On some elementary functions

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We gather here some material, 'well-known to the experts' — see the short historical discussion at the end — but not often found in the mass-market textbooks for high schoolers and undergraduates (who would benefit from it the most), that can be used to discuss e^x, $\sqrt{x}$ and $\log x$ rigorously in terms of simple inequalities and high school algebra of numbers, polynomials, and rational functions along with the completeness axiom of the real line:

(C) If $a_n \leqslant a_{n+1} \leqslant B$ in the number line $\mathbb{R}$ for some upper bound B and all $n \geqslant 1$, then the sequence a_n has a limit $A \in \mathbb{R}$ as $n \to \infty$.

Another standard way to formulate the completeness axiom (C) is to say that a bounded monotonic sequence in $\mathbb{R}$ has a finite limit in $\mathbb{R}$.

Most treatments of e^x, $\sqrt{x}$ and $\log x$ in high school mathematics are inadequate and largely circular in terms of logical development and ineffective for calculation, or else they require a lengthy development of more involved ideas such as continuity, differentiation, integration, and summation of power series.

To define e^x we will use and estimate $\left(1 + \frac{x}{n}\right)^n$ as $n \to \infty$ for x real. The standard treatment of this limit in elementary calculus is usually inadequate and circular (it relies on logarithms, often defined as the inverse of the exponential function, and L'Hospital's rule). Another elementary treatment of the same limit relies on the inequality between the arithmetic mean and the geometric mean of finitely but unboundedly many numbers, requiring nth roots (not easily defined and discussed before the introduction of e^x and $\log x$ or the notions of continuity and the intermediate value theorem). Our approach is similar, but we notice that in treating $\left(1 + \frac{x}{n}\right)^n$ it is easier to double n to $2n$ rather than augment n to $n + 1$. In particular, we will only require simple algebraic properties of polynomials and rational functions; we will not need even square roots, let alone nth roots.

In keeping with doubling, call as usual the numbers $1, 2, 4, \ldots, 2^m, \ldots$ *powers* of 2. Let $a_n(x) = \left(1 + \frac{x}{n}\right)^n$, $b_n(x) = \left(1 - \frac{x}{n}\right)^{-n} = \left(1 + \frac{x}{n - x}\right)^n$ for x real and $n \geqslant 1$ being a power of 2. Note that if $n > |x|$, then $a_n(x) > 0$ and $b_n(x) > 0$ since they are the squares of non-zero reals.

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Proposition 1:

(a) If x is real and $N \geqslant 1$ is a power of 2 so large that $\frac{x}{N} \geqslant -1$, then the sequence $a_n(x) = \left(1 + \frac{x}{n}\right)^n$ increases for any $n \geqslant N$ where n is a power of 2.

(b) In particular, if $x \geqslant -1$, then $a_n(x) = \left(1 + \frac{x}{n}\right)^n \geqslant 1 + x$ for any $n \geqslant 1$ where n is a power of 2.

(c) If x is real and $N \geqslant 1$ is a power of 2 so large that $\frac{x}{N} < 1$, then the sequence $b_n(x) = \left(1 - \frac{x}{n}\right)^{-n} = \left(1 + \frac{x}{n - x}\right)^n$ decreases for any $n \geqslant N$ where n is a power of 2.

(d) If B_n is a bounded sequence in $\mathbb{R}$, then $\left(1 + \frac{B_n}{n^2}\right)^n \to 1$ as $n \to \infty$ through powers of 2.

(e) If x is real and $N \geqslant 1$ is a power of 2 so large that $\frac{|x|}{N} < 1$, then

(i) $a_n(x) \leqslant b_n(x)$ for any $n \geqslant N$ where n is a power of 2, and

(ii) $\frac{a_n(x)}{b_n(x)} \to 1$ as $n \to \infty$ through powers of 2.

(f) If x, y are real, then $\frac{a_n(x)\, a_n(y)}{b_n(x + y)} \to 1$ as $n \to \infty$ through powers of 2.

Proof:

(a) We must show that if $n \geqslant N$ is a power of 2, then $a_{2n}(x) \geqslant a_n(x)$. Indeed,

$$a_{2n}(x) = \left(1 + \frac{x}{2n}\right)^{2n} = \left\{\left(1 + \frac{x}{2n}\right)^2\right\}^n = \left\{1 + \frac{x}{n} + \frac{x^2}{4n^2}\right\}^n$$

$$\geqslant \left\{1 + \frac{x}{n}\right\}^n = a_n(x),$$

where we note that if $\frac{x}{N} \geqslant -1$, then $\frac{x}{n} \geqslant -1$, the quantities in the curly brackets are non-negative, and the power function $t \to t^n$ increases for $t \geqslant 0$ and any $n \geqslant 1$ where n is a power of 2.

(b) This is the special case $a_n(x) \geqslant a_1(x)$ of (a) with $N = 1$.

(c) We must show that if $n \geqslant N$ is a power of 2, then $b_{2n}(x) \leqslant b_n(x)$. Indeed,

$$b_{2n} = \left(1 - \frac{x}{2n}\right)^{2n} = \left\{\left(1 - \frac{x}{2n}\right)^2\right\}^{-n} = \left\{1 - \frac{x}{n} + \frac{x^2}{4n^2}\right\}^{-n}$$

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$$\leqslant \left\{1 - \frac{x}{n}\right\}^{-n} = b_n(x),$$

where we note that if $\frac{x}{N} < 1$, then $\frac{x}{n} < 1$, the quantities in the curly brackets are positive, and the power function $t \to t^{-n}$ decreases for $t > 0$ and any $n \geqslant 1$ where n is a power of 2.

(d) Let $N \geqslant 1$ be a power of 2 so large that $\frac{|B_n|}{N} < 1$ for any power of 2, all $n \geqslant 1$. Then $\frac{|B_n|}{n} < 1$ also for all $n \geqslant N$ where n is a power of 2. Applying (b) we get $\left(1 + \frac{B_n}{n^2}\right)^n \geqslant 1 + \frac{B_n}{n} \geqslant 1 - \frac{N}{n}$ and $\left(1 - \frac{B_n}{n^2}\right)^n \geqslant 1 - \frac{B_n}{n} \geqslant 1 - \frac{N}{n}$ for any $n \geqslant N$ where n is a power of 2. So $1 - \frac{N}{n} \leqslant \left(1 + \frac{B_n}{n^2}\right)^n \leqslant \frac{1}{\left(1 - \frac{B_n}{n^2}\right)^n} \leqslant \frac{1}{1 - \frac{N}{n}}$ for $n > N$ where n is a power of 2. Making $n \to \infty$ through powers of 2 yields $\left(1 + \frac{B_n}{n^2}\right)^n \to 1$.

(e) Indeed, $\frac{a_n(x)}{b_n(x)} = \frac{(1 + \frac{x}{n})^n}{(1 - \frac{x}{n})^{-n}} = \left(1 - \frac{x^2}{n^2}\right)^n \leqslant 1$ and $\frac{a_n(x)}{b_n(x)} \to 1$ as $n \to \infty$ through powers of 2 by (d). We can also estimate the additive difference $b_n(x) - a_n(x)$ rather than its multiplicative analogue $\frac{a_n(x)}{b_n(x)}$. Indeed, as $\frac{a_n(x)}{b_n(x)} = \left(1 - \frac{x^2}{n^2}\right)^n \leqslant 1$, being the power of order $n \geqslant 1$ of a number in [0, 1], we have

$$0 \leqslant b_n(x) - a_n(x) = \left(1 - \frac{x}{n}\right)^{-n} - \left(1 + \frac{x}{n}\right)^n = \left(1 - \frac{x}{n}\right)^{-n}\left[1 - \left(1 - \frac{x^2}{n^2}\right)^n\right] \quad (1)$$

$$\leqslant \left(1 - \frac{x}{n}\right)^{-n}\left[1 - \left(1 - \frac{x^2}{n}\right)\right] = \left(1 - \frac{x}{n}\right)^{-n}\frac{x^2}{n} \leqslant \left(1 - \frac{x}{N}\right)^{-N}\frac{x^2}{n} \to 0 \quad (2)$$

as $n \to \infty$ through powers of 2, i.e. $b_n(x) - a_n(x) \to 0$. Here we applied (b) to the power in the square brackets at the end of (1) and (c) in the last inequality in (2).

(f) We can write $\frac{a_n(x)\,a_n(y)}{b_n(x + y)}$ as

$$\frac{a_n(x)\,a_n(y)}{b_n(x + y)} = \left\{\left(1 + \frac{x}{n}\right)\left(1 + \frac{y}{n}\right)\left(1 - \frac{x + y}{n}\right)\right\}^n = \left\{1 + \frac{B_n}{n^2}\right\}^n \to 1$$

as $n \to \infty$ through powers of 2 by (d), since the sequence

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$B_n = -(x + y)^2 + xy\left(1 - \frac{x + y}{n}\right)$ is bounded, for example, by $|B_n| \leqslant (x + y)^2 + |xy|(1 + |x + y|)$ for all $n \geqslant 1$.

Proposition 2: Let x be real and $N \geqslant 1$ a power of 2 so large that $|x| < N$. The sequences $a_n(x)$, $b_n(x)$ for $n > N$ where n is a power of 2 have a common limit $f(x) \in (0, \infty)$ where $a_n(x)$ is inceasing and converges to $f(x)$ from below and $b_n(x)$ is decreasing and converges to $f(x)$ from above for $x \in \mathbb{R}$ with $|x| < N$ as $n \to \infty$ through powers of 2.

Proof: This follows from Proposition 1(a, c, e) together with the completeness axiom (C).

The main characteristic properties of the (natural) exponential function $f(x)$ are the functional equation $f(x + y) = f(x)f(y)$ for all x, y real, and the tangent line estimate $f(x) \geqslant 1 + x$ for x real.

Proposition 3: The function $f : \mathbb{R} \to (0, \infty)$ defined in Proposition 2 satisfies

(a) the functional equation $f(x + y) = f(x)f(y)$ for all x, y real, and

(b) the estimate $f(x) \geqslant 1 + x$ for all $x \in \mathbb{R}$.

Proof:

(a) If we take a limit in Proposition 1(f) as $n \to \infty$ through powers of 2, then we get $\frac{f(x)f(y)}{f(x + y)} = 1$, i.e. $f(x + y) = f(x)f(y)$.

(b) If we take a limit in Proposition 1(b), then we get $f(x) \geqslant 1 + x$ for $x \geqslant -1$. This inequality is also true for $x < -1$, since then $f(x) \geqslant 0 \geqslant 1 + x$. (The fact that $f(x) \geqslant 0$ for all $x \in \mathbb{R}$ follows from Proposition 1 or from (a) above on writing $f(x) = f\left(\frac{x}{2} + \frac{x}{2}\right) = f\left(\frac{x}{2}\right)^2 \geqslant 0$.)

Proposition 4: If a function $f : \mathbb{R} \to (0, \infty)$ satisfies

(a) the functional equation $f(x + y) = f(x)f(y)$ for all x, y real, and

(b) the inequality $f(x) \geqslant 1 + x$ for all x real in an interval $|x| < \frac{1}{N}$, where $N \geqslant 1$ is a power of 2, then f coincides on $\mathbb{R}$ with the function defined in Proposition 2.

Proof: We show that $f(x) = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = \lim_{n\to\infty}\left(1 - \frac{x}{n}\right)^{-n}$ for $x \in \mathbb{R}$ as $n \to \infty$ through powers of 2. Indeed, $0 < f(0) = f(0 + 0) = f(0)^2$, so $f(0) = 1$, $1 = f(x + (-x)) = f(x)f(-x)$, i.e. $f(x) = \frac{1}{f(-x)}$ for $x \in \mathbb{R}$. Hence if $|x| < \frac{1}{N}$, then $1 - x \leqslant f(-x)$, whose reciprocal is $f(x) = \frac{1}{f(-x)} \leqslant \frac{1}{1 - x}$, i.e. $1 + x \leqslant f(x) \leqslant \frac{1}{1 - x}$ for $|x| < \frac{1}{N}$. If $x \in \mathbb{R}$, then there is a power $n \geqslant 1$ of 2 so large that $\frac{|x|}{n} < \frac{1}{N}$. Hence $f(x) = f\left(\frac{x}{n} + \dots + \frac{x}{n}\right) = f\left(\frac{x}{n}\right)^2$. Raising

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the inequality $0 < 1 + \frac{1}{x} \leqslant f\left(\frac{x}{n}\right) \leqslant \frac{1}{1 - \frac{x}{n}}$ to the nth power (an operation which preserves the directions of the inequalities), we obtain

$$a_n(x) = \left(1 + \tfrac{x}{n}\right)^n \leqslant f(x) \leqslant \left(1 - \tfrac{x}{n}\right)^{-n} = b_n(x).$$

Making $n \to \infty$ through powers of 2 gives $f(x) = \lim\limits_{n\to\infty} a_n(x) = \lim\limits_{n\to\infty} b_n(x)$, i.e. the function f equals on $\mathbb{R}$ the function defined in Proposition 2.

The above treatment of the exponential function e^x has the advantage that it is very primitive; it only uses high school algebra of numbers, polynomials, and rational functions. It does not use any notion of square roots or facts from calculus, other than the notion of the limit of a sequence of real numbers, and the completeness axiom (C), which seem unavoidable. The reader may enjoy imagining how the graph of e^x is approached by the right half $x > -n$ of the even power $a_n(x) = \left(1 + \frac{x}{n}\right)^n$ with its n-fold zero at $x = -n$ marching off to $-\infty$ (where $e^x = 0$ to infinite order) and by the left half $x < n$ of the rational function $b_n(x) = \left(1 + \frac{x}{n-x}\right)^n$ with no root but an n-fold pole at $x = n$ marching off to $+\infty$ (where $e^x = +\infty$ to infinite order).

If we admit the notion of the derivative, then we can easily check the major characterisation of the exponential function f in terms of the initial value problem $f'(x) = f(x)$ for all x real, and $f(0) = 1$.

Proposition 5: If a function $f : \mathbb{R} \to (0, \infty)$ satisfies the conditions of Proposition 4, then f has a derivative f' on $\mathbb{R}$, $f' = f$ on $\mathbb{R}$, and $f(0) = 1$.

Proof: We already know that $f(0) = 1$ and $1 + x \leqslant f(x) \leqslant \frac{1}{1-x}$ for $|x| < 1$ from the proof of Proposition 4. The above inequality enables us to find the derivative $f'(0) = \lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x}$. Indeed, the difference quotient above satisfies $\dfrac{1 + x - 1}{x} \leqslant \dfrac{f(x) - 1}{x} \leqslant \dfrac{\frac{1}{1-x} - 1}{x}$ for $0 < x < 1$ (with the inequalities reversed for $-1 < x < 0$), i.e. $1 \leqslant \dfrac{f(x) - 1}{x} \leqslant \dfrac{1}{1 - x}$ (with the inequalities reversed for $-1 < x < 0$), whose limit as $x \to 0$ exists and equals $f'(0) = 1$. As $f(x + y) = f(x)f(y)$ for all x, y real, we see on taking a y-partial derivative at $y = 0$ that $f'(x) = f(x)f(0) = f(x)$ for all $x \in \mathbb{R}$.

If we admit not only the notion of the derivative F' of a function $F : \mathbb{R} \to \mathbb{R}$, but also the fact that $F = 0$ on $\mathbb{R}$ if, and only if, $F(0) = 0$ and $F' = 0$ on $\mathbb{R}$, then we can easily show working solely with the initial value problem that a function F solving the initial value problem $F' = F$ on $\mathbb{R}$ and $F(0) = 1$ is unique by (i) comparing it with the exponential function f defined in Proposition 2 and by (ii) showing that F satisfies the conditions of Proposition 4.

Proposition 6: Let $F : \mathbb{R} \to \mathbb{R}$ be any function with $F' = F$ on $\mathbb{R}$. Then the following hold:

(a) The functional equation $F(0)F(x + y) = F(x)F(y)$ holds for all x, y real.

(b) If $f : \mathbb{R} \to \mathbb{R}$ satisfies the initial value problem $f' = f$ on $\mathbb{R}$, $f(0) = 1$, then F is of the form $F(x) = F(0)f(x)$ for all $x \in \mathbb{R}$.

(c) If $F(0) = 1$, then $G(x) = F(x) - 1 - x$ for x real satisfies $G(x) = \left[G\left(\frac{x}{2}\right) + \frac{x}{2}\right]^2 + 2G\left(\frac{x}{2}\right)$, $G(x) \geqslant 2G\left(\frac{x}{2}\right)$, $G(x) \geqslant 2^n G\left(\frac{x}{2^n}\right)$, $G'(0) = 0$, $G(x) \geqslant G'(0)x = 0$. Hence $G(x) \geqslant 0$ for x real, and F satisfies the conditions of Proposition 4.

Proof:

(a) We must show that the function $Z = F(0)F(x + y) - F(x)F(y)$ is identically zero. We keep the variable x and eliminate y by putting $x + y = z$, i.e. $Z = F(0)F(z) - F(x)F(z - x)$. To show that the latter form of Z vanishes, hold z fixed at any real value, vary x, and check that $Z|_{x=0} = 0$ and $\frac{\partial Z}{\partial x} = 0$. Indeed, $Z|_{x=0} = F(0)F(z) - F(0)F(z) = 0$, and

$$\frac{\partial Z}{\partial x} = -F'(x)F(z - x) + F(x)F'(z - x) = -F(x)F(z - x) + F(x)F(z - x) = 0.$$

(b) We must show that the function $Z = F(x)f(-x) - F(0)$ is identically zero. To that end, show that $Z(0) = 0$ and $Z' = 0$. Indeed, $Z(0) = F(0)f(0) - F(0) = 0$ and

$$Z'(x) = F'(x)f(-x) - F(x)f'(-x) = F(x)f(-x) - F(x)f(-x) = 0.$$

Then $0 = Z(x)f(x) = F(x)f(x)f(-x) - F(0)f(x) = F(x) - F(0)f(x)$, since $f(x)f(-x) = f(x - x) = f(0) = 1$ by (a).

(c) The function F satisfies the functional equation $F(x + y) = F(x)F(y)$ for x, y real by (a). To show that $F(x) \geqslant 1 + x$ for x real, we must check that $G(x) \geqslant 0$ for x real. Note that $G'(x) = F'(x) - 1 = F(x) - 1$, which vanishes for $x = 0$ since $G'(0) = F(0) - 1 = 0$. Write $F(x) = G(x) + 1 + x$ and put it into the functional equation $F(x) = F\left(\frac{x}{2}\right)^2$ to find that

$$G(x) + 1 + x = \left(\left[G\left(\tfrac{x}{2}\right) + \tfrac{x}{2}\right] + 1\right)^2 = \left[G\left(\tfrac{x}{2}\right) + \tfrac{x}{2}\right]^2 + 2G\left(\tfrac{x}{2}\right) + x + 1,$$

i.e. $G(x) = \left[G\left(\frac{x}{2}\right) + \frac{x}{2}\right]^2 + 2G\left(\frac{x}{2}\right)$. Hence $G(x) \geqslant 2G\left(\frac{x}{2}\right)$ for x real. Iterating yields

$$G(x) \geqslant 2G\left(\frac{x}{2}\right) \geqslant 4G\left(\frac{x}{4}\right) \geqslant 8G\left(\frac{x}{8}\right) \geqslant \ldots \geqslant 2^n G\left(\frac{x}{2^n}\right) \to G'(0)x = 0$$

as $n \to \infty$, i.e. $G(x) \geqslant 0$ for x real. Here we used the facts that

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$G(0) = F(0) - 1 = 0$, and $kG\left(\frac{x}{k}\right) = \frac{G\left(\frac{x}{k}\right) - G(0)}{\frac{1}{k} - 0}$ is a difference quotient for the function $G(tx)$ at $t = 0$, i.e. it converges as $k \to \infty$ (e.g. with $k = 2^n$) to the derivative $\left.\frac{\partial G(tx)}{\partial t}\right|_{t=0} = G'(tx)x\Big|_{t=0} = G'(0)x = 0$.

If we invert the functions $y = \left(1 + \frac{x}{n}\right)^n$ and $y = \left(1 - \frac{x}{n}\right)^{-n}$, then we need to use nth roots, which when n is a power of 2 are just iterated square roots, and obtain the candidates $A_n(y) = n\left(y^{1/n} - 1\right)$ and $B_n(y) = n\left(1 - y^{-1/n}\right)$ for the approximation of $\log y$, $y > 0$, $n \geqslant 1$ being a power of 2. If we know or admit that any $y > 0$ is a value of the exponential function $y = f(x)$ for some x real, then we can find a square root $\sqrt{y} > 0$ by $\sqrt{y} = f\left(\frac{x}{2}\right)$. Since the formulas for A_n and B_n to approximate the logarithm function feature square roots, we proceed to study the extraction of a square root using Newton's iteration (going back at least to the ancient Babylonians).

Proposition 7: For $x \geqslant 1$ fixed, define a sequence r_n by $r_0 = x$, $r_{n+1} = \frac{1}{2}\left(r_n + \frac{x}{r_n}\right)$ for $n \geqslant 0$. Then

(a) $r_n \geqslant 1$ and

(b) $r_n \leqslant x$ for all $n \geqslant 0$.

(c) Let $d_n = r_n^2 - x$ for $n \geqslant 0$. Then $d_n \geqslant 0$ for all $n \geqslant 0$.

(d) The sequence r_n is a bounded decreasing sequence, hence there is an $r \in \mathbb{R}$ with $r_n \to r$ as $n \to \infty$. This r satisfies that $r \geqslant 1$ and $r^2 = x$.

Proof: We prove (a) and (b) by induction on n. If $n = 0$, then $r_0 = x \geqslant 1$ satisfies the inequalities (a) and (b). We suppose that (a) and (b) are valid for a value of $n \geqslant 0$ and prove them for $n + 1$. We have

$$r_{n+1} - 1 = \tfrac{1}{2}\left(r_n - 1 + \frac{x}{r_n} - 1\right) = \tfrac{1}{2}\left[r_n - 1\right] + \frac{1}{2r_n}\left[x - r_n\right] \geqslant 0$$

and $x - r_{n+1} = \frac{1}{2}\left[x - r_n\right] + \frac{x}{2r_n}\left[r_n - 1\right] \geqslant 0$ since $r_n \leqslant x$, $r_n \geqslant 1$ by the induction hypothesis.

(c) We have $d_0 = r_0^2 - x = x^2 - x = x(x - 1) \geqslant 0$, and

$$d_{n+1} = r_{n+1}^2 - x = \left[\frac{1}{2}\left(r_n + \frac{x}{r_n}\right)\right]^2 - x = \frac{1}{4}\left(r_n^2 + 2x + \frac{x^2}{r_n^2} - 4x\right) = \frac{1}{4}\left(r_n - \frac{x}{r_n}\right)^2 \geqslant 0$$

for $n \geqslant 0$.

(d) We have

$$r_{n+1} - r_n = \frac{1}{2}\left(r_n + \frac{x}{r_n}\right) - r_n = \frac{1}{2}\left(\frac{x}{r_n} - r_n\right) = \frac{1}{2}\frac{x - r_n^2}{r_n} \leqslant 0$$

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for $n \geqslant 0$ by (a) and (c), i.e. $1 \leqslant r_{n+1} \leqslant r_n \leqslant x$. Hence the limit $r \in [1, x]$ of r_n exists by the completeness axiom (C). Making $n \to \infty$ in $r_{n+1} = \frac{1}{2}\left(r_n + \frac{x}{r_n}\right)$ gives $r = \frac{1}{2}\left(r + \frac{x}{r}\right)$, which boils down to $r^2 = x$.

Proposition 8: For any $x > 0$ real there is a unique $s > 0$ real with $s^2 = x$.

Proof: If $x \geqslant 1$, then we may take the $s = r$ given by Proposition 7. If $0 < x < 1$, then we may take $s = \frac{1}{r}$, where r is given for $\frac{1}{x} > 1$ by Proposition 7. As for the uniqueness, if $s > 0$ and $t > 0$ satisfy $s^2 = t^2$, then $0 = s^2 - t^2 = (s - t)(s + t)$, or $s - t = 0$ upon division by $s + t > 0$.

Proposition 9:

(a) If $y > 0$ and $n \geqslant 1$ is a power of 2, then
  (i) $A_{2n}(y) \leqslant A_n(y)$,
  (ii) $B_n(y) \leqslant B_{2n}(y)$, and
  (iii) $B_n(y) \leqslant A_n(y)$.

(b) If $y > 0$, then $y^{1/n} \to 1$, $n\left(y^{1/n} - 1\right)^2 \to 0$ and $n\left(y^{1/(2n)} - y^{-1/(2n)}\right)^2 \to 0$ as $n \to \infty$ through powers of 2.

(c) If $y > 0$, there is a common limit $g(y)$ such that $A_n(y)$ converges to $g(y)$ from above and $B_n(y)$ converges to $g(y)$ from below, as $n \to \infty$ through powers of 2.

(d) The function $g : (0, \infty) \to \mathbb{R}$ defined in (c) satisfies the functional equation (i) $g(xy) = g(x) + g(y)$ for all $x, y > 0$ and the (tangent line) inequality (ii) $g(y) \leqslant y - 1$ for $y > 0$.

Proof:

(a) We show that differences of the sides of the proposed inequalities are non-negative by completing various squares.

(i) $A_n(y) - A_{2n}(y) = n\left(y^{1/n} - 1\right) - 2n\left(y^{1/(2n)} - 1\right) = n\left(y^{1/n} - 2y^{1/(2n)} + 1\right)$
$= n\left(y^{1/(2y)} - 1\right)^2 \geqslant 0.$

(ii) $B_{2n}(y) - B_n(y) = 2n\left(1 - y^{-1/(2n)}\right) - n\left(1 - y^{-1/n}\right) = n\left(1 - 2y^{-1/(2n)} + y^{-1/n}\right)$
$= n\left(1 - y^{-1/(2n)}\right)^2 \geqslant 0.$ This also follows from (i) noting that $B_n(y) = -A_n(1/y)$.

(iii) $A_n(y) - B_n(y) = n\left(y^{1/n} - 1\right) - n\left(1 - y^{-1/n}\right) = n\left(y^{1/n} - 2 + y^{-1/n}\right)$
$= n\left(y^{1/(2n)} - y^{-1/(2n)}\right)^2 \geqslant 0.$

(b) By (a) the sequence $A_n(y) = n\left(y^{1/n} - 1\right)$ is bounded since $1 - \frac{1}{y} = B_1(y) \leqslant A_n(y) \leqslant A_1(y) = y - 1$, i.e. there is a constant $M = M(y)$ for each $y > 0$ such that $\left|n\left(y^{1/n} - 1\right)\right| < M$ for all $n \geqslant 1$ where n is a power of 2. Thus $\left|y^{1/n} - 1\right| < M/n \to 0$, $0 \leqslant n\left(y^{1/n} - 1\right)^2 < M^2/n \to 0$, and so

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$y^{1/n} \to 1$ and $n\left(y^{1/n} - 1\right)^2 \to 0$ as $n \to \infty$ through powers of 2. Then $n\left(y^{1/(2n)} - y^{-1/(2n)}\right)^2 = \dfrac{n\left(y^{1/n} - 1\right)^2}{y^{1/n}} \to \dfrac{0}{1} = 0$ as $n \to \infty$ through powers of 2 by the previous two limits.

(c) The existence of the limit $g(y)$ follows from the inequalities $B_1(y) \leqslant B_2(y) \leqslant \ldots \leqslant B_{2^n}(y) \leqslant A_{2^n}(y) \leqslant \ldots \leqslant A_4(y) \leqslant A_2(y) \leqslant A_1(y)$ in (a), the third part of (b), and the completeness axiom (C) of the real line.

(d) To verify the functional equation (i) $g(xy) = g(x) + g(y)$, consider

$$-g(xy) + g(x) + g(y) = \lim_{n\to\infty}\left[-2n\left((xy)^{1/(2n)} - 1\right) + n\left(x^{1/n} - 1\right) + n\left(y^{1/n} - 1\right)\right]$$

where

$$-2n\left((xy)^{1/(2n)} - 1\right) + n\left(x^{1/n} - 1\right) + n\left(y^{1/n} - 1\right)$$

$$= n\left(x^{1/n} - 2x^{1/(2n)}y^{1/(2n)} + y^{1/n}\right) = x^{1/n}n\left(1 - \left(\frac{y}{x}\right)^{1/(2n)}\right)^2 \to 1 \times 0 = 0$$

as $n \to \infty$ through powers of 2 by (b) and (c). To check the inequality (ii) $g(y) \leqslant y - 1$, take a limit of $A_n(y) \leqslant A_1(y) = y - 1$ for $y > 0$ as $n \to \infty$ through powers of 2.

Proposition 10: If a function $g : (0, \infty) \to \infty$ satisfies the functional equation (a) $g(xy) = g(x) + g(y)$ for $x, y > 0$, and the inequality (b) $g(y) \leqslant y - 1$ for $1/r < y < r$, where $r = 1 + 1/N$ and $N \geqslant 1$ is a power of 2, then g coincides on $(0, \infty)$ with the function defined in Proposition 9(c).

Proof: We show that $g(y) = \lim\limits_{n\to\infty} n\left(y^{1/n} - 1\right) = \lim\limits_{n\to\infty} n\left(1 - y^{-1/n}\right)$ for $y > 0$ as $n \to \infty$ through powers of 2. Indeed, $g(1) = g(1\cdot1) = g(1) + g(1)$, so $g(1) = 0$, and $0 = g(1) = g(y\cdot\frac{1}{y}) = g(y) + g\left(\frac{1}{y}\right)$, or, $g(y) = -g\left(\frac{1}{y}\right)$ for $y > 0$, and $y - 1 \geqslant g(y) \geqslant -\left(\frac{1}{y} - 1\right) = 1 - \frac{1}{y}$ for $\frac{1}{r} < y < r$. If $N_0 \geqslant 1$ is a power of 2 so large (depending on y) that $\frac{1}{r} < y^{1/n} < r$ for any power n of 2 with $n \geqslant N_0$, then $g(y) = g\left(y^{1/n}\cdot\ldots\cdot y^{1/n}\right) = ng\left(y^{1/n}\right)$ can be approximated from below and above as $B_n(y) = n\left(1 - \frac{1}{y^{1/n}}\right) \leqslant g(y) = ng\left(y^{1/n}\right) \leqslant n\left(y^{1/n} - 1\right) = A_n(y)$. Making $n \to \infty$ through powers of 2 concludes the proof.

If we admit the notion of the derivative, then we can easily check the major characterisation of the logarithm function g in terms of the initial value problem $g'(x) = \frac{1}{x}$ for all $x > 0$, and $g(1) = 0$.

Proposition 11: If a function $g : (0, \infty) \to \mathbb{R}$ satisfies the conditions of Proposition 10, then g has a derivative g' on $(0, \infty)$, and g solves the initial value problem $g'(x) = 1/x$ for $x > 0$, and $g(1) = 0$.

Proof: We already know that $g(1) = 0$ and $1 - \frac{1}{y} \leqslant g(y) \leqslant y - 1$ for $y > 0$ from the proof of Proposition 10. The above enable us to find the derivative $g'(1) = \lim_{y \to 1} \dfrac{g(y) - g(1)}{y - 1}$. Indeed, the difference quotient satisfies $\dfrac{1}{y} = \dfrac{1 - \frac{1}{y}}{y - 1} \leqslant \dfrac{g(y) - g(1)}{y - 1} \leqslant \dfrac{y - 1}{y - 1} = 1$ for $y > 1$ and with the inequalities reversed for $0 < y < 1$. Making $y \to 1$ yields $g'(1) = 1$. Taking the y-partial derivative at $y = 1$ of the functional equation $g(xy) = g(x) + g(y)$, $x, y > 0$, we get $g'(x)x = g'(1) = 1$, i.e. $g'(x) = \frac{1}{x}$ for $x > 0$.

The reader may enjoy imagining how the graph of $\log y$ is approached by the lower bounded function $A_n(y) = n\left(y^{1/n} - 1\right)$ from above, and by the upper bounded function $B_n(y) = n\left(1 - y^{-1/n}\right)$ from below as $n \to \infty$.

We now proceed to check directly that $f\left(g(y)\right) = y$ for $y > 0$ and $g\left(f(x)\right) = x$ for x real, i.e. that f and g are inverse functions of each other.

Proposition 12:

(a) If $r > 0$, $|u|, |v| \leqslant r$ in $\mathbb{R}$, and $n \geqslant 1$ is an integer, then $|u^n - v^n| \leqslant nr^{n-1}|u - v|$.

(b) If $y > 0$, then (i) $\left(2 - y^{-1/n}\right)^n \to y$ and (ii) $\left(2 - y^{1/n}\right)^{-n} \to y$ as $n \to \infty$ through powers of 2.

(c) If $y > 0$, then $f\left(g(y)\right) = y$.

(d) If x is real, then $g\left(f(x)\right) = x$.

Proof:

(a) We have $u^n - v^n = (u - v)\sum_{i=0}^{n-1} u^i v^{n-1-i}$. Applying the triangle inequality and replacing $|u|, |v|$ under the summation sign by r we get $|u^n - v^n| = |u - v|\sum_{i=0}^{n-1} r^{n-1} = nr^{n-1}|u - v|$. Note that if $r \geqslant 1$, then $|u^n - v^n| \leqslant nr^{n-1}|u - v| \leqslant nr^n|u - v|$.

(b) As we get (i) if we take the reciprocal of (ii) applied to $\frac{1}{y}$, it is enough to prove (ii). To that end, write $\left|y^{1/n} - 1\right| < \frac{M}{n}$ for some constant power of 2, $M = M(y)$, and all $n \geqslant 1$ where n is a power of 2. Rewrite the difference $D = \left(2 - y^{1/n}\right)^{-n} - y$ as $D = \left(\frac{1}{2 - y^{1/n}}\right)^n - \left(y^{1/n}\right)^n = u^n - v^n$, where $u = \frac{1}{2 - y^{1/n}}$, $v = y^{1/n}$. We give upper bounds for $|u|$ and $|v|$ as $2 - y^{1/n} \geqslant 2 - \left(1 + \frac{M}{n}\right) = 1 - \frac{M}{n}$, which is positive if n is large enough, $|u| = \frac{1}{2 - y^{1/n}} \leqslant \frac{1}{1 - \frac{M}{n}}$, and $|v| = y^{1/n} \leqslant 1 + \frac{M}{n}$. Let $r = \frac{1}{1 - \frac{M}{n}}$, and note that $r > 1 + \frac{M}{n}$, $|u| \leqslant r$, and $|v| \leqslant r$. Hence $|D| \leqslant \left|\frac{1}{2 - y^{1/n}} - y^{1/n}\right| \cdot nr^n$. Note that $r^n = b_n(M) \leqslant b_{2M}(M) = 4^M$ by Proposition 2(c) if $n \geqslant 2M$ is a power of 2, and

$$\frac{1}{2 - y^{1/n}} - y^{1/n} = \frac{1 - 2y^{1/n} + y^{2/n}}{2 - y^{1/n}} = \frac{\left(y^{1/n} - 1\right)^2}{2 - y^{1/n}} \leqslant \frac{M^2/n^2}{1 - \frac{M}{n}}.$$

So $|D| \leqslant \frac{M^2/n^2}{1 - \frac{M}{n}} \cdot n \cdot 4^M \to 0$ as $n \to \infty$ through powers of 2.

(c) We know that $n\left(1 - y^{-1/n}\right) \leqslant g(y) \leqslant n\left(y^{1/n} - 1\right)$ if $y > 0$ and $n \geqslant 1$ is a power of 2. As f is an increasing function, being the limit of increasing functions, we have $f\left(n\left(1 - y^{-1/n}\right)\right) \leqslant f\left(g(y)\right) \leqslant f\left(n\left(y^{1/n} - 1\right)\right)$. We also know that $\left(1 + \frac{x}{n}\right)^n \leqslant f(x) \leqslant \left(1 - \frac{x}{n}\right)^{-n}$ if x is real, and $n \geqslant 1$ is a power of 2 so large that $|x|/n < 1$. Applying this to $x = n\left(1 - y^{-1/n}\right)$ and $x = n\left(y^{1/n} - 1\right)$, as both choices of x have $x/n \to 0$ if $\to \infty$ through powers of 2, we obtain

$$\left(1 + \frac{n\left(1 - y^{-1/n}\right)}{n}\right)^n \leqslant f\left(g(y)\right) \leqslant \left(1 - \frac{n\left(y^{1/n} - 1\right)}{n}\right)^{-n},$$

i.e. $\left(2 - y^{-1/n}\right)^n \leqslant f\left(g(y)\right) \leqslant \left(2 - y^{1/n}\right)^{-n}$, the limit of which by (b) is $y \leqslant f\left(g(y)\right) \leqslant y$ as $n \to \infty$ through powers of 2, i.e. $f\left(g(y)\right) = y$ for $y > 0$.

(d) Suppose that n is a power of 2 so large that $|x|/n < 1$. Then we can apply the increasing functions $A_n(y) \geqslant g(y) \geqslant B_n(y)$ to the inequality $\left(1 + \frac{x}{n}\right)^n \leqslant f(x) \leqslant \left(1 - \frac{x}{n}\right)^{-n}$ as

$$B_n\left(\left(1 + \frac{x}{n}\right)^n\right) \leqslant B_n\left(f(x)\right) \leqslant g\left(f(x)\right) \leqslant A_n\left(f(x)\right) \leqslant A_n\left(\left(1 - \frac{x}{n}\right)^{-n}\right).$$

Thus $\quad n\left(1 - \frac{1}{1 + \frac{x}{n}}\right) \leqslant g\left(f(x)\right) \leqslant n\left(\frac{1}{1 - \frac{x}{n}} - 1\right)$, or $\quad \frac{x}{1 + \frac{x}{n}} \leqslant g\left(f(x)\right) \leqslant \frac{x}{1 - \frac{x}{n}}$.
Making $n \to \infty$ through powers of 2 yields $x \leqslant g\left(f(x)\right) \leqslant x$, i.e. $g\left(f(x)\right) = x$ for x real.

Logarithms were introduced in the early 1600s by Napier and Briggs (and Kepler and Bürgi) essentially using the iterated square roots as in A_n and B_n above. The exponential function (or antilogarithm) is probably as old as well. Its theory and notation go back at least to Euler, who also gave us the limit of a_n and b_n (for general $n \to \infty$ not just powers of 2). Square roots go back millennia at least to the ancient Babylonians using the method of Proposition 7. The tangent line inequality $(1 + x)^n \geqslant 1 + nx$ for $x > -1$ and $n \geqslant 1$ is called Bernoulli's inequality and is centuries old, and so is its application to studying the power, exponential, and logarithmic functions.

While most estimates in this paper are of sufficient accuracy, it is possible (and normal) to approximate more closely in Proposition 7. It is easy to check that the Newton iteration $r_0 = x$, $r_{n+1} = \frac{1}{2}\left(r_n + \frac{x}{r_n}\right)$ for $n \geqslant 0$ for a real or complex number x with $\text{Re}\, x > 0$ is conjugate to the iterated squaring map $\rho_0 = \frac{\sqrt{x} - 1}{\sqrt{x} + 1}$, $\rho_{n+1} = \rho_n^2$ for $n \geqslant 0$, i.e.

$$\rho_n = \rho_0^{2^n} = \left(\frac{\sqrt{x} - 1}{\sqrt{x} + 1}\right)^{2^n}$$

for $n \geqslant 0$ under the Möbius transformation $\rho = \frac{r - \sqrt{x}}{r + \sqrt{x}}$ that maps one square root $r = \sqrt{x}$ of x (with $\operatorname{Re}\sqrt{x} > 0$) to $\rho = 0$ and the other square root $r = -\sqrt{x}$ (with negative real part) to $\rho = \infty$, whose inverse transformation is $r = \sqrt{x}\,\frac{1+\rho}{1-\rho}$. Hence $r_n = \sqrt{x}\,\dfrac{1 + \rho_n}{1 - \rho_n} = \sqrt{x}\,\dfrac{1 + \rho_0^{2^n}}{1 - \rho_0^{2^n}}$. If $x > 0$, then $-1 < \rho_0 < 1$, and so $\rho_0^{2^n} \in [0, 1)$ for $n > 0$, and $0 \leqslant r_n - \sqrt{x} \leqslant 2\sqrt{x}\rho_0^{2^n} \to 0$ as $n \to \infty$. The reader may enjoy imagining that the complex plane of r is subdivided by the two square roots $\pm\sqrt{x}$ of $x \in \mathbb{C} - \{0\}$ into the perpendicular bisector of the two points $\pm\sqrt{x}$ and the two half planes that this bisector creates. If the initial value r_0 falls into the half plane of one square root $\sqrt{x}$, then the iterates stay in that half plane and converge to that value of $\sqrt{x}$. If the initial value r_0 falls onto the bisector, then it stays there and it usually undergoes a chaotic motion on the line. The bisector $|r - \sqrt{x}| = |r + \sqrt{x}|$ maps to the unit circle $|\rho| = 1$ and the iteration for an initial value r_0 on the bisector undergoes the doubling map of the circle. The half plane $|r - \sqrt{x}| < |r + \sqrt{x}|$ maps to the interior disc $|\rho| < 1$ and the iteration undergoes repeated squaring and falls quickly into the origin $\rho = 0$ which corresponds to convergence to $r = \sqrt{x}$ in the r half plane. The other half plane $|r - \sqrt{x}| > |r + \sqrt{x}|$ maps to the exterior disc $|\rho| > 1$ and the iteration undergoes repeated squaring and escapes quickly to $\rho = \infty$, which corresponds to convergence to $r = -\sqrt{x}$ in the r half plane.

The method of functional equations and functional inequalities goes back at least to Cauchy, who also wrote on doing analysis by algebra, calling it algebraic analysis; it has relevance even today. The method of doubling and halving is also standard and has been used many times over the centuries and is even used nowadays. The idea that it is easier to double than augment in the limits that we gave for the exponential and logarithm functions is also ancient. In the case of the logarithm it goes back to Napier and Briggs, and in a more modern form to the paper [1] of Hurwitz, which treats from the point of view of the iteration of one-variable holomorphic functions the extraction of square roots, finding logarithms and exponential values among other elementary functions. I read this paper of Hurwitz some 15 to 20 years ago. In the case of the exponential function, the doubling idea goes back at least to Hurwitz [1], Huntington [2], and Dunkel [3]. Neither of [2, 3] treats both functions e^x and $\log x$ and their inverse relations in detail, but treats one or the other and defines the other as its inverse function. A motivation for an elementary and minimalistic treatment of the first few transcendental functions as early as possible is that we can then use these functions in the teaching of calculus, avoiding some of the pitfalls of 'functionless calculus' in which the only examples to differentiate and integrate and otherwise study are rational functions and their algebraic functions.

The fact that none of the above development seems to have yet appeared in a textbook is responsible in part for its repeated rediscovery in one form or another, e.g. the case of the exponential function in [4] by Kemény, who credits Emil Artin with the idea of the doubling definition of the exponential function; [4] has no bibliographical references. In the era of online searching the reader may easily produce a hundred other references as well, especially to developments via calculus or differential equations. Were it possible meaningfully to machine search Euler's copious Latin, the author would not be surprised by a reference to the same ideas there as well.

None of the sources that I found give the entire treatment in one place and with as little background as above. None of the treatments of Hurwitz, Huntington, Dunkel, and Kemény seems to have trickled down to the mass-market textbooks for high schoolers and undergraduates in over a century. Some of the readers could check back a century from today to see if any of it will finally have found its way into a such a textbook. Do not hold your breath.

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