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Slip velocity over a perforated or patchy surface

C. POZRIKIDIS†

Department of Chemical Engineering, University of Massachusetts, 686 North Pleasant Street, Amherst, MA 01003, USA

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Shear flow over a solid surface containing perforations or patches of zero shear stress is discussed with a view to evaluating the slip velocity. In both cases, the functional dependence of the slip velocity on the solid fraction of the surface strongly depends on the surface geometry, and a universal law cannot be established. Numerical results for flow over a plate with circular or square perforations or patches of zero shear stress, and flow over a plate consisting of separated square or circular tiles corroborate the assertion.

Key words: boundary element method, slip velocity, Stokes flow

1. Introduction

The concept of macroscopic slip velocity over nominally solid surfaces containing regions of zero shear stress has gained increasing attention in recent years due to its relevance to the hydrodynamics of microfluidics devices and to the microphysics of superhydrophobic surfaces (e.g. Karniadakis & Beskok 2001; Zheng, Yu & Zhao 2005). From the viewpoint of mathematical analysis, the problem is similar to that of shear flow over a porous or perforated surface separating two flow domains, where a shear flow is imposed on one side and the fluid drifts with a uniform velocity on the other side. A common objective in both cases is the computation of the slip velocity shifting the streamwise velocity profile normal to the surface.

Tio & Sadhal (1994) acknowledged the occurrence of apparent slip in shear flow over a porous membrane and performed detailed calculations for a model configuration consisting of a zero-thickness plate perforated by periodic parallel slits. Longitudinal (unidirectional) and transverse (two-dimensional) configurations were considered where the fluid undergoes simple shear flow with a specified shear stress far above the plate and is quiescent far below the plate. In an earlier study, Philip (1972*a*,*b*) performed similar calculations for flow over a flat or cylindrical surface hosting solitary or periodic stripes of vanishing shear stress. In fact, the Tio & Sadhal (1994) and Philip (1972*a*,*b*) problems are equivalent, in that the flow in the second problem can be deduced from that in the first problem by a mere reflection with respect to the plate followed by superposition. In related efforts, Wang (2001) studied flow in a two-dimensional channel divided into two compartments by a slotted plane, where the motion is driven by the parallel translation of the walls. Other configurations involving flow over grated surfaces were considered more recently by Ng & Wang (2009).

† Email address for correspondence: cpozrikidis@ecs.umass.edu

Pozrikidis (2001) discussed shear flow over a particulate plate representing a finite-thickness membrane. The fluid velocity was restricted to either vanish or tend to an *a priori* unknown constant value far below the plate. A distinction was made between the slip velocity shifting the linear velocity far above the plate $\boldsymbol{U}_s$ and the uniform drift velocity established far below the plate $\boldsymbol{U}_d$. The direction of the slip and drift velocities is not necessarily parallel to that of the simple shear flow imposed above the plate due to the possible anisotropic geometrical structure of the plate. Because the difference $\boldsymbol{U}_s - \boldsymbol{U}_d$ is on the order of the plate thickness, the slip and drift velocities are identical only in the case of a zero-thickness plate. By adding to the flow over the particular plate its reflection with respect to the plate, we obtain shear flow over a surface containing patches of zero shear stress exhibiting macroscopic slip velocity $\boldsymbol{V}_s = \boldsymbol{U}_s + \boldsymbol{U}_d$.

To reconcile and unify the results of the previous authors, in § 2 we discuss the functional dependence of the slip velocity on the surface liquid fraction by summarizing and comparing the results of previous authors. New boundary element solutions demonstrating that the slip velocity strongly depends on the morphology of a surface are presented in § 3, and further comments are made in § 4.

2. Holes and tiles

In the configuration considered by Philip (1972*a*,*b*) and Tio & Sadhal (1994), far above a plate perforated by parallel slits, the velocity component parallel to the plate exhibits the asymptotic form $u_x \simeq \xi y + U_s$, where ξ is the shear rate, y is the distance from the plate and U_s is a slip velocity. By adding to this primary flow its reflection with respect to the plate, we obtain shear flow past a surface with zero shear stress over an array of slits. Detailed analysis yields an exact expression for the corresponding slip velocity,

$$V_s = 2U_s = -\frac{\delta}{2\pi}\,\xi L \ln\left(\cos\frac{\pi\phi_l}{2}\right), \tag{2.1}$$

where L is the period of the perforations, $\phi_l = (L-a)/L$ is the surface liquid fraction, a is the length of the solid segments and $L-a$ is the slit opening inside each period. The solid surface fraction is $\phi_s = 1-\phi_l = a/L$. The dimensionless coefficient δ has the value of two in the case of unidirectional longitudinal flow, and the value of unity in the case of two-dimensional transverse flow. In the limit of vanishing slit opening, $\phi_l \to 0$, we obtain

$$V_s \simeq \frac{\delta\pi}{16}\,\xi L\,\phi_l^2. \tag{2.2}$$

In the diametrically opposite limit of dominant slit opening, $\phi_s \to 0$, we obtain

$$V_s \simeq -\frac{\delta}{2\pi}\,\xi L\left(\ln\phi_s + \ln\frac{\pi}{2}\right). \tag{2.3}$$

This expression shows that the slip velocity diverges logarithmically with respect to the surface solid fraction. Physically, a one-dimensional array of thin wires is unable to sustain a longitudinal or transverse shear flow. A similar logarithmic divergence with respect to the solid fraction is observed in the permeability of an idealized two-dimensional porous medium consisting of parallel cylinders for longitudinal and transverse flow (Hasimoto 1959; Sparrow & Loeffler 1959).

Smith (1987) and Davis (1991) presented analytical solutions for shear flow past a solitary circular orifice on an infinite plane wall. Pozrikidis (2001) used their results to derive an expression for the slip velocity in the case of shear flow over a zero-thickness plate perforated by a large number of well-separated circular holes of radius b. By adding to this flow the reflection of the flow with respect to the plate, we obtain the slip velocity

$$V_s = 2U_s \simeq \frac{8}{9} n \xi b^3 = \frac{8}{9\pi} \xi b \, \phi_l, \tag{2.4}$$

where n is the surface number density, defined as the surface concentration of the holes; the surface liquid fraction is $\phi_l = n\pi b^2$. A factor of π was inadvertently omitted in the denominator of the first fraction in (54) of Pozrikidis (2001), and should be carried over to two subsequent equations. Expression (2.4) was recently rederived by Sbragaglia & Prosperetti (2007). If the circular holes are deployed on a square lattice with side length L, $n = 1/L^2$ and $\phi_l = \pi b^2/L^2$, yielding

$$V_s \simeq \frac{8}{9} \xi L \left(\frac{\phi_l}{\pi}\right)^{3/2}, \tag{2.5}$$

applicable in the limit of disappearing holes, $\phi_l \to 0$. Comparing (2.5) with (2.2), we observe that, in the limit $\phi_l \to 0$, the exponent of ϕ_l is higher in two-dimensional than in three-dimensional flow.

Pozrikidis (2001, 2005) carried out an asymptotic analysis and performed boundary-integral computations for an idealized membrane consisting of a doubly periodic array of spherical or spheroidal particles in three-dimensional flow, and a periodic array of cylinders in unidirectional and two-dimensional flow. In the case of a square array of spherical particles of radius a, the slip velocity for small solid fraction is

$$V_s = 2U_s \simeq \frac{1}{3\pi} \xi L \left(\frac{L}{a} + \frac{3}{4} c\right) = \frac{1}{3} \xi L \left(\frac{1}{\sqrt{\pi \phi_s}} + \frac{3}{4\pi} c\right), \tag{2.6}$$

where L is the lattice side length, $\phi_s = \pi a^2/L^2$ is the surface solid fraction of the projection of the particulate sheet in the xy plane, and $c = 5.85$ is a numerical coefficient. Comparing (2.6) with (2.3), we observe that, in the limit $\phi_s \to 0$, the divergence of the slip velocity with respect to ϕ_s is stronger in three-dimensional than in two-dimensional flow.

We conclude that the functional form of the slip velocity on the liquid or solid areal fraction strongly depends on the geometry of the surface, and a universal law cannot be established. To further demonstrate this strong dependence, we visualize flow over an idealized perforated surface consisting of a doubly periodic array of square plates with side length $2a$ separated by uniform gaps along the four sides. The surface solid fraction of the surface is $\phi_s = 4a^2/L^2$, where L is the lattice side length. For shear flow parallel to two square sides and perpendicular the other two square sides, in the limit $2a \to L$ and correspondingly $\phi_l \to 0$, we obtain a superposition of longitudinal and transverse shear flow over a one-dimensional array of slits separating nearly touching plates. Applying (2.2) for $\phi_l = 1 - 2a/L$ and $\delta = 2$ for the longitudinal flow and $\delta = 1$ for the transverse flow, neglecting the small overlapping gaps at the four corners of each plate, and then adding the individual slip velocities, we find

$$V_s \simeq \frac{3\pi}{16} \xi L \left(1 - \frac{2a}{L}\right)^2 = \frac{3\pi}{16} \xi L \, (1 - \sqrt{\phi_s})^2. \tag{2.7}$$

A similar formula can be derived for rectangular plates.

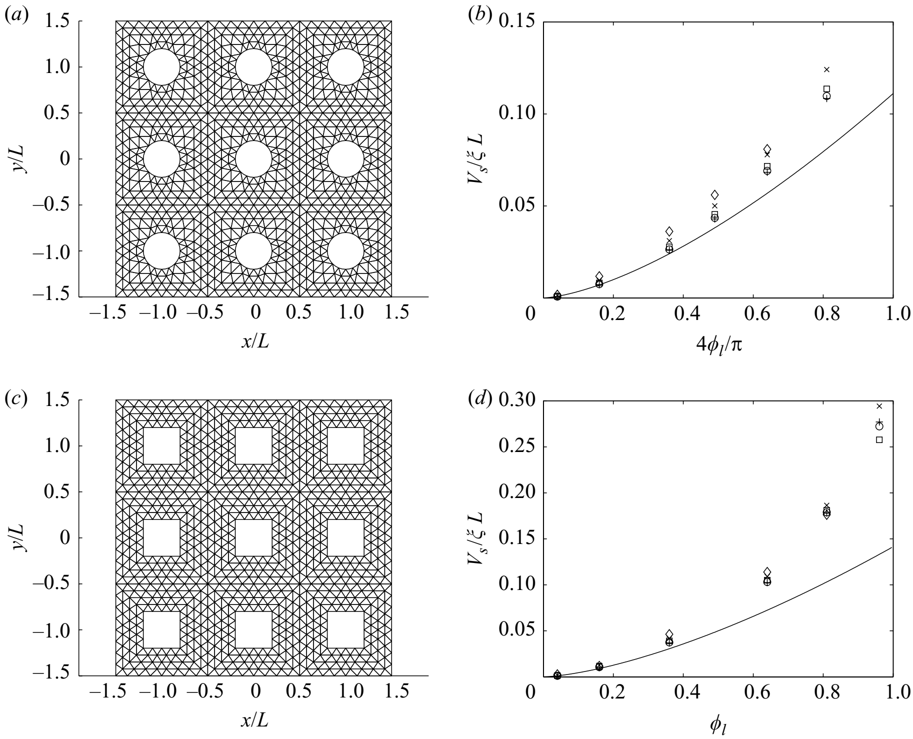

FIGURE 1. (a, b) Discretization of a plane wall perforated by a doubly periodic array of circular holes of radius $b = 0.2L$ into 192 boundary elements over each period, and dependence of the slip velocity on the surface liquid fraction. The curve in (b) represents the predictions of the asymptotic theory expressed by (2.4). (c, d) Same as (a, b) but for a doubly periodic array of square holes; a configuration with square holes of half side length $b = 0.2L$ is shown in (c). In (b, d), the diamonds, $\times$ symbols, squares and circles correspond to boundary element discretization into 12, 48, 192 and 768 boundary elements, and the crosses arise by numerical extrapolation.

3. Boundary element solutions

Boundary element solutions of the equations of Stokes flow were computed for shear flow over a flat plate perforated by a square array of circular or square holes (figure 1), and shear flow over a permeable wall consisting of a doubly periodic array of square or circular plates (figure 2). The integral formulation involving the doubly periodic Green's function of three-dimensional Stokes flow and the implementation of the numerical method are discussed by Pozrikidis (2001). Briefly, one period of the solid portion of the surface is divided into six-node boundary elements, and the boundary traction is computed by solving a Fredholm integral equation of the first kind supplemented by an integral constraint requiring that the force exerted on the boundaries balances the force imparted to the surface by the shear flow. An important feature of the formulation is that the slip velocity arises directly as part of the solution. Boundary element discretization was carried out by custom-made methods based on the successive subdivision of an eight or twelve element hard-coded pattern. The computation of the Green's function was expedited by the use of Ewald sums (Pozrikidis 1996).

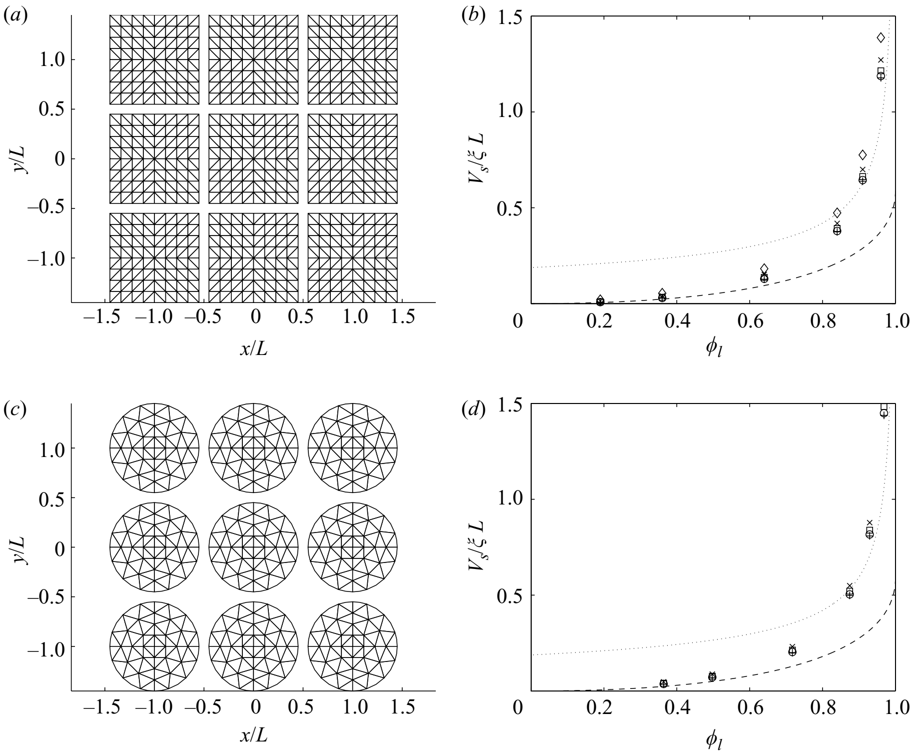

FIGURE 2. Same as figure 1, but for a doubly periodic array of (a, b) square, and (c, d) square plates with half side length $a = 0.45L$ are shown in (a), and circular plates with radius $a = 0.45L$ are shown in (c).

Figure 1(a, b) illustrates a boundary element discretization and presents numerical results for the slip coefficient in the case of shear flow over a flat plate perforated by a square array of circular holes. The various symbols in figure 1(b) represent results obtained by different levels of discretization, as explained in the caption. The numerical results at small liquid areal fractions are in excellent agreement with the predictions of the asymptotic theory expressed by (2.4), represented by the solid line in figure 1(b). As the surface liquid fraction tends to the maximum possible value of $\pi/4$ corresponding to touching holes, the slip velocity tends to a well defined limit. The asymptotic solution underestimates the slip velocity at finite volume fractions. Reliable numerical results could not be obtained for nearly touching holes due to the skewness of the boundary element distribution.

Figure 1(c, d) illustrates a boundary element discretization and presents numerical results for the slip coefficient in the case of shear flow over a flat plate perforated by a square array of square holes. The numerical results for small liquid areal fractions are in good agreement with the predictions of the asymptotic theory expressed by (2.4), even though the theory strictly applies for circular holes. As in the case of circular holes, reliable numerical results could not be obtained for liquid surface fraction near unity. However, the numerical results clearly indicate that the slip velocity diverges as $\phi_l \to 1$ due to the inability of the emerging wireframe to sustain the shear flow.

Figure 2(a, b) illustrates a boundary element discretization and presents numerical results for the slip coefficient in the case of shear flow over a plate consisting of a square array of square plates. The physical configuration shown in figure 2(*a*) is the mirror image of that shown in figure 1(*c*). As the plates tend to touch and correspondingly the surface liquid fraction ϕ_l tends to zero, the numerical results for the slip coefficient are in excellent agreement with the predictions of the asymptotic analysis expressed by (2.7), represented by the dashed line in figure 2(*b*). In the opposite limit where the plates shrink down to points and correspondingly ϕ_l tends to unity, the slip velocity diverges. The functional dependence of the slip velocity on ϕ_l in this limit is well described by (2.6) with $c = 0$, represented by the dotted line in figure 2(*b*). Figure 2(*c, d*) shows corresponding results for shear flow over a surface consisting of a square lattice of circular plates. The numerical results for the slip coefficient are well described by the asymptotic theories for small and large liquid surface fractions represented by the dashed and dotted lines.

4. Discussion

We have demonstrated that the slip velocity in shear flow over a perforated or loosely tiled surface strongly depends on the geometry of the surface, and a universal law cannot be established. In previous analysis and calculations discussed earlier in this paper, the flow is unidirectional or else the motion of the fluid is governed by the linear equations of Stokes flow applicable for low Reynolds numbers. Pozrikidis (2004) assessed the effect of fluid inertia by computing numerical solutions of the Navier–Stokes equation for two-dimensional transverse shear flow over a periodic array of cylinders using a finite-difference method combined with conformal mapping for generating an orthogonal grid. The results demonstrated that fluid inertia promotes the magnitude of the slip and drift velocities. In view of the nonlinearity of inertial flow, the significance of the boundary geometry is expected to be even more pronounced in the case of Navier–Stokes flow.

Tangential shear flow over a perforated surface is complementary to normal flow through a perforated surface. Sampson (1891) derived an exact solution for Stokes flow through a single hole in an infinite plane. Wang (1994) used eigenfunction expansions to study flow through a plane perforated by a double array of circular holes. His results show that the boundary geometry is an important factor for determining the pressure drop for a given flow rate.

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