

UNIVERSAL SUBGROUPS OF POLISH GROUPS

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Abstract. Given a class \mathcal{C} of subgroups of a topological group G , we say that a subgroup $H \in \mathcal{C}$ is a *universal \mathcal{C} subgroup* of G if every subgroup $K \in \mathcal{C}$ is a continuous homomorphic preimage of H . Such subgroups may be regarded as complete members of \mathcal{C} with respect to a natural preorder on the set of subgroups of G . We show that for any locally compact Polish group G , the countable power G^ω has a universal K_σ subgroup and a universal compactly generated subgroup. We prove a weaker version of this in the nonlocally compact case and provide an example showing that this result cannot readily be improved. Additionally, we show that many standard Banach spaces (viewed as additive topological groups) have universal K_σ and compactly generated subgroups. As an aside, we explore the relationship between the classes of K_σ and compactly generated subgroups and give conditions under which the two coincide.

§1. Introduction.

1.1. Background. The study of definable equivalence relations on Polish spaces has been one of the major threads of descriptive set theory for the past thirty years. In many cases, important equivalence relations arise from algebraic or combinatorial properties of the underlying Polish spaces. A common situation is that of a coset equivalence relation on a Polish group G . If $H \subseteq G$ is a subgroup, one defines the equivalence relation E_H by

$$x E_H y \iff y^{-1}x \in H.$$

Viewed as a subset of $G \times G$, E_H has the same topological complexity (Borel, analytic, etc.) as H and its equivalence classes are the left cosets of H . To give a concrete example, consider the equivalence relation E_0 on 2^ω , defined by

$$x E_0 y \iff (\forall^\infty n)(x(n) = y(n)).$$

Identifying 2^ω with the Polish group \mathbb{Z}_2^ω , one recognizes E_0 as the coset equivalence relation of the subgroup

$$\text{Fin} = \{x \in \mathbb{Z}_2^\omega : (\forall^\infty n)(x(n) = 0)\}.$$

Given equivalence relations E, F on a space X , one often asks whether or not there exists a definable map $f : X \rightarrow X$ reducing E to F , i.e., such that

$$(\forall x, y)(x E y \iff f(x) F f(y)).$$

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In this situation, “definable” is usually (though not always) interpreted to mean Baire- or Borel-measurable. (In the case that a Borel reduction exists, one writes $E \leq_B F$.)

Returning to the setting of groups, suppose that $H, K \subseteq G$ are subgroups of a Polish group G and $\varphi : G \rightarrow G$ is a group homomorphism such that

$$(\forall x)(x \in H \iff \varphi(x) \in K).$$

This in turn gives a reduction of E_H to E_K since, by the properties of group homomorphisms,

$$(\forall x, y)(y^{-1}x \in H \iff \varphi(y)^{-1}\varphi(x) \in K).$$

As mentioned above, one is generally interested in reducing maps which are at least Baire-measurable. Recall, however, that Baire-measurable homomorphisms of Polish groups are automatically continuous (see Kechris [14, 9.10] or Gao [8, Theorem 2.3.3]). Taken together, these observations motivate the following definition.

DEFINITION 1.1. Let G_1, G_2 be Polish groups. Suppose that $H \subseteq G_1$ and $K \subseteq G_2$ are subgroups. We say that H is *group-homomorphism reducible* to K if, and only if, there exists a continuous homomorphism $\varphi : G_1 \rightarrow G_2$ such that $\varphi^{-1}(K) = H$. We write $H \leq_g K$.

As mentioned above,

$$H \leq_g K \implies E_H \leq_B E_K. \tag{1.1}$$

In fact, many Borel reductions among coset equivalence relations derive from corresponding group-homomorphism reductions. Each of the Borel reductions $E_0 \leq_B E_1, E_2, E_3$ arises in this way.¹ The following is a representative example.

EXAMPLE 1.2. Recall from above that E_0 is the coset equivalence relation of the subgroup $\text{Fin} \subseteq \mathbb{Z}_2^\omega$. Consider the equivalence relation E_2 , where

$$x E_2 y \iff \sum_{x(n) \neq y(n)} \frac{1}{n+1} < \infty.$$

Notice that E_2 is the coset equivalence relation of the subgroup

$$H = \left\{ x \in \mathbb{Z}_2^\omega : \sum_{x(n) \neq 0} \frac{1}{n+1} < \infty \right\}.$$

A map witnessing the reduction $E_0 \leq_B E_2$ is

$$\varphi(x) = x(0) \frown x(1)^2 \frown x(2)^4 \frown x(3)^8 \frown \dots$$

In other words, φ copies the n th bit of x to a block of 2^n bits of $\varphi(x)$. Observe that φ is actually a continuous group homomorphism of \mathbb{Z}_2^ω and $\text{Fin} = \varphi^{-1}(H)$, i.e., $\text{Fin} \leq_g H$. (This follows since each nonzero bit of x increases $\sum \{ \frac{1}{n+1} : \varphi(x)(n) \neq 0 \}$ by more than $\frac{1}{2}$.)

¹ See Kanovei [13] for definitions of these equivalence relations.

In general, however, the converse of the implication (1.1) is false. Consider the following situation; suppose that H, K are normal subgroups of a group G and $H \leq_g K$, via φ . The map φ induces an injective homomorphism $\tilde{\varphi} : G/H \rightarrow G/K$, defined by $\tilde{\varphi}(\pi_H(x)) = \pi_K(\varphi(x))$, where π_H and π_K are the quotient maps onto G/H and G/K , respectively. This observation justifies the following two examples.

EXAMPLE 1.3. Let

$$H_2 = \{x \in \mathbb{Z}^\omega : (\forall n)(x(n) \text{ is divisible by } 2)\}$$

and

$$H_3 = \{x \in \mathbb{Z}^\omega : (\forall n)(x(n) \text{ is divisible by } 3)\}.$$

Note that $\mathbb{Z}^\omega/H_2 \cong \mathbb{Z}_2^\omega$ and $\mathbb{Z}^\omega/H_3 \cong \mathbb{Z}_3^\omega$. Thus $H_2 \not\leq_g H_3$ and $H_3 \not\leq_g H_2$, since there are no injective homomorphisms $\mathbb{Z}_2^\omega \rightarrow \mathbb{Z}_3^\omega$, or vice versa.

On the other hand, $E_{H_2} \leq_B E_{H_3}$ via the map $f : \mathbb{Z}^\omega \rightarrow \mathbb{Z}^\omega$ given by

$$f(x)(n) = \begin{cases} 0 & \text{if } x(n) \text{ is even,} \\ 1 & \text{if } x(n) \text{ is odd,} \end{cases}$$

for each $n \in \omega$. Similarly, $E_{H_3} \leq_B E_{H_2}$.

EXAMPLE 1.4. In [22], Christian Rosendal showed that the coset equivalence relation of the subgroup

$$\mathcal{B} = \{x \in \mathbb{Z}^\omega : (\exists M)(\forall n)(|x(n)| \leq M)\}$$

is Borel-complete for K_σ equivalence relations. (See Louveau-Rosendal [18] for another example of a complete K_σ equivalence relation.) In particular, $E_H \leq_B E_{\mathcal{B}}$, for each K_σ subgroup of \mathbb{Z}^ω . There are, however, K_σ subgroups which are not group-homomorphism reducible to \mathcal{B} . For example,

$$2\mathcal{B} = \{x \in \mathcal{B} : (\forall n)(x(n) \text{ is even})\} \not\leq_g \mathcal{B},$$

since $\mathbb{Z}^\omega/2\mathcal{B}$ has elements of order 2 and $\mathbb{Z}^\omega/\mathcal{B}$ has no elements of finite order.

Our work on \leq_g was motivated in part by the last example. In particular, we wondered if there would be an analog of Rosendal’s theorem for group-homomorphism reductions. In other words, are there \leq_g -complete K_σ subgroups?

Naturally, one can ask this question for classes besides K_σ . This suggests the following definition.

DEFINITION 1.5. Let G_1, G_2 be Polish groups and \mathcal{C} a class of subgroups of G_1 . We say that a subgroup K of G_2 is *universal for subgroups of G_1 in \mathcal{C}* if, and only if, for each subgroup $H \subseteq G_1$, with $H \in \mathcal{C}$, we have $H \leq_g K$.

In the case that $G_1 = G_2$ and $K \in \mathcal{C}$, we simply say that K is a *universal \mathcal{C} subgroup of G_1* .

In this context, the simplest classes to study are those for which membership of a subgroup H in the class is determined by the nature of a generating set for H . In the present paper, we will consider two classes of subgroups with this property, namely the classes of K_σ and compactly generated subgroups. In [2], we take up the study of Σ_n^1 and Π_1^1 subgroups.

1.2. Summary of results. Our main results concern the existence of universal K_σ and universal compactly generated subgroups in the countable powers and products of various Polish groups.

The following is our principal positive result.

THEOREM 1.6. *Let $(G_n)_{n \in \omega}$ be a sequence of locally compact Polish groups, each term of which occurs infinitely often (up to isomorphism). We have the following;*

1. $\prod_n G_n$ has a universal compactly generated subgroup.
2. $\prod_n G_n$ has a universal K_σ subgroup.

Although stated for products, Theorem 1.6 implies that the countable power of any locally compact group has universal K_σ and compactly generated subgroups, e.g., $\mathbb{Z}_2^\omega, \mathbb{Z}^\omega, \mathbb{R}^\omega, \mathbb{Q}^\omega$ (with the discrete topology on \mathbb{Q}), and \mathbb{T}^ω (where \mathbb{T} is the unit circle in \mathbb{C}).

For the case of groups which are not locally compact, we have an ‘‘approximation’’ of the last theorem.

THEOREM 1.7. *For every Polish group G , there is an F_σ subgroup $H \subseteq G^\omega$, which is universal for K_σ subgroups of G^ω , i.e., every K_σ subgroup of G^ω is a continuous homomorphic preimage of H .*

We prove Theorems 1.6 and 1.7 in Sections 4.2 and 5, respectively. In Section 6, we prove the following counterpoint to Theorem 1.7.

THEOREM 1.8. *There is no universal K_σ subgroup of S_∞^ω .*

Theorem 1.8 is, in essence, a ‘‘sharpness’’ result for Theorems 1.6 and 1.7, suggesting that neither can readily be improved.

Propositions 1.9 and 1.11 will allow us to apply Theorems 1.6 and 1.7 in a broader context, e.g., to groups which are not of the form $G^\mathbb{N}$.

PROPOSITION 1.9. *Suppose that G_1 and G_2 are topological groups such that there exist continuous injective homomorphisms $\varphi_1 : G_1 \rightarrow G_2$ and $\varphi_2 : G_2 \rightarrow G_1$. Let \mathcal{C} be a class of subgroups which is closed under continuous homomorphic images. If G_1 has a universal \mathcal{C} subgroup, then G_2 also has a universal \mathcal{C} subgroup.*

PROOF. Let $H \subseteq G_1$ be a universal \mathcal{C} subgroup of G_1 . We will show that $H' = \varphi_1(H)$ is a universal \mathcal{C} subgroup of G_2 . Indeed, fix a \mathcal{C} subgroup $K \subseteq G_2$. By the closure properties of \mathcal{C} , $\varphi_2(K)$ is a \mathcal{C} subgroup of G_1 . Let $\psi : G_1 \rightarrow G_1$ be a continuous endomorphism such that $\varphi_2(K) = \psi^{-1}(H)$. The injectivity of φ_1 and φ_2 implies that $K = (\varphi_1 \circ \psi \circ \varphi_2)^{-1}(H')$. This completes the proof. \dashv

REMARK 1.10. The hypothesis of Proposition 1.9 is a weak form of bi-embeddability, since the maps φ_1 and φ_2 need not have continuous inverses and hence need not be isomorphisms between their domains and ranges.

Since the class of F_σ subgroups is not closed under continuous homomorphic images, we require a stronger form of mutual embeddability to apply Theorem 1.7 to a larger class of groups. Proposition 1.11 follows by the same proof as Proposition 1.9.

PROPOSITION 1.11. *Suppose that G_1 and G_2 are Polish groups, each of which is isomorphic to a closed subgroup of the other. If G_1 has a universal F_σ subgroup for K_σ , then so does G_2 .*

As an example, consider the Polish group \mathfrak{c}_0^ω , where \mathfrak{c}_0 is the additive group of sequences of real numbers tending to zero (with the sup-metric). In the weak sense of Proposition 1.9, \mathfrak{c}_0^ω is bi-embeddable with \mathbb{R}^ω . (We give more details in Section 7.) Proposition 1.9 thus implies that \mathfrak{c}_0^ω has universal K_σ and universal compactly generated subgroups, even though \mathfrak{c}_0 is not a product of locally compact groups.

Consider the groups S_∞ and $\text{Aut}(\mathbb{Q})$, both of which embed their countable powers as closed subgroups. Theorem 1.7 and Proposition 1.11 together yield an F_σ subgroup of S_∞ (respectively, of $\text{Aut}(\mathbb{Q})$), which is universal for K_σ subgroups of S_∞ (respectively, of $\text{Aut}(\mathbb{Q})$).

In Section 7, we give more details of such examples and provide further applications of Theorem 1.6 in conjunction with Proposition 1.9. We will also show that a class of standard Banach and Hilbert spaces, viewed as complete topological groups, have universal K_σ and compactly generated subgroups.

Section 4.1 is a brief detour exploring the relationship between K_σ and compactly generated subgroups. We obtain the following result.

THEOREM 1.12. *Suppose that G is countable discrete group. Every K_σ subgroup of G^ω is compactly generated if, and only if, every subgroup of G is finitely generated.*

In particular, every K_σ subgroup of the countable power of a finite group is compactly generated. Likewise, in \mathbb{Z}^ω . Regarding this result, Arnold Miller [19] has shown that every K_σ subgroup of \mathbb{R}^ω is compactly generated. This is an interesting complement to the theorem above as \mathbb{R} is not discrete and, in general, countable subgroups of \mathbb{R} are not finitely generated.

In Section 8, we apply the method of Theorem 1.6 to demonstrate the existence of complete F_σ ideals on ω , with respect to a weak form of Rudin-Keisler reduction.

§2. Preliminaries and notation. The definitions and notation we use are standard and essentially identical to those in the references by Kechris [14], Kanovei [13], or Moschovakis [20]. We recall some key points below.

A *Polish space* is a separable space whose topology is compatible with a complete metric. A *topological group* is a topological space G equipped with a group operation and an inverse map, such that the group operation is continuous as a function $G^2 \rightarrow G$ and the inverse map is continuous as a function $G \rightarrow G$. Hence a *Polish group* is a topological group, the topology of which is Polish.

Except when working with specific groups, we will always use multiplicative notation for group operations. Henceforth, $\mathbf{1}$ will denote the identity element of a multiplicative group.

It is useful to have the notion of a *group word*. An n -ary group word w is a function taking n symbols as input and combining these symbols using multiplication and inverses. For example, $w(a, b, c) = b^{-1}ac^{-1}$ is a ternary group word. For an n -ary group word w and a topological group G , note that w induces a continuous function $G^n \rightarrow G$. When there is no ambiguity, we will sometimes write w for $w(a_1, \dots, a_n)$.

For $A \subseteq G$, we let $w[A]$ denote the set

$$\{w(x_1, \dots, x_m) : x_1, \dots, x_m \in A\}.$$

We let $\langle A \rangle$ denote the *subgroup generated by A* , i.e., the smallest (with respect to containment) subgroup of G which contains A . Equivalently,

$$\langle A \rangle = \bigcup \{w[A] : w \text{ is a group word}\}.$$

For subsets A, B of a group G and $g \in G$, we let AB denote the set $\{ab : a \in A \ \& \ b \in B\}$, gA denote $\{ga : a \in A\}$, and A^{-1} denote $\{a^{-1} : a \in A\}$. Likewise, define $A + B$ and $-A$, in the case of additive groups.

If x is any sequence, we let $x(n)$ denote the n th term (or *bit*) of x . We denote the length n initial segment of x by $x \upharpoonright n$. If $I \subset \omega$ is the interval $\{k, k + 1, \dots, k + m\}$, then $x \upharpoonright I$ denotes the finite sequence

$$(x(k), x(k + 1), \dots, x(k + m)).$$

For a set A of sequences, we let $A \upharpoonright n$ denote the set $\{x \upharpoonright n : x \in A\}$.

For finite sequences s, t , $s \hat{\ } t$ denotes the concatenation of s and t . If t is the length 1 sequence (a) , for some $a \in X$, we simply write $s \hat{\ } a$, for $s \hat{\ } t$.

If X is any set and $a \in X$, a^n denotes the finite sequence $(a, \dots, a) \in X^n$ and \bar{a} the infinite sequence $(a, a, \dots) \in X^\omega$.

If $T \subseteq X^{<\omega}$ is a tree, then $[T]$ denotes the set $\{x \in X^\omega : (\forall n)(x \upharpoonright n \in T)\}$. If $T \subseteq (X \times Y)^{<\omega}$ is a tree, then $p[T]$ denotes the set $\{x \in X^\omega : (\exists y \in Y^\omega)((x, y) \in [T])\}$.

For $\alpha, \beta \in \omega^\omega$, we write $\alpha \leq \beta$ to mean that $(\forall i)(\alpha(i) \leq \beta(i))$. Similarly, if $s, t \in \omega^k$, $s \leq t$ means that $s(i) \leq t(i)$, for each $i < k$.

Finally, if A is a subset of a topological space X , $\text{cl}(A)$ denotes the (topological) closure of A .

§3. A universal closed subgroup of \mathbb{Z}^ω . The following is our simplest result. Although it does not fit into the scheme outlined in Section 1.2, it provides an example of the type of “coding” we will use to produce universal subgroups.

THEOREM 3.1. *There is a universal closed subgroup of \mathbb{Z}^ω .*

PROOF. \mathbb{Z}^k is a free, finitely generated, abelian group. Hence all of its subgroups are also finitely generated (see Lang [16, 7.3]). In particular, there are only countably many subgroups of \mathbb{Z}^k . Enumerate them as G_0^k, G_1^k, \dots . For each n, k , let $I_n^k \subseteq \omega$ be an interval of length k , such that $\{I_n^k : n, k \in \omega\}$ partition ω .

Define a closed subgroup G of \mathbb{Z}^ω by

$$x \in G \iff (\forall k, n)(x \upharpoonright I_n^k \in G_n^k).$$

We will show that G is a universal closed subgroup. Let H be an arbitrary closed subgroup of \mathbb{Z}^ω . We show that $H \leq_g G$.

Let T be a pruned tree on \mathbb{Z} such that $H = [T]$. Note that, because T is pruned, $T \cap \mathbb{Z}^k$ is a subgroup of \mathbb{Z}^k , for each k . Given k , let n_k be such that $T \cap \mathbb{Z}^k = G_{n_k}^k$.

Define a continuous group homomorphism $\varphi : \mathbb{Z}^\omega \rightarrow \mathbb{Z}^\omega$ by

$$\varphi(x) \upharpoonright I_n^k = \begin{cases} x \upharpoonright k & \text{if } n = n_k, \\ 0^k & \text{otherwise.} \end{cases}$$

For $x \in \mathbb{Z}^\omega$ and $y = \varphi(x)$, we have

$$\begin{aligned} x \in H &\iff (\forall k)(x \upharpoonright k \in T \cap \mathbb{Z}^k) \\ &\iff (\forall k)(y \upharpoonright I_{n_k}^k \in G_{n_k}^k) \\ &\iff (\forall k, n)(y \upharpoonright I_n^k \in G_n^k) \\ &\iff \varphi(x) \in G. \end{aligned}$$

The third ‘ \iff ’ follows from the fact that, if $n \neq n_k$, then $y \upharpoonright I_n^k = 0^k \in G_n^k$. This shows that $H \leq_g G$. ⊣

If G is a finite group, then there are only finitely many subgroups of G^k , for each k . Thus we have the following corollary to the proof of Theorem 3.1.

COROLLARY 3.2. *If G is a finite group, then G^ω has a universal closed subgroup.*

§4. K_σ subgroups. In Section 4.1, we study the relationship between K_σ and compactly generated subgroups and prove Theorem 1.12. In Section 4.2, we produce universal K_σ and compactly generated subgroups in the direct product of any sequence of locally compact Polish groups, with infinitely often repeated factors, thereby proving Theorem 1.6.

4.1. K_σ vs. compactly generated subgroups. A compactly generated subgroup will always be K_σ . Examples of such subgroups in \mathbb{Z}^ω are

$$B = \{x : x \text{ is bounded}\}$$

(generated by the set of all 0-1 sequences) and

$$\text{Fin} = \{x : (\forall^\infty n)(x(n) = 0)\}$$

(generated by the set of 0-1 sequences with at most one nonzero bit).

In some cases, the classes of K_σ and compactly generated subgroups coincide. The following two theorems give a sufficient condition for this to be the case. In particular, they imply that every K_σ subgroup of \mathbb{Z}^ω is compactly generated.

THEOREM 4.1. *For a Polish group G , every K_σ subgroup of G is compactly generated if, and only if, every countable subgroup of G is compactly generated.²*

PROOF. The “only if” part follows from the fact that every countable subgroup is K_σ .

For the “if” part, suppose that $H = \bigcup_n K_n$ is a K_σ subgroup of G . Let $U_0 \supseteq U_1 \supseteq \dots$ be a neighborhood base at the identity element $\mathbf{1} \in G$, with the additional property that each $\text{cl}(U_{n+1}) \subseteq U_n$. For each n

$$\{xU_{n+1} : x \in K_n\}$$

covers K_n . By compactness, there exists a finite set $S_n \subseteq K_n$ such that

$$\{xU_{n+1} : x \in S_n\}$$

²For countable subgroups, note that compactly generated is not the same as finitely generated, e.g., $\mathbb{Q} \subseteq \mathbb{R}$ is generated by $\{\frac{1}{n} : n \in \omega\} \cup \{0\}$, but is not finitely generated.

still covers K_n . Now let

$$K_n^* = \bigcup_{x \in S_n} x^{-1}((x \text{cl}(U_{n+1})) \cap K_n).$$

First note that, as the finite union of translates of compact sets, K_n^* is compact. Also, $K_n^* \subseteq H$ and $\mathbf{1} \in K_n^* \subseteq \text{cl}(U_{n+1}) \subseteq U_n$. Furthermore, $K_n \subseteq \langle K_n^* \cup S_n \rangle$. Let $K^* = \bigcup_n K_n^*$. Then $K^* \subseteq H$ and

$$H = \langle K^* \cup \bigcup_n S_n \rangle.$$

We claim that K^* is compact. Indeed, suppose that $z_0, z_1, \dots \in K^*$. If there is n such that $z_j \in K_n^*$, for infinitely many j , then $(z_j)_{j \in \omega}$ has a subsequential limit in K_n^* , by compactness. On the other hand, suppose that there are only finitely many z_j in each K_n^* . Let $n_0 < n_1 < \dots$ and j_0, j_1, \dots be such that for each k , $z_{j_k} \in K_{n_k}^*$. Then for each k , $z_{j_k} \in U_{n_k}$. Hence $z_{j_k} \rightarrow \mathbf{1} \in K^*$, as $k \rightarrow \infty$.

Let $S \subseteq H$ be the subgroup generated by $\bigcup_n S_n$ (a countable subgroup). By assumption, S is compactly generated. Therefore, take a compact set $C \subseteq S$ with $S = \langle C \rangle$. Then H will be generated by the compact set $K^* \cup C$. \dashv

We now restate and prove Theorem 1.12.

THEOREM 1.12. *Suppose that G is a discrete group. Every K_σ subgroup of G^ω is compactly generated if, and only if, every subgroup of G is finitely generated. (In this latter case, G is automatically countable.)*

Groups of which every subgroup is finitely generated are known as *Noetherian groups*, or are said to have the *maximal condition on subgroups*. There is a rich class of groups with this property. For instance, all *polycyclic groups* (i.e., groups possessing a cyclic series) have the maximal property on subgroups (see [17, Theorem 1.3.1]). The class of polycyclic groups includes that of finitely generated nilpotent groups and hence that of finitely generated abelian groups.

Theorem 1.12 could thus be rephrased as: “If G is discrete, then every K_σ subgroup of G^ω is compactly generated iff G is Noetherian.”

PROOF OF THEOREM 1.12. First suppose that there is a subgroup H of G which is not finitely generated. Then $H^* = \{\bar{a} : a \in H\}$ is a K_σ subgroup of G^ω with no compact generating set.

Suppose now that every subgroup of G is finitely generated. We will show that every K_σ subgroup of G^ω is compactly generated. By Theorem 1.12, it suffices to show that every countable subgroup of G^ω is compactly generated. Fix a countable subgroup $C = \{x_1, x_2, \dots\}$. For each n , let $C_n = \{x \in C : x \upharpoonright n = \mathbf{1}^n\}$.

CLAIM 4.2. *For each n , there is a finite set $F_n \subseteq C_n$ such that if $x \in C_n$, then there exists a group word w in the elements of F_n such that $x \cdot w^{-1} \in C_{n+1}$.*

PROOF OF CLAIM. For each C_n there is a finite set $F_n \subseteq C_n$ such that $\{x(n) : x \in F_n\}$ generates $\{x(n) : x \in C_n\}$, since the latter is a subgroup of G .

This implies that, for each $x \in C_n$ there is a group word w in the elements of F_n such that $x(n) = w(n)$. Hence $x(n) \cdot w^{-1}(n) = \mathbf{1}$. On the other hand, $x \upharpoonright n = w \upharpoonright n = \mathbf{1}^n$, since $x, w \in C_n$. Thus

$$x \cdot w^{-1} \upharpoonright (n + 1) = \mathbf{1}^{n+1}.$$

In other words, $x \cdot w^{-1} \in C_{n+1}$. This proves the claim.

CLAIM 4.3. For each n there exists $\tilde{x}_n \in C_n$ and a group word w_n in the elements of $F_0 \cup \dots \cup F_{n-1}$ such that $x_n = \tilde{x}_n \cdot w_n$.

PROOF OF CLAIM. The argument is a finite induction. Let $w_{n,0}$ be a group word in the elements of F_0 , as in Claim 4.2, such that $x_n \cdot w_{n,0}^{-1} \in C_1$. Set $x_{n,1} = x_n \cdot w_{n,0}^{-1}$. Now let $w_{n,1}$ be a group word in the elements of F_1 such that $x_{n,1} \cdot w_{n,1}^{-1} \in C_2$ and define $x_{n,2} = x_{n,1} \cdot w_{n,1}^{-1}$. In general, we obtain $x_{n,i} \in C_i$ and group words $w_{n,i}$ in the elements of F_i such $x_{n,i+1} = x_{n,i} \cdot w_{n,i}^{-1} \in C_{i+1}$.

Let $\tilde{x}_n = x_{n,n}$ and $w_n = w_{n,n-1} \cdot \dots \cdot w_{n,0}$. Observe that w_n is a group word in the elements of $F_0 \cup \dots \cup F_{n-1}$, $\tilde{x}_n \in C_n$, and $x_n = \tilde{x}_n \cdot w_n$, as desired.

Claim 4.3 implies that each x_n is in the subgroup generated by \tilde{x}_n together with $F_0 \cup \dots \cup F_{n-1}$. Thus the set

$$\tilde{C} = \bigcup_n (\{\tilde{x}_n\} \cup F_n)$$

generates C .

It remains to check that \tilde{C} is compact. For each n , observe that there are only finitely many elements $x \in \tilde{C}$ such that $x(n) \neq \mathbf{1}$, since all such elements are contained in $\{\tilde{x}_i : i \leq n\} \cup F_0 \cup \dots \cup F_n$. Thus every infinite sequence of distinct elements of \tilde{C} must converge to $\bar{\mathbf{1}}$. This implies that every infinite sequence in \tilde{C} is either eventually constant or has a subsequence converging to $\bar{\mathbf{1}}$. \dashv

We enumerate a couple of direct consequences.

1. Every K_σ subgroup of \mathbb{Z}^ω is compactly generated. (Since every subgroup of \mathbb{Z} is singly generated.)
2. If G is a finite group, then every K_σ subgroup of G^ω is compactly generated.

For a Polish group G , even if there are noncompactly generated K_σ subgroups, we can still ask whether or not every K_σ subgroup is group-homomorphism reducible to a compactly generated one. The following two examples illustrate the range of possibilities.

EXAMPLE 4.4. Let $S = \bigoplus_\omega \mathbb{Z}$ be the direct sum of countably many copies of \mathbb{Z} . Unlike \mathbb{Z} , the countable group S is not finitely generated. Thus, with the discrete topology, S is K_σ , but not compactly generated. (In a discrete space, compact is the same as finite.)

By extension, not all K_σ subgroups of S^ω will be compactly generated. For example, $\{x \in S^\omega : x \text{ is a constant sequence}\}$. On the other hand, we will see that every K_σ subgroup is group-homomorphism reducible to a compactly generated one. We begin by showing that S^ω may be mapped one-to-one homomorphically into \mathbb{Z}^ω . Let $\varphi_n : S \rightarrow \mathbb{Z}$ be the projection map onto the n th coordinate. Define $\psi : S^\omega \rightarrow \mathbb{Z}^\omega$ by

$$\psi(x)(\langle m, n \rangle) = \varphi_n(x(m)),$$

where $\langle \cdot, \cdot \rangle : \omega^2 \leftrightarrow \omega$ is a fixed bijection. The map ψ is a continuous injective homomorphism whose range is the Π_3^0 subgroup

$$\{y \in \mathbb{Z}^\omega : (\forall m)(\forall^\infty n)((x(\langle m, n \rangle) = 0))\}.$$

Now let $H \subseteq S^\omega$ be any K_σ subgroup. The image $\psi(H) \subseteq \mathbb{Z}^\omega$ is also K_σ (because ψ is continuous) hence compactly generated by Theorem 1.12. Say $\psi(H) = \langle K \rangle$.

Let $i : \mathbb{Z}^\omega \rightarrow S^\omega$ be the natural “inclusion” map. Then $i(K) \subseteq S^\omega$ is compact and $H = (i \circ \psi)^{-1}(i(K))$, because $i \circ \psi$ is injective.

For example 4.5, we introduce some terminology. Suppose that H is a subgroup of an abelian group G (with additive notation) and $x \in H$. We say that x is *divisible in H* to mean that for each $n \in \omega$, there exists $y \in H$ such that $x = ny$. Note that for subgroups $H_1, H_2 \subseteq G$, if $\varphi : G \rightarrow G$ is a group homomorphism such that $\varphi^{-1}(H_2) = H_1$ and $x \in H_1$ is divisible in H_1 , then $\varphi(x) \in H_2$ is divisible in H_2 .

EXAMPLE 4.5. Consider the group \mathbb{Q} of rational numbers with the discrete topology. We will see that there are K_σ subgroups of \mathbb{Q}^ω that are not group-homomorphism reducible to any compactly generated subgroup.

We first claim that there are no nonzero divisible elements in a compactly generated subgroup of \mathbb{Q}^ω . Indeed, suppose that, on the contrary, H is generated by the compact set K and there is a nonzero element $x \in H$, with x divisible in H . Let $m \in \omega$ be such that $x(m) \neq 0$. Let

$$A = \{y(m) : y \in K\}.$$

Note that, since x is divisible in H , $x(m)$ will be divisible in $\langle A \rangle \subset \mathbb{Q}$. As K is compact and we have given \mathbb{Q} the discrete topology, A must be finite. Therefore, let $k \in \mathbb{Z}$ be such that $ka \in \mathbb{Z}$, for each $a \in A$. This implies that, for any $b \in \langle A \rangle$, we also have $kb \in \mathbb{Z}$. Let n be large enough that $\frac{k}{n}x(m) \notin \mathbb{Z}$. Thus $\frac{1}{n}x(m) \notin \langle A \rangle$, contradicting the divisibility of $x(m)$ in $\langle A \rangle$.

We now exhibit a K_σ subgroup which is not group-homomorphism reducible to any compactly generated subgroup. Consider the subgroup

$$\text{Fin} = \{x \in \mathbb{Q}^\omega : (\forall^\infty n)(x(n) = 0)\}.$$

Fin is K_σ and every element of Fin is divisible in Fin. Suppose that $\varphi : \mathbb{Q}^\omega \rightarrow \mathbb{Q}^\omega$ is a continuous homomorphism and H is a subgroup such that $\varphi^{-1}(H) = \text{Fin}$. In the first place, we have that $\ker \varphi \subseteq \text{Fin}$. Note, however, that $\ker \varphi \neq \text{Fin}$, since then we would have $\varphi \equiv 0$, because Fin is dense in \mathbb{Q}^ω . Hence, there exists $x \in \text{Fin}$ with $\varphi(x) \neq 0$. Since x is divisible in Fin, we have that $\varphi(x)$ is divisible in H and nonzero. Thus H cannot be compactly generated, by the comments above.

4.2. Universal subgroups. Before giving the proof of Theorem 1.6, we consider the special case of \mathbb{Z}^ω to illustrate the main ideas in a more straightforward setting.

4.2.1. *The case of \mathbb{Z}^ω .*

THEOREM 4.6. *There is a universal compactly generated subgroup of \mathbb{Z}^ω .*

PROOF. We essentially construct a \leq_g -complete compact subset of \mathbb{Z}^ω .

For each $m \in \omega$, let A_0^k, A_1^k, \dots list all finite subsets of \mathbb{Z}^k which contain 0^k and are such that $-A_j^k = A_j^k$. Let I_j^k ($k, j \in \omega$) partition ω , with each I_j^k an interval of length k . Define $K_0 \subset \mathbb{Z}^\omega$ by

$$x \in K_0 \iff (\forall k, j)(x \upharpoonright I_j^k \in A_j^k).$$

Note that K_0 is compact and $-K_0 = K_0$. Consider $\langle K_0 \rangle$ (the subgroup generated by K_0). We show that $\langle K_0 \rangle$ is universal for compactly generated subgroups of \mathbb{Z}^ω .

Suppose that $\langle K \rangle$ is any compactly generated subgroup. With no loss of generality, we assume that $-K = K$ and $\bar{0} \in K$. There is a pruned tree T on \mathbb{Z} such that $K = [T]$. Since K is compact, all levels of T must be finite. For each k , choose $\tau(k) \in \omega$ such that $A_{\tau(k)}^k = T \cap \mathbb{Z}^k$. Define a homomorphism $\varphi : \mathbb{Z}^\omega \rightarrow \mathbb{Z}^\omega$ by

$$\varphi(x) \upharpoonright I_j^k = \begin{cases} x \upharpoonright k & \text{if } j = \tau(k), \\ 0^k & \text{otherwise.} \end{cases}$$

Observe that $\varphi^{-1}(K_0) = K$. The following claim will complete the proof of this theorem.

CLAIM. $\varphi^{-1}(\langle K_0 \rangle) = \langle K \rangle$.

PROOF OF CLAIM. Suppose that $x \in \langle K \rangle$, with $x_1, \dots, x_m \in K$ such that $x = x_1 + \dots + x_m$. (Note that, since $-K = K$, all elements of $\langle K \rangle$ are finite sums of elements of K .) Then $\varphi(x_1), \dots, \varphi(x_m) \in K_0$ and hence $\varphi(x) = \varphi(x_1) + \dots + \varphi(x_m) \in \langle K_0 \rangle$.

Suppose, on the other hand, that $\varphi(x) \in \langle K_0 \rangle$, with $y_1, \dots, y_m \in K_0$ such that $\varphi(x) = y_1 + \dots + y_m$. (Again, because $-K_0 = K_0$, $\langle K_0 \rangle$ is the set of finite sums of members of K_0 .) We want $x_1, \dots, x_m \in K$ with $x = x_1 + \dots + x_m$.

For each $i \leq m$, let $v_i^k = y_i \upharpoonright I_{\tau(k)}^k$. Since each $y_i \in K_0$, the definition of K_0 implies that each

$$v_i^k \in A_{\tau(k)}^k = T \cap \mathbb{Z}^k.$$

Hence (because T is pruned) there exists $x_i^k \in K$ such that

$$x_i^k \upharpoonright k = v_i^k.$$

By the compactness of K , we may iteratively (for $i \leq m$) take convergent subsequences of $(x_i^k)_{k \in \omega}$ to obtain a common subsequence $k_0 < k_1 < \dots$ such that, for each $i \leq m$, $(x_i^{k_n})_{n \in \omega}$ is convergent, with limit $x_i \in K$. Finally, fix p and let $k_n \geq p$ be large enough that $x_i^{k_n} \upharpoonright p = x_i \upharpoonright p$, for each $i \leq m$. Thus

$$\begin{aligned} x \upharpoonright p &= \sum_{i \leq p} v_i^{k_n} \upharpoonright p \\ &= \sum_{i \leq p} x_i^{k_n} \upharpoonright p \quad (\text{because } k_n \geq p) \\ &= \sum_{i \leq p} x_i \upharpoonright p. \end{aligned}$$

As p was arbitrary, we have $x = \sum_{i \leq m} x_i \in \langle K \rangle$. This completes the proof. ⊢

COROLLARY 4.7. *There is a universal K_σ subgroup of \mathbb{Z}^ω .*

PROOF. Since every K_σ subgroup of \mathbb{Z}^ω is compactly generated by Theorem 1.12, Theorem 4.6 actually gives a universal K_σ subgroup of \mathbb{Z}^ω . ⊢

4.2.2. *Proof of Theorem 1.6.* In this sections, we prove the following theorem.

THEOREM 1.6. *Let $(G_n)_{n \in \omega}$ be a sequence of locally compact Polish groups, each term of which occurs infinitely often (up to isomorphism). We have the following;*

1. $\prod_n G_n$ has a universal compactly generated subgroup.
2. $\prod_n G_n$ has a universal K_σ subgroup.

Note that if every K_σ subgroup of $\prod_n G_n$ is reducible to a compactly generated subgroup, then part (1) of this result implies part (2). On the other hand, in Section 4.1, we saw examples of K_σ subgroups of Polish groups which do not reduce to compactly generated subgroups. In such cases, (1) and (2) remain distinct results.

As mentioned in the Section 1, the following is a corollary of Theorem 1.6.

COROLLARY 4.8. *If G is locally compact, then G^ω has universal compactly generated and K_σ subgroups.*

For most of the examples we consider in Section 7, we will only use the statement of Corollary 4.8.

Our key lemma in the proof of Theorem 1.6 is a restricted, but refined, version of Theorem 1.6(1). (Recall that for an m -ary group word w , we define $w[A] = \{w(x_1, \dots, x_m) : x_1, \dots, x_m \in A\}$.)

LEMMA 4.9. *Let G be a locally compact Polish group with identity element $\mathbf{1}$. There exists a compact set $K_0 \subseteq G^\omega$ with $\mathbf{1} \in K_0$ and the property that for each compact $K \subseteq G^\omega$, with $\mathbf{1} \in K$, there is a continuous group homomorphism $\varphi : G^\omega \rightarrow G^\omega$ such that, for each group word w ,*

$$\varphi^{-1}(w[K_0]) = w[K].$$

In particular, $\langle K_0 \rangle$ is a universal compactly generated subgroup of G^ω .

Basic notions. We begin with some notation and facts we will use in the proof of Lemma 4.9. From now on, fix a locally compact Polish group G , with identity element $\mathbf{1}$.

The following lemma gives a neighborhood base at $\mathbf{1}$ with the specific properties we require.

LEMMA 4.10. *There is a neighborhood base $\{U_k\}$ at $\mathbf{1}$ such that*

1. *Each U_k has compact closure.*
2. *$U_0 \supseteq U_1 \supseteq \dots$.*
3. *For each k , $U_k^{-1} = U_k$.*
4. *For each $k > 0$, $\text{cl}(U_k U_k) \subseteq U_{k-1}$.*

PROOF. We construct the U_k inductively. Let $V_0 \supseteq V_1 \supseteq \dots \ni \mathbf{1}$ be any “nested” neighborhood base at $\mathbf{1}$, such that $\text{cl}(V_0)$ is compact. (Such V_k exists, since G is locally compact.) Let $U_0 = V_0$. Suppose that $U_0 \supseteq \dots \supseteq U_k$ are given with the desired properties. By the continuity of the group operation, there is a neighborhood V of $\mathbf{1}$ such that $\text{cl}(VV) \subseteq V_k \cap U_k$. By the continuity of the map $(x, y) \mapsto x^{-1}y$, there is a neighborhood W of $\mathbf{1}$ such that $W^{-1}W \subseteq V$. Let $U_{k+1} = W^{-1}W$. Then $(U_{k+1})^{-1} = U_{k+1}$ and

$$\text{cl}(U_{k+1}U_{k+1}) \subseteq \text{cl}(VV) \subseteq U_k. \tag{1}$$

Fix a neighborhood base $\{U_k\}$, as in the lemma above. For $a, b \in G$, write $a \approx_k b$ (“ a k -approximates b ”) if, and only if, $a^{-1}b \in \text{cl}(U_k)$. Note that, by the properties of the U_k ,

1. $a \approx_k a$
2. $a \approx_k b \iff b \approx_k a$
3. $a \approx_k b \approx_k c \implies a \approx_{k-1} c$
4. $(a \approx_k b \ \& \ k' \leq k) \implies a \approx_{k'} b$
5. $\lim_n a_n = a \iff (\forall k)(\forall^\infty n)(a_n \approx_k a)$.

If $x, y \in G^\omega$ (or G^p), we will write $x \approx_k y$ to indicate that $x(i) \approx_k y(i)$, for each $i \in \omega$ (or $i < p$). Item 5 above implies that for $x, x_n \in G^\omega$

$$\lim_n x_n = x \iff (\forall p, k)(\forall^\infty n)(x_n \upharpoonright p \approx_k x \upharpoonright p). \tag{4.1}$$

Also note that, for each k and fixed $a_0 \in G$, the set

$$\{a \in G : a_0 \approx_k a\}$$

is compact.

Fix a countable dense set $D \subseteq G$, with $\mathbf{1} \in D$. Let $n \leq \omega$ be the cardinality of D , and $\# : D \leftrightarrow n$ be a bijection, with $\#\mathbf{1} = 0$.

For $x \in G^\omega$ and $k \in \omega$, we define a sequence $\beta_x^k \in D^\omega$ (which we call the *least k -approximation* of x) as follows; for each i , let $a_i \in D$ be the element with $\#a_i$ least such that $a_i \approx_k x(i)$. Define $\beta_x^k \in D^\omega$ by

$$(\forall i)(\beta_x^k(i) = a_i).$$

Given a closed set $K \subseteq G^\omega$ and $k \in \omega$, let

$$\mathcal{B}_k = \{\beta_x^k \upharpoonright k : x \in K\}.$$

Since K is closed, the statement (4.1) above implies that $x \in K$ if, and only if, $(\forall k)(\beta_x^k \upharpoonright k \in \mathcal{B}_k)$. We have the following fact.

LEMMA 4.11. *If $K \subseteq G^\omega$ is compact, then $\{\beta_x^k(n) : x \in K\}$ is finite, for each $k, n \in \omega$. In particular, each \mathcal{B}_k is finite.*

PROOF. Since K is compact, so is the set $A = \{x(n) : x \in K\} \subseteq G$. There is thus a finite set $F_n \subseteq D$ such that, for each $x \in K$, there is an $a \in F_n$ with $x(n) \approx_k a$. As $\beta_x^k(n)$ is the $\#$ -least element of D which k -approximates $x(n)$, we conclude that $\#\beta_x^k(n) \leq \max\{\#a : a \in F_n\}$, for each $x \in K$. Hence $\{\beta_x^k(n) : x \in K\}$ is finite.

This implies that each \mathcal{B}_k is finite, since $\mathcal{B}_k \subseteq \prod_{n < k} F_n$. ◻

Proof of Lemma 4.9 Fix a locally compact group G and let $D, \#$, and \approx_k be defined as above for G . For each $k \in \omega$, let $A_0^k, A_1^k, \dots \subseteq D^k$ be such that, for each k, j , we have

- A_j^k is finite.
- $\mathbf{1}^k \in A_j^k$.
- For each finite $A \subseteq D^k$, with $\mathbf{1}^k \in A$, there exists j such that $A = A_j^k$.

Let I_j^k (for $k, j \in \omega$) be intervals partitioning ω such that each I_j^k has length k . Define $K_0 \subseteq G^\omega$ by

$$x \in K_0 \iff (\forall k, j)(\exists u \in A_j^k)(u \approx_k x \upharpoonright I_j^k).$$

Note that K_0 is compact, since “ $u \approx_k x \upharpoonright I_j^k$ ” defines a compact subset of G^k and the existential quantifier is over a finite set. We shall show that $\langle K_0 \rangle$ has the property that for any compact $K \subseteq G^\omega$, containing $\bar{\mathbf{1}}$, there is a continuous homomorphism $\varphi : G^\omega \rightarrow G^\omega$ with

$$\varphi^{-1}(w[K_0]) = w[K],$$

for each group word w .

Let K be an arbitrary compact subset of G^ω , with $\bar{\mathbf{1}} \in K$. For each k , let

$$\mathcal{B}_k = \{\beta_x^k \upharpoonright k : x \in K\}$$

be as above. As we remarked in Lemma 4.11, the compactness of K implies that each \mathcal{B}_k is finite. Since $\bar{\mathbf{1}}$ is its own least k -approximation, each \mathcal{B}_k contains $\mathbf{1}^k$. For each $k \in \omega$, we may therefore choose $\tau(k) \in \omega$ such that

$$A_{\tau(k)}^k = \mathcal{B}_k.$$

Define a continuous group homomorphism $\varphi : G^\omega \rightarrow G^\omega$ by

$$\varphi(x) \upharpoonright I_j^k = \begin{cases} x \upharpoonright k & \text{if } j = \tau(k), \\ \mathbf{1}^k & \text{otherwise.} \end{cases}$$

Fix an m -ary group word w . The following two claims will complete the proof.

CLAIM 4.12. $x \in w[K] \implies \varphi(x) \in w[K_0]$.

PROOF OF CLAIM. Since φ is a group homomorphism, it will suffice to show that $x \in K \implies \varphi(x) \in K_0$. Suppose that $x \in K$. For each k , let

$$u_k = \beta_x^k \upharpoonright k \in \mathcal{B}_k = A_{\tau(k)}^k.$$

Hence $u_k \approx_k x \upharpoonright k = \varphi(x) \upharpoonright I_{\tau(k)}^k$. On the other hand, if $j \neq \tau(k)$, then $\varphi(x) \upharpoonright I_j^k = \mathbf{1}^k \in A_j^k$. Putting these together, we see that

$$(\forall k, j)(\exists u \in A_j^k)(u \approx_k \varphi(x) \upharpoonright I_j^k).$$

Thus $\varphi(x) \in K_0$. This completes the claim.

CLAIM 4.13. $\varphi(x) \in w[K_0] \implies x \in w[K]$.

PROOF OF CLAIM. Let $y_1, \dots, y_m \in K_0$ be such that $\varphi(x) = w(y_1, \dots, y_m)$. We will find $x_1, \dots, x_m \in K$ such that $x = w(x_1, \dots, x_m)$ and conclude that $x \in w[K]$.

For each k, i , let

$$v_i^k = y_i \upharpoonright I_{\tau(k)}^k$$

and let $u_i^k \in A_{\tau(k)}^k = \mathcal{B}_k$ be such that $u_i^k \approx_k v_i^k$. By the definition of \mathcal{B}_k , there exist $x_i^k \in K$ such that $u_i^k \approx_k x_i^k \upharpoonright k$, for each k and $i \leq m$. Since K is compact, we may take $k_0 < k_1 < \dots$ and $x_1, \dots, x_m \in K$ with $\lim_n x_i^{k_n} = x_i$, for each $i \leq m$.

Let $z_i^k = v_i^k \wedge \bar{\mathbf{1}}$. We claim that

$$\lim_n z_i^{k_n} = x_i.$$

Indeed, fix $p, r \in \omega$ and let M be large enough that whenever $k_n \geq M$, we have

$$x_i^{k_n} \upharpoonright r \approx_{p+2} x_i \upharpoonright r.$$

The existence of M follows from (4.1), since $\lim_n x_i^{k_n} = x_i$. We may assume that $M > r, p + 2$ and so if $k_n \geq M$, we have

$$\begin{aligned} z_i^{k_n} \upharpoonright r &= v_i^{k_n} \upharpoonright r \\ &\approx_{p+2} u_i^{k_n} \upharpoonright r \\ &\approx_{p+2} x_i^{k_n} \upharpoonright r \\ &\approx_{p+2} x_i \upharpoonright r. \end{aligned}$$

Hence $z_i^{k_n} \upharpoonright r \approx_p x_i \upharpoonright r$, for all $k_n \geq M$. As p, r were arbitrary, we conclude (again, by (4.1)) that $z_i^{k_n} \rightarrow x_i$ as $n \rightarrow \infty$.

We may now finish the claim. Observe that for fixed r and each $k_n > r$, we have

$$\begin{aligned} x \upharpoonright r &= (\varphi(x) \upharpoonright I_{\tau(k_n)}^{k_n}) \upharpoonright r \\ &= w(v_1^{k_n}, \dots, v_m^{k_n}) \upharpoonright r \\ &= w(z_1^{k_n}, \dots, z_m^{k_n}) \upharpoonright r. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ and using the fact that w induces a continuous function $G^{rm} \rightarrow G^r$ we conclude that

$$x \upharpoonright r = w(x_1, \dots, x_m) \upharpoonright r.$$

Since r was arbitrary, $x = w(x_1, \dots, x_m) \in w[K]$. This completes the proof. \dashv

Proof of main result. We first prove (1) of Theorem 1.6 and then prove (2) from (1).

PROOF OF THEOREM 1.6(1). Let $(G_n)_{n \in \omega}$ be a sequence of locally compact Polish groups, with each term occurring infinitely often up to isomorphism. This implies that $\prod_n G_n \cong \prod_n (G_0^\omega \times \dots \times G_n^\omega) \cong \prod_n G_n^\omega$. It will therefore suffice to show that there is a compactly generated subgroup of $\prod_n (G_0^\omega \times \dots \times G_n^\omega)$, which is universal for compactly generated subgroups of $\prod_n G_n^\omega$.

For each n , note that $G_0^\omega \times \dots \times G_n^\omega \cong (G_0 \times \dots \times G_n)^\omega$. As the direct product of finitely many locally compact groups, $G_0 \times \dots \times G_n$ itself is locally compact. Therefore, take compact sets $K_n \subseteq G_0^\omega \times \dots \times G_n^\omega$ with $\bar{1} \in K_n$, as in Lemma 4.9, such that, for any compact $K \subseteq G_0^\omega \times \dots \times G_n^\omega$ with $\bar{1} \in K$, there is a continuous endomorphism φ of $G_0^\omega \times \dots \times G_n^\omega$ such that $\varphi^{-1}(w[K_n]) = w[K]$, for each group word w .

Define a compact set $K_\infty \subseteq \prod_n (G_0^\omega \times \dots \times G_n^\omega)$ by

$$\xi \in K_\infty \iff (\forall n)(\xi(n) \in K_n).$$

We will show that $\langle K_\infty \rangle$ is universal for compactly generated subgroups of $\prod_n G_n^\omega$. Indeed, fix an arbitrary compactly generated subgroup $\langle K \rangle \subseteq \prod_n G_n^\omega$. We may assume that $\bar{1} \in K$. For each n , Lemma 4.9 gives an endomorphism φ_n of $G_0^\omega \times \dots \times G_n^\omega$ such that

$$\varphi_n^{-1}(w[K_n]) = w[K \upharpoonright (n+1)] = w[K] \upharpoonright (n+1), \tag{4.2}$$

for each group word w . (Recall here that $K \upharpoonright (n+1) = \{x \upharpoonright (n+1) : x \in K\} \subseteq G_0^\omega \times \dots \times G_n^\omega$.)

Define a continuous homomorphism $\varphi : \prod_n G_n^\omega \rightarrow \prod_n (G_0^\omega \times \dots \times G_n^\omega)$ by

$$\varphi(x)(n) = \varphi_n(x \upharpoonright (n+1)),$$

for each n . The following claim will complete the proof.

CLAIM. $\varphi^{-1}(\langle K_\infty \rangle) = \langle K \rangle$.

PROOF OF CLAIM. It suffices to show that, for each group word w ,

$$\varphi^{-1}(w[K_\infty]) = w[K]. \tag{4.3}$$

Fix a group word w . Armed with (4.2) and the fact that $w[K]$ is compact, we have

$$\begin{aligned} x \in w[K] &\iff (\forall n)(x \upharpoonright(n+1) \in w[K] \upharpoonright(n+1)) \\ &\iff (\forall n)(\varphi_n(x \upharpoonright(n+1)) \in w[K_n]) \\ &\iff (\forall n)(\varphi(x)(n) \in w[K_n]) \\ &\iff \varphi(x) \in w[K_\infty]. \end{aligned}$$

The third “ \iff ” follows from the definition of $\varphi(x)(n)$ as $\varphi_n(x \upharpoonright(n+1))$. This completes the proof. –

REMARK 4.14. In the proof above, (4.3) and the definition of K_∞ imply that the conclusion of Lemma 4.9 holds for $\prod_n G_n$, i.e., $\bar{\mathbf{1}} \in K_\infty$ and, for each compact $K \subseteq \prod_n G_n$ containing $\bar{\mathbf{1}}$, there is a continuous homomorphism $\varphi : \prod_n G_n \rightarrow \prod_n G_n$ with $\varphi^{-1}(w[K_\infty]) = w[K]$, for each group word w .

Considering the group word $w_0(a) = a$ and noting that $\langle K \rangle = \bigcup_w w[K]$, we obtain the following corollary to the proof of Theorem 1.6(1).

COROLLARY 4.15. *Suppose that $(G_n)_{n \in \omega}$ is a sequence of locally compact Polish groups, each term of which appears infinitely often. There exists a compact set $K_0 \subseteq \prod_n G_n$ such that $\bar{\mathbf{1}} \in K_0$ and for each compact $K \subseteq \prod_n G_n$ with $\bar{\mathbf{1}} \in K$, there is a continuous group homomorphism $\varphi : \prod_n G_n \rightarrow \prod_n G_n$ such that*

$$\varphi^{-1}(K_0) = K \text{ and } \varphi^{-1}(\langle K_0 \rangle) = \langle K \rangle.$$

We will use this in the next proof.

PROOF OF THEOREM 1.6(2). Fix a sequence $(G_n)_{n \in \omega}$ of locally compact Polish groups, as above. For each n , let $D_n \subseteq G_n$ be a countable dense set, containing the identity element $\mathbf{1}_n \in G_n$. For each n , fix an enumeration $\{x_0^n, x_1^n, \dots\}$ of D_n , with $x_0^n = \mathbf{1}_n$, and fix a neighborhood $U_n \ni \mathbf{1}_n$, with $\text{cl}(U_n)$ compact.

For each n and $x \in \prod_n G_n$, define $x^* \in \omega^\omega$ by

$$x^*(n) = \min\{i : (x_i^n)^{-1}x(n) \in \text{cl}(U_n)\},$$

for each $n \in \omega$. Define $u^* \in \omega^\omega$ analogously, for $u \in \prod_{i < n} G_i$. Observe that, by the argument of Lemma 4.11, if $K \subseteq \prod_n G_n$ is compact, then $\{x^* : x \in K\}$ has compact closure in ω^ω . Conversely, since each $\text{cl}(U_n)$ is compact, it follows that

$$\{x \in \prod_n G_n : x^* \leq \alpha\}$$

is compact, for each $\alpha \in \omega^\omega$.

For notational reasons, we will consider the group

$$G' = \prod_{\substack{n \in \omega \\ s \in \omega^{<\omega}}} (G_0 \times \dots \times G_{|s|-1}).$$

Note that n is a “dummy” index, serving only to produce infinitely many copies of the term inside the product. For the sake of clarity, we remark that $\zeta(n, s) \in G_0 \times \dots \times G_{|s|-1}$, for each n, s and $\zeta \in G'$.

Since each G_n is isomorphic to infinitely many other G_m , we have $G' \cong \prod_n G_n$. To prove our theorem, it will therefore suffice to produce a K_σ subgroup of G' , which is universal for K_σ subgroups of $\prod_n G_n$.

Let $K_0 \subseteq \prod_n G_n$ be as in Corollary 4.15. For each n , define

$$A_n = \{\xi \in G' : (\forall n' \geq n)(\forall s \in \omega^{<\omega})(\xi(n', s) \in K_0 \upharpoonright |s|)\}.$$

For each n , the subgroup $\langle A_n \rangle$ is F_σ . This follows from the fact that each A_n is the direct product of a compact set with factors of the form $G_0 \times \dots \times G_k$.

Define the set

$$\tilde{A} = \{\xi \in G' : (\forall^\infty (n, s))(\xi(n, s)^* \leq s)\}.$$

It follows that \tilde{A} is K_σ and hence $\langle \tilde{A} \rangle$ is as well. Let

$$H_0 = \langle \tilde{A} \rangle \cap \bigcup_n \langle A_n \rangle$$

and note that, since the term $\bigcup_n \langle A_n \rangle$ is an increasing union of subgroups, H_0 itself is a subgroup of G' . As the intersection of an F_σ set with a K_σ set, H_0 is K_σ . We will show that H_0 is universal for K_σ subgroups of $\prod_n G_n$.

Let $B = \bigcup_n B_n$ be an arbitrary K_σ subgroup of $\prod_n G_n$, with each B_n compact and $\bar{\mathbf{1}} \in B_0 \subseteq B_1 \subseteq \dots$. Take continuous endomorphisms ψ_n of $\prod_n G_n$ such that

$$\psi_n^{-1}(K_0) = B_n \quad \text{and} \quad \psi_n^{-1}(\langle K_0 \rangle) = \langle B_n \rangle,$$

for each n . Each $\psi_n(B_n)$ is compact. As noted above, this implies that the closure of $\{x^* : x \in \psi_n(B_n)\}$ is compact in ω^ω . Thus we may choose $\alpha_n \in \omega^\omega$ such that each α_n is increasing, $\alpha_0 \leq \alpha_1 \leq \dots$ and $x^* \leq \alpha_n$, for each $x \in \bigcup_{n' \leq n} \psi_{n'}(B_{n'})$. Define $\psi : \prod_n G_n \rightarrow G'$ by

$$\psi(x)(n, s) = \begin{cases} \psi_n(x) \upharpoonright p & \text{if } s = \alpha_{n+p} \upharpoonright p, \\ (\mathbf{1}_0, \dots, \mathbf{1}_{p-1}) & \text{otherwise,} \end{cases}$$

for each $n \in \omega$ and $s \in \omega^{<\omega}$ with $p = |s|$. It remains to show that $\psi^{-1}(H_0) = B$.

CLAIM 4.16. *If $\psi(x) \in H_0$, then $x \in B$.*

PROOF OF CLAIM. Let n be such that $\psi(x) \in \langle A_n \rangle$, with w a group word such that $\psi(x) \in w[A_n]$. For each p , if $s = \alpha_{n+p} \upharpoonright p$, we have

$$\begin{aligned} \psi_n(x) \upharpoonright p &= \psi(x)(n, s) \\ &\in \{\xi(n, s) : \xi \in w[A_n]\} \\ &= w[K_0 \upharpoonright p] \\ &= w[K_0] \upharpoonright p \end{aligned}$$

and hence $\psi_n(x) \in w[K_0]$, since the latter is closed. (As the continuous image of a compact set, $w[K_0]$ is compact.) This implies that $\psi_n(x) \in \langle K_0 \rangle$ and, since ψ_n reduces $\langle B_n \rangle$ to $\langle K_0 \rangle$, we conclude that $x \in \langle B_n \rangle \subseteq B$.

CLAIM 4.17. *If $x \in B$, then $\psi(x) \in H_0$.*

PROOF OF CLAIM. Suppose that $x \in B$, say $x \in B_{n_0}$. We first verify that $\psi(x) \in A_{n_0}$. Fix $n \geq n_0$ and $s \in \omega^{<\omega}$, with $p = |s|$. If $s \neq \alpha_{n+p} \upharpoonright p$, then $\psi(x)(n, s) = (\mathbf{1}_0, \dots, \mathbf{1}_{p-1}) \in K_0 \upharpoonright p$, since $\bar{\mathbf{1}} \in K_0$. On the other hand, if $s = \alpha_{n+p} \upharpoonright p$, then

$$\psi(x)(n, s) = \psi_n(x) \upharpoonright p \in K_0 \upharpoonright p,$$

since $\psi_n(B_{n_0}) \subseteq \psi_n(B_n) \subseteq K_0$, by assumption. As $n \geq n_0$ and s were arbitrary, we see that $\psi(x) \in A_{n_0}$.

It remains to see that $\psi(x) \in \langle \tilde{A} \rangle$. Naturally, it suffices to prove that $\psi(x) \in \tilde{A}$. We must show that, for all but finitely many pairs n, s ,

$$(\psi(x)(n, s))^* \leq s. \tag{4.4}$$

Fix n, s with $p = |s|$. If $s \neq \alpha_{n+p} \upharpoonright p$, then (4.4) follows, since $\psi(x)(n, s) = (\mathbf{1}_0, \dots, \mathbf{1}_{p-1})$ and $(\mathbf{1}_0, \dots, \mathbf{1}_{p-1})^* = 0^p$. If $s = \alpha_{n+p} \upharpoonright p$ and $n + p \geq n_0$, then $(\psi_n(x))^* \leq \alpha_{n+p}$, since $x \in B_{n_0} \subseteq B_n$ and $n \leq n + p$. Hence

$$\begin{aligned} (\psi(x)(n, s))^* &= \psi_n(x)^* \upharpoonright p \\ &\leq \alpha_{n+p} \upharpoonright p \\ &= s \end{aligned}$$

and (4.4) holds for n, s . We see that (4.4) only fails when $n + |s| < n_0$ and $s = \alpha_{n+|s|} \upharpoonright |s|$. There are only finitely many such pairs n, s .

We have shown that $\psi(x) \in \tilde{A}$ and hence $\psi(x) \in \tilde{A} \cap A_{n_0} \subseteq H_0$. This completes the proof. ⊖

§5. Universal F_σ subgroups for K_σ . Theorem 1.6 gives a universal K_σ subgroup of G^ω whenever G is locally compact. If G is arbitrary, we can still prove that there is an F_σ subgroup of G^ω , which is universal for K_σ subgroups of G^ω (Theorem 1.7 from Section 1). In Section 6, we will show that S_∞^ω does not have a universal K_σ subgroup, implying that this result cannot, in general, be improved.

We recall Theorem 1.7 and give its proof below.

THEOREM 1.7. *For every Polish group G , there is an F_σ subgroup $H \subseteq G^\omega$, which is universal for K_σ subgroups of G^ω .*

Before proceeding, it is worth mentioning a corollary of Theorem 1.7. Recall that a Polish group \mathbb{G} is *universal* if every Polish group is isomorphic to a closed subgroup of \mathbb{G} .

COROLLARY 5.1. *If \mathbb{G} is a universal Polish group, then there is an F_σ subgroup $H_0 \subseteq \mathbb{G}$ such that, for any K_σ subgroup H of a Polish group G , there is a continuous injective group homomorphism $\varphi : G \rightarrow \mathbb{G}$ such that $H = \varphi^{-1}(H_0)$.*

PROOF. Let \mathbb{G} be a universal Polish group and $\tilde{H}_0 \subseteq \mathbb{G}^\omega$ an F_σ subgroup which is universal for K_σ subgroups of \mathbb{G}^ω . By the universality of \mathbb{G} , we may identify \tilde{H}_0 with an F_σ subgroup $H_0 \subseteq \mathbb{G}$. Observe that, since \mathbb{G} itself is isomorphic to a closed subgroup of \mathbb{G}^ω , H_0 is universal for K_σ subgroups of \mathbb{G} .

Fix any Polish group G and $H \subseteq G$, a K_σ subgroup. Let $\pi : G \rightarrow \mathbb{G}$ be an isomorphic embedding. Note that $\pi(H)$ is a K_σ subgroup of \mathbb{G} and hence there is a continuous homomorphism $\varphi : \mathbb{G} \rightarrow \mathbb{G}$ such that $\varphi^{-1}(H_0) = \pi(H)$. Inspecting the proof of Theorem 1.7, it will become apparent that φ can be chosen to be injective. Since π is injective, it follows that $(\varphi \circ \pi)^{-1}(H_0) = H$. ⊖

It is a theorem of V. V. Uspenskii (see Kechris [14, 9.18]) that there are universal Polish groups. In particular, the homeomorphism group of the Hilbert cube is a universal Polish group.

Notation. For the sake of the next proof, we introduce some notation. If w is a group word, let w^{-1} denote the shortest group word such that $(w(a_0, \dots, a_m))^{-1} = w^{-1}(a_0, \dots, a_m)$, for any group elements a_0, \dots, a_m . (For example, if $w(a, b) = ab$, then $w^{-1}(a, b) = b^{-1}a^{-1}$.) If w_1, w_2 are group words, let w_1w_2 denote the concatenation of w_1 and w_2 , i.e., for any group elements a_0, \dots, a_m and b_0, \dots, b_n ,

$$w_1w_2(a_0, \dots, a_m, b_0, \dots, b_n) = w_1(a_0, \dots, a_m) \cdot w_2(b_0, \dots, b_n).$$

PROOF OF THEOREM 1.7. Fix a Polish group G , with compatible metric d and identity element $\mathbf{1}$. For $x \in G$ and $A \subseteq G$, let $\text{dist}(x, A) = \inf\{d(x, y) : y \in A\}$. Let \mathcal{B} be a countable topological basis for G and let \mathcal{F} denote the set of all finite families $F \subseteq \mathcal{B}$ such that $\mathbf{1} \in \bigcup F$.

Since our objective is to produce a subgroup of G^ω with the desired universality property, we will simplify notation by assuming that G is itself a countable power (i.e., $G \cong G^\omega$) and show that there is an F_σ subgroup of G^ω of which every K_σ subgroup of G is a continuous homomorphic preimage. Specifically, we will work with an isomorphic copy of G^ω in the form of $G^{\mathcal{F} \times \omega}$.

Define an F_σ subset H of $G^{\mathcal{F} \times \omega}$ by letting $\xi \in H$ if, and only if, there exist an $n \in \omega$ and an $(m + 1)$ -ary group word w such that, for each $F \in \mathcal{F}$ and $n' \geq n$,

$$\xi(F, n') \in \text{cl} \left(\left\{ w(x_0, \dots, x_m) : x_0, \dots, x_m \in \bigcup F \right\} \right).$$

Claims 5.2 and 5.3 will complete the proof.

CLAIM 5.2. *H is a subgroup of $G^{\mathcal{F} \times \omega}$.*

PROOF OF CLAIM. Let $\bar{\mathbf{1}}$ denote the identity element of $G^{\mathcal{F} \times \omega}$. We have $\bar{\mathbf{1}} \in H$, witnessed by $n = 0$ and the word $w(a) = a$, since $\mathbf{1} \in \bigcup F$, for each $F \in \mathcal{F}$.

Closure under taking inverses follows from the observation that if n and w witness $\xi \in H$, then n and w^{-1} will witness $\xi^{-1} \in H$.

Suppose that $\xi_1, \xi_2 \in H$. Let $n_1, n_2 \in \omega$ and w_1, w_2 be group words as in the definition of H , witnessing the membership of ξ_1 and ξ_2 in H , respectively. Let $n = \max\{n_1, n_2\}$ and $w = w_1w_2$. It follows from the definition of H that the pair n, w witnesses the membership of the product $\xi_1\xi_2$ in H .

CLAIM 5.3. *Every K_σ subgroup of G is a continuous homomorphic preimage of H .*

PROOF OF CLAIM. Fix a K_σ subgroup $\bigcup_n K_n$ of G , with each K_n compact and $\mathbf{1} \in K_0 \subseteq K_1 \subseteq \dots$. For each n, k , let $F_{n,k} \in \mathcal{F}$ be a finite $\frac{1}{k}$ -cover of K_n (i.e., each $U \in F_{n,k}$ has diameter less than $1/k$ and $K_n \subseteq \bigcup F_{n,k}$) with the property that $U \cap K_n \neq \emptyset$, for each $U \in F_{n,k}$. Define a continuous homomorphism $\psi : G \rightarrow G^{\mathcal{F} \times \omega}$ by

$$\psi(x)(F, n) = \begin{cases} x & \text{if } (\exists k)(F = F_{n,k}), \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

We wish to see that $\bigcup_n K_n = \psi^{-1}(H)$. Suppose first that $x \in K_n$ and hence $x \in K_{n'}$, for each $n' \geq n$. If w is the group word $w(a) = a$, then n and w witness $\psi(x) \in H$. Indeed, if $n' \geq n$ and, for some k , we have $F = F_{n',k}$, then $\psi(x)(F, n') = x \in K_{n'} \subseteq \bigcup F_{n',k}$. On the other hand, if, for every k , we have $F \neq F_{n',k}$, then $\psi(x)(F, n) = \mathbf{1} \in \bigcup F$, since $F \in \mathcal{F}$.

Suppose now that $\psi(x) \in H$, witnessed by n, w , where w is $(m + 1)$ -ary. This implies that, for each $k \in \omega$,

$$x = \psi(x)(F_{n,k}, n) \in \text{cl}(\{w(x_0, \dots, x_m) : x_0, \dots, x_m \in \bigcup F_{n,k}\}).$$

Hence, by the choice of $F_{n,k}$, there exist $x_0^k, \dots, x_m^k \in \bigcup F_{n,k}$ such that

1. $d(x, w(x_0^k, \dots, x_m^k)) \leq 1/k$ and
2. $\text{dist}(x_i^k, K_n) \leq 1/k$, for each $i \leq m$.

Since K_n is compact, condition 2 implies that there are elements $y_0, \dots, y_m \in K_n$ and a subsequence $k_0 < k_1 < \dots$ such that, for each $i \leq m$,

$$x_i^{k_p} \rightarrow y_i \text{ as } p \rightarrow \infty.$$

Taking the limit as $k \rightarrow \infty$ of both sides of expression 1 above shows that

$$x = w(y_0, \dots, y_m).$$

This completes the claim and proof. −

§6. The example of S_∞ . In this section, we prove that there is a countable power with no universal K_σ subgroup. In particular, we show that S_∞^ω has no universal K_σ subgroup (Theorem 1.8). This suggests that Theorem 1.6 cannot readily be expanded to a larger class of Polish groups. In some sense, the example of S_∞ also serves as a complement to Theorem 1.7, again suggesting that this may be a “best possible” result.

We state the main result of this section.

THEOREM 6.1. *There is no K_σ subgroup of S_∞^ω , which is universal for compactly generated subgroups of S_∞^ω .*

This theorem shows (in a strong way) that S_∞^ω has neither universal compactly generated nor universal K_σ subgroups. Since S_∞^ω embeds in S_∞ as a closed subgroup and *vice versa*, it follows from Proposition 1.11 that it will suffice to prove Theorem 6.1 for S_∞ , in place of S_∞^ω .

Recall that the topology on S_∞ is generated by the basic clopen sets

$$\mathcal{U}(s) = \{f \in S_\infty : s \subset f\},$$

where $s \in \omega^{<\omega}$ is an injection. Hence the sets $\mathcal{U}(\text{id} \upharpoonright n)$ form a neighborhood basis at the identity. Because we will refer to these open sets several times in what follows, we write \mathcal{U}_n for $\mathcal{U}(\text{id} \upharpoonright n)$.³

The fundamental elements of S_∞ are cycles. We use the notation $[a_1, \dots, a_n]$ for n -cycles and $[\dots, a_{-1}, a_0, a_1, \dots]$ for ∞ -cycles. For $\pi \in S_\infty$, we let

$$\text{supp}(\pi) = \{n : \pi(n) \neq n\} = \{n : \pi^{-1}(n) \neq n\}.$$

For $\pi \in S_\infty$, with $\text{supp}(\pi)$ finite, we will abuse notation and write $\mathcal{U}(\pi)$ for the basic open neighborhood $\mathcal{U}(\pi \upharpoonright \text{supp}(\pi))$.

For any $f \in \omega^\omega$ (viewed as a function $\omega \rightarrow \omega$) we write f^p for the p -fold composite of f with itself, e.g., $f^2 = f \circ f$. We will use this notation both for permutations of ω as well as arbitrary functions on ω .

³General background on the topology and characteristics of S_∞ can be found in Becker-Kechrin [1].

For each $\alpha \in \omega^\omega$, define

$$K_\alpha = \{f \in S_\infty : f, f^{-1} \leq \alpha\},$$

where “ $f \leq \alpha$ ” signifies that, for each $n \in \omega$, $f(n) \leq \alpha(n)$. Note that each K_α is compact in S_∞ and that every compact subset of S_∞ is contained in some K_α . Suppose that $H = \bigcup_n K_n$ is a K_σ subset of S_∞ , with each K_n compact. To show that H is not universal for compactly generated subgroups of S_∞ , it will suffice to find a compact set $K \subseteq S_\infty$ such that no homomorphic image of K is contained in H . For this, it is enough to assume that each K_n has the form K_{β_n} , for some $\beta_n \in \omega^\omega$. We therefore show

THEOREM 6.2. *Given $\beta_0, \beta_1, \dots \in \omega^\omega$, there exists $\alpha \in \omega^\omega$ such that, for each continuous injective group homomorphism $\Phi : S_\infty \rightarrow S_\infty$, we have $\Phi(K_\alpha) \not\subseteq \bigcup_n K_{\beta_n}$.*

REMARK 6.3. It follows from Dixon-Neumann-Thomas [11] that every endomorphism of S_∞ is continuous. Theorem 6.2 could thus be equivalently stated for arbitrary endomorphisms of S_∞ .⁴

We require a few lemmas.

LEMMA 6.4. *If Φ is a continuous endomorphism of S_∞ and $\ker(\Phi) \neq S_\infty$, then Φ is injective.*

PROOF. Since Φ is continuous, $\ker(\Phi)$ is a closed normal subgroup of S_∞ . On the other hand, it is a theorem of Schreier-Ulam that the only normal subgroups of S_∞ are $\{\text{id}\}$, the infinite alternating group, the group of finite support permutations and S_∞ itself. Of these, only $\{\text{id}\}$ and S_∞ are closed. ⊥

Noting that every K_σ subgroup of S_∞ is a proper subgroup, Lemma 6.4 implies that a group-homomorphism reduction between K_σ subgroups of S_∞ must be injective. (This follows from the fact that, if $A = \varphi^{-1}(B)$, then $\ker \varphi \subseteq A$.)

LEMMA 6.5. *Suppose that $\alpha \in \omega^\omega$ is such that $(\forall n)(\alpha(n) \geq n)$. If $f \in K_\alpha$ and $s \subset f$ is a finite injection, then there is a finite support permutation $\pi \in K_\alpha$ such that $s \subset \pi$.*

PROOF. Let S be the set of cycles $\sigma \subset f$ such that $\text{supp}(\sigma)$ intersects the domain or range of s . Since s is a finite function, S is a finite set of disjoint cycles. For each ∞ -cycle $\tau \in S$, we will define a finite cycle $\tau^* \in K_\alpha$ such that τ^* agrees with τ on $\text{dom}(s) \cup \text{ran}(s)$. Write τ as

$$[\dots a_{-1}, a_0, a_1, \dots].$$

Let $n_0, n_1 \in \mathbb{Z}$ be such that $n_0 \leq n_1$ and if $a_i \in \text{dom}(s) \cup \text{ran}(s)$, for some i , then $n_0 \leq i \leq n_1$. By taking n_1 large enough, we may assume that $a_{n_1} > a_{n_0}$. Let $m \leq n_0$ be large enough that $a_m < a_{n_1}$ and $a_{m-1} > a_{n_1}$. (Note that we have strict inequalities, since τ is an ∞ -cycle and hence all a_i are distinct.) Define

$$\tau^* = [a_m, \dots, a_{n_0}, \dots, a_{n_1}].$$

We will verify that $\tau^* \in K_\alpha$, i.e., $\tau^*, (\tau^*)^{-1}$ are both bounded by α . Since $\tau \subset f \in K_\alpha$, we know that τ, τ^{-1} are bounded by α . Hence we need only check

⁴See Rosendal’s survey article [23] for further discussion of such “automatic continuity” results.

that $a_{n_1} \leq \alpha(a_m)$ and $a_m \leq \alpha(a_{n_1})$, since τ^* agrees with τ , except at a_{n_1} . That $a_{n_1} \leq \alpha(a_m)$ follows from

$$a_{n_1} < a_{m-1} = \tau^{-1}(a_m) \leq \alpha(a_m).$$

(We are using the fact that $\tau^{-1} \leq \alpha$.) On the other hand, $a_m \leq \alpha(a_{n_1})$ follows from the fact that

$$a_m < a_{n_1} \leq \alpha(a_{n_1}),$$

by our assumption that $(\forall n)(\alpha(n) \geq n)$.

We may now define the desired π as in the statement of the lemma. Let π be the product of all finite cycles in \mathcal{S} together with all τ^* , for ∞ -cycles $\tau \in \mathcal{S}$. ⊖

LEMMA 6.6. *If Φ is a continuous endomorphism of S_∞ and $\alpha, \beta_m \in \omega^\omega$ are such that $(\forall n)(\alpha(n) \geq n)$, $\Phi(K_\alpha) \subseteq \bigcup_n K_{\beta_m}$ and $\{\beta_m : m \in \omega\}$ is closed under composition, then there exist $n, m \in \omega$ such that*

$$\Phi(\mathcal{U}_n \cap K_\alpha) \subseteq K_{\beta_m}.$$

PROOF. The following claim is the core of the proof.

CLAIM. *There exists a finite support permutation $\pi \in K_\alpha$ and $m \in \omega$ such that $\Phi(\mathcal{U}(\pi) \cap K_\alpha) \subseteq K_{\beta_m}$.*

PROOF OF CLAIM. Let C be the compact set $\Phi(K_\alpha)$. Applying the Baire Category Theorem to C , it follows that exists $m \in \omega$ such that $K_{\beta_m} \cap C$ is non-meager relative to C . As K_{β_m} is closed, this implies that there exists a nonempty open set $\mathcal{V} \subseteq S_\infty$ such that $\mathcal{V} \cap C \subseteq K_{\beta_m}$. Let $\mathcal{U} = \Phi^{-1}(\mathcal{V})$. Since $\mathcal{U} \cap K_\alpha \neq \emptyset$, there is a finite injection $s : \omega \rightarrow \omega$ with $\mathcal{U}(s) \subseteq \mathcal{U}$ and $\mathcal{U}(s) \cap K_\alpha \neq \emptyset$. Lemma 6.5 thus yields a finite support permutation $\pi \in K_\alpha$ such that $\mathcal{U}(\pi) \subseteq \mathcal{U}$. Hence

$$\Phi(\mathcal{U}(\pi) \cap K_\alpha) \subseteq \Phi(\mathcal{U} \cap K_\alpha) \subseteq (\mathcal{V} \cap C) \subseteq K_{\beta_m}.$$

This completes the claim.

Suppose that π, m are as in the claim, such that $\Phi(\mathcal{U}(\pi) \cap K_\alpha) \subseteq K_{\beta_m}$. If n is an upper bound for the support of π , then $\pi^{n!} = \text{id}$. Given $g \in S_\infty$, note that $g \in \mathcal{U}(\pi) \cap K_\alpha$ iff $g = \pi \circ f$, for some $f \in \mathcal{U}_n \cap K_\alpha$. With this in mind, fix an arbitrary $\pi \circ f \in \mathcal{U}(\pi) \cap K_\alpha$ (where $f \in \mathcal{U}_n \cap K_\alpha$) and observe that

$$\pi^{n!-1} \circ \pi \circ f = f$$

and hence $\Phi(f)$ is the composite of $n!$ elements of K_{β_m} , since $\Phi(\pi), \Phi(\pi \circ f) \in K_{\beta_m}$. On the other hand, any composite of $n!$ elements of K_{β_m} is bounded by the $n!$ -fold composite of β_m with itself. As we assumed that $\{\beta_m : m \in \omega\}$ is closed under composition, we conclude that $\Phi(\mathcal{U}_n \cap K_\alpha) \subseteq K_{\beta_r}$, for an appropriate $r \in \omega$. This completes the proof. ⊖

LEMMA 6.7. *For each m , let $\beta_m : \omega \rightarrow \omega$ be an increasing function. There exists $\alpha \in \omega^\omega$ such that, for each $n, m \in \omega$, there is no continuous injective group homomorphism Φ of S_∞ with $\Phi(\mathcal{U}_n \cap K_\alpha) \subseteq K_{\beta_m}$.*

PROOF. For the sake of the present proof, if $f \in S_\infty$, we define a chain of roots of length n for f to be a sequence $f_0, \dots, f_n \in S_\infty$ such that $f_0 = f$ and $f_j^2 = f_{j-1}$, for each $1 \leq j \leq n$.

Suppose that $f \in K_\alpha$ is a product of disjoint 2^k -cycles, for some $k \geq 1$. If $n \in \text{supp}(f)$ (i.e., $f(n) \neq n$), then f has no chain of roots in K_α , of length greater than $\alpha(n)$. This follows from the fact that, were f_0, \dots, f_p a chain of roots of length $p > \alpha(n)$, then at least one f_j is not a member of K_α , as $f_0(n), \dots, f_p(n)$ are all distinct. Recall here that “square-roots” of products of disjoint 2^k -cycles are obtained by interleaving terms of distinct cycles to form permutations containing products of disjoint 2^{k+1} -cycles. This is a consequence of the fact that, if σ is a 2^m -cycle, for some m , then σ^2 is a product of two disjoint 2^{m-1} -cycles.

As an example,

$$f_0 = [0, 1][2, 3][4, 5][6, 7]$$

$$f_1 = [0, 2, 1, 3][4, 6, 5, 7]$$

$$f_2 = [0, 4, 2, 6, 1, 5, 3, 7]$$

is a chain of roots for f_0 , of length 2. We remark that this behavior does not apply to cycles of other lengths. For instance, in the case of 3-cycles, one has $[1, 3, 2]^2 = [1, 2, 3]$.

Let $\alpha \in \omega^\omega$ be such that, for each $k \in \omega$, the permutation

$$f_k = [k, k + 1][k + 2, k + 3] \dots$$

has a chain of roots in K_α of length at least

$$\max_{i \leq k} (\beta_i^{2k+2}(k)) + 1.$$

We may further assume that $(\forall k)(\alpha(k) \geq k + 2)$.

Suppose, towards a contradiction, that Φ is a continuous endomorphism of S_∞ with $\Phi(\mathcal{U}_n \cap K_\alpha) \subseteq K_{\beta_m}$, for some $\alpha \in \omega^\omega$ and $m, n \in \omega$. For simplicity, write $\beta_m = \beta$. Let $a \in \omega$ be least such that $\Phi([n, n + 1, n + 2])(a) \neq a$.

For each $k \geq n + 2$, we have that

$$\Phi([n, n + 1, k]) = \Phi([n + 2, k]) \circ \Phi([n, n + 1, n + 2]) \circ \Phi([n + 2, k]).$$

Observe that

$$[n + 2, k] = [n + 2, n + 3][n + 3, n + 4] \dots [k - 1, k][k - 2, k - 1] \dots [n + 2, n + 3]$$

and hence $[n + 2, k]$ is a product of fewer than $2k$ members of K_α , since each $[j, j + 1]$ is in K_α . Thus $\Phi([n + 2, k])$ is a product of fewer than $2k$ members of K_β . (Since each $\Phi([j, j + 1]) \in K_\beta$, for each $j \geq n$.) In particular, $\Phi([n + 2, k])$ is bounded by β^{2k} , the $2k$ -fold composite of β with itself. (This follows in part from the fact that β was assumed to be increasing.) Hence we have

$$\Phi([n + 2, k])^{-1}(a) = \Phi([n + 2, k])(a) \leq \beta^{2k}(a)$$

and thus there exists $b_0 \in \text{supp}(\Phi([n, n + 1, k]))$ with $b_0 \leq \beta^{2k}(a)$. Specifically, $b_0 = \Phi([n + 2, k])^{-1}(a)$.

As noted above, the choice of α guarantees that each $[j, j + 1] \in K_\alpha$. Hence

$$[k, k + 1][k + 2, k + 3] \dots \in K_\alpha$$

and thus

$$h = \Phi([k, k + 1][k + 2, k + 3] \dots) \in K_\beta.$$

Observe now that

$$\begin{aligned} &\Phi([n, n + 1, k, k + 1][k + 2, k + 3][k + 4, k + 5] \dots) \quad (*) \\ &= \Phi([n, n + 1, k][k, k + 1][k + 2, k + 3] \dots) \\ &= \Phi([n, n + 1, k]) \circ \Phi([k, k + 1][k + 2, k + 3] \dots) \\ &= \Phi([n, n + 1, k]) \circ h. \end{aligned}$$

As can be seen from the line marked (*), this permutation has order 4, while $\Phi([n, n + 1, k])$ has order 3. Thus $\text{supp}(h)$ must intersect each orbit of $\Phi([n, n + 1, k])$, as otherwise the permutation above will contain a 3-cycle and not be of order 4. In particular, $\text{supp}(h)$ contains an element of the orbit of b_0 under $\Phi([n, n + 1, k])$. In other words, $\text{supp}(h)$ contains an element of the set

$$\{b_0, \Phi([n, n + 1, k])(b_0), \Phi([n, n + 1, k])^{-1}(b_0)\}.$$

Note that while $b_0 \leq \beta^{2k}(a)$, we have

$$\begin{aligned} \Phi([n, n + 1, k])(b_0) &= \Phi([n + 2, k][n, n + 1, n + 2][n + 2, k])(b_0) \\ &= \Phi([n + 2, k]) \circ \Phi([n, n + 1, n + 2]) \circ \Phi([n + 2, k])(b_0) \\ &= \Phi([n + 2, k]) \circ \Phi([n, n + 1, n + 2])(a) \\ &\leq \beta^{2k+1}(a). \end{aligned}$$

A similar argument shows that $\Phi([n, n + 1, k])^{-1}(b_0) \leq \beta^{2k+1}(a)$.

This implies that $\text{supp}(h)$ contains an element $b_1 \leq \beta^{2k+1}(a)$. We now conclude that h has no chain of roots in K_β , of length greater than $\beta(b_1) \leq \beta^{2k+2}(a)$. (Again, we are using the fact that β is increasing to obtain this inequality.)

On the other hand, if $k \geq m, a$, then

$$\beta^{2k+2}(a) \leq \beta^{2k+2}(k) = \beta_m^{2k+2}(k) \leq \max_{i \leq k}(\beta_i^{2k+2}(k))$$

and $[k, k + 1][k + 2, k + 3] \dots$ has a chain of roots in K_α of length at least

$$\max_{i \leq k}(\beta_i^{2k+2}(k)) + 1.$$

This is a contradiction, since Φ maps K_α into K_β and, being a homomorphism, must preserve chains of roots. ⊥

We may now complete the proof of Theorem 6.2.

PROOF OF THEOREM 6.2. Suppose that $\beta_0, \beta_1, \dots \in \omega^\omega$ are given. With no loss of generality, we may assume that $\{\beta_m : m \in \omega\}$ is closed under compositions and that each β_m is strictly increasing. (Making these assumptions only enlarges the K_σ set $\bigcup_m K_{\beta_m}$. Also, these two assumptions do not conflict as the composite of increasing functions remains increasing.)

Let $\alpha \in \omega$ be as in Lemma 6.7, for $\{\beta_m : m \in \omega\}$. Here we may assume that $\alpha(n) \geq n$, for each n . If there is a continuous endomorphism Φ of S_∞ such that $\Phi(K_\alpha) \subseteq \bigcup_m K_{\beta_m}$, then Lemma 6.6 yields m, n such that $\Phi(\mathcal{U}_n \cap K_\alpha) \subseteq K_{\beta_m}$. This contradicts the properties of α . ⊥

§7. Examples.

7.1. Basic examples. We restate a proposition from the Section 1 which will be our main tool in this section.

PROPOSITION 1.9. *Suppose that G_1 and G_2 are topological groups such that there exist continuous injective homomorphisms $\varphi_1 : G_1 \rightarrow G_2$ and $\varphi_2 : G_2 \rightarrow G_1$. Let \mathcal{C} be a class of subgroups which is closed under continuous homomorphic images. If G_1 has a universal \mathcal{C} subgroup, then G_2 also has a universal \mathcal{C} subgroup.*

The following examples are direct applications of Proposition 1.9, together with Theorem 1.6.

EXAMPLE 7.1. Let $c_0 \subset \mathbb{R}^\omega$ be the subgroup

$$\{x \in \mathbb{R}^\omega : \lim_n x(n) = 0\}.$$

Recall that c_0 is a separable Banach space (hence a Polish group) when equipped with the sup-norm (denoted by $\|\cdot\|_{\text{sup}}$). Let \mathcal{C} be either the class of compactly generated or K_σ subgroups. Since c_0 is nowhere locally compact, Theorem 1.6 does not immediately give a universal \mathcal{C} subgroup of c_0^ω . Nonetheless, we shall see that c_0^ω has a universal \mathcal{C} subgroup.

The Banach space topology on c_0 refines the subspace topology inherited from \mathbb{R}^ω . To see this, suppose that $U = I_0 \times \cdots \times I_{k-1} \times \mathbb{R}^\omega$ is a basic open set in \mathbb{R}^ω (where $I_0, \dots, I_{k-1} \subseteq \mathbb{R}$ are bounded open intervals) and $x_0 \in U \cap c_0$. Let $\varepsilon > 0$ be small enough that, for each $n < k$, the open interval $(x_0(n) - \varepsilon, x_0(n) + \varepsilon)$ is contained in I_n . If

$$B = \{x \in c_0 : \|x - x_0\|_{\text{sup}} < \varepsilon\},$$

then B is open in c_0 and $x \in B \subseteq U \cap c_0$. Hence $U \cap c_0$ is open with respect to the Banach space topology on c_0 . This implies that the inclusion map $c_0 \rightarrow \mathbb{R}^\omega$ is a continuous injective homomorphism and hence so is the inclusion $c_0^\omega \rightarrow \mathbb{R}^{\omega \times \omega} \cong \mathbb{R}^\omega$.

To apply Proposition 1.9, we also need to check that there is a continuous injective homomorphism of \mathbb{R}^ω into c_0^ω . Indeed, this is witnessed by the map $\varphi : \mathbb{R}^\omega \rightarrow c_0^\omega$, where

$$\varphi(x)(n) = (x(n), 0, 0, \dots).$$

By Proposition 1.9 we conclude that c_0^ω has a universal \mathcal{C} subgroup, since \mathbb{R}^ω does.

REMARK 7.2. In the previous example, we do not claim that \mathbb{R}^ω is isomorphic to a subgroup of c_0 , nor *vice versa*, as these would be false statements.

By similar arguments using the fact that the Banach space topologies of $\ell^p, \ell^\infty, c \subset \mathbb{R}^\omega$ refine their subspace topologies, we can also conclude that the groups $(\ell^p)^\omega, (\ell^\infty)^\omega$, and c^ω contain universal subgroups for the classes of K_σ and compactly generated subgroups. The case of $(\ell^\infty)^\omega$ is interesting, because ℓ^∞ (with the sup-norm) is complete, but not separable, i.e., not a Polish space.⁵

It is also worth mentioning the case of ℓ^2 . Since ℓ^2 is a separable Hilbert space and, by Corollary 5.5 in Conway [5], all separable Hilbert spaces (over \mathbb{R}) are isomorphic, we have that all separable Hilbert spaces are isomorphic to ℓ^2 . The comments above thus imply the following.

⁵Definitions of the Banach spaces ℓ^p, ℓ^∞ , and c may be found in Conway [5].

PROPOSITION 7.3. *The countable power of every separable Hilbert space (over \mathbb{R}) contains universal K_σ and compactly generated subgroups.*

REMARK 7.4. The arguments above apply equally to \mathbb{C} in place of \mathbb{R} . (I.e., \mathbb{C}^ω also has universal subgroups in these two classes.) Thus the proposition above applies to complex Hilbert spaces as well.

The following example shows the existence of universal subgroups in another nonseparable topological group.

EXAMPLE 7.5. Let S be a separable space and $C(S)$ be the additive group of continuous real-valued functions on S , with the topology of uniform convergence. The group $C(S)$ is metrizable, but not separable if S is not compact. A compatible metric is

$$\rho(f, g) = \sup\{\min\{|f(x) - g(x)|, 1\} : x \in S\}.$$

The distance function ρ is the so-called “uniform metric” on $C(S)$.⁶

Let $A \subseteq S$ be a countable dense set. Consider the Polish group \mathbb{R}^A , equipped with the product topology, i.e., $\mathbb{R}^A \cong \mathbb{R}^\omega$. The map $\psi : C(S) \rightarrow \mathbb{R}^A$ defined by

$$f \mapsto f \upharpoonright A$$

is a group homomorphism. To see that ψ is continuous it suffices to check that $\psi^{-1}(U)$ contains a neighborhood of the zero function whenever U is a basic neighborhood of $\bar{0}$. Given a basic neighborhood $U \ni \bar{0}$, we may assume that, for some finite set $F \subseteq A$ and $\varepsilon > 0$,

$$U = \{x \in \mathbb{R}^A : (\forall a \in F)(|x(a)| < \varepsilon)\}.$$

Let $\mathcal{F} = \{f \in C(S) : (\forall a \in F)(f(a) = 0)\}$ and take

$$\mathcal{V} = \bigcup_{f \in \mathcal{F}} \{g \in C(S) : \rho(f, g) < \varepsilon\}.$$

As the union of open sets, \mathcal{V} is open in $C(S)$ and $\mathcal{V} \subseteq \psi^{-1}(U)$. Also, ψ is injective, because A is dense and thus $f \upharpoonright A = g \upharpoonright A$ implies $f = g$. It follows that $C(S)^\omega$ may be mapped into $\mathbb{R}^{A \times \omega} \cong \mathbb{R}^\omega$ as well, via a continuous injective group homomorphism.

Finally, note that \mathbb{R}^ω embeds in $C(S)^\omega$ (as a closed subgroup in this case) via the map $\varphi : \mathbb{R}^\omega \rightarrow C(S)^\omega$, where $\varphi(x)(n)$ is the constant function $f \equiv x(n)$. Proposition 1.9 now lets us conclude that $C(S)^\omega$ contains universal compactly generated and K_σ subgroups.

As noted in Kechris [14, §12.E], every separable Banach space is isomorphic to a closed subspace of $C(2^\omega)$. By Example 7.1, we therefore have

PROPOSITION 7.6. *Let \mathcal{C} be either the classes of compactly generated or the class of K_σ subgroups. There is a subgroup $H_0 \subseteq C(2^\omega)^\omega$, with $H_0 \in \mathcal{C}$, such that for any separable Banach space \mathfrak{B} and any subgroup $H \subseteq \mathfrak{B}^\omega$ in \mathcal{C} , there is a continuous group homomorphism $\varphi : \mathfrak{B}^\omega \rightarrow C(2^\omega)^\omega$ such that $H = \varphi^{-1}(H_0)$.*

Example 7.7 relates directly to Theorem 1.6.

⁶See Munkres [21, p. 266].

EXAMPLE 7.7. Let $(G_n)_{n \in \omega}$ be a sequence of locally compact Polish groups. Consider $\bigoplus_n G_n$ with the subspace topology from $\prod_n G_n$. Although separable, the direct sum $\bigoplus_n G_n$ is, in general, not Polishable.⁷

The product $\prod_n G_n^\omega$ is isomorphic to a closed subgroup of $(\bigoplus_n G_n)^\omega$. Furthermore, $(\bigoplus_n G_n)^\omega$ is isomorphic to the Π_3^0 subgroup

$$\{\xi : (\forall k)(\forall^\infty n)(\xi(n)(k) = \mathbf{1}_n)\}$$

of $\prod_n G_n^\omega$. Theorem 1.6 together with Proposition 1.9 therefore implies that $(\bigoplus_n G_n)^\omega$ has universal compactly generated and K_σ subgroups.

7.2. Separable Banach spaces. In this section, we show that every separable infinite-dimensional Banach space with an unconditional basis (we give the definition below) has universal compactly generated and K_σ subgroups. The key facts will be Proposition 1.9 along with the following.

THEOREM 7.8. *The Banach space c_0 has universal compactly generated and universal K_σ subgroups.*

In each case, we obtain the desired universal subgroup of c_0 by “shrinking” an appropriate universal subgroup of \mathbb{R}^ω . Note that we could also prove these facts directly by modifying the proof of Theorem 1.6. We begin with a lemma.

LEMMA 7.9. *Suppose $\alpha : \omega \rightarrow \mathbb{R}^+$ is such that $\lim_n \alpha(n) = 0$. If $F \subseteq c_0$ is closed and $|x(n)| \leq \alpha(n)$, for each $x \in F$ and $n \in \omega$, then F is compact in c_0 .*

PROOF. Suppose that $(x_i)_{i \in \omega}$ is a sequence of elements of F . Let i_0, i_1, \dots be a subsequence such that $(x_{i_n}(k))_{n \in \omega}$ is convergent, for each $k \in \omega$. Such a subsequence may be obtained by successively choosing subsequences to guarantee that $(x_{i_n}(j))_{n \in \omega}$ is Cauchy for all $j \leq k$ and taking $(i_n)_{n \in \omega}$ to be a pseudo-intersection of these subsequences. Let $x \in c_0$ be given by $x(k) = \lim_n x_{i_n}(k)$, for each k . Note that $|x(k)| \leq \alpha(k)$, for each $k \in \omega$.

To see that $\|x_{i_n} - x\|_{\text{sup}} \rightarrow 0$, as $n \rightarrow \infty$, fix $\varepsilon > 0$ and let k_0 be large enough that $|\alpha(k)| < \frac{\varepsilon}{2}$, for each $k \geq k_0$. Let n_0 be large enough that $|x_{i_n}(k) - x(k)| < \varepsilon$, for each $n \geq n_0$ and $k < k_0$. It follows that $\|x_{i_n} - x\|_{\text{sup}} < \varepsilon$, for each $n \geq n_0$. \dashv

PROOF OF THEOREM 7.8. We consider each of the statements in Theorem 7.8 separately.

Compactly generated subgroups. Let $\langle K \rangle \subseteq \mathbb{R}^\omega$ be a universal compactly generated subgroup of \mathbb{R}^ω . (Such a subgroup exists by Theorem 1.6(1).) With no loss of generality, we assume that the compact set K contains $\hat{0}$. For each $n \in \omega$, let

$$\beta(n) = \max\{|x(i)| : i \leq n\} + 1.$$

and let

$$\tilde{K} = \{(1/\beta)x : x \in K\},$$

where $(1/\beta)x \in \mathbb{R}^\omega$ is such that $((1/\beta)x)(n) = (1/\beta(n)) \cdot x(n)$, for each $n \in \omega$. We claim that \tilde{K} is still a universal compactly generated subgroup of \mathbb{R}^ω . Indeed,

⁷To see this with $G_n = \mathbb{R}^n$, suppose that \mathcal{T} is a Polish topology on $\bigoplus_n \mathbb{R}^n$. By the Baire Category Theorem, there is an n such that \mathbb{R}^n is \mathcal{T} -nonmeager in $\bigoplus_n \mathbb{R}^n$. Being a subgroup, \mathbb{R}^n is thus open in $\bigoplus_n \mathbb{R}^n$, by Pettis’ theorem. This gives a contradiction to separability, since \mathbb{R}^n has uncountable index in $\bigoplus_n \mathbb{R}^n$.

if $\psi : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$ is a homomorphism reduction of a compactly generated subgroup H to $\langle K \rangle$, then $\tilde{\psi}$ defined by $\tilde{\psi}(x)(n) = (1/\beta(n))\psi(x)(n)$ reduces H to \tilde{K} .

Thus, by replacing K with \tilde{K} , we may assume that if $x \in K$ and $n \in \omega$, then $|x(n)| \leq 1$.

Let $\{I_{n,p} : n, p \in \omega\}$ be intervals partitioning ω such that each $I_{n,p}$ has length n . Define $K' \subseteq \mathbb{R}^\omega$ by

$$x \in K' \iff (\forall n, p)(x \upharpoonright I_{n,p} \in (1/np)K \upharpoonright n),$$

where $(1/np)K \upharpoonright n$ denotes the set of scalar multiples by $(1/np)$ of elements of $K \upharpoonright n$. It follows from Lemma 7.9 (and our assumptions about the “boundedness” of elements of K) that K' is compact in e_0 .

We will show that $\langle K' \rangle$ is a universal compactly generated subgroup of e_0 . Indeed, fix an arbitrary compact $A \subseteq e_0$. Since A is also compact in \mathbb{R}^ω , there is a continuous group homomorphism $\varphi : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$, such that $\langle A \rangle = \varphi^{-1}(\langle K \rangle)$.⁸

For each $n \in \omega$, let $\tau(n) \in \omega \setminus \{0\}$ be such that, for every $x \in [-1, 1]^\omega$ and $i < n$, we have $|\varphi(x)(i)| \leq \tau(n)$. (Such $\tau(n)$ exists by the compactness of $[-1, 1]^\omega$ and the continuity of φ .) Define $\psi : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$ by

$$\psi(x) \upharpoonright I_{n,p} = \begin{cases} (1/np)\varphi(x) \upharpoonright n & \text{if } p = \tau(n)^2, \\ 0^n & \text{otherwise.} \end{cases}$$

CLAIM 7.10. $\psi(e_0) \subseteq e_0$.

PROOF OF CLAIM. Note that all continuous group homomorphisms of \mathbb{R}^ω are automatically linear, hence ψ is linear. Thus, to prove the claim, it will suffice to show that $\psi(x) \in e_0$, for all $x \in e_0$ with $\|x\|_{\text{sup}} \leq 1$. Fix such an x and an $\varepsilon > 0$. For $i \in \omega$, $\psi(x)(i) \neq 0$ only if $i \in I_{n,\tau(n)^2}$, for some n . For $i \in I_{n,\tau(n)^2}$, we have

$$\begin{aligned} |\psi(x)(i)| &\leq (1/n\tau(n)^2) \max_{j < n} |\varphi(x)(j)| \\ &\leq 1/n\tau(n). \end{aligned}$$

Thus $|\psi(x)(i)| \geq \varepsilon$ only if $i \in I_{n,\tau(n)^2}$ and $1/n\tau(n) \geq \varepsilon$. There are only finitely many such i .

CLAIM 7.11. For each $x \in e_0$, we have $x \in \langle A \rangle \iff \psi(x) \in \langle K' \rangle$.

PROOF OF CLAIM. To prove the claim, it will suffice to show that $\psi(x) \in \langle K' \rangle \iff \varphi(x) \in \langle K \rangle$, since we already have $x \in \langle A \rangle \iff \varphi(x) \in \langle K \rangle$.

Fix a group word w ,

$$\begin{aligned} \varphi(x) \in w[K] &\iff (\forall n)(\varphi(x) \upharpoonright n \in w[K] \upharpoonright n) \\ &\iff (\forall n)(\psi(x) \upharpoonright I_{n,\tau(n)^2} \in (1/n\tau(n)^2)(w[K] \upharpoonright n)) \\ &\iff \psi(x) \in w[K']. \end{aligned}$$

The first and last “ \iff ” use the fact that $w[K]$ is closed (since K is compact). As w was arbitrary, this completes the claim and proof.

⁸As noted earlier the Banach space topology of e_0 refines the subspace topology inherited from \mathbb{R}^ω and hence compactness is “preserved upwards.”

K_σ **subgroups.** Let $H = \bigcup_n K_n$ be a universal K_σ subgroup of \mathbb{R}^ω , as given by Theorem 1.6(1). We may assume that

$$(\bar{0} \in K_0) \text{ and } (\forall n)(-K_n = K_n \text{ and } K_n + K_n \subseteq K_{n+1}). \tag{7.1}$$

Let $\lambda : \omega \rightarrow \mathbb{R}^+$ be given by

$$\lambda(n) = \max(\{|x(i)| : i \leq n \ \& \ (\exists m \leq n)(x \in K_m)\}) + 1$$

and define $\tilde{K}_n = \{(1/\lambda)x : x \in K_n\}$. Observe that, for each n and $x \in \tilde{K}_n$, the bits of x are uniformly bounded in absolute value, i.e., there is a constant c such that $|x(i)| \leq c$, for each $i \in \omega$. (Note that $|x(i)| \leq 1$ if $x \in \tilde{K}_n$ and $i \geq n$.) By the same reasoning as above, the subgroup $\bigcup \tilde{K}_n$ is a universal K_σ subgroup of \mathbb{R}^ω and retains property (7.1) above.

Replacing each K_n with \tilde{K}_n , we assume that for each n and $x \in K_n$, the bits of x are uniformly bounded.

Let $\{I_{m,p} : m, p \in \omega\}$ be a family of intervals partitioning ω such that each $I_{m,p}$ has length m . Define $K'_n \subseteq \mathbb{R}^\omega$ by

$$x \in K'_n \iff (\forall m, p)(x \upharpoonright I_{m,p} \in (1/mp)K_n \upharpoonright m)$$

and let $H' = \bigcup K'_n$. Again, Lemma 7.9 implies that each K'_n is compact in e_0 . Observe that property (7.1) holds for the K'_n as well. In particular, H' is a subgroup of \mathbb{R}^ω . We will show that H' is in fact a universal K_σ subgroup of e_0 .

Let $A = \bigcup_n A_n$ be an arbitrary K_σ subgroup of e_0 . Again, A is still K_σ in \mathbb{R}^ω . Hence there is a continuous homomorphism $\varphi : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$ such that $\varphi^{-1}(H) = A$. Let $\tau(m) \in \omega \setminus \{0\}$ be such that, for each $x \in [-1, 1]^\omega$ and $i < m$, we have $|\varphi(x)(i)| \leq \tau(m)$. Define $\psi : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$ by

$$\psi(x) \upharpoonright I_{m,p} = \begin{cases} (1/mp)\varphi(x) \upharpoonright m & \text{if } p = \tau(m)^2, \\ 0^m & \text{otherwise.} \end{cases}$$

As in previous case, it follows that $\psi(e_0) \subseteq e_0$. Finally, to see that $\psi^{-1}(H') = A$, it will suffice to show that

$$(\forall x \in e_0)(\forall n)(\psi(x) \in K'_n \iff \varphi(x) \in K_n).$$

To see this, observe that, for each n ,

$$\begin{aligned} \psi(x) \in K'_n &\iff (\forall m)(\psi(x) \upharpoonright I_{m,\tau(m)^2} \in (1/m\tau(m)^2)K_n \upharpoonright m) \\ &\iff (\forall m)(\varphi(x) \upharpoonright m \in K_n \upharpoonright m) \\ &\iff \varphi(x) \in K_n. \end{aligned} \tag{†}$$

We now proceed to the main result of this section. The following definition may be found at the beginning of Gowers-Maurey [9].

DEFINITION 7.12. Let \mathfrak{B} be an infinite-dimensional Banach space (over \mathbb{R}). An *unconditional basis* for \mathfrak{B} is a set $\{e_n\}_{n \in \omega} \subseteq \mathfrak{B}$ such that

1. Each e_n is a unit vector,
2. For each $x \in \mathfrak{B}$, there is a unique sequence $a_0, a_1, \dots \in \mathbb{R}$ with $x = \sum_{n \in \omega} a_n e_n$ (convergence in norm), and
3. Any permutation of $\{e_n\}_{n \in \omega}$ still has the previous property.

The following fact (also mentioned in [9]) gives a useful property of unconditional bases.

PROPOSITION 7.13 ([9], Theorem 1). *If $\{e_n\}_{n \in \omega}$ is an unconditional basis for a Banach space \mathfrak{B} , then there is a constant C such that for each $x = \sum_{n \in \omega} a_n e_n \in \mathfrak{B}$ and $(\varepsilon_n)_{n \in \omega} \in [-1, 1]^\omega$, we have*

$$\left\| \sum_{n \in \omega} \varepsilon_n a_n \right\| \leq C \left\| \sum_{n \in \omega} a_n e_n \right\|.$$

The following lemma is consequence of this fact.

LEMMA 7.14. *If \mathfrak{B} is an infinite-dimensional Banach space with an unconditional basis, then there are continuous linear maps $T_1 : \mathfrak{B} \rightarrow c_0$ and $T_2 : c_0 \rightarrow \mathfrak{B}$.*

PROOF. Let $\{e_n\}_{n \in \omega}$ be an unconditional basis for \mathfrak{B} , with C as in Proposition 7.13.

We first show the existence of the map $T_1 : \mathfrak{B} \rightarrow c_0$. Define $T_1 : \mathfrak{B} \rightarrow c_0$ by $T_1(\sum_n a_n e_n) = (a_n)_{n \in \omega}$. Since the sum $\sum_n a_n e_n$ is convergent, the sequence of partial sums is Cauchy. Hence the norm of the n th term converges to 0. It follows that T_1 maps \mathfrak{B} into c_0 . We must now see that T_1 is continuous. Since T_1 is linear, it will suffice to show that T_1 is continuous at the zero element of \mathfrak{B} . Fix $x = \sum_n a_n e_n \in \mathfrak{B}$. For each n , let $\varepsilon_n = 1$ and $\varepsilon_k = 0$, for $k \neq n$, and observe that

$$|a_n| = \|a_n e_n\| = \left\| \sum_{n \in \omega} \varepsilon_n a_n \right\| \leq C \|x\|.$$

Thus $\|T_1(x)\|_{\text{sup}} \leq C \|x\|$, showing that T_1 is continuous at $0 \in \mathfrak{B}$.

We now proceed to the second claim. Define $T_2 : c_0 \rightarrow \mathfrak{B}$ by $T_2((a_n)_{n \in \omega}) = \sum_n \frac{a_n}{2^n} e_n$. Since $(a_n)_{n \in \omega}$ is a bounded sequence, this latter sum is always well-defined. To see that T_2 is continuous, observe that, if $\|(a_n)_{n \in \omega}\|_{\text{sup}} \leq 1$, then by Proposition 7.13

$$\|T_2((a_n)_{n \in \omega})\| = \left\| \sum_n \frac{a_n}{2^n} e_n \right\| \leq C \left\| \sum_n \frac{1}{2^n} e_n \right\|.$$

Thus T_2 is a bounded linear map and hence continuous. ◻

Combining this lemma with Proposition 1.9, we obtain the following theorem.

THEOREM 7.15. *Let \mathfrak{B} be an infinite-dimensional Banach space with an unconditional basis. Then \mathfrak{B} has universal compactly generated and K_σ subgroups.*

REMARK 7.16. To put this theorem in context, recall that (among many others) all ℓ^p spaces ($1 \leq p < \infty$) have unconditional bases. On the other hand, Per Enflo [6] and later Gowers-Maurey [9] showed that there exist separable Banach spaces with no unconditional bases. Ferenczi [7] explores properties of certain Banach spaces which fail (in a very strong way) to have unconditional bases. A consequence of Ferenczi’s work is that there are relatively few endomorphisms of so-called ”hereditarily indecomposable” spaces.

The following serves as an addendum to the last theorem.

THEOREM 7.17. *The following Banach spaces (viewed as topological groups) have universal compactly generated and K_σ subgroups:*

1. ℓ^∞ ,
2. $C(X)$, if X is infinite, Polish and compact, and
3. $m C_0(X)$, if X is infinite, Polish and locally compact.

REMARK 7.18. In general, the spaces listed in this theorem may not have unconditional bases (ℓ^∞ is not even separable) and so Theorem 7.15 does not necessarily apply.

PROOF OF THEOREM 7.17. In each case, we will apply Proposition 1.9 and Theorem 7.8.

1. The injection $e_0 \rightarrow \ell^\infty$ is via the inclusion map, while the injection $\ell^\infty \rightarrow e_0$ is by means of the map $(a_n)_{n \in \omega} \mapsto ((1/n)a_n)_{n \in \omega}$.

2. Let $\{x_n\}_{n \in \omega}$ be a discrete sequence of distinct points in X . For each n , let $f_n \in C(X)$ have sup-norm 1 and be such that $f_n(x_n) = 1$ and $f_n(x_k) = 0$, if $k \neq n$. Such functions exist by the Tietze Extension Theorem. Then e_0 may be one-to-one homomorphically mapped into $C(X)$ via the continuous function $x \mapsto \sum_{n \in \omega} (x(n)/2^n) f_n$.

Let $\{y_n\}_{n \in \omega}$ be a countable dense subset of X . Then $C(X)$ is injected into e_0 via the map $f \mapsto ((1/n)f(y_n))_{n \in \omega}$.

3. Use the same functions as in 2. ◻

7.3. A negative example. The following example gives our only instances of perfect Polish groups without universal subgroups in either of the classes we consider. The key fact is that any nontrivial group homomorphism of \mathbb{R}^n is in fact an automorphism.

EXAMPLE 7.19. By Theorem 1.6 there is a universal K_σ subgroup of \mathbb{R}^ω . On the other hand, we shall see that there is no universal K_σ subgroup of \mathbb{R}^n , for $n \in \omega$. First, if $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous group homomorphism, then φ is automatically a linear transformation. To see this, observe that, since φ is a group homomorphism, one can show that $\varphi(qr) = q\varphi(r)$, for any $q \in \mathbb{Q}$ and $r \in \mathbb{R}^n$. One then concludes that $\varphi(ar) = a\varphi(r)$, for any $a \in \mathbb{R}$, by the density of \mathbb{Q} in \mathbb{R} and the continuity of φ .

Towards a contradiction, suppose that $H_0 \subseteq \mathbb{R}^n$ is a universal K_σ subgroup of \mathbb{R}^n . Let $\tilde{A}, \tilde{B} \subsetneq \mathbb{R}^n$ be nontrivial K_σ subgroups such that \tilde{A} is countable and \tilde{B} is uncountable. Let

$$A = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \in \tilde{A} \ \& \ x_2 = x_3 = \dots = x_n = 0\}$$

and

$$B = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \in \tilde{B} \ \& \ x_2 = x_3 = \dots = x_n = 0\}.$$

A and B are K_σ subgroups of \mathbb{R}^n that contain no linear (over \mathbb{R}) subspaces of \mathbb{R}^n other than $\{0^n\}$. Let φ_A, φ_B be continuous endomorphisms of \mathbb{R}^n reducing A, B to H_0 . As φ_A and φ_B are actually linear transformations, $\ker \varphi_A$ and $\ker \varphi_B$ are linear subspaces of \mathbb{R}^n . Since φ_A and φ_B are reductions between subgroups, we must have that $\ker \varphi_A \subseteq A$ and $\ker \varphi_B \subseteq B$, in particular, both kernels are trivial. As operators on a finite dimensional vector space, the maps φ_A and φ_B are injective iff they are surjective. It follows that both maps are in fact automorphisms. Thus A and B have the same cardinality, a contradiction.

By the same reasoning, there are no universal compactly generated subgroups of \mathbb{R}^n . In fact, no class of subgroups of \mathbb{R}^n which contains members of different cardinalities can possess a universal member.

§8. An application to ideals. Recall that an *ideal* on ω is a set $\mathcal{I} \subseteq \mathcal{P}(\omega)$ that is closed under finite unions and closed downwards (i.e., if $x \subseteq y \in \mathcal{I}$, then $x \in \mathcal{I}$). Also recall that $\mathcal{P}(\omega)$ becomes a Polish group when equipped with the addition operation

$$x \Delta y = (x \setminus y) \cup (y \setminus x).$$

In particular, every ideal is a subgroup of $\mathcal{P}(\omega)$, since $x \Delta y \subseteq x \cup y$, for $x, y \subseteq \omega$.

By identifying each $x \subseteq \omega$ with its characteristic function, one can regard $(\mathcal{P}(\omega), \Delta)$ as $(\mathbb{Z}_2^\omega, +)$. With this identification, the relation $x \subseteq y$ agrees with the pointwise $x \leq y$. We use the latter when dealing with \mathbb{Z}_2^ω to avoid confusion with the “ \subset ” (extension) relation on $\mathbb{Z}_2^{<\omega}$.

In this section, we study the following weak form of Rudin-Keisler reduction.

DEFINITION 8.1. For ideals \mathcal{I}, \mathcal{J} on ω , we write $\mathcal{I} \leq_{\text{RK}}^+ \mathcal{J}$ if, and only if, there is a subset $A \subseteq \omega$ and a function $\beta : A \rightarrow \omega$ such that $x \in \mathcal{I} \iff \beta^{-1}(x) \in \mathcal{J}$, for each $x \subseteq \omega$.⁹

Theorem 8.5 will use the method of Theorem 1.6 to show that there is a \leq_{RK}^+ -complete F_σ ideal. In a personal communication, Michael Hrušák has informed us that, though unpublished, this result is already known to him, albeit in a slightly different form.¹⁰

The only difference between \leq_{RK}^+ and the usual Rudin-Keisler order is that the reducing map in the case of \leq_{RK}^+ need not be defined on all of ω . As with Rudin-Keisler reduction, if $\mathcal{I} \leq_{\text{RK}}^+ \mathcal{J}$ and \mathcal{J} is an ideal, then \mathcal{I} is an ideal as well. We call a map β as in the definition above a *weak RK-reduction*. Observe that the map

$$x \mapsto \beta^{-1}(x)$$

defines a continuous homomorphism of $\mathcal{P}(\omega)$ (equivalently, of \mathbb{Z}_2^ω). This implies that, for ideals \mathcal{I}, \mathcal{J} , if $\mathcal{I} \leq_{\text{RK}}^+ \mathcal{J}$, then automatically $\mathcal{I} \leq_{\text{g}} \mathcal{J}$.

REMARK 8.2. The Rudin-Keisler ordering was originally developed to study ultrafilters (see [24], [4], and [3] for early work on the subject).

Before proceeding, we verify that \leq_{RK}^+ is indeed weaker than \leq_{RK} . Consider the following example.

EXAMPLE 8.3. For $x \subseteq \omega$, let

$$\text{Fin}(x) = \{y \in \mathcal{P}(\omega) : y \text{ is finite and } y \subseteq x\}.$$

With this notation, the ideal Fin is $\text{Fin}(\omega)$. If x is infinite, then any bijection $\beta : x \longleftrightarrow \omega$ witnesses $\text{Fin} \leq_{\text{RK}}^+ \text{Fin}(x)$. On the other hand, if $x \neq \omega$, then $\text{Fin} \not\leq_{\text{RK}} \text{Fin}(x)$. To see this, suppose otherwise and let $\beta : \omega \rightarrow \omega$ be such for each

⁹We use the notation \leq_{RK}^+ as a parallel with \leq_{RB} versus \leq_{RB}^+ . See Kanovei [13, pp. 41-42] for definitions.

¹⁰See Hrušák [10, 5.4] for a similar result.

$y \subseteq \omega, y \in \text{Fin} \iff \beta^{-1}(y) \in \text{Fin}(x)$. Let $a \in \omega \setminus x$ and let $b = \beta(a)$. We have $\{b\} \in \text{Fin}$, but $\beta^{-1}(\{b\}) \notin \text{Fin}(x)$, since $a \in \beta^{-1}(\{b\})$ and $a \notin x$.

We also remark on the fact that \leq_g is weaker than \leq_{RK}^+ .

EXAMPLE 8.4. Consider

$$H = \{x \subseteq \omega : \text{either } 0, 1 \in x \text{ or both } 0, 1 \notin x\}$$

and the ideal Fin . Both are subgroups of $(\mathcal{P}(\omega), \Delta)$ and $H \leq_g \text{Fin}$, via the map $\varphi : \mathcal{P}(\omega) \rightarrow \mathcal{P}(\omega)$ defined by

$$\varphi(x) = \begin{cases} \emptyset & \text{if } 0, 1 \in x \text{ or both } 0, 1 \notin x, \\ \omega & \text{otherwise.} \end{cases}$$

It is easier to see that this is a group homomorphism by viewing $\mathcal{P}(\omega)$ as \mathbb{Z}_2^ω . With this identification, φ is given by

$$\varphi(x)(n) = x(0) + x(1),$$

for all $x \in \mathbb{Z}_2^\omega$ and $n \in \omega$.

On the other hand, we cannot have $H \leq_{\text{RK}}^+ \text{Fin}$, since this would imply that H is an ideal.

We now proceed to the main result of this section.

THEOREM 8.5. *There is a \leq_{RK}^+ -complete F_σ ideal in \mathbb{Z}_2^ω .*

REMARK 8.6. 1. This result can be interpreted in the context of *Borel cardinality*, as discussed in Kechris [15]. That is, Theorem 8.5 implies the existence of an F_σ ideal \mathcal{J} on ω such that $\mathcal{P}(\omega)/\mathcal{J}$ has maximum Borel cardinality among F_σ factor algebras of $\mathcal{P}(\omega)$, i.e., every factor algebra $\mathcal{P}(\omega)/\mathcal{I}$ by an F_σ ideal \mathcal{I} embeds in $\mathcal{P}(\omega)/\mathcal{J}$ via a map with a Borel lifting to $\mathcal{P}(\omega)$.

2. It is worth noting that the statement “ $\mathcal{I} \leq_{\text{RK}}^+ \mathcal{J}$ ” is much stronger than the statement “ $\mathcal{P}(\omega)/\mathcal{I}$ embeds in $\mathcal{P}(\omega)/\mathcal{J}$ ”. Indeed, Just-Krawczyk [12] showed that, assuming the Continuum Hypothesis, all factor algebras of the form $\mathcal{P}(\omega)/\mathcal{I}$, with \mathcal{I} an F_σ ideal, are isomorphic.

3. Since every ideal on ω is a subgroup of the compact group \mathbb{Z}_2^ω , Theorem 1.12 implies that every F_σ (i.e., K_σ) ideal is compactly generated. Since the downward closure of a compact set is also compact, we conclude that every F_σ ideal on ω is the set of finite unions of elements of a downward closed compact subset of $\mathcal{P}(\omega)$.

PROOF OF THEOREM 8.5. For $k \in \omega$ and $s \in \omega^{<\omega}$, let A_s^k be subsets of \mathbb{Z}_2^k such that

- Each A_s^k is closed downward, i.e., $u \leq v \in A_s^k \implies u \in A_s^k$.
- If $A \subseteq \mathbb{Z}_2^k$ is closed downward and $A \supseteq A_s^k$, then there exists i such that $A = A_{s \smallfrown i}^k$.

For each k, j , let I_j^k be an interval in ω of length k , such that the I_j^k partition ω . Define $A \subseteq \mathbb{Z}_2^\omega$ by

$$x \in A \iff (\exists n)(\forall k, s)(|s| \geq n \implies x \upharpoonright I_s^k \in A_s^k).$$

Observe that A is F_σ and hence so is the ideal \mathcal{I}_0 , generated by A . Note that A is already closed downward and thus \mathcal{I}_0 is the set of finite unions of elements of A . We will show that \mathcal{I}_0 is \leq_{RK}^+ -complete among F_σ ideals.

Let $\mathcal{I} = \bigcup_n F_n$ be an arbitrary F_σ ideal. We may assume that $F_0 \subseteq F_1 \subseteq \dots$ and that each F_n is closed downward. (Since the downward closure of a closed set is also closed.) For each k , choose $\alpha_k \in \omega^\omega$ such that for each n ,

$$F_n \upharpoonright k = A_{\alpha_k \upharpoonright n}^k.$$

Let $S = \bigcup_{s \subset \alpha_k} I_s^k$. We will define a weak RK-reduction $\beta : S \rightarrow \omega$ which will witness $\mathcal{I} \leq_{\text{RK}}^+ \mathcal{I}_0$. For each I_s^k , with $s \subset \alpha_k$, if i is the j th element of I_s^k , we set $\beta(i) = j$. We can rewrite the map $x \mapsto \beta^{-1}(x)$ in a way that will be easier to work with. Observe that

$$\beta^{-1}(x) \upharpoonright I_s^k = \begin{cases} x \upharpoonright k & \text{if } s \subset \alpha_k, \\ 0^k & \text{otherwise.} \end{cases}$$

The following two claims will complete the proof.

CLAIM 8.7. *If $x \in \mathcal{I}$, then $\beta^{-1}(x) \in \mathcal{I}_0$.*

PROOF OF CLAIM. Suppose that $x \in \mathcal{I}$, with $x \in F_{n_0}$. This implies that, for each k and $s \subset \alpha_k$, with $n = |s| \geq n_0$, we have

$$\begin{aligned} \beta^{-1}(x) \upharpoonright I_s^k &= x \upharpoonright k \\ &\in F_n \upharpoonright k \\ &= A_{\alpha_k \upharpoonright n}^k. \end{aligned}$$

If $s \not\subset \alpha_k$, then $\beta^{-1}(x) \upharpoonright I_s^k = 0^k \in A_s^k$, since A_s^k is closed downwards. Putting these two cases together, we see that

$$(\forall k, s)(|s| \geq n_0 \implies \beta^{-1}(x) \upharpoonright I_s^k \in A_s^k).$$

Hence $\beta^{-1}(x) \in A \subseteq \mathcal{I}_0$. This proves our first claim.

CLAIM 8.8. *If $\beta^{-1}(x) \in \mathcal{I}_0$, then $x \in \mathcal{I}$.*

PROOF OF CLAIM. Suppose that $\beta^{-1}(x) \in \mathcal{I}_0$ and $y_1, \dots, y_m \in A$ are such that $\beta^{-1}(x) = y_1 \cup \dots \cup y_m$. We will find $x_1, \dots, x_m \in \mathcal{I}$ such that $x = x_1 \cup \dots \cup x_m$. Let n be such that for each $i \leq m$,

$$(\forall k, s)(|s| \geq n \implies y_i \upharpoonright I_s^k \in A_s^k).$$

Let $v_i^k = y_i \upharpoonright I_{\alpha_k \upharpoonright n}^k$. For each k and all $i \leq m$, $v_i^k \in A_{\alpha_k \upharpoonright n}^k = F_n \upharpoonright k$. Hence there exists $x_i^k \in F_n$ such that $v_i^k = x_i^k \upharpoonright k$. By repeated use of the compactness of \mathbb{Z}_2^ω , we choose a subsequence $k_0 < k_1 < \dots$ and $x_i \in F_n$ such that, for each $i \leq m$,

$$\lim_{p \rightarrow \infty} x_i^{k_p} = x_i.$$

To check that $x = x_1 \cup \dots \cup x_m$, observe that, for each fixed ℓ and p with $k_p \geq \ell$, we have

$$\begin{aligned} x \upharpoonright \ell &= (\beta^{-1}(x) \upharpoonright I_{\alpha_{k_p} \upharpoonright n}^{k_p}) \upharpoonright \ell \\ &= (v_1^{k_p} \cup \dots \cup v_m^{k_p}) \upharpoonright \ell \\ &= (x_1^{k_p} \cup \dots \cup x_m^{k_p}) \upharpoonright \ell. \end{aligned}$$

Taking the limit as $p \rightarrow \infty$, we see that

$$x \upharpoonright \ell = (x_1 \cup \dots \cup x_m) \upharpoonright \ell.$$

Since ℓ was arbitrary, we must have $x = x_1 \cup \dots \cup x_m$. This shows that $x \in \mathcal{I}$ and completes the proof. \dashv

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