Proceedings of the Edinburgh Mathematical Society (2021) **64**, 1002–1027
doi:10.1017/S0013091521000742

EXTRAPOLATION TO WEIGHTED MORREY SPACES WITH VARIABLE EXPONENTS AND APPLICATIONS

KWOK-PUN HO

Department of Mathematics and Information Technology, The Education University of Hong Kong, 10 Lo Ping Road, Tai Po, Hong Kong, China (vkpho@eduhk.hk)

(Received 2 September 2020; first published online 9 November 2021)

Abstract This paper establishes the mapping properties of pseudo-differential operators and the Fourier integral operators on the weighted Morrey spaces with variable exponents and the weighted Triebel–Lizorkin–Morrey spaces with variable exponents. We obtain these results by extending the extrapolation theory to the weighted Morrey spaces with variable exponents. This extension also gives the mapping properties of Calderón–Zygmund operators on the weighted Hardy–Morrey spaces with variable exponents and the wavelet characterizations of the weighted Hardy–Morrey spaces with variable exponents.

Keywords: Morrey spaces; Triebel–Lizorkin spaces; Hardy–Morrey spaces; extrapolation; pseudo-differential operators; Fourier integral operators; Calderón–Zygmund operators; wavelets

2020 *Mathematics subject classification:* Primary: 42B20; 42B35; 46E30; 47G30; 35S30
Secondary: 42B30; 42C40

1. Introduction

The main theme of this paper is the mapping properties of pseudo-differential operators and Fourier integral operators on the weighted Morrey spaces with variable exponents.

The Morrey spaces were introduced by Morrey in [43] to study the solutions of some quasi-linear elliptic partial differential equations. The Morrey spaces are important generalizations of Lebesgue spaces. The studies of Morrey spaces had been extended to the generalized Morrey spaces [44], the weighted Morrey spaces [21, 35, 48] and the Morrey spaces with variable exponents [1, 18, 19, 25].

The weighted Morrey spaces with variable exponents were introduced and studied in [20, 42]. Moreover, the mapping properties of singular integral operators and the Riesz potentials were obtained in [19] and [42], respectively.

In this paper, we further extend the studies in [20, 42] to establish the mapping properties of pseudo-differential operators and Fourier integral operators on weighted Morrey spaces with variable exponents. We obtain our main results by extending the well-known extrapolation theory introduced by Rubio de Francia [51–52] to the weighted Morrey spaces with variable exponents.

Our studies also cover the weighted Triebel–Lizorkin–Morrey spaces with variable exponents and the weighted Hardy–Morrey spaces with variable exponents. They are generalizations of the Triebel–Lizorkin–Morrey spaces [57, 62] and the Hardy–Morrey spaces [24, 32]. We establish the mapping properties of pseudo-differential operators and Fourier integral operators on the weighted Triebel–Lizorkin–Morrey spaces with variable exponents. We also obtain the mapping properties of the Calderón–Zygmund operators on the weighted Morrey spaces with variable exponents and establish the wavelet characterizations of the weighted Morrey spaces with variable exponents.

This paper is organized as follows. The definition of weighted Lebesgue spaces with variable exponents and the boundedness of the maximal function on these spaces are recalled in § 2. The main results for the weighted Morrey spaces with variable exponents, the weighted Triebel–Lizorkin–Morrey spaces with variable exponents and the weighted Hardy–Morrey spaces with variable exponents are established in § 3, § 4 and § 5, respectively.

2. Definitions and preliminaries

Let $\mathcal{M}$ denote the class of Lebesgue measurable functions. Let $\mathcal{S}$ and $\mathcal{S}'$ be the class of Schwartz functions and tempered distributions, respectively. For any $x \in \mathbb{R}^n$ and $r > 0$, write $B(x,r) = \{y : |y-x| < r\}$. Define $\mathbb{B} = \{B(x,r) : x \in \mathbb{R}^n, r > 0\}$.

Let $p \in (0,\infty)$ and $\omega : \mathbb{R}^n \to (0,\infty)$, the weighted Lebesgue space $L^p(\omega)$ consists of all $f \in \mathcal{M}$ satisfying

$$\|f\|_{L^p(\omega)} = \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx\right)^{\frac{1}{p}} < \infty.$$

For any Lebesgue measurable function $p(\cdot) : \mathbb{R}^n \to (0,\infty]$, define

$$p_+ = \mathrm{esssup}_{x\in\mathbb{R}^n} p(x) \quad \text{and} \quad p_- = \mathrm{essinf}_{x\in\mathbb{R}^n} p(x)$$

and

$$\mathbb{R}^{p(\cdot)}_\infty = \{x \in \mathbb{R}^n : p(x) = \infty\}.$$

When $p(\cdot) : \mathbb{R}^n \to [1,\infty]$, define the conjugate function $p'(\cdot)$ by $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.

We recall the definitions of the Lebesgue spaces with variable exponents and the weighted Lebesgue spaces with variable exponents from [11, Definitions 3.1.2, 3.2.1 and (5.8.1)].

Definition 2.1. Let $p(\cdot) : \mathbb{R}^n \to (0,\infty]$ and $\omega : \mathbb{R}^n \to (0,\infty)$ be Lebesgue measurable functions. The Lebesgue space with variable exponent consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx + \left\|\frac{f\chi_{\mathbb{R}^{p(\cdot)}_\infty}}{\lambda}\right\|_{L^\infty} \le 1\right\} < \infty.$$

https://doi.org/10.1017/S0013091521000742 Published online by Cambridge University Press

The weighted Lebesgue space with variable exponent consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{L^{p(\cdot)}_\omega} = \|f\omega\|_{L^{p(\cdot)}} < \infty.$$

In particular, when $p(\cdot) = p$, $p \in (0, \infty)$, is a constant function, we have $L^p_\omega = L^p(\omega^p)$. Notice that for any unbounded Lebesgue measurable set E with $|E| < \infty$, $\int_E \omega(x)^p dx$ is not necessarily bounded. Therefore, L^p_ω is not a Banach function space with respect to the Lebesgue measure. For the definition of Banach function space, the reader is referred to [2, Chapter 1, Definitions 1.1 and 1.3] and [11, Definition 2.7.7].

Theorem 2.1. *Let $p(\cdot) : \mathbb{R}^n \to [1, \infty]$ be a Lebesgue measurable function. Then, $L^{p(\cdot)}$ is a Banach function space and*

$$\|f\|_{L^{p'(\cdot)}} = \sup\left\{\left|\int_{\mathbb{R}^n} f(x)g(x)dx\right| : g \in L^{p(\cdot)},\ \|g\|_{L^{p(\cdot)}} \le 1\right\}.$$

That is, the associate space of $L^{p(\cdot)}$ is $L^{p'(\cdot)}$.

The reader is referred to [11, Theorem 3.2.13] for the proof of the above result. Notice that $L^{p(\cdot)}_\omega$ is not necessarily a Banach function space, see [11, p.192-193].

In view of the above theorem, we have the following Hölder inequality for the weighted Lebesgue spaces with variable exponents

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx = \int_{\mathbb{R}^n} |f(x)\omega(x)g(x)\omega(x)^{-1}|dx \le 2\|f\|_{L^{p(\cdot)}_\omega}\|g\|_{L^{p'(\cdot)}_{\omega^{-1}}}. \tag{2.1}$$

Additionally, Theorem 2.1 gives

$$\begin{aligned}
&\|f\|_{L^{p'(\cdot)}_{\omega^{-1}}} = \|f\omega^{-1}\|_{L^{p'(\cdot)}} \\
&\le C \sup\left\{\left|\int_{\mathbb{R}^n} f(x)\omega(x)^{-1}g(x)\omega(x)dx\right| : g\omega \in L^{p(\cdot)},\ \|g\omega\|_{L^{p(\cdot)}} \le 1\right\} \\
&\le C \sup\left\{\left|\int_{\mathbb{R}^n} f(x)g(x)dx\right| : g \in L^{p(\cdot)}_\omega,\ \|g\|_{L^{p(\cdot)}_\omega} \le 1\right\}
\end{aligned}$$

for some $C > 0$. That is, the associate space of $L^{p(\cdot)}_\omega$ with respect to the Lebesgue measure dx is $L^{p'(\cdot)}_{\omega^{-1}}$.

We recall the class of weight functions associated with $L^{p(\cdot)}_\omega$ from [8, Definition 1.4].

Definition 2.2. Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ and $\omega : \mathbb{R}^n \to (0, \infty)$ be Lebesgue measurable functions. We write $\omega \in A_{p(\cdot)}$ if there exists a constant $C > 0$ such that for any $B \in \mathbb{B}$,

$$\|\chi_B\omega\|_{L^{p(\cdot)}}\|\chi_B\omega^{-1}\|_{L^{p'(\cdot)}} \le C|B|.$$

We also recall the definition of the Muckenhoupt weight functions $\mathcal{A}_p$ because they are important ingredient in the extrapolation theory. A positive locally integrable function

https://doi.org/10.1017/S0013091521000742 Published online by Cambridge University Press

ω belongs to $\mathcal{A}_p$ if it satisfies

$$[\omega]_{\mathcal{A}_p} = \sup_{B\in\mathbb{B}} \left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{-\frac{p'}{p}}dx\right)^{\frac{p}{p'}} < \infty$$

where $p' = \frac{p}{p-1}$. A locally integrable function $\omega : \mathbb{R}^n \to [0,\infty)$ is said to be an $\mathcal{A}_1$ weight if there is a constant $C > 0$ such that for any $B \in \mathbb{B}$,

$$\frac{1}{|B|}\int_B \omega(y)dy \le C\omega(x), \quad a.e.\, x \in B.$$

The infimum of all such C is denoted by $[\omega]_{\mathcal{A}_1}$. We define $\mathcal{A}_\infty = \cup_{p\ge1}\mathcal{A}_p$.

When $p(\cdot) = p$, $p \in [1,\infty)$, we have $\omega \in A_{p(\cdot)} \Leftrightarrow \omega^p \in \mathcal{A}_p$.

The following is an important class of exponent functions used in the function spaces with variable exponents.

Definition 2.3. A continuous function g on $\mathbb{R}^n$ is locally log-Hölder continuous if there exists $c_{log} > 0$ such that

$$|g(x) - g(y)| \le \frac{c_{log}}{-\log(|x-y|)}, \quad \forall x, y \in \mathbb{R}^n,\ |x-y| < \frac{1}{2}. \tag{2.2}$$

We denote the class of locally log-Hölder continuous function by $C^{\log}_{loc}(\mathbb{R}^n)$.

Furthermore, a continuous function g is globally log-Hölder continuous if $g \in C^{\log}_{loc}(\mathbb{R}^n)$ and there exist $g_\infty \in \mathbb{R}$ and $c_\infty > 0$ so that

$$|g(x) - g_\infty| \le \frac{c_\infty}{\log(e + |x|)}, \quad \forall x \in \mathbb{R}^n. \tag{2.3}$$

The class of globally log-Hölder continuous function is denoted by $C^{\log}(\mathbb{R}^n)$.

According to [8, Theorem 1.5], we have the following boundedness result of the Hardy–Littlewood maximal operator on the weighted Lebesgue spaces with variable exponents.

Theorem 2.2. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < p_- \le p_+ < \infty$. If $\omega \in A_{p(\cdot)}$, then the Hardy–Littlewood maximal operator* M *is bounded on $L^{p(\cdot)}_\omega$. Moreover, the Hardy–Littlewood maximal operator* M *is also bounded on $L^{p'(\cdot)}_{\omega^{-1}}$.*

The results given in [8, Theorem 1.5] are presented for the weighted Lebesgue space with variable exponent $L^{p(\cdot)}_\omega$. It is easy to see that $\omega \in A_{p(\cdot)} \Leftrightarrow \omega^{-1} \in A_{p'(\cdot)}$. Moreover, for any $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < p_- \le p_+ < \infty$, we have $p'(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < p'_- \le p'_+ < \infty$. Consequently, [8, Theorem 1.5] also yields the boundedness for the Hardy–Littlewood maximal operator on $L^{p'(\cdot)}_{\omega^{-1}}$.

3. Weighted Morrey spaces with variable exponents

The main results for the weighted Morrey spaces with variable exponents are obtained in this section. We establish the extrapolation theory for the weighted Morrey spaces

https://doi.org/10.1017/S0013091521000742 Published online by Cambridge University Press

with variable exponents. This extrapolation theory yields the boundedness of pseudo-differential operators and Fourier integral operators on the weighted Morrey spaces with variable exponents.

We start with the definitions of weighted Morrey spaces with variable exponents.

Definition 3.1. Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $0 < p_- \leq p_+ < \infty$, $\omega : \mathbb{R}^n \to (0, \infty)$ and $u(\cdot) : (0, \infty) \to (0, \infty)$ be Lebesgue measurable functions. The weighted Morrey space with variable exponent $M^u_{p(\cdot),\omega}$ consists of all $f \in \mathcal{M}$ satisfying

$$\|f\|_{M^u_{p(\cdot),\omega}} = \sup_{y\in\mathbb{R}^n, r>0} \frac{1}{u(\|\chi_{B(y,r)}\|_{L^{p(\cdot)}_\omega})} \|\chi_{B(y,r)} f\|_{L^{p(\cdot)}_\omega} < \infty. \tag{3.1}$$

To simplify the presentation, for the rest of this paper, we write $u(y, r) = u(\|\chi_{B(y,r)}\|_{L^{p(\cdot)}_\omega})$, $y \in \mathbb{R}^n$, $r > 0$.

When $p(\cdot) = p$, $1 < p < \infty$, is a constant function, $u(y, r) = |B(y, r)|^{\frac{1}{p} - \frac{1}{q}}$, $1 \leq q \leq p < \infty$ and $\omega \equiv 1$, $M^u_{p(\cdot),\omega}$ becomes the classical Morrey space M^q_p.

When $\omega \equiv 1$, the weighted Morrey space with variable exponent becomes the Morrey spaces with variable exponents [1, 18, 19, 25]. When $p(\cdot) = p$, $p \in (0, \infty)$, $M^u_{p(\cdot),\omega}$ reduces to the weighted Morrey spaces $M^u_{p,\omega}$ [21, 35, 48].

For the mapping properties of the singular integral operators and the Riesz potentials on the weighted Morrey spaces with variable exponents, the reader is referred to [20, 42].

We now introduce the weighted block spaces with variable exponents. We need to use the boundedness of the Hardy–Littlewood maximal operator on the weighted block spaces with variable exponents to establish the extrapolation theory on the weighted Morrey spaces with variable exponents.

Definition 3.2. Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < p_- \leq p_+ < \infty$, $\omega \in A_{p(\cdot)}$ and $u(\cdot) : (0, \infty) \to (0, \infty)$ be a Lebesgue measurable function. For any Lebesgue measurable function b, we write $b \in \mathfrak{b}^u_{p(\cdot),\omega}$ if supp$b \subseteq B(x_0, r)$ for some $x_0 \in \mathbb{R}^n$ and $r > 0$ and

$$\|b\|_{L^{p(\cdot)}_\omega} \leq \frac{1}{u(x_0, r)}.$$

The block space $\mathfrak{B}^u_{p(\cdot),\omega}$ is defined as

$$\mathfrak{B}^u_{p(\cdot),\omega} = \left\{ \sum_{k=1}^{\infty} \lambda_k b_k : \sum_{k=1}^{\infty} |\lambda_k| < \infty \text{ and } b_k \in \mathfrak{b}^\omega_{p(\cdot),u} \right\}. \tag{3.2}$$

The block space $\mathfrak{B}^u_{p(\cdot),\omega}$ is endowed with the norm

$$\|f\|_{\mathfrak{B}^u_{p(\cdot),\omega}} = \inf \left\{ \sum_{k=1}^{\infty} |\lambda_k| \text{such that } f = \sum_{k=1}^{\infty} \lambda_k b_k \text{ a.e.} \right\}. \tag{3.3}$$

We now present some supporting results for $M^u_{p(\cdot),\omega}$ and $\mathfrak{B}^u_{p'(\cdot),\omega^{-1}}$. The reader is referred to [38] for the corresponding results on the Morrey type spaces. We have the Hölder inequalities for $M^u_{p(\cdot),\omega}$ and $\mathfrak{B}^u_{p'(\cdot),\omega^{-1}}$.

https://doi.org/10.1017/S0013091521000742 Published online by Cambridge University Press

Lemma 3.1. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < p_- \le p_+ < \infty$, $\omega \in A_{p(\cdot)}$ and $u(\cdot) : (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. We have a constant $C > 0$ such that*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \le C\|f\|_{M^u_{p(\cdot),\omega}} \|g\|_{\mathfrak{B}^u_{p'(\cdot),\omega^{-1}}}.$$

Proof. Let $f \in M^u_{p(\cdot),\omega}$ and $g \in \mathfrak{B}^u_{p'(\cdot),\omega^{-1}}$. For any $\epsilon > 0$, we have $g = \sum_{k=1}^{\infty} \lambda_k b_k$ with $\operatorname{supp} b_k \subseteq B(x_k, r_k) = B_k$, $\|b_k\|_{L^{p(\cdot)}_\omega} \le \frac{1}{u(x_k,r_k)}$ and

$$\sum_{k=1}^{\infty} |\lambda_k| \le (1+\epsilon)\|g\|_{\mathfrak{B}^u_{p'(\cdot),\omega^{-1}}}.$$

The Hölder inequality gives

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)b_k(x)|dx &\le C\|f\chi_{B_k}\|_{L^{p(\cdot)}_\omega} \|b_k\|_{L^{p'(\cdot)}_{\omega^{-1}}} \\ &= \frac{C}{u(x_k,r_k)} \|f\chi_{B_k}\|_{L^{p(\cdot)}_\omega} u(x_k,r_k)\|b_k\|_{L^{p'(\cdot)}_{\omega^{-1}}} \le C\|f\|_{M^u_{p(\cdot),\omega}}.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)g(x)|dx &\le \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |f(x)b_k(x)|dx \le C\|f\|_{M^u_{p(\cdot),\omega}} \sum_{k=1}^{n} |\lambda_k| \\ &\le C\|f\|_{M^u_{p(\cdot),\omega}} \|g\|_{\mathfrak{B}^u_{p'(\cdot),\omega^{-1}}}.\end{aligned}$$

□

The following proposition gives criteria for the membership $f \in M^u_{p(\cdot),\omega}$.

Proposition 3.2. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < p_- \le p_+ < \infty$, $\omega \in A_{p(\cdot)}$ and $u(\cdot) : (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. If $f \in \mathcal{M}$ satisfies*

$$\sup_{b \in \mathfrak{b}^u_{p'(\cdot),\omega^{-1}}} \int_{\mathbb{R}^n} |f(x)b(x)|dx < \infty,$$

then $f \in M^u_{p(\cdot),\omega}$.

Proof. For any $g \in L^{p'(\cdot)}_{\omega^{-1}}$ with $\|g\|_{L^{p'(\cdot)}_{\omega^{-1}}} \le 1$ and $B = B(x_0, r)$, $x_0 \in \mathbb{R}^n$ and $r > 0$, define $b_{g,B} = \frac{1}{u(x_0,r)} g\chi_B$. We find that $b_{g,B} \in \mathfrak{b}^u_{p'(\cdot),\omega^{-1}}$.

Since the associate space of $L^{p'(\cdot)}_{\omega^{-1}}$ with respect to the Lebesgue measure is $L^{p(\cdot)}_\omega$, we have

$$\begin{aligned}\sup_{\substack{g \in L^{p'(\cdot)}_{\omega^{-1}} \\ \|g\|_{L^{p'(\cdot)}_{\omega^{-1}}} \le 1}} \int_{\mathbb{R}^n} |f(x)b_{g,B}(x)|dx &= \frac{1}{u(x_0,r)} \sup_{\substack{g \in L^{p'(\cdot)}_{\omega^{-1}} \\ \|g\|_{L^{p'(\cdot)}_{\omega^{-1}}} \le 1}} \int_{\mathbb{R}^n} |f(x)g(x)\chi_B(x)|dx \\ &= \frac{1}{u(x_0,r)} \|\chi_{B(x_0,r)} f\|_{L^{p(\cdot)}_\omega}.\end{aligned}$$

https://doi.org/10.1017/S0013091521000742 Published online by Cambridge University Press

By taking the supremum over $B(x_0,r)\in\mathbb{B}$, we obtain

$$\|f\|_{M^u_{p(\cdot),\omega}}=\sup_{B(x_0,r)\in\mathbb{B}}\frac{1}{u(x_0,r)}\|\chi_{B(x_0,r)}f\|_{L^{p(\cdot)}_\omega}\le\sup_{b\in\mathfrak{b}_{X',u}}\int_{\mathbb{R}^n}|f(x)b(x)|dx<\infty.$$

Therefore, $f\in M^u_{p(\cdot),\omega}$. □

Next, we have the norm conjugate formula for the weighted Morrey spaces with variable exponents.

Proposition 3.3. *Let $p(\cdot)\in C^{\log}(\mathbb{R}^n)$ with $1<p_-\le p_+<\infty$, $\omega\in A_{p(\cdot)}$ and $u(\cdot):(0,\infty)\to(0,\infty)$ be a Lebesgue measurable function. There exist constants $C_0,C_1>0$ such that for any $f\in M^u_{p(\cdot),\omega}$,*

$$C_0\|f\|_{M^u_{p(\cdot),\omega}}\le\sup_{b\in\mathfrak{b}^u_{p'(\cdot),\omega^{-1}}}\int_{\mathbb{R}^n}|f(x)b(x)|dx\le C_1\|f\|_{M^u_{p(\cdot),\omega}}.\tag{3.4}$$

Proof. In view of the definition of $M^u_{p(\cdot),\omega}$, we have a $B(x_0,r)\in\mathbb{B}$ such that

$$\|f\|_{M^u_{p(\cdot),\omega}}\le 2\frac{1}{u(x_0,r)}\|f\chi_{B(x_0,r)}\|_{L^{p(\cdot)}_\omega}.\tag{3.5}$$

As the associate space of $L^{p(\cdot)}_\omega$ with respect to the Lebesgue measure is $L^{p'(\cdot)}_{\omega^{-1}}$, we have a $g\in L^{p'(\cdot)}_{\omega^{-1}}$ with $\|g\|_{L^{p'(\cdot)}_{\omega^{-1}}}\le 1$ such that

$$\|f\chi_{B(x_0,r)}\|_{L^{p(\cdot)}_\omega}\le 2\int_{B(x_0,r)}|f(x)g(x)|dx.\tag{3.6}$$

Define $b(x)=\frac{1}{u(x_0,r)}\chi_{B(x_0,r)}(x)g(x)$. We have $\operatorname{supp} b\subseteq B(x_0,r)$ and $\|b\|_{L^{p'(\cdot)}_{\omega^{-1}}}\le\frac{1}{u(x_0,r)}$. Therefore, $b\in\mathfrak{b}^u_{p'(\cdot),\omega^{-1}}$. Consequently, (3.5) and (3.6) yield

$$C_0\|f\|_{M^u_{p(\cdot),\omega}}\le\int_{\mathbb{R}^n}|f(x)b(x)|dx\le C_1\sup_{b^*\in\mathfrak{b}^u_{p'(\cdot),\omega^{-1}}}\int_{\mathbb{R}^n}|f(x)b^*(x)|dx$$

for some $C_0,C_1>0$. The above inequality gives the first inequality in (3.4). The second inequality in (3.4) follows from Lemma 3.1 and the fact that $\|b\|_{\mathfrak{B}^u_{p'(\cdot),\omega^{-1}}}\le 1$ for any $b\in\mathfrak{b}^u_{p'(\cdot),\omega^{-1}}$. □

We now introduce a class of weight function for the study of the boundedness of the Hardy–Littlewood maximal operator on the weighted block spaces with variable exponents.

Definition 3.3. Let $p(\cdot)\in C^{\log}(\mathbb{R}^n)$ with $1<p_-\le p_+<\infty$ and $\omega\in A_{p(\cdot)}$. We say that a Lebesgue measurable function, $u(\cdot):(0,\infty)\to(0,\infty)$, belongs to $u\in\mathbb{W}_{p(\cdot),\omega}$ if there

https://doi.org/10.1017/S0013091521000742 Published online by Cambridge University Press

exists a constant $C > 0$ such that for any $x \in \mathbb{R}^n$ and $r > 0$, u fulfils

$$C \leq u(r), \quad r \geq 1, \tag{3.7}$$

$$r \leq Cu(r), \quad r < 1, \tag{3.8}$$

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}_\omega}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}_\omega}} u(\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}_\omega}) \leq Cu(\|\chi_{B(x,r)}\|_{L^{p(\cdot)}_\omega}). \tag{3.9}$$

The reader may consult [47, Proposition 2.7] and [46, Definition 1.1] for some similar classes of weight functions used to study the generalized Morrey spaces and the generalized weighted Morrey spaces, respectively.

When $p(\cdot) = p$, $p \in (1, \infty)$ is a constant function, we write $\mathbb{W}_{p(\cdot),\omega} = \mathbb{W}_{p,\omega}$.

For instance, when $\theta \in (0, 1)$, $u_\theta(r) = r^\theta$ fulfils (3.7) and (3.8). Notice that Definition 2.2 assures that for any $B \in \mathbb{B}$, $\|\chi_B\|_{L^{p(\cdot)}_\omega} = \|\chi_B \omega\|_{L^{p(\cdot)}} < \infty$. Therefore, the inequality in (3.9) is well defined.

Furthermore, (3.7) assures that

$$1 \leq \|\chi_{B(x,r)}\|_{L^{p(\cdot)}_\omega} \Rightarrow C \leq u(x, r), \quad x \in \mathbb{R}^n, \ r > 0. \tag{3.10}$$

Similarly, (3.8) guarantees that

$$\|\chi_{B(x,r)}\|_{L^{p(\cdot)}_\omega} \leq 1 \Rightarrow \|\chi_{B(x,r)}\|_{L^{p(\cdot)}_\omega} \leq u(x, r), \quad x \in \mathbb{R}^n, \ r > 0. \tag{3.11}$$

Let $w \in \mathcal{A}_\infty$, $p(\cdot) = p$, $p \in (1, \infty)$ and $\omega = w^{1/p}$. Recall that for any $w \in \mathcal{A}_\infty$, there exist constants $\epsilon_0, C > 0$ such that for any $B \in \mathbb{B}$ and measurable subset A of B, we have

$$\frac{w(A)}{w(B)} \leq C \left(\frac{|A|}{|B|}\right)^{\epsilon_0}, \tag{3.12}$$

see [17, Theorem 9.3.3 (d)]. We find that (3.12) gives

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^p_\omega}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^p_\omega}} \frac{u_\theta(x, 2^{j+1}r)}{u_\theta(x, r)} &= \sum_{j=0}^{\infty} \left(\frac{\|\chi_{B(x,r)}\|_{L^p_\omega}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^p_\omega}}\right)^{1-\theta} \\ &\leq C \sum_{j=0}^{\infty} 2^{-(j+1)n\epsilon_0} < \infty.\end{aligned}$$

Thus, $u_\theta \in \mathbb{W}_{p,\omega}$.

The condition (3.9) is used to obtain the boundedness of the Hardy–Littlewood maximal operator on $\mathfrak{B}^u_{p(\cdot),\omega}$.

Proposition 3.4. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < p_- \leq p_+ < \infty$, $\omega \in A_{p(\cdot)}$ and $u(\cdot) : (0, \infty) \to (0, \infty)$ be a Lebesgue measurable function. If u satisfies (3.7) and (3.8), then for any $B \in \mathbb{B}$, $\chi_B \in M^u_{p(\cdot),\omega}$.*

Proof. Let $B \in \mathbb{B}$, $x \in \mathbb{R}^n$ and $r > 0$. Whenever $\|\chi_{B(x,r)}\|_{L^{p(\cdot)}_\omega} \geq 1$, (3.10) gives

$$\frac{1}{u(x,r)}\|\chi_B\chi_{B(x,r)}\|_{L^{p(\cdot)}_\omega} \leq \frac{1}{u(x,r)}\|\chi_B\|_{L^{p(\cdot)}_\omega} \leq C\|\chi_B\|_{L^{p(\cdot)}_\omega} \tag{3.13}$$

for some $C > 0$. When $\|\chi_{B(x,r)}\|_{L^{p(\cdot)}_\omega} < 1$, (3.11) assures that

$$\frac{1}{u(x,r)}\|\chi_B\chi_{B(x,r)}\|_{L^{p(\cdot)}_\omega} \leq \frac{1}{u(x,r)}\|\chi_{B(x,r)}\|_{L^{p(\cdot)}_\omega} \leq C. \tag{3.14}$$

Consequently, (3.13) and (3.14) yield

$$\|\chi_B\|_{M^u_{p(\cdot),\omega}} = \sup_{B(x,r)\in\mathbb{B}} \frac{1}{u(x,r)}\|\chi_B\chi_{B(x,r)}\|_{L^{p(\cdot)}_\omega} < C + C\|\chi_B\|_{L^{p(\cdot)}_\omega}.$$

Therefore, $\chi_B \in M^u_{p(\cdot),\omega}$. □

The preceding theorem shows that for any u satisfies (3.7) and (3.8), $M^u_{p(\cdot),\omega}$ is non-trivial. Furthermore, Definition 3.2 and Lemma 3.1 guarantee that for any $f \in M^u_{p(\cdot),\omega}$ and $B \in \mathbb{B}$, we have $\int_B |f(x)|dx \leq C\|f\|_{M^u_{p(\cdot),\omega}}$. Therefore, this inequality and Proposition 3.4 assure that $M^u_{p(\cdot),\omega}$ is a ball Banach function space. For the definition of ball Banach function space, see [59, Definition 2.2].

We are now ready to show that the Hardy–Littlewood maximal operator is bounded on $\mathfrak{B}^u_{p'(\cdot),\omega^{-1}}$.

Theorem 3.5. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < p_- \leq p_+ < \infty$, $\omega \in A_{p(\cdot)}$ and $u(\cdot) : (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. If $u \in \mathbb{W}_{p(\cdot),\omega}$, then the Hardy–Littlewood maximal operator M is bounded on $\mathfrak{B}^u_{p'(\cdot),\omega^{-1}}$.*

Proof. Let $b \in \mathfrak{b}^u_{p'(\cdot),\omega^{-1}}$ with $\operatorname{supp} b \subseteq B(x_0, r)$, $x_0 \in \mathbb{R}^n$, $r > 0$. For any $k \in \mathbb{N} \cup \{0\}$, write $B_k = B(x_0, 2^k r)$, $d_0 = \chi_{B_1}\mathrm{M}(b)$ and $d_k = \chi_{B_{k+1}\backslash B_k}\mathrm{M}(b)$, $k \in \mathbb{N}$. Thus, $\operatorname{supp} d_k \subseteq B_{k+1}\backslash B_k$ and $\mathrm{M}(b) = \sum_{k=0}^\infty d_k$.

Theorem 2.2 assures that

$$\|d_0\|_{L^{p'(\cdot)}_{\omega^{-1}}} \leq \|\mathrm{M}(b)\|_{L^{p'(\cdot)}_{\omega^{-1}}} \leq C\|b\|_{L^{p'(\cdot)}_{\omega^{-1}}} \leq C\frac{1}{u(x_0,r)}$$

for some $C > 0$. Thus, $d_0 = Ce_0$ for some $C > 0$ with $e_0 \in \mathfrak{b}^u_{p'(\cdot),\omega^{-1}}$.

Next, (2.1) gives

$$\begin{aligned} d_k = \chi_{B_{k+1}\backslash B_k}\mathrm{M}(b) &\leq \chi_{B_{k+1}\backslash B_k}\frac{1}{2^{kn}r^n}\int_{B(x_0,r)}|b(x)|dx \\ &\leq C\chi_{B_{k+1}\backslash B_k}\frac{1}{2^{kn}r^n}\|b\|_{L^{p'(\cdot)}_{\omega^{-1}}}\|\chi_{B_0}\|_{L^{p(\cdot)}_\omega}. \end{aligned}$$

https://doi.org/10.1017/S0013091521000742 Published online by Cambridge University Press

Consequently, as $\omega \in A_{p(\cdot)}$, Definition 2.2 guarantees that

$$\begin{aligned}\|d_k\|_{L^{p'(\cdot)}_{\omega^{-1}}} &\le C\|\chi_{B_{k+1}\setminus B_k}\|_{L^{p'(\cdot)}_{\omega^{-1}}}\frac{1}{2^{kn}r^n}\|b\|_{L^{p'(\cdot)}_{\omega^{-1}}}\|\chi_{B_0}\|_{L^{p(\cdot)}_{\omega}}\\ &\le C\|\chi_{B_{k+1}}\|_{L^{p'(\cdot)}_{\omega^{-1}}}\frac{1}{2^{kn}r^n}\|b\|_{L^{p'(\cdot)}_{\omega^{-1}}}\|\chi_{B_0}\|_{L^{p(\cdot)}_{\omega}}\\ &\le C_0\frac{\|\chi_{B_0}\|_{L^{p(\cdot)}_{\omega}}}{\|\chi_{B_{k+1}}\|_{L^{p(\cdot)}_{\omega}}}\frac{u(x_0,2^{k+1}r)}{u(x_0,r)}\frac{1}{u(x_0,2^{k+1}r)}\end{aligned}$$

for some $C_0>0$.

Define $e_k=\frac{1}{\gamma_k}d_k$ where

$$\gamma_k = C_0\frac{\|\chi_{B_0}\|_{L^{p(\cdot)}_{\omega}}}{\|\chi_{B_{k+1}}\|_{L^{p(\cdot)}_{\omega}}}\frac{u(x_0,2^{k+1}r)}{u(x_0,r)}. \tag{3.15}$$

We see that $e_k\in\mathfrak{b}^u_{p'(\cdot),\omega^{-1}}$ with $\operatorname{supp} d_k\subseteq B_{k+1}$. Since (3.9) asserts that

$$\sum_{k=1}^{\infty}\gamma_k = C_0\sum_{k=1}^{\infty}\frac{\|\chi_{B_0}\|_{L^{p(\cdot)}_{\omega}}}{\|\chi_{B_{k+1}}\|_{L^{p(\cdot)}_{\omega}}}\frac{u(x_0,2^{k+1}r)}{u(x_0,r)}<\infty.$$

Therefore,

$$\mathrm{M}(b)=\sum_{k=1}^{\infty}\gamma_k e_k\in\mathfrak{B}^u_{p'(\cdot),\omega^{-1}} \quad\text{with}\quad \|\mathrm{M}(b)\|_{\mathfrak{B}^u_{p'(\cdot),\omega^{-1}}}\le C \tag{3.16}$$

for some $C>0$ independent of b.

For any $g\in\mathfrak{B}^u_{p'(\cdot),\omega^{-1}}$, we have $g=\sum_{j=1}^{\infty}\lambda_j b_j$ with $b_j\in\mathfrak{b}^u_{p'(\cdot),\omega^{-1}}$ and $\sum_{j=1}^{\infty}|\lambda_j|\le 2\|g\|_{\mathfrak{B}^u_{p'(\cdot),\omega^{-1}}}$. Let $\{e_{j,k}\}$ and $\{\gamma_{j,k}\}$ be defined by (3.15) and (3.16) with b replaced by b_j. We have $\mathrm{M}(b_j)=\sum_{k=1}^{\infty}\gamma_{j,k}e_{j,k}$. The sub-linearity of M yields

$$\mathrm{M}(g)\le\sum_{j=1}^{\infty}|\lambda_j|\,\mathrm{M}(b_j)=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\lambda_j|\gamma_{j,k}e_{j,k}.$$

Since

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\lambda_j|\gamma_{j,k}\le C\sum_{j=1}^{\infty}|\lambda_j|\le C\|g\|_{\mathfrak{B}^u_{p'(\cdot),\omega^{-1}}},$$

we find that $B=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\lambda_j|\gamma_{j,k}e_{j,k}\in\mathfrak{B}^u_{p'(\cdot),\omega^{-1}}$.

Write

$$G=\begin{cases}\mathrm{M}(g)/B, & B\ne 0,\\ 0, & B=0.\end{cases}$$

Obviously,

$$\mathrm{M}(g)=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\lambda_j|\gamma_{j,k}g_{j,k}$$

where $g_{j,k}=Ge_{j,k}$. Since $|G|\le 1$, $g_{j,k}\in\mathfrak{b}^u_{p'(\cdot),\omega^{-1}}$ with $\operatorname{supp} g_{j,k}\subseteq\operatorname{supp} e_{j,k}$. Thus, $\|\mathrm{M}(g)\|_{\mathfrak{B}^u_{p'(\cdot),\omega^{-1}}}\le\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\lambda_j|\gamma_{j,k}\le C\|g\|_{\mathfrak{B}^u_{p'(\cdot),\omega^{-1}}}$. □

https://doi.org/10.1017/S0013091521000742 Published online by Cambridge University Press

A similar result of Theorem 3.5 is obtained in [58]. We use the preceding theorem to establish the extrapolation theory for $M^u_{p(\cdot),\omega}$. Let $p_0 \in (0,\infty)$, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < p_- \le p_+ < \infty$, $\omega \in A_{p(\cdot)}$ and $u(\cdot) : (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. The operator $\mathcal{R}$ is defined by

$$\mathcal{R}h = \sum_{k=0}^{\infty} \frac{\mathrm{M}^k(h)}{2^k \| \mathrm{M}^k \|_{\mathfrak{B}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}} \to \mathfrak{B}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}}}, \quad h \in L^1_{loc},$$

where M^k is the k iterations of the operator M. Notice that the operator $\mathcal{R}$ is depending on $p_0, p(\cdot), \omega$ and u.

Proposition 3.6. *Let $p_0 \in (0,\infty)$, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $p_0 < p_- \le p_+ < \infty$, $\omega \in A_{p(\cdot)}$ and $u(\cdot) : (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. If $\omega^{p_0} \in A_{p(\cdot)/p_0}$ and $u^{p_0} \in \mathbb{W}_{p(\cdot)/p_0,\omega^{p_0}}$, then $\mathcal{R}$ is well defined on $\mathfrak{B}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}$ and satisfies*

$$|h(x)| \le \mathcal{R}h(x), \tag{3.17}$$

$$\|\mathcal{R}h\|_{\mathfrak{B}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}} \le 2\|h\|_{\mathfrak{B}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}}, \quad h \in \mathfrak{B}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}, \tag{3.18}$$

$$[\mathcal{R}h]_{\mathcal{A}_1} \le 2\| \mathrm{M} \|_{\mathfrak{B}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}} \to \mathfrak{B}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}}, \quad h \in \mathfrak{B}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}. \tag{3.19}$$

Proof. According to Theorem 3.5, the Hardy–Littlewood maximal operator is bounded on $\mathfrak{B}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}$. Therefore, $\mathcal{R}$ is well defined on $\mathfrak{B}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}$.

In view of the definition of $\mathcal{R}$, (3.17) and (3.18) are valid. Since M is a sublinear operator, for any $h \in \mathfrak{B}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}$, we get

$$\begin{aligned}\mathrm{M}(\mathcal{R}h) &\le \sum_{k=0}^{\infty} \frac{\mathrm{M}^{k+1}(h)}{2^k \| \mathrm{M}^k \|_{\mathfrak{B}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}} \to \mathfrak{B}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}}} \\ &\le 2\| \mathrm{M} \|_{\mathfrak{B}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}} \to \mathfrak{B}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}} \mathcal{R}h.\end{aligned}$$

Consequently, $\mathcal{R}h \in \mathcal{A}_1$ and (3.19) is valid. □

We now establish the extrapolation theory for the weighted Morrey spaces with variable exponents.

Theorem 3.7. *Let $p_0 \in (0,\infty)$, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $p_0 < p_- \le p_+ < \infty$, $\omega^{p_0} \in A_{p(\cdot)/p_0}$ and $u^{p_0} \in \mathbb{W}_{p(\cdot)/p_0,\omega^{p_0}}$. Suppose that $\mathcal{F}$ be a family of pairs of non-negative Lebesgue*

https://doi.org/10.1017/S0013091521000742 Published online by Cambridge University Press

measurable functions such that for every

$$v \in \{\mathcal{R}h : h \in \mathfrak{b}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}\},$$

we have

$$\int_{\mathbb{R}^n} f(x)^{p_0} v(x)dx \le C \int_{\mathbb{R}^n} g(x)^{p_0} v(x)dx < \infty, \quad (f,g) \in \mathcal{F} \tag{3.20}$$

where C is independent of f and g. For any $(f,g) \in \mathcal{F}$ with $g \in M^u_{p(\cdot),\omega}$, we have

$$\|f\|_{M^u_{p(\cdot),\omega}} \le C\|g\|_{M^u_{p(\cdot),\omega}}. \tag{3.21}$$

Proof. Let $f \in \mathcal{M}$ with $(f,g) \in \mathcal{F}$ for some $g \in M^u_{p(\cdot),\omega}$. For any $h \in \mathfrak{b}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}$, (3.20) with $v = \mathcal{R}h$ and (3.17) assure that

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)|^{p_0}|h(x)|dx &\le \int_{\mathbb{R}^n} |f(x)|^{p_0}\mathcal{R}h(x)dx \\ &\le \int_{\mathbb{R}^n} |g(x)|^{p_0}\mathcal{R}h(x)dx \\ &\le C\||g|^{p_0}\|_{M^{u^{p_0}}_{p(\cdot)/p_0,\omega^{p_0}}}\|\mathcal{R}h\|_{\mathfrak{B}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}}.\end{aligned}$$

Thus, (3.18) gives

$$\int_{\mathbb{R}^n} |f(x)|^{p_0}|h(x)|dx \le C\|g\|^{p_0}_{M^u_{p(\cdot),\omega}}\|h\|_{\mathfrak{B}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}}$$

for some $C > 0$.

Furthermore, we have

$$\begin{aligned}&\sup\left\{\int_{\mathbb{R}^n} |f(x)|^{p_0}|h(x)|dx,\ h \in \mathfrak{b}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}},\right\} \\ &\le C\|g\|^{p_0}_{M^u_{p(\cdot),\omega}}.\end{aligned}$$

Proposition 3.3 guarantees that $f \in M^u_{p(\cdot),\omega}$ and

$$\begin{aligned}\|f\|^{p_0}_{M^u_{p(\cdot),\omega}} &= \||f|^{p_0}\|_{M^{u^{p_0}}_{p(\cdot)/p_0,\omega^{p_0}}} \\ &= \sup\left\{\int_{\mathbb{R}^n} |f(x)|^{p_0}|h(x)|dx,\ h \in \mathfrak{b}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}\right\} \\ &\le C\|g\|^{p_0}_{M^u_{p(\cdot),\omega}}\end{aligned}$$

for some $C > 0$. □

The above theorem is a refined version of the general extrapolation theory [51–52] by using the idea from [28]. The general extrapolation theory requires the validity of (3.20) for all $\omega \in A_1$ while Theorem 3.7 only requires the validity of (3.20) for

https://doi.org/10.1017/S0013091521000742 Published online by Cambridge University Press

$\omega \in \{\mathcal{R}h : h \in \mathfrak{b}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}\}$. This relaxation gives us extra flexibility to obtain the boundedness of operators without using the density argument.

Theorem 3.7 yields the boundedness of pseudo-differential operators and Fourier integral operators on $M^u_{p(\cdot),\omega}$.

Definition 3.4. Let $m \in \mathbb{R}$. A smooth function $a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is a symbol of order m if for any multi-indices α, β, we have a constant $C > 0$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \le C(1+|\xi|)^{m-\beta}.$$

Let a be a symbol of order m. The pseudo-differential operator with symbol a is defined as

$$T_a f(x) = \int_{\mathbb{R}^n} a(x,\xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi, \quad f \in \mathcal{S}$$

where $\hat{f}$ is the Fourier transform of f.

In view of [41, Theorem 2.12], we have the following weighted norm inequality for the pseudo-differential operators of order 0.

Theorem 3.8. *Let $p \in (1,\infty)$, $\omega \in \mathcal{A}_p$ and a be a symbol of order 0. The pseudo-differential operator T_a is bounded on $L^p(\omega)$.*

We are now ready to present the boundedness of pseudo-differential operators on the weighted Morrey spaces with variable exponents. Notice that in view of [60], T_a is bounded on $\mathcal{S}'$. Thus, T_a is well defined on $M^u_{p(\cdot),\omega}$.

Theorem 3.9. *Let $p_0 \in (0,\infty)$, $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be a symbol of order 0, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $p_0 < p_- \le p_+ < \infty$. If $\omega^{p_0} \in A_{p(\cdot)/p_0}$ and $u^{p_0} \in \mathbb{W}_{p(\cdot)/p_0,\omega^{p_0}}$, then T_a is bounded on $M^u_{p(\cdot),\omega}$.*

Proof. For any $f \in M^u_{p(\cdot),\omega}$ and $h \in \mathfrak{b}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}$ Lemma 3.1 and (3.18) guarantee that

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)|^{p_0}\mathcal{R}h(x)dx &\le \||f|^{p_0}\|_{M^{u^{p_0}}_{p(\cdot)/p_0,\omega^{p_0}}}\|\mathcal{R}h\|_{\mathfrak{B}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}} \\ &\le C\|f\|^{p_0}_{M^u_{p(\cdot),\omega}}\|h\|_{\mathfrak{B}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}} < \infty.\end{aligned}$$

That is,

$$M^u_{p(\cdot),\omega} \hookrightarrow \bigcap_{h\in\mathfrak{b}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}} L^{p_0}(\mathcal{R}h). \tag{3.22}$$

Define $\mathcal{F}_0 = \{(|T_a f|, |f|) : f \in M^u_{p(\cdot),\omega}\}$. For any

$$v \in \left\{\mathcal{R}h : h \in \mathfrak{b}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}\right\},$$

(3.22) ensures that $M^u_{p(\cdot),\omega} \hookrightarrow L^{p_0}(v)$. Theorem 3.8 guarantees that (3.20) is valid for $\mathcal{F}_0$. Theorem 3.7 asserts that $\|T_a f\|_{M^u_{p(\cdot),\omega}} \le C\|f\|_{M^u_{p(\cdot),\omega}}$, $\forall f \in M^u_{p(\cdot),\omega}$, for some $C > 0$. $\square$

https://doi.org/10.1017/S0013091521000742 Published online by Cambridge University Press

Corollary 3.10. *Let $p \in (1,\infty)$, $\omega^p \in \mathcal{A}_p$ $u \in \mathbb{W}_{p,\omega}$ and $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be a symbol of order* 0*. The pseudo-differential operator T_a is bounded on the weighted Morrey space $M^u_{p,\omega}$.*

Proof. In view of the left openness property of Muckenhoupt weight functions [17, Corollary 9.2.6], we have a $p_0 \in (1,p)$ such that $\omega^p \in \mathcal{A}_{p/p_0}$. Whenever $p(\cdot) = p$, $p \in (1,\infty)$ is a constant function, we have $\omega^{p_0} \in A_{p(\cdot)/p_0}$ if and only if $\omega^p \in \mathcal{A}_{p/p_0}$. Thus, we have $\omega^{p_0} \in A_{p(\cdot)/p_0}$.

Since $u \in \mathbb{W}_{p,\omega}$, (3.9) gives

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^p_\omega}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^p_\omega}} u(x, 2^{j+1}r) \leq Cu(x,r).$$

As $p_0 > 1$, we find that

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p/p_0}_{\omega^{p_0}}}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p/p_0}_{\omega^{p_0}}}} u(x, 2^{j+1}r)^{p_0} &= \sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|^{p_0}_{L^p_\omega}}{\|\chi_{B(x,2^{j+1}r)}\|^{p_0}_{L^p_\omega}} u(x, 2^{j+1}r)^{p_0} \\ &\leq \left(\sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^p_\omega}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^p_\omega}} u(x, 2^{j+1}r) \right)^{p_0} \leq Cu(x,r)^{p_0}.\end{aligned}$$

Therefore, $u^{p_0} \in \mathbb{W}_{p/p_0,\omega^{p_0}}$. Theorem 3.9 yields the boundedness of T_a on the weighted Morrey space $M^u_{p,\omega}$. □

One of the essential components in the proof of [41, Theorem 2.12] is the density of $\mathcal{S}$ in $L^p(\omega)$. As $\mathcal{S}$ is not necessary dense in $M^u_{p(\cdot),\omega}$, the argument in [41, Theorem 2.12] cannot directly apply to obtain the boundedness of T_a on $M^u_{p(\cdot),\omega}$. We can overcome this obstacle because we have the embedding (3.22) and Theorem 3.7 requires the validity of 3.20 for all $\omega \in \{\mathcal{R}h : h \in \mathfrak{b}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}\}$ only.

There are a number of extensions on the study of the boundedness of pseudo-differential operators on weighted Lebesgue spaces [40, 49, 53, 54, 68]. By using Theorem 3.7, we can extend the results in [40, 49, 53, 68] to the weighted Morrey spaces with variable exponents. For brevity, we omit the details and leave them to the readers.

We turn to the study of Fourier integral operators. We recall some definitions and notions from [12].

Definition 3.5. Let $m \in \mathbb{R}$ and $0 \leq \rho \leq 1$. A Lebesgue measurable function $a(x,\xi)$ which is smooth in the frequency variable ξ and bounded in the spatial variable x, is said to belong to $L^\infty S^m_\rho$ if for all multi-indices α, a satisfies

$$\sup_{\xi \in \mathbb{R}^n} (1+|\xi|^2)^{\frac{-m+\rho|\alpha|}{2}} \|\partial^\alpha_\xi a(\cdot,\xi)\|_{L^\infty(\mathbb{R}^n)} < \infty.$$

Comparing to the classical symbol class $S^m_{\rho,\delta}$ for the pseudo-differential operators [60, Chapter VII, Section 1.1 (3)], Definition 3.5 gives a class of symbols where smoothness of the x variable is not required.

https://doi.org/10.1017/S0013091521000742 Published online by Cambridge University Press

Let $\varphi(x,\xi)$ be a smooth function and homogeneous of degree 1 in the frequency variable and the amplitude $a(x,\xi) \in L^\infty S^m_\rho$ where $m \in \mathbb{R}$ and $0 \le \rho \le 1$. The Fourier integral operator associated with a and φ is defined as

$$T_{a,\varphi}f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} a(x,\xi)\hat{f}(\xi)d\xi.$$

We now state some conditions for the study of Fourier integral operators.

Definition 3.6. Let $k \in \mathbb{N} \cup \{0\}$. A real-valued function $\varphi(x,\xi)$ belongs to the class $L^\infty\Phi^k$ if it is homogeneous of degree 1 and smooth on $\mathbb{R}^n\backslash\{0\}$ in the frequency variable ξ, bounded measurable in the spatial variable x and for all multi-indices $|\alpha| \ge k$, it satisfies

$$\sup_{\xi\in\mathbb{R}^n\backslash\{0\}} |\xi|^{-1+|\alpha|} \|\partial^\alpha_\xi \varphi(\cdot,\xi)\|_{L^\infty} < \infty.$$

Definition 3.7. A real-valued function φ satisfies the rough non-degeneracy condition if it is C^1 on $\mathbb{R}^n\backslash\{0\}$ in the frequency variable ξ, bounded measurable in the spatial variable x and there exists a constant $c > 0$ such that for all $x, y \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n\backslash\{0\}$, it satisfies

$$|\partial_\xi\varphi(x,\xi) - \partial_\xi\varphi(y,\xi)| \ge c|x-y|.$$

We have the following boundedness result for $T_{a,\varphi}$ on weighted Lebesgue spaces [12, Theorem 3.11].

Theorem 3.11. *Let $1 < p < \infty$, $\rho \in [0,1]$ and $a \in L^\infty S^{-(\frac{n+1}{2})\rho+n(\rho-1)}$. Suppose that a and φ satisfy either*

(1) a is compactly supported in the spatial variable x and the phase function $\varphi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n\backslash\{0\})$ is positively homogeneous of degree 1 in the frequency variable ξ and satisfies $\det\partial^2_{x\xi}\varphi(x,\xi) \neq 0$ and $\operatorname{rank}\partial^2_{\xi\xi}\varphi(x,\xi) = n-1$ or

(2) $\varphi(x,\xi) - \langle x,\xi\rangle \in L^\infty\Phi^1$, φ satisfies the rough non-degeneracy condition and $|\det_{n-1}\partial^2_{\xi\xi}\varphi(\xi,\xi)| \ge c > 0$.

Then for any $\omega \in \mathcal{A}_p$, the Fourier integral operator $T_{a,\varphi}$ is bounded on $L^p(\omega)$.

We now present the boundedness of the Fourier integral operator $T_{a,\varphi}$ on $M^u_{p(\cdot),\omega}$.

Theorem 3.12. *Let $\rho \in [0,1]$ and $a \in L^\infty S^{-(\frac{n+1}{2})\rho+n(\rho-1)}$. Suppose that a and φ satisfy either (1) or (2) in Theorem 3.11.*

Let $p_0 \in (1,\infty)$, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $p_0 < p_- \le p_+ < \infty$. If $\omega^{p_0} \in A_{p(\cdot)/p_0}$ and $u^{p_0} \in \mathbb{W}_{p(\cdot)/p_0,\omega^{p_0}}$, then $T_{a,\varphi}$ is bounded on $M^u_{p(\cdot),\omega}$.

By using Theorem 3.11 instead of Theorem 3.8, the proof of the preceding theorem follows from the proof of Theorem 3.9. Thus, for brevity, we omit the details.

Corollary 3.13. *Let $\rho \in [0,1]$ and $a \in L^\infty S^{-(\frac{n+1}{2})\rho+n(\rho-1)}$. Suppose that a and φ satisfy either (1) or (2) in Theorem 3.11. Let $p \in (1,\infty)$ and $\omega^p \in \mathcal{A}_p$ $u \in \mathbb{W}_{p,\omega}$. The Fourier integral operator $T_{a,\varphi}$ is bounded on the weighted Morrey space $M^u_{p,\omega}$.*

https://doi.org/10.1017/S0013091521000742 Published online by Cambridge University Press

4. Weighted Triebel–Lizorkin–Morrey spaces with variable exponents

In this section, we extend the study in the previous section to the weighted Triebel–Lizorkin–Morrey spaces with variable exponents. For any $f \in \mathcal{S}'$, let us denote the Fourier transform of f by $\hat{f}$.

Definition 4.1. Let $\alpha \in \mathbb{R}$, $1 \le q < \infty$, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $0 < p_- \le p_+ < \infty$, $\omega : \mathbb{R}^n \to (0,\infty)$ and $u(\cdot) : (0,\infty) \to (0,\infty)$ be Lebesgue measurable functions. The weighted Triebel–Lizorkin–Morrey space with variable exponent $F^{\alpha,q,u}_{p(\cdot),\omega}$ consists of those $f \in \mathcal{S}'$ satisfying

$$\|f\|_{F^{\alpha,q,u}_{p(\cdot),\omega}} = \left\| \left(\sum_{j=0}^{\infty} (2^{j\alpha}|f * \varphi_j|)^q \right)^{\frac{1}{q}} \right\|_{M^u_{p(\cdot),\omega}} < \infty, \tag{4.1}$$

where $\varphi_0 \in \mathcal{S}$ satisfies

$$\operatorname{supp} \hat{\varphi}_0 \subseteq \{x \in \mathbb{R}^n : |x| \le 2\} \quad \textit{and} \quad |\hat{\varphi}_0(\xi)| \ge C, \, |x| \le 3/2 \tag{4.2}$$

and $\varphi_j(x) = 2^{jn}\varphi(2^j x)$, $j \ge 1$ where $\varphi \in \mathcal{S}$ satisfying

$$\operatorname{supp} \hat{\varphi} \subseteq \{x \in \mathbb{R}^n : 1/2 \le |x| \le 2\} \tag{4.3}$$

$$|\hat{\varphi}(\xi)| \ge C, \quad 3/5 \le |x| \le 5/3 \tag{4.4}$$

for some $C > 0$.

The above definition is a special case of the Triebel–Lizorkin type spaces defined in [37]. In particular, when $\omega \equiv 1$, the weighted Triebel–Lizorkin–Morrey space with variable exponent becomes the inhomogeneous version of the Triebel–Lizorkin–Morrey space with variable exponent [23, Definition 6.6] with $q(\cdot)$ being a constant function. When $\omega \equiv 1$ and $p(\cdot)$ is a constant function, then $F^{\alpha,q,u}_{p(\cdot),\omega}$ reduces to the inhomogeneous version of the Triebel–Lizorkin–Morrey spaces studied in [61, 62]. Moreover, when $\omega \equiv 1$ and $u \equiv 1$, $F^{\alpha,q,u}_{p(\cdot),\omega}$ is the inhomogeneous version of the variable Triebel–Lizorkin spaces introduced in [67, Definition 2].

Let $p, q \in (0,\infty)$, $\alpha \in \mathbb{R}$ and $\omega : \mathbb{R}^n \to (0,\infty)$, the weighted Triebel–Lizorkin space $F^{\alpha,q}_p(\omega)$ consists if all $f \in \mathcal{S}'$ satisfying

$$\|f\|_{F^{\alpha,q}_p(\omega)} = \left\| \left(\sum_{j=0}^{\infty} (2^{j\alpha}|f * \varphi_j|)^q \right)^{\frac{1}{q}} \right\|_{L^p(\omega)} < \infty$$

where φ_0 and φ satisfy (4.2) and (4.3), see [5, 6] and [13, p.124]. We see that whenever $p(\cdot) = p$, $p \in (0,\infty)$ is a constant function, $F^{\alpha,q,u}_{p(\cdot),\omega} = F^{\alpha,q}_p(\omega^p)$.

We first show that the definition of the weighted Triebel–Lizorkin–Morrey space with variable exponent is independent of the choice of functions φ_0, φ satisfying (4.2)–(4.4).

Theorem 4.1. *Let $\alpha \in \mathbb{R}$, $1 \le q < \infty$, $p_0 \in (0,\infty)$, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $p_0 < p_- \le p_+ < \infty$. If $\omega^{p_0} \in A_{p(\cdot)/p_0}$ and $u^{p_0} \in \mathbb{W}_{p(\cdot)/p_0,\omega^{p_0}}$. Let $\varphi_0, \psi_0 \in \mathcal{S}$ satisfy (4.2) and $\varphi, \psi \in \mathcal{S}$*

https://doi.org/10.1017/S0013091521000742 Published online by Cambridge University Press

satisfy (4.3) and (4.4). There exist constants $C_0, C_1 > 0$ such that for any $f \in \mathcal{S}'$, we have

$$C_0 \left\| \left(\sum_{j=0}^{\infty} (2^{j\alpha} |f * \varphi_j|)^q \right)^{\frac{1}{q}} \right\|_{M^u_{p(\cdot),\omega}} \leq \left\| \left(\sum_{j=0}^{\infty} (2^{j\alpha} |f * \psi_j|)^q \right)^{\frac{1}{q}} \right\|_{M^u_{p(\cdot),\omega}} \leq C_1 \left\| \left(\sum_{j=0}^{\infty} (2^{j\alpha} |f * \varphi_j|)^q \right)^{\frac{1}{q}} \right\|_{M^u_{p(\cdot),\omega}}. \tag{4.5}$$

Proof. In view of [13, Remark 2.6 and Proposition 10.14], for any $\omega \in \mathcal{A}_1$, we have

$$C_0 \left\| \left(\sum_{j=0}^{\infty} (2^{j\alpha} |f * \varphi_j|)^q \right)^{\frac{1}{q}} \right\|_{L^p(\omega)} \leq \left\| \left(\sum_{j=0}^{\infty} (2^{j\alpha} |f * \psi_j|)^q \right)^{\frac{1}{q}} \right\|_{L^p(\omega)} \leq C_1 \left\| \left(\sum_{j=0}^{\infty} (2^{j\alpha} |f * \varphi_j|)^q \right)^{\frac{1}{q}} \right\|_{L^p(\omega)}. \tag{4.6}$$

Notice that the results given in [13, Remark 2.6 and Proposition 10.14] are for the homogeneous Triebel–Lizorkin spaces, as stated at [13, Section 12], the results also apply to the inhomogeneous Triebel–Lizorkin spaces.

We denote the weighted Triebel–Lizorkin–Morrey space with variable exponent generated by (φ_0, φ) and (ψ_0, ψ) by $F^{\alpha,q,u}_{p(\cdot),\omega}(\varphi)$ and $F^{\alpha,q,u}_{p(\cdot),\omega}(\psi)$, respectively.

The embedding (3.22) guarantees that for any $f \in F^{\alpha,q,u}_{p(\cdot),\omega}(\psi)$

$$\left(\sum_{j=0}^{\infty} (2^{j\alpha} |f * \psi_j|)^q \right)^{\frac{1}{q}} \in \bigcap_{h \in \mathfrak{b}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}} L^{p_0}(\mathcal{R}h).$$

Define

$$\mathcal{F} = \left\{ \left(\left(\sum_{j=0}^{\infty} (2^{j\alpha} |f * \varphi_j|)^q \right)^{\frac{1}{q}}, \left(\sum_{j=0}^{\infty} (2^{j\alpha} |f * \psi_j|)^q \right)^{\frac{1}{q}} \right) : f \in F^{\alpha,q,u}_{p(\cdot),\omega}(\psi) \right\}.$$

The first inequality in (4.6) shows that (3.20) is valid for $\mathcal{F}$. Theorem 3.7 yields a constant $C_0 > 0$ such that for any $f \in F^{\alpha,q,u}_{p(\cdot),\omega}(\psi)$

$$C_0 \left\| \left(\sum_{j=0}^{\infty} (2^{j\alpha} |f * \varphi_j|)^q \right)^{\frac{1}{q}} \right\|_{M^u_{p(\cdot),\omega}} \leq \left\| \left(\sum_{j=0}^{\infty} (2^{j\alpha} |f * \psi_j|)^q \right)^{\frac{1}{q}} \right\|_{M^u_{p(\cdot),\omega}}.$$

We establish the first inequality in (4.9). The second inequality in (4.9) follows similarly. □

We study the boundedness of pseudo-differential operators on $F^{\alpha,q,u}_{p(\cdot),\omega}$. Let L be a non-negative integer, $r \in (0,\infty)\backslash\mathbb{N}$ and $\delta \in [0,1]$. Suppose that a satisfies

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leq \frac{1}{(1+|\xi|)^{|\alpha|}},\ |\alpha| \leq L, \tag{4.7}$$

$$|\partial_x^\beta \partial_\xi^\alpha a(x+y,\xi) - \partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leq \frac{|y|^{r-[r]}}{(1+|\xi|)^{|\alpha|-\delta r}},\ |\alpha| \leq L,\ |\beta| = [r] \tag{4.8}$$

where $[r]$ is the integral part of r.

According to [54, Theorem 1 and Remaek 3], we have the boundedness result for the pseudo-differential operators on the weighted Triebel–Lizorkin spaces.

Theorem 4.2. *Let $q \in (1,2]$ and a satisfies (4.7) and (4.8) with $L = [n/q]+1$. If $\alpha \in ((\delta-1)r, r)$, $q \leq p < \infty$ and $\omega \in \mathcal{A}_{p/q}$, then T_a is bounded on $F^{\alpha,q}_p(\omega)$.*

We now extend the boundedness of T_a to $F^{\alpha,q,u}_{p(\cdot),\omega}$.

Theorem 4.3. *Let $q \in (1,2]$, $\alpha \in ((\delta-1)r, r)$ and a satisfies (4.7) and (4.8) with $L = [n/q]+1$, $p_0 \in [q,\infty)$, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $p_0 < p_- \leq p_+ < \infty$. If $\omega^{p_0} \in A_{p(\cdot)/p_0}$ and $u^{p_0} \in \mathbb{W}_{p(\cdot)/p_0,\omega^{p_0}}$, then T_a is bounded on $F^{\alpha,q,u}_{p(\cdot),\omega}$.*

Proof. In view of Theorem 4.2, for any $\omega \in \mathcal{A}_1 \subseteq \mathcal{A}_{p_0/q}$, the pseudo-differential operator T_a is bounded on $F^{\alpha,q}_{p_0}(\omega)$. In view of (3.19) and the embedding (3.22) guarantees that

$$F^{\alpha,q,u}_{p(\cdot),\omega} \hookrightarrow \bigcap_{h \in \mathfrak{b}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}} F^{\alpha,u}_{p_0}(\mathcal{R}h), \tag{4.9}$$

we are allowed to apply Theorem 3.7 with

$$\mathcal{F} = \left\{ \left(\left(\sum_{j=0}^{\infty} (2^{j\alpha}|T_a f * \varphi_j|)^q \right)^{\frac{1}{q}}, \left(\sum_{j=0}^{\infty} (2^{j\alpha}|f * \varphi_j|)^q \right)^{\frac{1}{q}} \right) : f \in F^{\alpha,q,u}_{p(\cdot),\omega} \right\}$$

and obtain the boundedness of T_a on $F^{\alpha,q,u}_{p(\cdot),\omega}$. □

As a special case of Theorem 4.3, we have the boundedness of the pseudo-differential operator T_a on the weighted Triebel–Lizorkin–Morrey spaces.

Corollary 4.4. *Let $q \in (1,2]$ and a satisfies (4.7) and (4.8) with $L = [n/q]+1$. If $\alpha \in ((\delta-1)r, r)$, $q \leq p < \infty$, $\omega^p \in \mathcal{A}_{p/q}$ and $u \in \mathbb{W}_{p,\omega}$, then T_a is bounded on $F^{\alpha,q,u}_{p,\omega}$.*

Proof. As $\omega^p \in \mathcal{A}_{p/q}$, the left openness property of the Muckenhoupt weighted functions yields a $p_0 \in (q,\infty)$ such that $\omega^p \in \mathcal{A}_{p/p_0}$. Therefore, $\omega^{p_0} \in A_{p(\cdot)/p_0}$ where $p(\cdot) \equiv p$. The proof of the membership $u^{p_0} \in \mathbb{W}_{p/p_0,\omega^{p_0}}$ is the same as the proof in Corollary 3.10. Thus, Theorem 4.3 yields the boundedness of T_a. □

https://doi.org/10.1017/S0013091521000742 Published online by Cambridge University Press

There are some other results on the boundedness of pseudo-differential operators on the weighted Triebel–Lizorkin spaces such as [4, Section 3.4 (c)]. We can use the method in Theorem 4.2 to extend the result in [4, Section 3.4 (c)] to $M^u_{p(\cdot),\omega}$. For brevity, we leave the details to the readers.

At the end of this section, we consider the boundedness of Fourier integral operators on $F^{\alpha,q,u}_{p(\cdot),\omega}$.

Theorem 4.5. *Let $\alpha\in\mathbb{R}$, $1\le q<\infty$, $p\in(1,\infty)$, $\omega\in\mathcal{A}_p$, $a(\xi)$ is a symbol of order $-\frac{n+1}{2}$ and $\varphi\in C^\infty(\mathbb{R}^n\backslash\{0\})$ is a positive homogeneous function of degree 1. If $|\det_{n-1}\partial^2_{\xi\xi}\varphi(\xi)|\ge c>0$ and for any $|\alpha|\ge 1$, there is a constant $C>0$ such that*

$$|\partial^\alpha_\xi\varphi(\xi)|\le C|\xi|^{1-|\alpha|},\quad \xi\in\mathbb{R}^n\backslash\{0\}, \tag{4.10}$$

then the Fourier integral operator

$$Tf(x)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{i\varphi(\xi)+ix\cdot\xi}a(\xi)\hat{f}(\xi)d\xi$$

is bounded on $F^{\alpha,q}_p(\omega)$.

The reader is referred to [12, Theorem 4.1.4] for the proof of the above theorem. In view of the embedding (4.9) and the preceding theorem, T is well defined on $F^{\alpha,q,u}_{p(\cdot),\omega}$.

The above theorem and Theorem 3.7 yield the boundedness of the Fourier integral operator T on $F^{\alpha,q,u}_{p(\cdot),\omega}$.

Theorem 4.6. *Let $\alpha\in\mathbb{R}$, $1\le q<\infty$, $a(\xi)$ is a symbol of order $-\frac{n+1}{2}$ and $\varphi\in C^\infty(\mathbb{R}^n\backslash\{0\})$ is a positive homogeneous function of degree 1. satisfying $|\det_{n-1}\partial^2_{\xi\xi}\varphi(\xi)|\ge c>0$ and (4.10). Suppose that $p(\cdot)\in C^{\log}(\mathbb{R}^n)$. If the exists a $p_0\in(1,\infty)$ such that $p_0<p_-\le p_+<\infty$, $\omega^{p_0}\in A_{p(\cdot)/p_0}$ and $u^{p_0}\in\mathbb{W}_{p(\cdot)/p_0,\omega^{p_0}}$, then the Fourier integral operator T is bounded on $F^{\alpha,q,u}_{p(\cdot),\omega}$.*

As the proof of the preceding theorem is similar to the proof of Theorem 4.3, for simplicity, we omit the details.

Corollary 4.7. *Let $\alpha\in\mathbb{R}$, $1\le q<\infty$, $a(\cdot)$ is a symbo0l of order $-\frac{n+1}{2}$ and $\varphi\in C^\infty(\mathbb{R}^n\backslash\{0\})$ is a positive homogeneous function of degree 1. satisfying $|\det_{n-1}\partial^2_{\xi\xi}\varphi(\xi)|\ge c>0$ and (4.10). Let $p\in(1,\infty)$ and $\omega^p\in\mathcal{A}_p$ $u\in\mathbb{W}_{p,\omega}$. The Fourier integral operator T is bounded on the weighted Triebel–Lizorkin–Morrey space $F^{\alpha,q,u}_{p,\omega}$.*

5. Weighted Hardy–Morrey spaces with variable exponents

In this section, we study the weighted Hardy–Morrey spaces with variable exponents by using the extrapolation theory. This approach had been used in [27, 30, 31] for the weighted Hardy spaces with variable exponents, the Orlicz-slice Hardy spaces and the Hardy local Morrey spaces with variable exponents.

We begin with the definition of the weighted Hardy–Morrey spaces with variable exponents. Let $\mathcal{F} = \{\|\cdot\|_{\alpha_i,\beta_i}\}$ be any finite collection of semi-norms on $\mathcal{S}$ and

$$\mathcal{S}_{\mathcal{F}} = \{\psi \in \mathcal{S} : \|\psi\|_{\alpha_i,\beta_i} \leq 1, \text{ for all } \|\cdot\|_{\alpha_i,\beta_i} \in \mathcal{F}\}.$$

For any $f \in \mathcal{S}'$, write

$$\mathcal{M}_{\mathcal{F}} f(x) = \sup_{\psi \in \mathcal{S}_{\mathcal{F}}} \sup_{t>0} |(f * \psi_t)(x)|$$

where for any $t > 0$, write $\psi_t(x) = t^{-n}\psi(x/t)$.

Definition 5.1. Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$, $\omega : \mathbb{R}^n \to (0,\infty)$ and $u : (0,\infty) \to (0,\infty)$ be Lebesgue measurable functions. The weighted Hardy–Morrey space with variable exponent $\mathcal{H}^u_{p(\cdot),\omega}$ consists of all $f \in \mathcal{S}'$ satisfying

$$\|f\|_{\mathcal{H}^u_{p(\cdot),\omega}} = \|\mathcal{M}_{\mathcal{F}} f\|_{M^u_{p(\cdot),\omega}} < \infty.$$

When $\omega \equiv 1$, the weighted Hard–Morrey spaces with variable exponents become the Hardy–Morrey spaces with variable exponents [25, 29]. When $u \equiv 1$, the weighted Hardy Morrey spaces with variable exponents reduce to the Hardy spaces with variable exponents [26, 27]. When $p(\cdot) = p$, $p \in (0,1]$, is a constant function, $\mathcal{H}^u_{p(\cdot),\omega}$ becomes the weighted Hardy–Morrey space $\mathcal{H}^u_{p,\omega}$. For $p \in (0,1]$ and $\omega : \mathbb{R}^n \to (0,\infty)$, the weighted Hardy space $H^p(\omega)$ consists of all $f \in \mathcal{S}'$ satisfies

$$\|f\|_{H^p(\omega)} = \|\mathcal{M}_{\mathcal{F}} f\|_{L^p(\omega)} < \infty.$$

We recall the definition of regular Calderón–Zygmund operators from [15, Definitions 2.1, 2.2 and 2.6].

Definition 5.2. We say that T is a singular integral operator with kernel $K(x,y)$ if for every bounded function f with compact support

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\,dy, \quad x \in \mathbb{R}^n \backslash \mathrm{supp} f.$$

If T is bounded on L^p for some $p \in (1,\infty)$, T is called as a Calderón–Zygmund operator.

Let $\gamma > 0$. We say that K is γ-regular with respect to y if for every multi-index α with $|\alpha| < \gamma$,

$$|\partial^\alpha_y K(x,y)| \leq C|x-y|^{-n-|\alpha|}$$

and for those multi-index satisfying $\alpha = [\gamma]$

$$|\partial^\alpha_y K(x,y) - \partial^\alpha_y K(x,z)| \leq C\frac{|y-z|^{\gamma-|\alpha|}}{|x-y|^{n+\gamma}}$$

whenever $|y-z| \leq \frac{1}{2}|x-y|$.

The following theorem gives the mapping properties of the Calderón–Zygmund operator on the weighted Hardy spaces.

https://doi.org/10.1017/S0013091521000742 Published online by Cambridge University Press

Theorem 5.1. *Let $r \in (0,1]$, $\gamma > n(\frac{1}{r}-1)$, $\epsilon > 0$ and $\omega \in \mathcal{A}_1$. If T is a Calderón–Zygmund operator with kernel γ-regular with respect to y and ϵ-regular with respect to x, then $T : H^r(\omega) \to L^r(\omega)$ is bounded.*

The reader is referred to [15, Theorem 2.8] for the proof of Theorem 5.1.

Theorem 5.2. *Let $p_0 \in (0,1)$, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $p_0 < p_- \le p_+ < \infty$, $\omega^{p_0} \in A_{p(\cdot)/p_0}$ and $u^{p_0} \in \mathbb{W}_{p(\cdot)/p_0,\omega^{p_0}}$.*

Let $\gamma > n(\frac{1}{p_0}-1)$ and $\epsilon > 0$. If T is a Calderón–Zygmund operator with kernel γ-regular with respect to y and ϵ-regular with respect to x, then $T : \mathcal{H}^u_{p(\cdot),\omega} \to M^u_{p(\cdot),\omega}$ is bounded.

Proof. Whenever $f \in \mathcal{H}^u_{p(\cdot),\omega}$, according to (3.22), we have

$$\mathcal{M}_{\mathcal{F}} f \in M^u_{p(\cdot),\omega} \hookrightarrow \bigcap_{h \in \mathfrak{b}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}} L^{p_0}(\mathcal{R}h). \tag{5.1}$$

That is,

$$\mathcal{H}^u_{p(\cdot),\omega} \hookrightarrow \bigcap_{h \in \mathfrak{b}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}} H^{p_0}(\mathcal{R}h). \tag{5.2}$$

Consequently, (3.19) and Theorem 5.1 show that for any

$$v \in \left\{\mathcal{R}h : h \in \mathfrak{b}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}\right\},$$

we have a constant $C > 0$ such that for any $f \in \mathcal{H}^u_{p(\cdot),\omega}$

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0} v(x)dx \le C \int_{\mathbb{R}^n} (\mathcal{M}_{\mathcal{F}} f(x))^{p_0} v(x)dx.$$

Applying Theorem 3.7 with

$$\mathcal{F} = \{(|Tf|, \mathcal{M}_{\mathcal{F}} f) : f \in \mathcal{H}^u_{p(\cdot),\omega}\},$$

we obtain a constant $C > 0$ such that for any $f \in \mathcal{H}^u_{p(\cdot),\omega}$, we have

$$\|Tf\|_{M^u_{p(\cdot),\omega}} \le C\|\mathcal{M}_{\mathcal{F}} f\|_{M^u_{p(\cdot),\omega}} = C\|f\|_{\mathcal{H}^u_{p(\cdot),\omega}}. \quad \blacksquare$$

In particular, we have the mapping properties of the Calderón-Zygmund operators on the weighted Hardy–Morrey spaces. □

Corollary 5.3. *Let $p \in (0,1]$ and $\omega^p \in \mathcal{A}_1$. If there exists a $p_0 \in (0,p)$ such that $u^{p_0} \in \mathbb{W}_{p/p_0,\omega^{p_0}}$ and T is a Calderón–Zygmund operator with kernel γ-regular with respect to y and ϵ-regular with respect to x, where $\gamma > n(\frac{1}{p_0}-1)$ and $\epsilon > 0$, then $T : \mathcal{H}^u_{p,\omega} \to M^u_{p,\omega}$ is bounded.*

https://doi.org/10.1017/S0013091521000742 Published online by Cambridge University Press

Proof. For any $p_0 \in (0,p)$ $\omega^p \in \mathcal{A}_1 \subset \mathcal{A}_{p/p_0}$. Consequently, we have $\omega^{p_0} \in A_{p(\cdot)/p_0}$ where $p(\cdot) \equiv p$. Therefore, the conditions in Theorem 5.2 are fulfilled and, hence, the Calderón-Zygmund operator $T : \mathcal{H}^u_{p,\omega} \to M^u_{p,\omega}$ is bounded. □

The mapping properties of Calderón–Zygmund operators on the weighted Morrey spaces with variable exponents were established in [20, Theorem 4.3].

In [15], the boundedness of the Calderón–Zygmund operators is used to establish the wavelet characterization of weighted Hardy spaces. We also have the wavelet characterization for the weighted Hardy–Morrey spaces with variable exponents. We obtain this characterization by solely using Theorem 3.7. For the wavelet characterizations of the Besov–Morrey spaces and the Triebel–Lizorkin–Morrey spaces, the reader is referred to [56].

We recall some definitions from wavelet theory. A function $\psi \in L^2(\mathbb{R})$ is an orthonormal wavelet if the system

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \quad j,k \in \mathbb{Z}$$

is an orthonormal basis for $L^2(\mathbb{R})$.

For any wavelet ψ, write

$$\mathcal{W}_\psi f = \left(\sum_{j,k\in\mathbb{Z}} 2^j |\langle f, \psi_{j,k}\rangle|^2 \chi_{I_{j,k}} \right)^{1/2}$$

where $I_{j,k} = [2^{-j}k, 2^{-j}(k+1)]$.

We have the following wavelet characterization of the weighted Hardy spaces [16, Theorem 4.2].

Theorem 5.4. *Let $\alpha \geq 1$, $\frac{1}{\alpha} \leq p$, $\omega \in \mathcal{A}_1$ and $\psi \in C^{[\alpha]}$. If there exist $C, r, \epsilon > 0$ such that*

$$\int_{\mathbb{R}} x^k \psi(x) dx = 0, \quad k = 0, 1 \cdots, [\alpha] - 1, \tag{5.3}$$

$$|\psi(x)| \leq C(1 + |x|)^{-[\alpha]-r-1}, \quad \forall x \in \mathbb{R}, \tag{5.4}$$

$$|\psi^{(k)}(x)| \leq C(1 + |x|)^{-\alpha-\epsilon}, \quad \forall x \in \mathbb{R}, \quad \text{and} \quad k = 0, 1 \cdots, [\alpha], \tag{5.5}$$

then there exist $C_0 > C_1 > 0$ such that

$$C_0 \|f\|_{H^p(\omega)} \leq \|\mathcal{W}_\psi f\|_{L^p(\omega)} \leq C_1 \|f\|_{H^p(\omega)}.$$

For any $w \in \mathcal{A}_1$, Theorem 5.4 guarantees that the linear functional $l_{\psi_{j,k}}(f) = \langle f, \psi_{j,k}\rangle$ is bounded on the weighted Hardy space $H^p(w)$. In view of the embedding (5.2), we see that the functional $l_{\psi_{j,k}}$ is also bounded on $\mathcal{H}^u_{p(\cdot),\omega}$. Thus, $\langle f, \psi_{j,k}\rangle$ and $\mathcal{W}_\psi f$ are well defined on $\mathcal{H}^u_{p(\cdot),\omega}$. For the dual spaces of the weighted Hardy spaces, the reader is referred to [14, Theorem II.4.4].

https://doi.org/10.1017/S0013091521000742 Published online by Cambridge University Press

We give the wavelet characterizations of the weighted Hardy–Morrey spaces with variable exponents in the following theorem.

Theorem 5.5. *Let $\alpha \geq 1$ and $\psi \in C^{[\alpha]}$. Suppose that ψ satisfies (5.3)-(5.5). Let $p_0 \in (\alpha^{-1}, 1)$, $p(\cdot) \in C^{\log}(\mathbb{R})$ with $p_0 < p_- \leq p_+ < \infty$. If $\omega^{p_0} \in A_{p(\cdot)/p_0}$ and $u^{p_0} \in \mathbb{W}_{p(\cdot)/p_0,\omega^{p_0}}$, then there exist $C_0 > C_1 > 0$ such that for any $f \in \mathcal{H}^u_{p(\cdot),\omega}$*

$$C_0\|f\|_{\mathcal{H}^u_{p(\cdot),\omega}} \leq \|\mathcal{W}_\psi f\|_{M^u_{p(\cdot),\omega}} \leq C_1\|f\|_{\mathcal{H}^u_{p(\cdot),\omega}}. \tag{5.6}$$

Proof. For any $f \in \mathcal{H}^u_{p(\cdot),\omega}$ with $\mathcal{W}_\psi f \in M^u_{p(\cdot),\omega}$, (3.22) assures that

$$\mathcal{W}_\psi f \in \bigcap_{h \in \mathfrak{b}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}} L^{p_0}(\mathcal{R}h).$$

Therefore, for any

$$v \in \left\{\mathcal{R}h : h \in \mathfrak{b}^{u^{p_0}}_{(p(\cdot)/p_0)',\omega^{-p_0}}\right\},$$

Theorem 5.4 shows that

$$\int_{\mathbb{R}} (\mathcal{M}_{\mathcal{F}} f(x))^{p_0} v(x)dx \leq C \int_{\mathbb{R}} (\mathcal{W}_\psi f(x))^{p_0} v(x)dx.$$

By applying Theorem 3.7 with

$$\mathcal{F} = \{(\mathcal{M}_{\mathcal{F}} f, \mathcal{W}_\psi f) : f \in \mathcal{H}^u_{p(\cdot),\omega}\},$$

we obtain a constant $C_0 > 0$ such that for any $f \in \mathcal{H}^u_{p(\cdot),\omega}$,

$$C_0\|f\|_{\mathcal{H}^u_{p(\cdot),\omega}} \leq \|\mathcal{W}_\psi f\|_{M^u_{p(\cdot),\omega}}.$$

We establish the first inequality in (5.6). The proof of the second inequality in (5.6) follows similarly. □

The following wavelet characterization of the weighted Hardy–Morrey space is a special case of Theorem 5.5.

Corollary 5.6. *Let $\alpha \geq 1$, $\psi \in C^{[\alpha]}$, $p \in (\alpha^{-1}, 1]$ and $\omega^p \in \mathcal{A}_1$. Suppose that ψ satisfies (5.3) and (5.5). If there exists a $p_0 \in (\alpha^{-1}, p)$ such that $u^{p_0} \in \mathbb{W}_{p/p_0,\omega^{p_0}}$, then there exist constants $C_0, C_1 > 0$ such that for any $f \in \mathcal{H}^u_{p,\omega}$*

$$C_0\|f\|_{\mathcal{H}^u_{p,\omega}} \leq \|\mathcal{W}_\psi f\|_{M^u_{p,\omega}} \leq C_1\|f\|_{\mathcal{H}^u_{p,\omega}}.$$

Acknowledgements. The author thanks the reviewers for their valuable suggestions which improve the presentation of this paper.

https://doi.org/10.1017/S0013091521000742 Published online by Cambridge University Press

References

1. A. Almeida, J. Hasanov and S. Samko, Maximal and potential operators in variable exponent Morrey spaces, *Georgian Math. J.* **15** (2008), 1–15.
2. C. Bennett and R. Sharpley, *Interpolations of operators* (Academic Press, Orlando, 1988).
3. H.-Q. Bui, Weighted Hardy spaces, *Math. Nachr.* **103** (1981), 45–62.
4. H.-Q. Bui, Weighted Besov and Triebel spaces: interpolation by the real method, *Hiroshima Math. J.* **12** (1982), 581–605.
5. H.-Q. Bui, Characterizations of weighted Besov and Triebel-Lizorkin spaces via temperatures, *J. Funct. Anal.* **55** (1984), 39–62.
6. H.-Q. Bui, M. Palusyński and M. Taibleson, A maximal function characterization of weighted Besov-Lipschitz and Triebel-Lizorkin spaces, *Studia Math.* **119** (1996), 219–246.
7. A. Caetanoa and H. Kempka, Variable exponent Triebel-Lizorkin-Morrey spaces, *J. Math. Anal. Appl.* **484** (2020), 123712.
8. D. Cruz-Uribe, A. Fiorenza and C. Neugebauer, Weighted norm inequalities for the maximal operator on variable Lebesgue spaces, *J. Math. Anal. Appl.* **394** (2012), 744–760.
9. D. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue spaces: foundations and harmonic analysis* (Birkhäuser/Springer, Basel, 2013).
10. Y. Deng and S. Long, Pseudodifferential operators on weighted Hardy spaces, *J. Funct. Space* **2020** (2020), 7154125.
11. L. Diening, P. Harjulehto, P. Hästö and M Ružička, Lebesgue and Sobolev spaces with variable exponent, Lecture Notes in Mathematics, Volume 2017 (Springer-Verlag, Berlin, 2011).
12. D. Ferreira and W. Staubach, Global and local regularity of Fourier integral operators on weighted and unweighted spaces, *Memoirs Amer. Math. Soc.* **1074** (2014), 1–65.
13. M. Frazier and B. Jawerth, A discrete transform and decompositions of distribution spaces, *J. Funct. Anal.* **93** (1990), 34–170.
14. J. García-Cuerva, Weighted H^p spaces, *Dissertations Math.* **162** (1979), 1–63.
15. J. García-Cuerva and K. Kazarian, Calderón-Zygmund operators and unconditional bases of weighted Hardy spaces, *Studia Math.* **109** (1994), 255–276.
16. J. García-Cuerva and J. Martell, Wavelet characterization of weighted spaces, *J. Gemo. Anal.* **11** (2001), 241–264.
17. L. Grafakos, *Modern Fourier analysis* (Springer-Verlag, New York, 2009).
18. V. Guliyev and S. Samko, Maximal potential and singular operators in the generalized variable exponent Morrey spaces on unbounded sets, *J. Math. Sci. (N. Y.)* **193** (2013), 228–248.
19. V. Guliyev, J. Hasanov and S. Samko, Boundedness of the maximal, potential and singular operators in the generalized variable exponent Morrey spaces, *Math. Scand.* **107** (2010), 285–304.
20. V. Guliyev, J. Hasanov and X. Badalov, Maximal and singular integral operators and their commutators on generalized weighted Morrey spaces with variable exponent, *Math. Ineq. Appl.* **21** (2018), 41–61.
21. D. Haroske and L. Skrzypczak, Embeddings of weighted Morrey spaces, *Math. Nachr.* **290** (2016), 1066–1086.
22. K.-P. Ho, Littlewood-Paley spaces, *Math. Scand.* **108** (2011), 77–102.
23. K.-P. Ho, Vector-valued singular integral operators on Morrey type spaces and variable Triebel-Lizorkin-Morrey spaces, *Ann. Acad. Sci. Fenn. Math.* **37** (2012), 375–406.

24. K.-P. Ho, Atomic decompositions of weighted Hardy-Morrey spaces, *Hokkaido Math. J.* **42** (2013), 131–157.
25. K.-P. Ho, Atomic decomposition of Hardy-Morrey spaces with variable exponents, *Ann. Acad. Sci. Fenn. Math.* **40** (2015), 31–62.
26. K.-P. Ho, Atomic decompositions of weighted Hardy spaces with variable exponents, *Tohoku Math. J. (2)* **69** (2017), 383–413.
27. K.-P. Ho, Sublinear operators on weighted Hardy spaces with variable exponents, *Forum Math.* **31** (2019), 607–617.
28. K.-P. Ho, Boundedness of operators and inequalities on Morrey-Banach spaces, *Publ. Res. Inst. Math. Sci.* (2021) (to appear).
29. K.-P. Ho, Calderón-Zygmund operators. Bochner-Riesz means and parametric Marcinkiewicz integrals on Hardy-Morrey spaces with variable exponents, *Kyoto J. Math.* (2021)(to appear).
30. K.-P. Ho, Operators on Orlicz-slice spaces and Orlicz-slice Hardy spaces, *J. Math. Anal. Appl.* **503** (2021), 125279.
31. K.-P. Ho, Singular integral operators and sublinear operators on Hardy local Morrey spaces with variable exponents, *Bull. Sci. Math.* **171** (2021), 103033.
32. H. Jia and H. Wang, Decomposition of Hardy-Morrey spaces, *J. Math. Anal. Appl.* **354** (2009), 99–110.
33. Y. Jiao, T. Zhao and d. Zhou, Variable martingale Hardy-Morrey spaces, *J. Math. Anal. Appl.* **484** (2020), 123722.
34. V. Kokilashvili and A. Meskhi, Boundedness of maximal and singular operators in Morrey spaces with variable exponent, *Armen. J. Math.* **1** (2008), 18–28.
35. V. Kokilashvili and S. Samko, Maximal and fractional operators in weighted $L^{p}(x)$ spaces, *Rev. Mat. Iberoamericana* **20** (2004), 493–515.
36. Y. Komori and S. Shirai, Weighted Morrey spaces and a singular integral operator, *Math. Nachr.* **282** (2009), 219–231.
37. Y. Liang, Y. Sawano, T. Ullrich, D. Yang and W. Yuan, A new framework for generalized Besov-type and Triebel-Lizorkin-type spaces, *Dissertationes Math. (Rozprawy Mat.)* **489** (2013), 114.
38. M. Mastyło, Y. Sawano and H. Tanaka, Morrey type space and its Köthe dual space, *Bull. Malays. Math. Soc.* **41** (2018), 1181–1198.
39. A. Mazzucato, Besov-Morrey spaces: function space theory and applications to non-linear PDE, *Trans. Amer. Math. Soc.* **355** (2003), 1297–1364.
40. N. Michalowski, D. Rule and W. Staubach, Weighted norm inequalities for pseudo-differential operators defined by amplitudes, *J. Funct. Anal.* **258** (2010), 4183–4209.
41. N. Miller, Weighted Sobolev spaces and pseudodifferential operators with smooth symbols, *Trans. Am. Math. Soc.* **269** (1982), 91–109.
42. Y. Mizuta and T. Shimomura, Weighted Morrey spaces of variable exponent and Riesz potentials, *Math. Nachr.* **288** (2015), 984–1002.
43. C. B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, *Trans. Amer. Math. Soc.* **43** (1938), 126–166.
44. E. Nakai, Hardy–Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces, *Math. Nachr.* **166** (1994), 95–103.
45. E. Nakai, G. Sadasue and Y. Sawano, Martingale Morrey-Hardy and Campanato-Hardy Spaces, *J. Funct. Spaces* **2013** (2013), 690258. 14 pages.
46. S. Nakamura, Generalized weighted Morrey spaces and classical operators, *Math. Nachr.* **289** (2016), 2235–2262.
47. S. Nakamura, T. Noi and Y. Sawano, Generalized Morrey spaces and trace operator, *Science China Math.* **59** (2016), 281–336.

48. S. Nakamura and Y. Sawano, The singular integral operator and its commutator on weighted Morrey spaces, *Collect. Math.* **68** (2017), 145–174.
49. S. Nishigiki, Weighted norm inequalities for certain pseudo-differential operators, *Tokyo J. Math.* **7** (1984), 129–140.
50. J. Rubio de Francia, Factorization and extrapolation of weights, *Bull. Amer. Math. Soc. (N.S.)* **7** (1982), 393–395.
51. J. Rubio de Francia, A new technique in the theory of A_p weights, Topics in modern harmonic analysis, Volume I, II (Turin/Milan, 1982), pp. 571–579. Ist. Naz. Alta Mat. Francesco Severi, Rome, 1983.
52. J. Rubio de Francia, Factorization theory and A_p weights, *Amer. J. Math.* **106** (1984), 533–547.
53. S. Sato, A note on weighted estimates for certain classes of pseudo-differential operators, *Rocky Mountain J. Math.* **35** (2005), 267–284.
54. S. Sato, Non-regular pseudo-differential operators on the weighted Triebel-Lizorkin spaces, *Tohoku Math. J. (2)* **59** (2007), 323–339.
55. Y. Sawano, Wavelet characterization of Besov-Morrey and Triebel-Lizorkin-Morrey spaces, *Funct. Approx. Comment. Math.* **38** (2008), 93–107.
56. Y. Sawano, A note on Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces, *Acta Math. Sin. (Engl. Ser.)* **25** (2009), 1223–1242.
57. Y. Sawano and H. Tanaka, Decompositions of Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces, *Math. Z.* **257** (2007), 871–905.
58. Y. Sawano and H. Tanaka, Predual spaces of Morrey spaces with nondoubling measures, *Tokyo J. Math.* **32** (2009), 471–486.
59. Y. Sawano, K.-P. Ho, D. Yang and S. Yang, Hardy spaces for ball quasi-Banach function spaces, *Dissertationes Mathematicae* **525** (2017), 1–102.
60. E. Stein, *Harmonic analysis* (Real-variable methods, orthogonality, and oscillatory integrals (Princeton, NJ, Princeton University Press, 1993).
61. J.-O. Strömberg and A. Torchinsky, Weighted Hardy spaces, Lecture Notes in Mathematics, Vol. 1381 (1989).
62. L. Tang and J. Xu, Some properties of Morrey type Besov-Triebel spaces, *Math. Nachr.* **278** (2005), 904–917.
63. J. Tao, D. C. Yang and D. Y. Yang, Boundedness and compactness characterizations of Cauchy integral commutators on Morrey spaces, *Math. Methods Appl. Sci.,* **42** (2019), 1631–1651.
64. R. Torres, Boundedness results for operators with singular kernels on distribution spaces, *Memoirs Amer. Math. Soc.* **442** (1991).
65. H. Triebel, Interpolation theory, function spaces, differential operators, North-Holland Math. Library, Volume 18 (North-Holland, Amsterdam, 1978).
66. H. Triebel, Theory of function spaces, Monographs in Math., Volume 78 (Birkhäuser, Basel 1983).
67. J. Xu, Variable Besov and Triebel–Lizorkin spaces, *Ann. Acad. Sci. Fenn. Math.* **33** (2008), 511–522.
68. K. Yabuta, Weighted norm inequalities for pseudodifferential operators, *Osaka J. Math.* **23** (1986), 703–723.
69. C. Zorko, Morrey spaces, *Proc. Amer. Math. Soc.* **98** (1986), 586–592.