CONSTRUCTION OF SPECIAL EDGE-CHROMATIC GRAPHS

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1. Introduction. The configuration formed by N points in general position in space, together with the $\binom{N}{2}$ lines joining them in pairs will be called an N-clique. The N-clique is coloured by assigning to each edge exactly one colour from a set of t possible colours. A theorem due to Ramsey [4] ensures the existence of a least integer $M(q_1, q_2, \ldots, q_t)$ such that if $N \geq M$, any such colouring of the N-clique must contain either a q_1-clique entirely of colour 1, or a q_2-clique of colour 2, ..., or a q_t-clique of colour t. Another proof of Ramsey's theorem is given by Ryser [5].

Definition 1: A $(q_1, q_2, \ldots, q_t)$-colouring of the N-clique is an assignment of colours to the edges of the N-clique as above in which there are no q_i-cliques of colour i $(i = 1, 2, \ldots, t)$.

Ramsey's theorem then states that there exists a least integer $M(q_1, q_2, \ldots, q_t)$ such that for $N \geq M$ there is no $(q_1, q_2, \ldots, q_t)$-colouring of the N-clique. The evaluation of these least integers M is a difficult combinatorial problem. The results in section 2 establish upper bounds for the M's, but except in the first few cases these upper bounds appear to be loose. Furthermore, to establish that M is the least such integer, a $(q_1, q_2, \ldots, q_t)$-colouring of the (M-1)-clique must be constructed.

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Greenwood and Gleason [3] have evaluated M(3, 3), M(3,4), M(3,5), M(4,4) and M(3,3,3). For these numbers the upper bounds of section 2 are tight, and colourings of the (M-1)-cliques were constructed using finite field residue theory. However, this method does not readily extend to higher numbers because, for this construction to give a colouring of the N-clique there must be a field on N elements in which (-1) is a residue of the proper type.

In this paper, a new method for constructing $(q_1, q_2, \ldots, q_t)$-colourings is described and illustrated. By means of it are obtained a (3,7)-colouring of the 21-clique, a (4,5)-colouring of the 24-clique, and a (3,8)-colouring of the 26-clique. These colourings are not obtainable by the method of Greenwood and Gleason. Thus lower bounds of 22 for M(3,7), 25 for M(4,5), and 27 for M(3,8) are established. The method of "regular colourings" is easily programmed for a high-speed electronic computer, so that regular colourings may be found even when the number of possibilities to be considered becomes very large.

2. Preliminary Results. Lemmas 1, 2 and 5 are given by Greenwood and Gleason [3]. Lemma 5 was originally proved in a paper by Erdős and Szekeres [2]. The result has been attributed to Szekeres by Erdős [1]. Lemma 3 is implicit in [3], and a weaker form of Lemma 4 is stated there.

LEMMA 1. $M(q_1, q_2, \ldots, q_t)$ is invariant under a permutation of the q's.

LEMMA 2. Since a 2-clique is a line, $M(2, q_2, \ldots, q_t) = M(q_2, \ldots, q_t)$, and in particular $M(2, q_2) = M(q_2) = q_2$.

LEMMA 3. In a $(q_1, q_2, \ldots, q_t)$-colouring of the N-clique, there are at most $M(q_1-1, q_2, \ldots, q_t) - 1$ lines of colour 1 from any point.

Proof. Suppose there exists a point A which is joined by colour 1 to $M(q_1-1, q_2, \ldots, q_t)$ or more points. These points must contain either a (q_1-1)-clique of colour 1, or a

q_2-clique of colour 2, ..., or a q_t-clique of colour t. Should they contain a (q_1-1)-clique of colour 1, it will be joined to A entirely by edges of colour 1, giving a q_1-clique of colour 1. Hence there is a q_i-clique of colour i for some $i = 1, 2, \ldots,$ or t, and no $(q_1, q_2, \ldots, q_t)$-colouring exists.

LEMMA 4. $M(q_1, q_2, \ldots, q_t) \leq M(q_1-1, q_2, \ldots, q_t) + M(q_1, q_2-1, q_3, \ldots, q_t) + M(q_1, q_2, \ldots, q_{t-1}, q_t-1) - t + 2$.

Proof. By lemma 3, for a $(q_1, q_2, \ldots, q_t)$-colouring there are at most $[M(q_1-1, q_2, \ldots, q_t) - 1] + [M(q_1, q_2-1, q_3, \ldots, q_t) - 1] + \ldots + [M(q_1, q_2, \ldots, q_t-1) - 1] = N$ lines from any point; so the number of points for a $(q_1, q_2, \ldots, q_t)$-colouring is at most $N + 1$. If there are $N + 2$ or more points, there is no $(q_1, q_2, \ldots, q_t)$-colouring; so $M(q_1, q_2, \ldots, q_t) \leq N + 2$.

LEMMA 5. $M(q_1, q_2) \leq M(q_1-1, q_2) + M(q_1, q_2-1)$ with strict inequality if both terms on the right hand side are even.

3. Regular Colourings. Space the N vertices of the N-clique equidistantly about the circumference of a circle so that all edges of the graph become chords of the circle. An s-line, or edge of length s, is an edge of the graph which cuts off a minor arc of the circle containing $s - 1$ interior vertices $(s=1, 2, \ldots, [\frac{N}{2}])$.

In attempting to construct $(q_1, q_2, \ldots, q_t)$-colourings of the N-clique, it is reasonable to look first for colourings of a very regular type. Such colourings should be fairly easy to find, and these may permit evaluation of some M's without a detailed consideration of all possible configurations. Accordingly a regular colouring is defined as follows:

Definition 2: A regular $(q_1, q_2, \ldots, q_t)$-colouring is a

$(q_1, q_2, \ldots, q_t)$-colouring in which, for each s, all edges of length s have the same colour.

It follows by Ramsey's theorem that there exists a greatest integer $L(q_1, q_2, \ldots, q_t)$ such that there is a regular $(q_1, q_2, \ldots, q_t)$-colouring of the $L(q_1, q_2, \ldots, q_t)$-clique, but there is no regular $(q_1, q_2, \ldots, q_t)$-colouring of the N-clique for any $N > L$. Lemmas 6, 7 and 8 follow immediately from the definition of L.

LEMMA 6. $L(q_1, q_2, \ldots, q_t)$ is invariant under a permutation of the q's.

LEMMA 7. $L(2, q_2, \ldots, q_t) = L(q_2, \ldots, q_t)$ and $L(2, q_2) = L(q_2) = q_2 - 1$.

LEMMA 8. $L(q_1, q_2, \ldots, q_t) \leq M(q_1, q_2, \ldots, q_t) - 1$.

If there exists a $(q_1, q_2, \ldots, q_t)$-colouring of the N-clique, a $(q_1, q_2, \ldots, q_t)$-colouring of the (N-1)-clique can always be found by removing any point (and all lines from it) from the colouring of the N-clique. It is interesting that there may be a regular $(q_1, q_2, \ldots, q_t)$-colouring of the N-clique, yet none for the (N-1)-clique. For example, there is a regular (4, 4)-colouring of the 17-clique, but there is no regular (4, 4)-colouring of the 16-clique.

All colourings constructed by the method of Greenwood and Gleason using a field on p elements (p prime) will necessarily be regular colourings. In this construction, vertices are named by field elements, and a line joining two vertices is coloured red if the two vertices have a numerical difference which is an nth power residue of p. In order that the order of differencing shall not matter, -1 must be an nth power residue of p. This insures that if s is an nth power residue, so is p-s, and the resulting colouring will be regular. This is not necessarily so for colourings constructed using

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fields on p^n elements ($n \neq 1$). For example, the (3, 3, 3)-colouring obtained in [3] using the field on 2^4 elements is not a regular colouring.

Table 1 lists all $L(q_1, q_2)$ values presently known. Some of these are found in sections 4 and 5. Those marked with an asterisk will be discussed in future papers, along with the results $L(3, 3, 3) = 14$ and $L(3, 3, 4) = 29$. The author is presently investigating $L(4, 6)$, $L(5, 5)$ and $L(3, 3, 3, 3)$ using the University of Waterloo's IBM 7040 computer.

$q_2 \backslash q_1$	2	3	4	5	6	7	8	9
2	1	2	3	4	5	6	7	8
3	2	5	8	13	16	21	26	35*
4	3	8	17	24	$\geq$29*			
5	4	13	24					
6	5	16	$\geq$29*					
7	6	21						
8	7	26						
9	8	35*						

Table 1

Known values of $L(q_1, q_2)$

Table 2 lists all $M(q_1, q_2)$ values presently known, together with upper bounds for some unknown M's. It is also known [3] that $M(3, 3, 3) = 17$. Results marked with one or two asterisks are not proved in this paper, and will be considered in future papers. Those results marked with two asterisks are used in the proofs of theorems 6, 7 and 8.

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$q_2 \backslash q_1$	2	3	4	5	6	7	8	9
2	2	3	4	5	6	7	8	9
3	3	6	9	14	18**	$\leq$24**	$\leq$31**	$\leq$38*
4	4	9	18	$\leq$30**	$\leq$47*			
5	5	14	$\leq$30**	$\leq$59*				
6	6	18**	$\leq$47*					
7	7	$\leq$24**						
8	8	$\leq$31**						
9	9	$\leq$38*						

Table 2

Known values of $M(q_1, q_2)$

4. Evaluation of L(3, 3), L(3, 4), L(3, 5), L(3, 6) and L(4, 4). The method of proof is illustrated for L(3, 5) in theorem 3. All other proofs in this section and in section 5 are essentially the same, and so are omitted.

THEOREM 1. L(3, 3) = 5.

Proof. By lemma 5, $M(3, 3) \leq M(2, 3) + M(3, 2) = 6$; $L(3, 3) \leq 5$ by lemma 8. A regular (3, 3)-colouring of the 5-clique may be constructed by making 1-lines red and 2-lines blue. Hence L(3, 3) = 5, and M(3, 3) = 6.

THEOREM 2. L(3, 4) = 8.

Proof. By lemma 5, $M(3, 4) < M(2, 4) + M(3, 3) = 10$; $L(3, 4) \leq 8$ by lemma 8. Colour 1-lines and 4-lines red, 2-lines and 3-lines blue to get a regular (3, 4)-colouring of the 8-clique. Thus L(3, 4) = 8 and M(3, 4) = 9.

THEOREM 3. L(3, 5) = 13.

Proof. By lemmas 5 and 8, $M(3, 5) \leq 14$ and $L(3, 5) \leq 13$. By lemma 3, for a (3, 5)-colouring there are at most $M(2, 5) - 1 = 4$ red lines and $M(3, 4) - 1 = 8$ blue lines from any point. Hence, for a (3, 5)-colouring of the 13-clique there are exactly 4 red and 8 blue lines from each point. For a regular (3, 5)-colouring of the 13-clique there must be two types of lines red, and four types blue.

The following are all possibilities for choices of red classes: 12, 13, 14, 15, 16, 23, 24, 25, 26, 34, 35, 36, 45, 46, 56. One of these pairs xy is required such that no triangle is made up of lines of lengths x and y only, but every 5-clique contains a line of length x or y.

As a triangle is a closed figure, the lengths of its sides (measured in the same direction about the circle) add up to N. For the purpose of regular colourings, any triangle may be represented by the lengths of its sides, and all possible triangles may be listed by writing down all partitions of 13 into three positive parts. To check for red triangles, it is then necessary to identify 13-s with s. (An edge must have the same length whether measured in a clockwise or counterclockwise direction.) All possible triangles are:

1-1-11 ≡ 1-1-2 ;	1-2-10 ≡ 1-2-3 ;	1-3-9 ≡ 1-3-4 ;
1-4-8 ≡ 1-4-5 ;	1-5-7 ≡ 1-5-6 ;	1-6-6 ;
2-2-9 ≡ 2-2-4 ;	2-3-8 ≡ 2-3-5 ;	2-4-7 ≡ 2-4-6 ;
2-5-6 ;	3-3-7 ≡ 3-3-6 ;	3-4-6 ;
3-5-5 ;	4-4-5 .	

Colourings 12, 16, 24, 36, 35, 45 may now be ruled out as these give red triangles. The remaining colourings do not.

In the same way all possible pentagons (for the purpose of regular colourings) may be listed by writing down all partitions of 13 into five positive parts, and identifying 13-s with s.

1-1-1-1-9 ≡ 1-1-1-1-4 ;	1-1-1-2-8 ≡ 1-1-1-2-5 ;
1-1-1-3-7 ≡ 1-1-1-3-6 ;	1-1-1-4-6 ;
1-1-1-5-5 ;	1-1-2-2-7 ≡ 1-1-2-2-6 ;
1-1-2-3-6 ;	1-1-2-4-5 ;
1-1-3-3-5 ;	1-1-3-4-4 ;
1-2-2-2-6 ;	1-2-2-3-5 ;
1-2-2-4-4 ;	1-2-3-3-4 ;
1-3-3-3-3 ;	2-2-2-2-5 ;
2-2-2-3-4 ;	2-2-3-3-3 .

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Now consider, for example, the colouring 15. This gives no red triangles, and it will be shown that every 5-clique has a side of length 1 or 5. Every 5-clique has an "outer" pentagon, and the lengths of sides of this pentagon will be given by a permutation of the numbers in one of the above partitions. Every such partition contains 1 or 5 except 2-2-2-3-4 and 2-2-3-3-3, so the only possible blue 5-cliques are ones with "outer" pentagons of these types. But for any ordering of 2-2-2-3-4 there is a side of length 2 adjacent to the side of length 3, and thus there is a diagonal of length 5. Similarly 2-2-3-3-3 gives a diagonal of length 5, so there are no blue 5-cliques.

Thus colouring 1- and 5- lines red and others blue gives a regular (3,5)-colouring of the 13-clique. $L(3,5) = 13$ and $M(3,5) = 14$. Other regular colourings are 23 and 46. All remaining colourings give blue 5-cliques. For example, 2-2-2-2-5 has no edges or diagonals of length 1 or 3; so 13 is not a regular (3,5)-colouring of the 13-clique.

THEOREM 4. $L(4,4) = 17$.

Proof. By lemmas 5 and 8, $M(4,4) \leq 18$ and $L(4,4) \leq 17$. Colour 1-, 2-, 4-, and 8- lines red in the 17-clique, and all other lines blue. This gives a regular (4,4)-colouring; $L(4,4) = 17$ and $M(4,4) = 18$.

THEOREM 5. $L(3,6) = 16$.

Proof. By lemma 5, $M(3,6) < M(2,6) + M(3,5) = 6+14$. Thus $M(3,6) \leq 19$ and $L(3,6) \leq 18$. A regular (3,6)-colouring of the 16-clique is obtained by making 1-, 3-, and 8- lines red. By checking all possible regular colourings it is easily verified that there are no regular (3,6)-colourings of the 17-clique or 18-clique. Therefore $L(3,6) = 16$.

The author has proved elsewhere that there are no (3,6)-colourings of the 18-clique, and that there are (3,6)-colourings of the 17-clique, but that these colourings contain two different types of points - some 4-valent and some 5-valent in red. Therefore $M(3,6) = 18$ [Table 2]. This is the first case in which the lower bound obtained using regular colourings is not tight, since $L(3,6) = 16$ and $M(3,6) = 18$.

5. Further Results. For all cases considered in section 4, colourings were constructed by Greenwood and Gleason [3]. In this section, regular colourings are used to obtain some configurations not obtainable by their methods, and thus some new lower bounds for unknown M's are obtained.

THEOREM 6. $L(3,7) = 21$, $M(3,7) \geq 22$.

Proof. By lemma 5, $M(3,7) \leq M(2,7) + M(3,6) = 25$, so $L(3,7) \leq 24$. A regular (3, 7)-colouring of the 21-clique is obtained by making 1-, 3-, and 8- lines red, and all others blue. It can be shown that there are no (3, 7)-colourings of the 24-clique, so that $M(3,7) \leq 24$ [see Table 2]. An exhaustive check of all possibilities shows that there are no regular (3, 7)-colourings of the 22-clique or 23-clique. Thus $L(3,7) = 21$ and $M(3,7) \geq 22$.

THEOREM 7. $L(3,8) = 26$, $M(3,8) \geq 27$.

Proof. Using the result that $M(3,7) \leq 24$, lemma 5 gives $M(3,8) \leq 31$ so that $L(3,8) \leq 30$. A check of all possible regular colourings shows that there are none for the 27, 28, 29, or 30-clique. Making 1-, 3-, 8-, and 13- lines red gives a regular (3, 8)-colouring of the 26-clique. Thus $L(3,8) = 26$ and $M(3,8) \geq 27$.

THEOREM 8. $L(4,5) = 24$, $M(4,5) \geq 25$.

Proof. By lemma 5, $M(4,5) < M(3,5) + M(4,4) = 14 + 18$, and $L(4,5) \leq 30$. It can be shown that there are no (4, 5)-colourings of the 30-clique, and in fact $M(4,5) \leq 30$ [see table 2]. It is easily verified that colouring 1-, 2-, 4-, 8-, and 9- lines red gives a regular (4, 5)-colouring of the 24-clique. Furthermore, by checking all the possibilities it can be seen that there are no regular (4, 5)-colourings of the 25-, 26-, 27-, 28-, or 29- cliques. $L(4,5) = 24$. This check was carried out by means of the IBM 7040 computer at the University of Waterloo with a programme written in Fortran IV. Total computer time was about 20 minutes.

6. Summary and Conclusion. A special type of colouring was defined for edge chromatic graphs. By means of the construction method described, lower bounds may be obtained for unknown Ramsey numbers, and this was done for $M(3,7)$,

M(3, 8) and M(4, 5). It is possible to obtain lower bounds for further numbers by this method, although this requires the use of computers because of the large numbers of possibilities.

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