure μ_u^x on F_x, such that*

$$\mu_s^x(Z^-(\lambda, x)) = t^{-d(\lambda)}a(r(\lambda)) \quad \textit{and} \quad \mu_u^x(Z^+(\lambda, x)) = t^{-d(\lambda)}b(s(\lambda))$$

for all $\lambda \in \Lambda$. The restriction of μ to $Z(x(0), 0)$ is $\mu_s^x \times \mu_u^x$ after identifying $E_x \times F_x$ with $Z(x(0), 0)$ as in Proposition 3.3. Moreover for $p \in \mathbb{N}^k$ we have

$$\mu_s^x = t^p \mu_s^{\sigma^p x} \circ \sigma^p \quad \textit{and} \quad \mu_u^x = t^p \mu_u^{\sigma^{-p} x} \circ \sigma^{-p} \tag{13}$$

on E_x and F_x, respectively.

Proof. The first assertion is clear. For the next part it suffices to consider cylinder sets of the form $Z(\lambda, n)$ where $\lambda = \lambda^-\lambda^+ \in \Lambda$ with $r(\lambda^+) = x(0)$ and $n = -d(\lambda^-)$. After identifying $Z^+(\lambda^+, x) \times Z^-(\lambda^-, x)$ with $Z(\lambda, -d(\lambda^-))$ (as in Proposition 3.3) we have

$$\begin{aligned}
\mu_s^x \times \mu_u^x(Z(\lambda, -d(\lambda^-))) &= \mu_s^x(Z^-(\lambda^-, x))\mu_u^x(Z^+(\lambda^+, x)) \\
&= t^{-d(\lambda^-)}a(r(\lambda^-))t^{-d(\lambda^+)}b(s(\lambda^+)) \\
&= t^{-d(\lambda)}a(r(\lambda))b(s(\lambda)) \\
&= \mu(Z(\lambda, -d(\lambda^-))),
\end{aligned}$$

hence the restriction of μ to $Z(x(0), 0)$ is $\mu_s^x \times \mu_u^x$ as required. From the definitions it is straightforward to verify that for $p \in \mathbb{N}^k$

$$\sigma^p Z^-(\lambda, x) = Z^-(\lambda x(0, p), \sigma^p x) \quad \text{and} \quad \sigma^{-p} Z^+(\lambda, x) = Z^+(x(-p, 0)\lambda, \sigma^{-p} x).$$

Hence,

$$\begin{aligned}
\mu_s^{\sigma^p x}(\sigma^p Z^-(\lambda, x)) &= t^{-p}\mu_s^x(Z^-(\lambda, x)), \\
\mu_u^{\sigma^{-p} x}(\sigma^{-p} Z^+(\lambda, x)) &= t^{-p}\mu_u^x(Z^+(\lambda, x)),
\end{aligned}$$

equations (13) then follow on E_x and F_x respectively. □

Note that for $x \in \Lambda^\Delta$ we have that $E_x = \{y : \pi(y) = \pi(x)\}$; evidently, if $\pi(y) = \pi(x)$ then $E_y = E_x$ and $\mu_s^y = \mu_s^x$. For $z = \pi(x)$, let $\mu_{s,0}^z$ denote the extension of μ_s^x to Λ^Δ ($\mu_{u,0}^z$ is defined similarly). Observe that $\mu_{s,0} = \{\mu_{s,0}^z : z \in \Lambda^\Omega\}$ is a full π-system. The continuity of the system follows from the fact that

$$\mu_{s,0}(\chi_{Z(\lambda,n)})(z) = t^{-d(\lambda)}a(r(\lambda))$$

is locally constant. By Propositions 2.7 and 3.4 we obtain a Haar system

$$\tilde{\mu}_{s,0} = \{\tilde{\mu}_{s,0}^x : x \in \Lambda^\Delta\}$$

for $G_{s,0}$. Recall from Proposition 3.4 that for $p \geq 0$, we have $G_{s,p} = \Lambda^\Delta \star_{\pi \circ \sigma^p} \Lambda^\Delta$.

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PROPOSITION 4.4. *Let Λ be an irreducible k-graph with Λ^0 finite. For $p \geq 0$ and $x, y \in \Lambda^\Delta$ if $\pi \circ \sigma^p x = \pi \circ \sigma^p y$, then*

$$\sigma^{-p} E_{\sigma^p x} = \sigma^{-p} E_{\sigma^p y} \quad \textit{and} \quad \mu_{\mathrm{s}}^{\sigma^p x} \circ \sigma^p = \mu_{\mathrm{s}}^{\sigma^p y} \circ \sigma^p. \tag{14}$$

For $z = \pi \circ \sigma^p(x)$, let $\mu_{\mathrm{s},p}^z$ denote the extension of the measure $t^p \mu_{\mathrm{s}}^{\sigma^p x} \circ \sigma^p$ from $\sigma^{-p} E_{\sigma^p x}$ to Λ^Δ. Then $\mu_{\mathrm{s},p} = \{\mu_{\mathrm{s},p}^z : z \in \Lambda^\Omega\}$ is a full $\pi \circ \sigma^p$-system and we obtain a Haar system $\tilde{\mu}_{\mathrm{s},p} = \{\tilde{\mu}_{\mathrm{s},p}^x : x \in \Lambda^\Delta\}$ for $G_{\mathrm{s},p}$. If $0 \leq p \leq q$, the restriction of $\tilde{\mu}_{\mathrm{s},q}^x$ to $G_{\mathrm{s},p}$ agrees with $\tilde{\mu}_{\mathrm{s},p}^x$ for all $x \in \Lambda^\Delta$. We thereby obtain a Haar system for G_{s}, $\tilde{\mu}_{\mathrm{s}} = \{\tilde{\mu}_{\mathrm{s}}^x : x \in \Lambda^\Delta\}$. A Haar system for G_{u} is obtained in a similar way.

Proof. Note that for $p \geq 0$ and $x, y \in \Lambda^\Delta$, $y \in \sigma^{-p} E_{\sigma^p x}$ if and only if $\pi \circ \sigma^p x = \pi \circ \sigma^p y$; hence, formulae (14) hold. By arguments similar to those given immediately prior to the statement of this proposition, $\mu_{\mathrm{s},p}$ is a full $\pi \circ \sigma^p$-system and $\tilde{\mu}_{\mathrm{s},p}$ is a Haar system for $G_{\mathrm{s},p}$. The compatibility of the Haar systems now follows from a short calculation involving (13). The Haar systems $\mu_{\mathrm{s},p}$ for $p \geq 0$ may therefore be patched together to give a Haar system for G_{s}: $\tilde{\mu}_{\mathrm{s}} = \{\tilde{\mu}_{\mathrm{s}}^x : x \in \Lambda^\Delta\}$. □

Now we may define the stable and unstable algebras associated with an irreducible k-graph Λ with finitely many vertices

$$S = S(\Lambda) = C^*(G_{\mathrm{s}}(\Lambda)) \quad \text{and} \quad U = U(\Lambda) = C^*(G_{\mathrm{u}}(\Lambda)).$$

For $n \in \mathbb{Z}^k$, the map $\sigma^n \times \sigma^n$ yields an automorphism of the stable and unstable equivalence relations but it rescales the Haar systems by (13); indeed

$$\tilde{\mu}_{\mathrm{s}}^x \circ (\sigma^{-n} \times \sigma^{-n}) = t^n \tilde{\mu}_{\mathrm{s}}^{\sigma^n x} \quad \text{and} \quad \tilde{\mu}_{\mathrm{u}}^x \circ (\sigma^{-n} \times \sigma^{-n}) = t^{-n} \tilde{\mu}_{\mathrm{u}}^{\sigma^n x}. \tag{15}$$

This induces actions $\beta_{\mathrm{s}}, \beta_{\mathrm{u}}$ of $\mathbb{Z}^k$ on both S and U given for $n \in \mathbb{Z}^k$ by

$$\beta_{\mathrm{s}}^n(f)(x, y) = t^n f(\sigma^{-n}(x), \sigma^{-n}(y)) \quad \text{where } f \in C_{\mathrm{c}}(G_{\mathrm{s}}),$$
$$\beta_{\mathrm{u}}^n(f)(x, y) = t^{-n} f(\sigma^{-n}(x), \sigma^{-n}(y)) \quad \text{where } f \in C_{\mathrm{c}}(G_{\mathrm{u}}),$$

and extending by continuity to the completions.

The measure μ on Λ^Δ gives rise to a densely-defined trace on S as follows.

PROPOSITION 4.5. *Let Λ be an irreducible k-graph with Λ^0 finite, μ the Parry measure and G_{s} the stable groupoid. For $f \in C_{\mathrm{c}}(G_{\mathrm{s}})$, set*

$$\tau_{\mathrm{s}}(f) = \int_{\Lambda^\Delta} f(x, x)\, d\mu(x).$$

Then τ_{s} is a densely-defined trace on $C^*(G_{\mathrm{s}})$. A densely-defined trace τ_{u} on $C^*(G_{\mathrm{u}})$ is defined similarly. Moreover, for $n \in \mathbb{Z}^k$ we have

$$\tau_{\mathrm{s}} \circ \beta_{\mathrm{s}}^n = t^n \tau_{\mathrm{s}} \quad \textit{and} \quad \tau_{\mathrm{s}} \circ \beta_{\mathrm{u}}^n = t^{-n} \tau_{\mathrm{u}}. \tag{16}$$

Proof. Formulae (16) follow from (15). We show that τ_{s} is a densely-defined trace; the case of τ_{u} is similar. It suffices to show that τ_{s} satisfies the trace property (τ_{s} is clearly densely-defined, linear and positive). For $f, g \in C_{\mathrm{c}}(G_{\mathrm{s}})$, there is $p \geq 0$ such that the supports of f and g are contained in $G_{s,p}$ (by Proposition 3.4, $\{G_{s,p} : p \geq 0\}$ forms an open cover of G_s). Since μ decomposes locally as a product measure as in Proposition 4.3 with $t^p \mu_s^{\sigma^p x} \circ \sigma^p$ in place of μ_s^x, there is a measure $\nu_{s,p}$ on Λ^Ω such that

$$\int_{\Lambda^\Delta} h(x)\, d\mu(x) = \int_{\Lambda^\Omega} \mu_{s,p}(h)\, d\nu_{s,p}$$

for all $h \in C(\Lambda^\Delta)$. It follows that

$$\begin{aligned}\tau_s(fg) &= \int_{\Lambda^\Delta} (fg)(x,x)\, d\mu(x) = \int_{\Lambda^\Omega} \int (fg)(y,y)\, d\mu_{s,p}^z(y)\, d\nu_{s,p}(z) \\ &= \int_{\Lambda^\Omega} \tau_{\mu_{s,p}}(fg)\, d\nu_{s,p} = \int_{\Lambda^\Omega} \tau_{\mu_{s,p}}(gf)\, d\nu_{s,p} = \tau_s(gf),\end{aligned}$$

where the penultimate equality follows from Proposition 2.7. □

Let Λ be an irreducible k-graph with finitely many vertices. The Ruelle algebras associated with Λ are defined to be the corresponding crossed products (cf. [**PtS, Pt2**])

$$R_s = R_s(\Lambda) = S(\Lambda) \times_{\beta_s} \mathbb{Z}^k \quad \text{and} \quad R_u = R_u(\Lambda) = U(\Lambda) \times_{\beta_u} \mathbb{Z}^k.$$

We express the Ruelle algebras as C^*-algebras of the semidirect product groupoids $G_s \times \mathbb{Z}^k$ and $G_u \times \mathbb{Z}^k$. The unit space is identified with Λ^Δ via the map $x \mapsto ((x,x),0)$. The structure maps are given by

$$r((x,y),n) = x, \quad s((x,y),n) = \sigma^n y, \quad ((x,y),n)((\sigma^n y, \sigma^n z), m) = ((x,z), n+m).$$

The structure maps of $G_u \times \mathbb{Z}^k$ are defined similarly.

LEMMA 4.6. *If Λ is an irreducible k-graph with Λ^0 finite, then both $G_s \times \mathbb{Z}^k$ and $G_u \times \mathbb{Z}^k$ have Haar systems. Moreover, we have $R_s \cong C^*(G_s \times \mathbb{Z}^k)$ and $R_u \cong C^*(G_u \times \mathbb{Z}^k)$.*

Proof. Let ϑ be the measure on $\mathbb{Z}^k$ given by $\vartheta(\{n\}) = t^{-n}$; then a direct computation using (15) shows that $\{\tilde{\mu}_s^x \times \vartheta : x \in \Lambda^\Delta\}$ is a Haar system for $G_s \times \mathbb{Z}^k$. □

Remark 4.7. Let Λ be an irreducible k-graph with Λ^0 finite, then by Remark 3.5 we have $G_u(\Lambda) = G_s(\Lambda^{op})$ and $G_s(\Lambda) = G_u(\Lambda^{op})$. Note that Λ^{op} is also irreducible and $(\Lambda^{op})^0 = \Lambda^0$ is finite. We have $U(\Lambda) = S(\Lambda^{op})$ and $S(\Lambda) = U(\Lambda^{op})$, similarly $R_u(\Lambda) = R_s(\Lambda^{op})$ and $R_s(\Lambda) = R_u(\Lambda^{op})$. Henceforth, we focus our attention on the stable case.

Remark 4.8. The asymptotic C^*-algebra may also be defined,

$$A = A(\Lambda) = C^*(G_a(\Lambda)).$$

Note that since G_a is r-discrete, it has a Haar system consisting of counting measures. The asymptotic Ruelle algebra is defined as the crossed product

$$R_a = R_a(\Lambda) = A(\Lambda) \times_{\beta_a} \mathbb{Z}^k = C^*(G_a(\Lambda) \times \mathbb{Z}^k).$$

Suppose that Λ is irreducible and Λ^0 is finite. With notation as in Remark 3.6,

$$\mu(Z(\lambda, -m)) = t^{-2m} a(r(\lambda)) b(s(\lambda)) = t^{-2m} a(r(\nu)) b(s(\nu)) = \mu(Z(\nu, -m)),$$

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hence μ is invariant under G_a. Thus we may define a unital trace on A by

$$\tau_a(f) = \int f(x, x)\, d\mu(x)$$

for $f \in C_c(G_a)$. The $\mathbb{Z}^k$ action σ on Λ^Δ induces an action $\sigma \times \sigma$ on G_a and, hence, we get an action $\beta_a : \mathbb{Z}^k \to \operatorname{Aut} A$. Since τ_a is invariant under β_a, we also obtain a trace on R_a.

5. *Morita equivalence*

In this section we prove our main results. If Λ is irreducible and Λ^0 is finite, we show that the stable algebra S is strongly Morita equivalent to $C^*(\Lambda)^\alpha$ and that R_s is strongly Morita equivalent to $C^*(\Lambda)$. Hence, if Λ satisfies the aperiodicity condition, then R_s is a stable Kirchberg algebra.

We begin by stating a groupoid equivalence result that will be useful in both cases. This result is no doubt well known to the experts but, as we are unable to find an explicit reference, we provide a proof.

Let Γ be a locally compact Hausdorff groupoid. Given a right principal Γ-space Y, one may construct the imprimitivity groupoid $Y \star_\Gamma Y^{\text{op}}$ where Y^{op} is the corresponding left principal Γ-space. By [**MW2**, Theorem 3.5] (see also [**M**, Theorem 5.31]), Y implements an equivalence between the imprimitivity groupoid and Γ in the sense of [**MRW**, Definition 2.1].

Let X be a locally compact Hausdorff space and let $\psi : X \to \Gamma^0$ be a continuous open surjection. Set

$$Z = X \star \Gamma = \{(x, \gamma) : x \in X, \gamma \in \Gamma, \psi(x) = r(\gamma)\}. \tag{17}$$

We define a right action of Γ on Z as follows: $s : Z \to \Gamma^0$ is given by $s(x, \gamma) = s(\gamma)$ and the map $Z \star \Gamma \to Z$ by $((x, \gamma_1), \gamma_2) = (x, \gamma_1\gamma_2)$. There is a corresponding left action of Γ on $Z^{\text{op}} = \Gamma \star X$.

LEMMA 5.1. *With the above structure maps, Z is a right principal Γ-space. Moreover the imprimitivity groupoid $Z \star_\Gamma Z^{\text{op}}$ is isomorphic to*

$$\Gamma^\psi := \{(x, \gamma, y) : x, y \in X, \gamma \in \Gamma, \psi(x) = r(\gamma), \psi(y) = s(\gamma)\},$$

equipped with the relative topology. Therefore Z implements an equivalence between the groupoids Γ and Γ^ψ.

Proof. To show that Z is a right principal Γ-space, we must show that the action is free and proper. The action is clearly free (because the action of a groupoid on itself is free). It suffices to show that the map $Z \star \Gamma \to Z \times Z$, given by

$$((x, \gamma_1), \gamma_2) \mapsto ((x, \gamma_1\gamma_2), (x, \gamma_1)),$$

is a homeomorphism onto a closed set (see [**MW2**, Lemma 2.2]). This follows from a similar fact for the right action of a groupoid on itself. By [**MW2**, Theorem 3.5], Z is a groupoid equivalence between the imprimitivity groupoid $Z \star_\Gamma Z^{\text{op}}$ and Γ. The isomorphism from $Z \star_\Gamma Z^{\text{op}}$ to Γ^ψ is given by the map

$$((x, \gamma_1), (\gamma_2, y)) \mapsto (x, \gamma_1\gamma_2, y).$$

The result now follows from this identification. □

The construction of Γ^{ψ} above appears in [**Ku1**, Proposition 5.7] (though in a more specialized setting). Recall that the restriction map $\pi : \Lambda^{\Delta} \to \Lambda^{\Omega}$ is a continuous open surjection.

LEMMA 5.2. *Let Λ be a k-graph and G_s be the stable groupoid associated to Λ. Then the map $(x, y) \mapsto (x, (\pi(x), 0, \pi(y)), y)$ gives an isomorphism $G_s \cong (\Gamma_\Lambda)^{\pi}$.*

Proof. Recall that $(x, y) \in G_{s,m}$ precisely when $\pi(\sigma^m x) = \pi(\sigma^m y)$. For $m \geq 0$ the given map restricts to a homeomorphism from $G_{s,m}$ to $(\Gamma_{\Lambda,m})^{\pi}$ where $\Gamma_{\Lambda,m}$ is the subgroupoid of Γ_Λ formed by those $(x, 0, y)$ where $\sigma^m(x) = \sigma^m(y)$. Since the topology on Γ_Λ is equivalent to the inductive limit topology from these subgroupoids, the given map is a homeomorphism. It is routine to check that the map is a groupoid morphism. □

The groupoid equivalence (in the sense of [**MRW**]) between G_s and Γ_Λ now follows.

THEOREM 5.3. *The space $Z = \Lambda^{\Delta} \star \Gamma_\Lambda$ is a (G_s, Γ_Λ)-equivalence. In particular, G_s is amenable in the sense of [**AR**]. Moreover, if Λ is irreducible and Λ^0 is finite, then the stable algebra S is strongly Morita equivalent to $C^*(\Lambda)^{\alpha}$ and, therefore, is an AF-algebra.*

Proof. The first part follows from Lemmas 5.1 and 5.2. That G_s is amenable now follows from [**AR**, Theorem 2.2.17]. If Λ is irreducible and Λ^0 is finite, then by Proposition 4.4, G_s has a Haar system (Γ_Λ has a Haar system consisting of counting measures) so that by [**MRW**, Theorem 2.8], S is strongly Morita equivalent to $C^*(\Gamma_\Lambda) = C^*(\Lambda)^{\alpha}$. The final assertion then follows from [**KP**, Lemma 3.2]. □

We could have deduced that $S = C^*(G_s)$ is AF more directly. It follows from the fact that

$$C^*(G_s) = \lim_{m\to\infty} C^*(G_{s,m})$$

and that $C^*(G_{s,m})$ is strongly Morita equivalent to the Abelian AF-algebra $C(\Lambda^{\Omega})$ for each m (see Propositions 2.7 and 3.4). If Λ is primitive then we can say more.

COROLLARY 5.4. *Let Λ be a primitive k-graph, then $C^*(\Lambda)^{\alpha}$ is a simple AF-algebra and, hence, so is S.*

Proof. Suppose that Λ is primitive, then by Standing Assumption 2.3, Λ^0 is finite. Moreover, there is an $n > 0$ (i.e. all coordinates are positive) such that for every $u, v \in \Lambda^0$ there is $\lambda \in \Lambda^n$ with $u = s(\lambda)$ and $v = r(\lambda)$. It follows that all the entries of the matrix $|\Lambda^n|$ are positive. Since the sequence $\{jn : j \in \mathbb{N}\}$ is cofinal in $\mathbb{N}^k$, we have that $C^*(\Lambda)^{\alpha} = \lim_{j\to\infty} \mathcal{F}_{jn}$. The multiplicity matrix of the inclusions may be identified with $|\Lambda^n|$ and the result now follows from [**B**, Corollary 3.5]. □

Analogous assertions hold for U when Λ is replaced by Λ^{op} (see Remark 4.7).

LEMMA 5.5. *Let Λ be a k-graph, G_s be the associated stable groupoid and $G_s \times \mathbb{Z}^k$ be the semidirect product groupoid (see Lemma 4.6). Then the map*

$$\varphi : ((x, y), n) \mapsto (x, (\pi(x), n, \pi(\sigma^n y)), \sigma^n y)$$

gives an isomorphism $G_s \times \mathbb{Z}^k \cong (\mathcal{G}_\Lambda)^{\pi}$.

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Proof. That $(\pi(x), n, \pi(\sigma^n y)) \in \mathcal{G}_\Lambda$ follows from (9), so φ is well defined and evidently injective. Given $(x, (\pi(x), n, \pi(z)), z) \in (\mathcal{G}_\Lambda)^\pi$, we have

$$(x, (\pi(x), n, \pi(z)), z) = \varphi((x, \sigma^{-n} z), n),$$

hence φ is surjective. Recall that $G_s \times \mathbb{Z}^k$ is given the product topology and observe that the restriction of φ to $G_s \times \{0\}$ agrees with the homeomorphism defined in Lemma 5.2. Similarly, the restriction to $G_s \times \{n\}$ is a homeomorphism onto the set

$$\{(x, (\pi(x), n, \pi(y)), y) : x, y \in \Lambda^\Delta, \sigma^\ell \pi(x) = \sigma^m \pi(y), n = \ell - m\}.$$

The reader is invited to check that the map is a groupoid morphism. □

Recall that a k-graph is said to satisfy the *aperiodicity condition* if for every vertex v there is $x \in \Lambda^\Omega$ with $x(0) = v$ which is not eventually periodic (see [**KP**, Definition 4.1]). Let $\mathcal{N}$ denote the bootstrap class of C^*-algebras to which the UCT applies (see [**RSc**]).

THEOREM 5.6. *The space $Z = \Lambda^\Delta \star \mathcal{G}_\Lambda$ is a $(G_s \times \mathbb{Z}^k, \mathcal{G}_\Lambda)$-equivalence. Therefore, if Λ is irreducible and Λ^0 is finite, then the stable Ruelle algebra R_s is strongly Morita equivalent to $C^*(\Lambda)$ and, hence, is nuclear and lies in the bootstrap class $\mathcal{N}$. If, in addition, Λ satisfies the aperiodicity condition, then R_s is simple, stable and purely infinite. Hence, the isomorphism class of R_s is completely determined by $K_*(R_s) = K_*(C^*(\Lambda))$.*

Proof. The first part follows from Lemmas 5.1 and 5.5. If Λ is irreducible and Λ^0 is finite, then by Lemma 4.6, $G_s \times \mathbb{Z}^k$ has a Haar system ($\mathcal{G}_\Lambda$ has a Haar system consisting of counting measures) so that by [**MRW**, Theorem 2.8], R_s is strongly Morita equivalent to $C^*(\mathcal{G}_\Lambda) = C^*(\Lambda)$. By [**KP**, Theorem 5.5], R_s is nuclear and lies in the bootstrap class $\mathcal{N}$ (since strong Morita equivalence preserves these properties). If Λ is irreducible then it is clearly cofinal and if, in addition, Λ satisfies the aperiodicity condition, it follows from [**KP**, Proposition 4.8] that $C^*(\Lambda)$ is simple. For every vertex $v \in \Lambda$, there is a morphism $\lambda \in \Lambda$ with $d(\lambda) \neq 0$ such that $r(\lambda) = s(\lambda) = v$ and so $C^*(\Lambda)$ is purely infinite by [**KP**, Proposition 4.9]. By Zhang's dichotomy, a simple purely infinite C^*-algebra is either unital or stable (see [**Z**, Theorem 1.2]); since R_s is not unital it must be stable. The Kirchberg–Phillips Theorem applies and the isomorphism class of R_s is completely determined by $K_*(R_s)$ (see [**Ki**, Theorem C] and [**Ph**, Corollary 4.2.2]). □

An analogous assertion holds for R_u when Λ is replaced by Λ^{op}. The aperiodicity condition is necessary in the statement of the above theorem. There is an irreducible 2-graph Λ with one vertex which is not aperiodic—every path has period $(1, -1)$. Furthermore, $C^*(\Lambda) \cong \mathcal{O}_2 \otimes C(\mathbb{T})$ is neither simple nor purely infinite (see [**KP**, Example 6.1]).

The restriction of Theorem 5.6 to the case $k = 1$ is certainly well known, but we have been unable to find a reference.

COROLLARY 5.7. *Let $A \in M_n(\mathbb{N})$ be irreducible and R_s be the stable Ruelle algebra of the associated Markov shift. Then R_s is strongly Morita equivalent to $\mathcal{O}_A$.*

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Remark 5.8. Suppose that Λ is an irreducible k-graph with Λ^0 finite. Then the $2k$-graph $\Lambda \times \Lambda^{\mathrm{op}}$ is irreducible and $(\Lambda \times \Lambda^{\mathrm{op}})^0 = \Lambda^0 \times (\Lambda^{\mathrm{op}})^0$ is finite. We have $(\Lambda \times \Lambda^{\mathrm{op}})^\Delta = \Lambda^\Delta \times (\Lambda^{\mathrm{op}})^\Delta$ and

$$G_{\mathrm{s}}(\Lambda \times \Lambda^{\mathrm{op}}) = G_{\mathrm{s}}(\Lambda) \times G_{\mathrm{s}}(\Lambda^{\mathrm{op}}) \cong G_{\mathrm{s}}(\Lambda) \times G_{\mathrm{u}}(\Lambda),$$

hence

$$S(\Lambda \times \Lambda^{\mathrm{op}}) \cong S(\Lambda) \otimes U(\Lambda).$$

Moreover, $A(\Lambda)$ is strongly Morita equivalent to $S(\Lambda) \otimes U(\Lambda)$ as in [**Pt1**, Theorem 3.1]. The 'same' argument applies: define a map $\phi : \Lambda^\Delta \to \Lambda^\Delta \times (\Lambda^{\mathrm{op}})^\Delta$ by $x \mapsto (x, x^{\mathrm{op}})$, then $N = \phi(\Lambda^\Delta)$ is an abstract transversal of the groupoid $G_{\mathrm{s}}(\Lambda \times \Lambda^{\mathrm{op}})$ in the sense of [**MRW**, Example 2.7]. Furthermore, $G_a(\Lambda)$ is isomorphic to the reduction $G_{\mathrm{s}}(\Lambda \times \Lambda^{\mathrm{op}})_N^N$ (for $x \sim_a y$ if and only if $x \sim_{\mathrm{s}} y$ and $x^{\mathrm{op}} \sim_{\mathrm{s}} y^{\mathrm{op}}$; that is, $(x, x^{\mathrm{op}}) \sim_{\mathrm{s}} (y, y^{\mathrm{op}})$). It follows that $A(\Lambda)$ is an AF-algebra and if Λ is primitive then $A(\Lambda)$ is simple. However, $R_a(\Lambda)$ is not purely infinite since it has a trace (see Remark 4.8).

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