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Heat equation with singular potential and singular data

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Nets of Schrödinger C_0-semigroups $(S_\varepsilon)_\varepsilon$ with the polynomial growth with respect to ε are used for solving the Cauchy problem $(\partial_t - \Delta)U + VU = f(t, U)$, $U(0, x) = U_0(x)$ in a suitable generalized function algebra (or space), where V and U_0 are singular generalized functions while f satisfies a Lipschitz-type condition. The existence of distribution solutions is proved in appropriate cases by the means of white noise calculus as well as classical energy estimates.

1. Introduction

Nets of C_0-semigroups depending on a small parameter ε with the polynomial growth rate with respect to ε are used in solving a class of heat equations with singular coefficients and data. Singular coefficients (viewed as generalized functions) of a partial differential equation (PDE) are regularized to become nets of smooth functions depending on $\varepsilon \in (0, 1)$. A PDE regularized in such a way is then solved by means of an appropriate net of semigroups. The net of solutions obtained in this way represents a generalized function. Moreover, *a priori* bounds imply that a net of solutions contains a convergent sequence leading to a distributional solution of a linear or semilinear equation with singular data and/or potential.

The framework for our analysis constitutes various Colombeau-type generalized function spaces and algebras [9]. They contain embedded distributions and, with the notion of association (instead of the strict equality), naturally extend notions of the weak limit and the equality in distribution theory. We also refer to [4, 10, 20, 22] for the properties of Colombeau-type algebras (and distributions embedded therein) and their use in a theory of PDEs. A global theory of generalized functions can be found in [13, 14]. Weak solutions and their local symmetries are the subject of [12]. For the classical theory of parabolic PDEs we refer the reader to [18].

In § 2 (and the appendix) we introduce generalized semigroups which map algebras of generalized functions into themselves. Such ideas appear in [6]. Appropriate equivalence relations of nets of C_0-semigroups $(S_\varepsilon)_\varepsilon$ and nets of infinitesimal generators $(A_\varepsilon)_\varepsilon$ are introduced and it is shown that they uniquely determine each other.

Generalized function spaces and algebras and generalized semigroups allows us to work in a framework where other types of equations and singularities can also be studied.

The solution concept for the equation $\partial_t G = AG$ is defined through the existence of the limit

$$\varepsilon^{-a} \sup_{t\in[0,T)} \|\partial_t G_\varepsilon - A_\varepsilon G_\varepsilon\|_{L^2} \to 0, \quad \varepsilon \to 0,$$

for every $a > 0$, where $(A_\varepsilon)_\varepsilon$ is an appropriate net of operators representing an infinitesimal generator A. Another approach, with uniformly continuous semigroups, is made in [19] and it is closely related to the theory of regularized derivatives [23].

Note that the analysis of families of semigroups and corresponding families of resolvents and infinitesimal generators dates back to Trotter [29] and has been used since by many authors. Here our global theory of generalized semigroups is interpreted through the nets of semigroups related to Schrödinger operators $\Delta - V_\varepsilon$, $\varepsilon \in (0,1)$. Such nets determine the generalized C_0-semigroups used in solving linear and semilinear parabolic equations $\partial_t U - (\Delta - V)U = f$, $U|_{t=0} = U_0$, with a singular potential V and singular initial data U_0.

Concerning semilinear parabolic equations with singularities and potential $V = 0$, [8] gave the stimulus for many papers in this direction. Let us now mention the papers of Kato [15], Kato and Ponce [16], Kozono and Yamazaki [17] and Biagioni *et al*. [7]. In general, in these papers, conditions on the growth order of a nonlinear term (for example, $f(u) = u|u|^p$, $V \equiv 0$) and the order of singularity of the initial data lead to a unique global solvability in an appropriate Kato-type space. For instance, in [7], the Cauchy problem for a semilinear parabolic equation $\partial_t u - \Delta u + g(u) = 0$, $t > 0$, $x \in \mathbb{R}^n$, is considered, where $g(u)$ is a locally Lipschitz real-valued function and the initial data are singular, i.e. they belong to the strong dual of the Banach space $C_b^k(\mathbb{R}^n) \subset C^k(\mathbb{R}^n)$ of functions with bounded derivatives up to order k. This problem in the case when $n = 1$, $g(u) = u^3$ is solved in [11] by use of an algebra of generalized functions, and the classical solutions are recovered when the initial data are L^p-functions.

In the present paper some classes of linear and semilinear Cauchy problems are solved in the generalized function space $\mathcal{G}_{C^1,W^2}([0,T) : \mathbb{R}^n)$, $T > 0$ (for $n \leqslant 3$ it is an algebra), which is a generalization of the space $C^1([0,T) : W^2(\mathbb{R}^n))$. The initial data U_0 and potential V are taken to be elements of generalized function spaces $\mathcal{G}_{W^2}(\mathbb{R}^n)$ and $\mathcal{G}_{W^1}(\mathbb{R}^n)$, generalizations of Sobolev spaces $W^2(\mathbb{R}^n)$ and $W^1(\mathbb{R}^n)$, respectively. The use of nets in Sobolev spaces admits singular initial data as embedded singular distributions. They can have the form

$$U_0 = \sum_{i=0}^{p} f_i^{(i)}, \quad f_i \in L^2,\ i = 0,1,\dots,p,$$

or

$$\sum_{i=0}^{p}\sum_{j=1}^{s} \delta^{(i)}(\cdot - x_j).$$

Theorem 3.2, a special case of theorem 3.10, gives a unique solution to

$$(\partial_t - \Delta)U + VU = 0, \quad U(0,x) = U_0(x),\ x \in \mathbb{R}^n, \tag{1.1}$$

where the potential V and initial data U_0 are singular distributions, for example, the delta distribution or its powers. In the cases $V = \delta^m$, $V = \delta^\alpha \in \mathcal{G}_{W^{1,\infty}}(\mathbb{R}^n)$, $m \in \mathbb{N}$,

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$\alpha \in (0,1)$ (for $m \neq 1$ and $\alpha \in (0,1)$ these generalized functions are not embedded Schwartz distributions), equation (1.1), with $U_0 \in \mathcal{G}_{W^2}(\mathbb{R}^n)$, has a unique solution $U(t,x) \in \mathcal{G}_{C^1,W^2}([0,T) \times \mathbb{R}^n)$. If $n > 1$, $V = \delta^m$ and $(U_\varepsilon)_\varepsilon$ is a representative of the solution to equation (1.1), $(U_\varepsilon)_\varepsilon$ has a subsequence $(U_{\varepsilon_\nu}(t,x))_{\nu\in\mathbb{N}}$ converging in $L^2([0,T) \times \mathbb{R}^n)$ weakly to $\tilde{U}(t,x) = \mathrm{e}^{-\Delta t}U_0(x)$, the solution to equation (1.1) with $V = 0$. This is the first of our two main theorems, theorem 3.3. Its proof is closely related to notions of Brownian motion, hitting times and polars [25]. Since a point in $\mathbb{R}$ is not a polar set, in the case $n = 1$ and $V = \delta^\alpha$, $\alpha \in (0,1)$, we have the same result as in theorem 3.3 but proved in a different way (proposition 3.5). In the case where $V = \delta \in \mathcal{G}_{W^{1,\infty}}(\mathbb{R}^1)$ and $U_0 \in L^2(\mathbb{R}^n)$, there exists a $L^2([0,T) \times \mathbb{R}^n)$-convergent subsequence $(U_{\varepsilon_\nu}(t,x))_{\nu\in\mathbb{N}}$, but we do not find that it is a distributional solution to (1.1) with $V = 0$ (proposition 3.6).

We refer to [2,3] and the references in [2] for singular perturbations of the operator $-\Delta$, for example $\Delta U(x) - \alpha U(0)\delta(x)$, and the corresponding approximating procedure with a net $(\Delta - \langle\delta_\varepsilon, \cdot\rangle\delta_\varepsilon)_\varepsilon$ leading to Schrödinger-type equations. Here we underline the difference between this procedure and that where we use the net of operators $(\Delta - \delta_\varepsilon)_\varepsilon$.

Let $f(t,u) \in C^1([0,t) \times \mathbb{R})$. The semi-linear Cauchy problem

$$(\partial_t - \Delta)U + VU = f(t,U), \qquad U(0,x) = U_0(x),$$

where $V \in \mathcal{G}_{W^{1,\infty}}$, $U_0 \in \mathcal{G}_{W^2}$ and f is of global Lipschitz class, is considered in the last section. In our second main theorem, theorem 3.10, the existence and the uniqueness of a generalized function solution is proved. With $U_0 = \delta$ (V and f satisfy the quoted assumptions) in example 3.11 (see also example 3.8), taking L^1 spaces instead of L^2 spaces and using the appropriate compactness arguments, we obtain the existence of a net of solutions involving a distributional solution.

2. Spaces and algebras of generalized functions

2.1. Generalized function algebra

Denote by Ω an open set of $\mathbb{R}^n$ and by $W^{r,s}(\Omega)$, $r \in \mathbb{N}_0$, $1 \leqslant s \leqslant \infty$, the Sobolev space of functions with all distributional derivatives of order $|\alpha| \leqslant r$ in $L^s(\Omega)$, equipped with the usual norm [1]. We simply write $W^r(\Omega)$ if $s = 2$. We refer to [5] and [21] for the algebras $\mathcal{G}_{L^p,L^q}$ and $\mathcal{G}_{L^p}$.

Notation $f_\varepsilon = \mathcal{O}(\varepsilon^a)$, $f : (0,1) \to \mathbb{C}$, means that $|f_\varepsilon| \leqslant C\varepsilon^a$, $0 < \varepsilon < \varepsilon_0$, for some constants $C > 0$ and $\varepsilon_0 \in (0,1)$. In that case, we say that $(f_\varepsilon)_\varepsilon$ has a moderate bound, or it is of moderate growth, or simply moderate. A net of functions $(g_\varepsilon)_\varepsilon$ in some Banach space $(B, \|\cdot\|_B)$ is called moderate (of moderate growth) if the above estimate holds for $(\|g_\varepsilon\|_B)_\varepsilon$.

Definition 2.1. $\mathcal{E}_{C^1,W^2}([0,T) : \mathbb{R}^n)$ (respectively, $\mathcal{N}_{C^1,W^2}([0,T) : \mathbb{R}^n)$, $T > 0$, is the vector space of nets $(G_\varepsilon)_\varepsilon$,

$$G_\varepsilon \in C^0([0,T) : W^2(\mathbb{R}^n)) \cap C^1((0,T) : L^2(\mathbb{R}^n)), \quad \varepsilon \in (0,1),$$

with the property that there exists $a \in \mathbb{R}$ (respectively, for every $a \in \mathbb{R}$) such that

$$\max\Big\{\sup_{t\in[0,T)} \|G_\varepsilon(t)\|_{W^2}, \sup_{t\in(0,T)} \|\partial_t G_\varepsilon(t)\|_{L^2}\Big\} = \mathcal{O}(\varepsilon^a), \quad \varepsilon \to 0. \tag{2.1}$$

Quotient space

$$\mathcal{G}_{C^1,W^2}([0,T):\mathbb{R}^n)=\mathcal{E}_{C^1,W^2}([0,T):\mathbb{R}^n)/\mathcal{N}_{C^1,W^2}([0,T):\mathbb{R}^n)$$

is a Colombeau-type vector space.

Dropping the condition on $\partial_t G_\varepsilon$ in (2.1), one obtains the spaces $\mathcal{E}_{C^0,W^2}([0,T):\mathbb{R}^n)$, $\mathcal{N}_{C^0,W^2}([0,T):\mathbb{R}^n)$ and $\mathcal{G}_{C^0,W^2}([0,T):\mathbb{R}^n)$.

The following lemma is a consequence of Sobolev-type inequalities, i.e. of the fact that, for only $n\leqslant 3$, the inclusion mapping from $W^2(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ is continuous.

LEMMA 2.2. *If $n\leqslant 3$, then $\mathcal{E}_{C^1,W^2}([0,T):\mathbb{R}^n)$ is an algebra with respect to the pointwise multiplication and $\mathcal{N}_{C^1,W^2}([0,T):\mathbb{R}^n)$ is an ideal of $\mathcal{E}_{C^1,W^2}([0,T):\mathbb{R}^n)$. In particular, $\mathcal{G}_{C^1,W^2}([0,T):\mathbb{R}^n)$ is an algebra. The same assertion holds for the spaces*

$$\mathcal{E}_{C^0,W^2}([0,T):\mathbb{R}^n),\qquad \mathcal{N}_{C^0,W^2}([0,T):\mathbb{R}^n)\quad\textit{and}\quad \mathcal{G}_{C^0,W^2}([0,T):\mathbb{R}^n).$$

Substituting the W^2-norm with the L^2-norm in (2.1) we obtain the spaces

$$\mathcal{E}_{C^0,L^2}([0,T):\mathbb{R}^n),\quad \mathcal{N}_{C^0,L^2}([0,T):\mathbb{R}^n)\quad\text{and}\quad \mathcal{G}_{C^0,L^2}([0,T):\mathbb{R}^n).$$

The canonical embedding $\iota_{L^2}:\mathcal{G}_{C^1,W^2}([0,T):\mathbb{R}^n)\to\mathcal{G}_{C^0,L^2}([0,T):\mathbb{R}^n)$ is defined by $\iota_{L^2}((G_\varepsilon)_\varepsilon+\mathcal{N}_{C^0,W^2}([0,T):\mathbb{R}^n))=(G_\varepsilon)_\varepsilon+\mathcal{N}_{C^0,L^2}([0,T):\mathbb{R}^n)$.

Now we define the space $\mathcal{G}_{W^{r,s}}(\mathbb{R}^n)$, $r\in\mathbb{N}_0$, $1\leqslant s\leqslant\infty$. $\mathcal{E}_{W^{r,s}}(\mathbb{R}^n)$ (respectively, $\mathcal{N}_{W^{r,s}}(\mathbb{R}^n)$) as the space of nets $(G_\varepsilon)_\varepsilon$ of functions $G_\varepsilon\in W^{r,s}(\mathbb{R}^n)$, $\varepsilon\in(0,1)$, with the property that there exists $a\in\mathbb{R}$ (respectively, for every $a\in\mathbb{R}$) such that

$$\|G_\varepsilon\|_{W^{r,s}(\mathbb{R}^n)}=\mathcal{O}(\varepsilon^a),\quad \varepsilon\to 0.$$

Both spaces are vector spaces and $\mathcal{N}_{W^{r,s}}(\mathbb{R}^n)$ is a subspace of $\mathcal{E}_{W^{r,s}}$. Thus, in this case, the Colombeau-type vector space is defined by

$$\mathcal{G}_{W^{r,s}}(\mathbb{R}^n)=\mathcal{E}_{W^{r,s}}(\mathbb{R}^n)/\mathcal{N}_{W^{r,s}}(\mathbb{R}^n).$$

We shall use the fact that $\mathcal{G}_{W^2}(\mathbb{R}^n):=\mathcal{G}_{W^{2,2}}(\mathbb{R}^n)$ is a multiplicative algebra if $n\leqslant 3$.

DEFINITION 2.3. An element $V\in\mathcal{G}_{W^{r,s}}(\mathbb{R}^n)$ is of logarithmic type if it has a representative $(V_\varepsilon)_\varepsilon\in\mathcal{E}_{W^{r,s}}(\mathbb{R}^n)$ with the property

$$\|V_\varepsilon\|_{W^{r,s}(\mathbb{R}^n)}=\mathcal{O}(\log\varepsilon^{-1}),\quad \varepsilon\to 0.$$

2.2. Generalized semigroups

Proofs of assertions in this subsection are given in the appendix. Various technical conditions in definitions 2.4 and 2.8 are related to the expected assertion of theorem 2.9: 'different generalized infinitesimal generators define different generalized C_0-semigroups'.

Let $(E,\|\cdot\|)$ be a Banach space and let $\mathcal{L}(E)$ be the space of all linear continuous mappings $E\to E$.

DEFINITION 2.4. $\mathcal{S}E_M([0,\infty):\mathcal{L}(E))$ is the space of nets $(S_\varepsilon)_\varepsilon$ of strongly continuous mappings $S_\varepsilon:[0,\infty)\to\mathcal{L}(E)$, $\varepsilon\in(0,1)$ with the property that, for every $T>0$, there exists $a\in\mathbb{R}$ such that

$$\sup_{t\in[0,T)}\|S_\varepsilon(t)\|=\mathcal{O}(\varepsilon^a),\quad \varepsilon\to 0. \tag{2.2}$$

$\mathcal{S}N([0,\infty):\mathcal{L}(E))\subset\mathcal{S}E_M([0,\infty):\mathcal{L}(E))$ is a space with the following properties.

(i)
$$\sup_{t\in[0,T)}\|N_\varepsilon(t)\|=\mathcal{O}(\varepsilon^b),\quad \varepsilon\to 0, \tag{2.3}$$
for every $b\in\mathbb{R}$ and $T>0$.

(ii) There exist $t_0>0$ and $a\in\mathbb{R}$ such that
$$\sup_{t<t_0}\left\|\frac{N_\varepsilon(t)}{t}\right\|=\mathcal{O}(\varepsilon^a),\quad \varepsilon\to 0. \tag{2.4}$$

(iii) There exists a net $(W_\varepsilon)_\varepsilon$ in $\mathcal{L}(E)$ and $\varepsilon_0\in(0,1)$ such that
$$\lim_{t\to 0}\frac{N_\varepsilon(t)}{t}x=W_\varepsilon x,\quad x\in E,\ \varepsilon<\varepsilon_0. \tag{2.5}$$

(iv)
$$\|W_\varepsilon\|=\mathcal{O}(\varepsilon^b),\quad \varepsilon\to 0,\ \text{for every } b>0. \tag{2.6}$$

PROPOSITION 2.5. *$\mathcal{S}E_M([0,\infty):\mathcal{L}(E))$ is an algebra with respect to the composition and $\mathcal{S}N([0,\infty):\mathcal{L}(E))$ is an ideal of $\mathcal{S}E_M([0,\infty):\mathcal{L}(E))$.*

We define a Colombeau-type algebra as the factor algebra

$$\mathcal{S}G([0,\infty):\mathcal{L}(E))=\mathcal{S}E_M([0,\infty):\mathcal{L}(E))/\mathcal{S}\mathcal{N}([0,\infty):\mathcal{L}(E)).$$

Elements of $\mathcal{S}G([0,\infty):\mathcal{L}(E))$ will be denoted by $S=[S_\varepsilon]$, where $(S_\varepsilon)_\varepsilon$ is a representative of the class.

DEFINITION 2.6. $S\in\mathcal{S}G([0,\infty):\mathcal{L}(E))$ is called a generalized C_0-semigroup if it has a representative $(S_\varepsilon)_\varepsilon$ such that, for some $\varepsilon_0>0$, S_ε is a C_0-semigroup for every $\varepsilon<\varepsilon_0$.

In the following we will use only representatives $(S_\varepsilon)_\varepsilon$ of a generalized C_0-semigroup, S_ε being a C_0-semigroup itself for ε small enough.

PROPOSITION 2.7. *Let $(S_\varepsilon)_\varepsilon$ and $(\tilde S_\varepsilon)_\varepsilon$ be representatives of a generalized C_0-semigroup S, with infinitesimal generators A_ε, $\varepsilon<\varepsilon_0$, and $\tilde A_\varepsilon$, $\varepsilon<\tilde\varepsilon_0$, respectively, where ε_0 and $\tilde\varepsilon_0$ correspond (in the sense of definition 2.6) to $(S_\varepsilon)_\varepsilon$ and $(\tilde S_\varepsilon)_\varepsilon$, respectively.*

Then $D(A_\varepsilon)=D(\tilde A_\varepsilon)$, for every $\varepsilon<\bar\varepsilon_0=\min\{\varepsilon_0,\tilde\varepsilon_0\}$ and $A_\varepsilon-\tilde A_\varepsilon$ can be extended to an element of $\mathcal{L}(E)$, again denoted by $A_\varepsilon-\tilde A_\varepsilon$.

Moreover, $\|A_\varepsilon-\tilde A_\varepsilon\|=\mathcal{O}(\varepsilon^a)$, $\varepsilon\to 0$, for every $a\in\mathbb{R}$.

Now we define infinitesimal generators of generalized C_0-semigroups. Denote by $\mathcal{A}$ the set of pairs $((A_\varepsilon)_\varepsilon, (D(A_\varepsilon))_\varepsilon)$, where A_ε is a closed linear operator on E with a dense domain $D(A_\varepsilon) \subset E$, for every $\varepsilon \in (0,1)$. We introduce an equivalence relation in $\mathcal{A}$:

$$((A_\varepsilon)_\varepsilon, (D(A_\varepsilon))_\varepsilon) \sim ((\tilde{A}_\varepsilon)_\varepsilon, (D(\tilde{A}_\varepsilon))_\varepsilon)$$

if there exists $\varepsilon_0 \in (0,1)$ such that $D(A_\varepsilon) = D(\tilde{A}_\varepsilon)$, for every $\varepsilon < \varepsilon_0$, and for every $a \in \mathbb{R}$ there exist $C_a > 0$ and $\varepsilon_a \leqslant \varepsilon_0$ such that

$$\|(A_\varepsilon - \tilde{A}_\varepsilon)x\| \leqslant C_a \varepsilon^a \|x\|, \quad x \in D(A_\varepsilon),\ \varepsilon \leqslant \varepsilon_a. \tag{2.7}$$

Since A_ε has a dense domain in E, $R_\varepsilon := A_\varepsilon - \tilde{A}_\varepsilon$ can be extended to an operator in $\mathcal{L}(E)$, denoted again by R_ε, satisfying $\|R_\varepsilon\| = \mathcal{O}(\varepsilon^a)$, $\varepsilon \to 0$, for every $a \in \mathbb{R}$. (Such a net of operators $(R_\varepsilon)_\varepsilon$ will be called a zero operator net.)

We denote by A the corresponding element of the quotient space $\mathcal{A}/\sim$. Due to proposition 2.7, the following definition makes sense.

Definition 2.8. $A \in \mathcal{A}/\sim$ is an infinitesimal generator of a generalized C_0-semigroup $S = [S_\varepsilon]$ if there exists a representative $(A_\varepsilon)_\varepsilon$ of A and $\varepsilon_0 \in (0,1)$ such that A_ε is the infinitesimal generator of S_ε, for every $\varepsilon < \varepsilon_0$.

Theorem 2.9. *Let S and $\tilde{S}$ be generalized C_0-semigroups with infinitesimal generators A and $\tilde{A}$, respectively. If $A = \tilde{A}$, then $S = \tilde{S}$.*

Remark 2.10. Let the assumptions of definition 2.4 hold. Moreover, assume a stronger assumption than (2.2), i.e. there exist $M > 0$, $a \in \mathbb{R}$ and $\varepsilon_0 \in (0,1)$ such that

$$\|S_\varepsilon(t)\| \leqslant M\varepsilon^a \mathrm{e}^{\alpha_\varepsilon t}, \quad \varepsilon < \varepsilon_0,\ t \geqslant 0,$$

where $0 < \alpha_\varepsilon < \alpha$, for some $\alpha > 0$. Then we obtain the corresponding subalgebra of $\mathcal{SG}([0,\infty) : \mathcal{L}(E))$. For this subalgebra we can formulate the Hille–Yosida theorem in the usual way.

For the whole algebra of generalized C_0-semigroups $\mathcal{SG}([0,\infty) : \mathcal{L}(E))$ a Hille–Yosida-type theorem is still an open problem.

In the sections which follow, we use only semigroups of Schrödinger-type operators.

Example 2.11 (semigroups of Schrödinger-type operators). Let $V \in \mathcal{G}_{W^{1,\infty}}(\mathbb{R}^n)$ be of logarithmic type. Then differential operators $A_\varepsilon u = (\Delta - V_\varepsilon)u$, $u \in W^2(\mathbb{R}^n)$, $\varepsilon < 1$, are infinitesimal generators of C_0-semigroups S_ε, $\varepsilon < 1$, and $(S_\varepsilon)_\varepsilon$ is a representative of a generalized C_0-semigroup $S \in \mathcal{SG}([0,\infty) : \mathcal{L}(L^2(\mathbb{R}^n)))$.

Let $\varepsilon < 1$. Operator A_ε is the infinitesimal generator of the corresponding C_0-semigroup $S_\varepsilon : [0,\infty) \to \mathcal{L}(L^2(\mathbb{R}^n))$ defined by the Feynman–Kac formula:

$$S_\varepsilon(t)\psi(x) = \int_\Omega \exp\biggl(-\int_0^t V_\varepsilon(\omega(s))\,\mathrm{d}s\biggr)\psi(\omega(t))\,\mathrm{d}\mu_x(\omega), \quad t \geqslant 0,\ x \in \mathbb{R}^n, \tag{2.8}$$

for $\psi \in L^2(\mathbb{R}^n)$, where $\Omega = \prod_{t\in[0,\infty)} \overline{\mathbb{R}^n}$ ($\overline{\mathbb{R}^n}$ is completion of $\mathbb{R}^n$) and μ_x is the Wiener measure concentrated at $x \in \mathbb{R}^n$ (see [26, 28]).

The assumption on V implies that there exists $C > 0$ such that

$$\begin{aligned}|S_\varepsilon(t)\psi(x)| &\leqslant \exp\Big(t \sup_{s\in\mathbb{R}^n} |V_\varepsilon(s)|\Big)\int_\Omega |\psi(\omega(t))|\,\mathrm{d}\mu_x(\omega)\\ &\leqslant \varepsilon^{Ct}(4\pi t)^{-n/2}\int_{\mathbb{R}^n}\exp\bigg(-\frac{|x-y|^2}{4t}\bigg)|\psi(y)|\,\mathrm{d}y\\ &= \varepsilon^{Ct}E_n(t,\cdot) * |\psi(y)|,\end{aligned}$$

for every $t > 0$, $x \in \mathbb{R}^n$ and $\varepsilon < 1$.

Recall that the heat kernel is given by

$$E_n(t,x) = \frac{1}{(4\pi t)^{n/2}}\exp\bigg(-\frac{x^2}{4t}\bigg), \quad t > 0,\ x \in \mathbb{R}^n,$$

and its $L^1(\mathbb{R}^n)$-norm equals 1 for every $t > 0$. By the Young inequality,

$$|S_\varepsilon(t)\psi| \leqslant \mathrm{e}^{Ct}\|E_n(t,\cdot)\|_{L^1(\mathbb{R}^n)}\|\psi\|_{L^2(\mathbb{R}^n)}, \quad t > 0,\ \varepsilon < 1.$$

Therefore, there exists $M > 0$ such that

$$\sup_{t\in[0,T)}\|S_\varepsilon(t)\psi\|_{L^2} \leqslant M\varepsilon^{CT}\|\psi\|_{L^2}, \quad \varepsilon < 1,$$

for every T, i.e. $(S_\varepsilon(t))_\varepsilon$, $t \in [0,\infty)$, satisfies relation (2.2) and $S = [S_\varepsilon] \in \mathcal{S}G([0,\infty) : \mathcal{L}(L^2(\mathbb{R}^n)))$.

In the following we will assume that $E = L^2$. For elements of $\mathcal{G}_{L^2}$, the action of a generalized C_0-semigroup $S = [S_\varepsilon]$ is defined by

$$SG = [S_\varepsilon(t)G_\varepsilon], \quad [G_\varepsilon] \in \mathcal{G}_{L^2}.$$

3. Heat equations with singular potentials and data

This section is devoted to solving a class of heat equations with singular potentials and singular data. First, let us note that the multiplication of elements $G \in \mathcal{G}_{W^{1,\infty}}(\mathbb{R}^n)$ and $W \in \mathcal{G}_{C^1,W^2}([0,T) : \mathbb{R}^n)$ gives an element in $\mathcal{G}_{C^1,W^1}([0,T) : \mathbb{R}^n)$: if $(G_\varepsilon)_\varepsilon \in \mathcal{E}_{W^{1,\infty}}(\mathbb{R}^n)$ and $(W_\varepsilon)_\varepsilon \in \mathcal{E}_{C^1,W^2}([0,T) : \mathbb{R}^n)$, then $(G_\varepsilon W_\varepsilon)_\varepsilon \in \mathcal{E}_{C^1,W^1}([0,T) : \mathbb{R}^n)$. Similarly, if $(G_\varepsilon)_\varepsilon \in \mathcal{N}_{W^{1,\infty}}(\mathbb{R}^n)$ or $(W_\varepsilon)_\varepsilon \in \mathcal{N}_{C^1,W^2}([0,T) : \mathbb{R}^n)$, then the product $(G_\varepsilon W_\varepsilon)_\varepsilon$ belongs to the ideal $\mathcal{N}_{C^1,W^1}([0,T) : \mathbb{R}^n)$. Thus the multiplication of $V \in \mathcal{G}_{W^{1,\infty}}(\mathbb{R}^n)$ and $U \in \mathcal{G}_{C^1,W^2}([0,T) : \mathbb{R}^n)$ in

$$\partial_t U(t,x) - \Delta U(t,x) + V(x)U(t,x) = 0, \quad U(0,x) = U_0(x). \tag{3.1}$$

makes sense.

Definition 3.1. Let A be represented by a net $(A_\varepsilon)_\varepsilon$ of operators $A_\varepsilon : W^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $\varepsilon < \varepsilon_0$. $G \in \mathcal{G}_{C^1,W^2}([0,T) : \mathbb{R}^n)$, $T > 0$, is said to be a solution to $\partial_t G = AG$ with initial data $G|_{t=0} = G_0 \in \mathcal{G}_{W^2(\mathbb{R}^n)}$ if

$$\sup_{t\in(0,T)}\|\partial_t G_\varepsilon(t,\cdot) - A_\varepsilon G_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^a), \quad \text{for every } a \in \mathbb{R}$$

and the restriction of G to the line $t = 0$ equals $G_0 \in \mathcal{G}_{W^2(\mathbb{R}^n)}$.

The solution is unique if for any other solution $\tilde{G} \in \mathcal{G}_{C^1,W^2}([0,T) : \mathbb{R}^n)$, with the same initial data

$$\iota_{L^2}(G) = \iota_{L^2}(\tilde{G}).$$

The next theorem is a special case of theorem 3.10.

THEOREM 3.2. *Let $V \in \mathcal{G}_{W^{1,\infty}}(\mathbb{R}^n)$ be of logarithmic type, $U_0 = [U_{0\varepsilon}] \in \mathcal{G}_{W^2}(\mathbb{R}^n)$ and $S = [S_\varepsilon]$ be defined as in example 2.11. Let $T > 0$.*

Then $U = SU_0 \in \mathcal{G}_{C^1,W^2}([0,T) : \mathbb{R}^n)$ given by the representative $U_\varepsilon(t,x) = S_\varepsilon(t)u_{0\varepsilon}(x)$, $t \geqslant 0$, $x \in \mathbb{R}^n$, $\varepsilon < 1$, is the unique solution to (3.1) in the sense of definition 3.1.

Note that, if in (3.1) V_ε is substituted by $V_\varepsilon + R_\varepsilon$, $(R_\varepsilon)_\varepsilon \in \mathcal{N}_{W^{1,\infty}}$, we have the same generalized solution.

3.1. Powers of the generalized delta function as a potential

Let $(\phi_\varepsilon)_\varepsilon$ be a net of mollifiers

$$\phi_\varepsilon = \varepsilon^{-n}\phi(\cdot/\varepsilon), \quad \varepsilon \in (0,1), \tag{3.2}$$

where $\phi \in C_0^\infty(\mathbb{R}^n)$, $\int \phi(x)\,\mathrm{d}x = 1$ and $\phi(x) \geqslant 0$, $x \in \mathbb{R}^n$. It represents the generalized delta function $\delta = [\phi_\varepsilon] \in \mathcal{G}(\mathbb{R}^n)$.

Put $A_\varepsilon = \Delta - \phi_\varepsilon$ and $\tilde{A}_\varepsilon = \Delta - \tilde{\phi}_\varepsilon$, $\varepsilon < 1$. Different ϕ_ε values (with the prescribed properties on ϕ above) define different infinitesimal generators. Let us show this. The equality of infinitesimal generators would imply that

$$\|(A_\varepsilon - \tilde{A}_\varepsilon)u\|_{L^2}^2 = \varepsilon^{-2n}\int_{\mathbb{R}^n} |\phi(y) - \tilde{\phi}(y)|^2 |u(\varepsilon y)|^2\,\mathrm{d}t \leqslant C_a\varepsilon^a\|u\|_{L^2}^2, \quad \varepsilon < 1$$

for every $a > 0$ (and corresponding $C_a > 0$). Thus, it follows that $\phi = \tilde{\phi}$.

Let $m \in \mathbb{N}$. We will use $\delta^m = [\phi_\varepsilon^m]_{m\in\mathbb{N}}$ as the definition of mth power of $\delta \in \mathcal{G}(\mathbb{R}^n)$. Let

$$A_{\varepsilon,m}u = (\Delta - \phi_\varepsilon^m)u, \quad u \in W^2(\mathbb{R}^n), \quad \varepsilon < 1.$$

$A_{\varepsilon,m}$ is the infinitesimal generator of the semigroup

$$S_{\varepsilon,m} : [0,\infty) \to \mathcal{L}(L^2(\mathbb{R}^n)), S_\varepsilon(t) = \exp((\Delta - \phi_\varepsilon^m)t), \quad t \geqslant 0 \text{ (cf. [11])}.$$

It follows that $(S_{\varepsilon,m})_\varepsilon$ is a representative of a generalized C_0-semigroup $S_m \in \mathcal{L}G([0,\infty) : \mathcal{L}(L^2(\mathbb{R}^n)))$.

Let $\varepsilon < 1$ and $\psi \in L^2(\mathbb{R}^n)$. We know that $S_{\varepsilon,m}\psi$ is given by

$$S_{\varepsilon,m}(t)\psi(x) = \int_\Omega \exp\biggl(-\int_0^t \phi_\varepsilon^m(\omega(s))\,\mathrm{d}s\biggr)\psi(\omega(t))\,\mathrm{d}\mu_x(\omega), \quad x \in \mathbb{R}^n,\ t \geqslant 0. \tag{3.3}$$

Since $\phi_\varepsilon(x) \geqslant 0$, $x \in \mathbb{R}^n$, $\varepsilon < 1$, it follows that the set $\{S_{\varepsilon,m} : \varepsilon \in (0,1), t \geqslant 0\}$ is bounded in $\mathcal{L}(L^2(\mathbb{R}^n))$ (not only moderately). Thus, (2.2) holds for $(S_{\varepsilon,m})_\varepsilon$.

Our goal is to prove the following theorem, where the assumption $n \geqslant 2$ is crucial.

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THEOREM 3.3. *Let $n \geqslant 2$, $m \in \mathbb{N}$, $T > 0$ and $U_0 \in W^2(\mathbb{R}^n)$. Then*

$$U_{\varepsilon,m}(t,x) = \int_{\Omega} \exp\left(-\int_0^t \varepsilon^{-mn}\phi^m\left(\frac{\omega(s)}{\varepsilon}\right) \mathrm{d}s\right) U_0(\omega(t))\,\mathrm{d}\mu_x(\omega),$$
$$t \geqslant 0, \quad x \in \mathbb{R}^n, \quad \varepsilon < 1, \tag{3.4}$$

defines a representative of a solution $U \in \mathcal{G}_{C^1,W^2}([0,T) : \mathbb{R}^n)$ to

$$\partial_t U(t,x) - \Delta U(t,x) + \delta^m(x)U(t,x) = 0, \quad U(0,x) = U_0(x). \tag{3.5}$$

The solution is unique in the sense of definition 3.1.

Moreover, for every $t > 0$, the net in (3.4) converges to

$$\tilde{U}(t,\cdot) = \mathrm{e}^{-\Delta t}U_0(\cdot) \quad \text{in } L^1(\Omega,\mu_x) \tag{3.6}$$

uniformly in $x \in K \subset\subset \mathbb{R}^n$ (this notation means that the closure of K is compact).

For the proof of theorem 3.3 we need several notions and properties of n-dimensional Brownian motions. Recall that the hitting time τ_A of a subset A of $\mathbb{R}^n$ is defined by $\tau_A = \inf\{t > 0 : \omega(t) \in A\}$ ($t_A = \infty$ if $\omega(t) \notin A$ for all $t > 0$). We refer to [25, ch. 1, § 2], for the elementary properties of hitting times. Recall, a Borel set A is said to be polar if

$$\mu_x(\{\omega \in \Omega : \omega(t) \in A \text{ for some } t < \infty\}) = 0.$$

We will use the fact that every one-point set is polar for $n \geqslant 2$. This is not true for $n = 1$ and that is the essential reason for different results in the cases $n \geqslant 2$ and $n = 1$.

Let $B_\varepsilon = \{x \in \mathbb{R}^n : \|x\| \leqslant \varepsilon\}$, $B = B_1$. Take $\varepsilon \in (0,1)$, $t > 0$ and define

$$W_{B_\varepsilon}(t) = \{\tau_{B_\varepsilon} < t\} = \{\omega : \omega(s) \in B_\varepsilon \text{ for some } s \in (0,t)\}, \quad W_{B_\varepsilon} = \bigcup_{t>0} W_{B_\varepsilon}(t).$$

Clearly, $W_{B_\varepsilon}(t) \subset W_{B_\varepsilon}(s)$, $0 < t \leqslant s$. Note that

$$W_{B_\varepsilon}(s) \setminus W_{B_\varepsilon}(t) = \{t \leqslant \tau_{B_\varepsilon} < s\}, \quad 0 < t < s$$

and

$$W_{B_\varepsilon}(s) \setminus W_{B_\varepsilon}(t) \subset W_{B_\varepsilon} \setminus W_{B_\varepsilon}(t) \subset \{t - 1 < \tau_{B_\varepsilon}\}, \quad s > t > 1.$$

LEMMA 3.4.

(1) *For every compact subset K of $\mathbb{R}^n$ and $\varepsilon < 1$ there exists $C_\varepsilon > 0$ such that $\mu_x(W_{B_\varepsilon}) \leqslant C_\varepsilon$.*

(2) $\lim_{\varepsilon\to 0} \sup_{x\in K} \mu_x(W_{B_\varepsilon}) = 0$.

Proof. (1) By proposition 2.2 in [25], for every compact subset K of $\mathbb{R}^n$ and $m \in \mathbb{N}$ there exists $t_m > 0$ such that

$$\mu_x(W_{B_\varepsilon} \setminus W_{B_\varepsilon}(t_m)) < 2^{-m}, \quad x \in K. \tag{3.7}$$

Choose an increasing sequence $(t_m)_m$ such that $t_{m+1} > t_m + 1$, $m \in \mathbb{N}$ and that (3.7) holds for every $m \in \mathbb{N}$.

We have

$$W_{B_\varepsilon} = W_{B_\varepsilon}(t_m) \cup \bigcup_{k=m}^{\infty} (W_{B_\varepsilon}(t_{k+1}) \setminus W_{B_\varepsilon}(t_k)),$$

and

$$\mu_x(W_{B_\varepsilon}) = \mu_x(W_{B_\varepsilon}(t_m)) + \sum_{k=m}^{\infty} \mu_x(W_{B_\varepsilon}(t_{k+1}) \setminus W_{B_\varepsilon}(t_k)), \quad x \in K.$$

Now the assertion in (1) follows by proposition 2.6 in [25] and by (3.7).

(2) Since the intersection of balls is the polar set $\{0\}$, the assertion in (2) follows by (1). □

Proof of theorem 3.3. Let $t > 0$, $\varepsilon < 1$. We will use the notation of lemma 3.4. Then the complement of W_{B_ε} is given by $CW_{B_\varepsilon} = \{\omega : |\omega(s)| \geqslant \varepsilon, 0 < s \leqslant t\}$. We have

$$\begin{aligned} U_{\varepsilon,m}(t,x) &= \int_\Omega \exp\biggl(-\int_0^t \varepsilon^{-mn}\phi^m\biggl(\frac{\omega(s)}{\varepsilon}\biggr)\,\mathrm{d}s\biggr) U_0(\omega(t))\,\mathrm{d}\mu_x(\omega) \\ &= I_{1\varepsilon}(t,x) + I_{2\varepsilon}(t,x), \end{aligned}$$

where

$$\begin{aligned} I_{1\varepsilon}(t,x) &= \int_{W_{B_\varepsilon}(t)} \exp\biggl(-\int_0^t \varepsilon^{-mn}\phi^m\biggl(\frac{\omega(s)}{\varepsilon}\biggr)\,\mathrm{d}s\biggr) U_0(\omega(t))\,\mathrm{d}\mu_x(\omega), \\ I_{2\varepsilon}(t,x) &= \int_{CW_{B_\varepsilon}(t)} \exp\biggl(-\int_0^t \varepsilon^{-mn}\phi^m\biggl(\frac{\omega(s)}{\varepsilon}\biggr)\,\mathrm{d}s\biggr) U_0(\omega(t))\,\mathrm{d}\mu_x(\omega). \end{aligned}$$

We have $I_{1\varepsilon} \to 0$ uniformly on compact sets $K \subset \mathbb{R}^n$ as $\varepsilon \to 0$, because

$$\mu_x(W_{B_\varepsilon}(t)) \to 0 \quad \text{as } \varepsilon \to 0$$

uniformly for $x \in K$. Since

$$\begin{aligned} &\int_{CW_{B_\varepsilon}(t)} \exp\biggl(-\int_0^t \varepsilon^{-mn}\phi^m\biggl(\frac{\omega(s)}{\varepsilon}\biggr)\,\mathrm{d}s\biggr) U_0(\omega(t))\,\mathrm{d}\mu_x(\omega) \\ &\qquad \leqslant \int_{CW_{B_\varepsilon}(t)} U_0(\omega(t))\,\mathrm{d}\mu_x(\omega), \quad x \in \mathbb{R}^n \ (t > 0 \text{ is fixed}) \end{aligned}$$

and $CW_{B_\varepsilon}(t) \supset CW_{B_\varepsilon} \to \Omega$, as $\varepsilon \to 0$, we have

$$U_{\varepsilon,m}(t,x) \to \int_\Omega U_0(\omega(t))\,\mathrm{d}\mu_x(\omega) = \tilde{U}(t,x) \quad \text{as } \varepsilon \to 0$$

in $L^1(\Omega, \mu_x)$ uniformly on compact sets $K \subset \mathbb{R}^n$. □

Powers of the generalized delta function, δ^α, $\alpha \in (0,1)$, are defined in this paper by

$$\delta^\alpha = [(\phi_\varepsilon)^\alpha * \phi_\varepsilon], \quad \varepsilon \in (0,1). \tag{3.8}$$

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The reason for introducing (3.8) is simple: When $\alpha \in (0,1)$, functions ϕ_ε^α, $\varepsilon < 1$ are not smooth, in general. Note that the generalized function $[\phi_\varepsilon * \phi_\varepsilon]$ is only associated with the generalized delta function $\delta = [\phi_\varepsilon]$, rather than being equal to it ($\int(\phi_\varepsilon * \phi_\varepsilon - \phi_\varepsilon)\psi \, \mathrm{d}t \to 0$ as $\varepsilon \to 0$, for every $\psi \in \mathcal{D}$).

Since one-point sets are not polar for $n = 1$, we cannot use the same arguments as in the case $n \geqslant 2$.

PROPOSITION 3.5. *Let $\alpha \in (0,1)$, $T > 0$ and $U_0 \in W^2(\mathbb{R})$. Then we define by*

$$U_\varepsilon(t,x) = \int_\Omega \exp\biggl(-\int_0^t (\phi_\varepsilon)^\alpha * \phi_\varepsilon(\omega(s)) \, \mathrm{d}s\biggr) U_0(\omega(t)) \, \mathrm{d}\mu_x(\omega), \qquad t > 0, \quad x \in \mathbb{R}, \quad \varepsilon < 1, \tag{3.9}$$

a representative of a solution $U(t,x) \in \mathcal{G}_{C^1,W^2}([0,T) \times \mathbb{R})$ to

$$\partial_t U(t,x) - \Delta U(t,x) + \delta^\alpha(x) U(t,x) = 0, \quad U(0,x) = U_0(x). \tag{3.10}$$

The solution is unique in the sense of definition 3.1.

Net (3.9) has a subsequence $(U_{\varepsilon_\nu,\alpha}(t,x))_{\nu \in \mathbb{N}}$, converging to

$$\tilde{U}(t,x) = \mathrm{e}^{-\Delta t} U_0(x), \quad t \geqslant 0, \ x \in \mathbb{R},$$

in the weak topology of $L^2([0,T) \times \mathbb{R})$.

Proof. Note that functions in $W^2(\mathbb{R})$ are continuous and bounded. The solution to equation (3.10) has a representative

$$U_{\varepsilon,\alpha}(t,x) = S_{\varepsilon,\alpha}(t) U_0(x), \quad t \in [0,T), \ x \in \mathbb{R}, \ \varepsilon < 1,$$

where $S_{\varepsilon,\alpha}(t)$ is the semigroup generated by $A_{\varepsilon,\alpha} = \Delta - (\phi_\varepsilon)^\alpha * \phi_\varepsilon$. Thus, (3.9) holds. As in theorem 3.2, $U_{\varepsilon,\alpha} \in C^0([0,T) : W^2(\mathbb{R})) \cap C^1((0,T) : L^2(\mathbb{R}))$, for every $T > 0$ and $\varepsilon < 1$. Since $\{U_{\varepsilon,\alpha}; \varepsilon < 1\}$ is bounded in $L^2([0,T) \times \mathbb{R})$ and hence is relatively compact with respect to the weak topology, we find that there exists a sequence $(\varepsilon_\nu)_{\nu \in \mathbb{N}}$ such that

$$U_{\varepsilon_\nu,\alpha}(t,x) = S_{\varepsilon_\nu,\alpha}(t) U_0(x) \to U(t,x), \quad \varepsilon_\nu \to 0,$$

in the sense of weak topology in $L^2([0,T) \times \mathbb{R})$. Let $x \in \mathbb{R}$, $t \in (0,T)$ and $\varepsilon < 1$. Using Duhamel's principle we have

$$\begin{aligned} U_{\varepsilon,\alpha}(t,x) &= \int_{\mathbb{R}} E_n(t, x-y) U_0(y) \, \mathrm{d}y + \int_0^t \int_{\mathbb{R}} E_n(t-s, x-y) \\ &\qquad \times \frac{1}{\varepsilon^\alpha} \int_{\mathbb{R}} \phi^\alpha\biggl(\frac{y}{\varepsilon} - u\biggr) \phi(u) \, \mathrm{d}u \, U_{\varepsilon,\alpha}(s,y) \, \mathrm{d}y \, \mathrm{d}s, \\ &= \int_{\mathbb{R}} E_n(t, x-y) U_{0\varepsilon}(y) \, \mathrm{d}y + \int_0^t \int_{\mathbb{R}} E_n(t-s, x-y\varepsilon) \varepsilon^{1-\alpha} \\ &\qquad \times \int_{\mathbb{R}^n} \phi^\alpha(y-u) \phi(u) \, \mathrm{d}u \, U_{\varepsilon,\alpha}(s, y\varepsilon) \, \mathrm{d}y \, \mathrm{d}s. \end{aligned} \tag{3.11}$$

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We will show that the last term in (3.11) tends to zero as $\varepsilon \to 0$ in the sense of weak convergence. This proves that $(U_{\varepsilon_\nu,\alpha}(t,x))_\nu$ weakly converges to $\mathrm{e}^{-\Delta t}U_0(x)$ in $L^2([0,T)\times\mathbb{R})$.

The Sobolev lemma and the assumption given on the initial data and ϕ imply that

$$\sup_{t\in(0,T)} \|U_{\varepsilon,\alpha}(t,\cdot)\|_{L^\infty} < \infty.$$

Let $\psi \in \mathcal{D}((0,T)\times\mathbb{R})$. Put

$$J_{\varepsilon_\nu} = \varepsilon_\nu^{1-\alpha} \int_0^T \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} \frac{1}{(4\pi(t-s))^{1/2}} \exp\biggl(-\frac{(x-\varepsilon_\nu y)^2}{4(t-s)}\biggr) \times \biggl(\int_{\mathbb{R}} \phi^\alpha(y-u)\phi(u)\,\mathrm{d}u\biggr) U_{\varepsilon_\nu,\alpha}(s,\varepsilon_\nu y)\,\mathrm{d}y\,\mathrm{d}s\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t, \quad \varepsilon<1, \quad \nu\in\mathbb{N}.$$

The Fubini–Tonelli theorem and a suitable change of variables imply that

$$\begin{aligned} J_{\varepsilon_\nu} &= \varepsilon^{1-\alpha} \int_0^T \int_{\mathbb{R}} \biggl(\int_0^t \int_{\mathbb{R}} \frac{1}{(4\pi s)^{1/2}} \exp\biggl(-\frac{(x-\varepsilon_\nu y)^2}{4s}\biggr) \\ &\qquad \times \biggl(\int_{\mathbb{R}} \phi^\alpha(y-u)\phi(u)\,\mathrm{d}u\biggr) \psi(t,x) U_{\varepsilon_\nu,\alpha}(t-s,\varepsilon_\nu y)\,\mathrm{d}x\,\mathrm{d}s\biggr)\,\mathrm{d}y\,\mathrm{d}t \\ &=: \int_0^T \int_{\mathbb{R}} \varepsilon^{1-\alpha} W_{\varepsilon_\nu,\alpha}(y,t)\,\mathrm{d}y\,\mathrm{d}t, \quad \varepsilon<\varepsilon_0, \quad \nu\in\mathbb{N}. \end{aligned}$$

Now,

$$\begin{aligned} |W_{\varepsilon_\nu,\alpha}(y,t)| &\leqslant C_1 \int_0^t \int_{\mathbb{R}} \frac{1}{(4\pi s)^{1/2}} \exp\biggl(-\frac{(x-\varepsilon_\nu y)^2}{4s}\biggr) \\ &\qquad \times |\psi(t,x)||U_{\varepsilon_\nu}(t-s,\varepsilon_\nu y)|\,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant C \int_0^t \int_{\mathbb{R}} \frac{1}{(4\pi s)^{1/2}} \exp\biggl(-\frac{(x-\varepsilon_\nu y)^2}{4s}\biggr)\,\mathrm{d}x\,\mathrm{d}s = Ct, \end{aligned}$$

for $\varepsilon<\varepsilon_0$, $\nu\in\mathbb{N}$, $y\in\mathbb{R}$ and $t\in(0,T)$, since we have already proved that $\{U_{\varepsilon,\alpha}, \varepsilon<\varepsilon_0\}$ is bounded.

Then Lebesgue's dominated convergence theorem gives $J_{\varepsilon_\nu}\to 0$ as $\nu\to\infty$. This proves that $(U_{\varepsilon_\nu,\alpha}(t,x))_\nu$ converges to $\mathrm{e}^{-\Delta t}U_0(x)$ in the sense of weak convergence in $L^2([0,T\times\mathbb{R}))$. □

Using the arguments of weak compactness as at the beginning of the proof of proposition 3.5, we have the following assertion.

Proposition 3.6. *Let $T>0$, $\delta\in\mathcal{G}_{W^{1,\infty}}(\mathbb{R})$ be defined by (3.2) and $U_0\in W^2(\mathbb{R})$. Let U be a solution to equation (3.1) with $V=\delta$. Then there is a decreasing sequence $(\varepsilon_\nu)_{\nu\in\mathbb{N}}$ converging to zero and $U_1\in L^2([0,T)\times\mathbb{R})$ such that*

$$U_{\varepsilon_\nu}(t,x) = S_{\varepsilon_\nu}(t)U_0(x) \to U_1(t,x),$$

in the sense of the weak topology in $L^2([0,T)\times\mathbb{R})$, $T>0$.

($S_\varepsilon(t)$ is the semigroup generated by $A_\varepsilon = \Delta - \phi_\varepsilon$, $\varepsilon < 1$.)

Note that we do not find that U_1 is equal to $\mathrm{e}^{-t\Delta}U_0$ as in proposition 3.5.

REMARK 3.7. Theorem 3.3 holds with

$$V = \sum_{i=1}^{m} a_i \delta^{i+\alpha_i}(x), \quad m \in \mathbb{N}, \quad a_i > 0 \quad \text{and} \quad \alpha_i \in (0,1)$$

with appropriate definitions of the powers $\delta^{i+\alpha_i}$, $i = 1, \dots, m$. Also, appropriate generalizations of propositions 3.5 and 3.6 can be made in this sense.

EXAMPLE 3.8. Consider

$$\partial_t U(t,x) - \Delta U(t,x) + V(x)U(t,x) = 0, \quad U(0,x) = \delta(x), \tag{3.12}$$

where $V \in W^{1,\infty}(\mathbb{R}^n)$. With $(\phi_\varepsilon)_\varepsilon$ as in (3.2), we have a net of approximated solutions

$$U_\varepsilon(t,x) = \int_\Omega \exp\biggl(-\int_0^t V(\omega(s))\,\mathrm{d}s\biggr)\phi_\varepsilon(\omega(t))\,\mathrm{d}\mu_x(\omega), \quad t \geqslant 0,\ x \in \mathbb{R}^n,\ \varepsilon < 1.$$

Assume $n > 1$. We will show that there exists a sequence $(U_{\varepsilon_\nu})_{\nu\in\mathbb{N}}$ converging to

$$U \in L^q_{\mathrm{loc}}((0,T),\mathbb{R}^n), \quad 1 \leqslant q < \frac{n}{n-1}$$

in $L^q_{\mathrm{loc}}((0,T),\mathbb{R}^n)$, such that $\partial_t U = (\Delta - V)U$ in $\mathcal{D}'((0,T),\mathbb{R}^n)$.

Let $\varepsilon < 1$ be fixed. Then $U_\varepsilon \in C^0([0,T) : L^1(\mathbb{R}^n)) \cap C^1((0,T) : L^1(\mathbb{R}^n))$ and $U_\varepsilon(t,\cdot) \in W^{2,1}(\mathbb{R}^n)$ for every $t > 0$. Again, by the estimates which are to follow, we will prove that $(tU_\varepsilon)_\varepsilon$ is bounded in $W^{1,1}((0,T)\times\mathbb{R}^n)$. We will give uniform estimates with respect to ε. There exists $C > 0$ such that

$$|U_\varepsilon(t,x)| \leqslant \int_\Omega |\phi_\varepsilon(\omega(t))|\,\mathrm{d}\mu_x(\omega) = C(4\pi t)^{-n/2}\int_{\mathbb{R}^n}\exp\biggl(-\frac{|x-y|^2}{4t}\biggr)\phi_\varepsilon(y)\,\mathrm{d}y,$$

for every $t > 0$, $x \in \mathbb{R}^n$ and $\varepsilon < 1$. Therefore,

$$\|U_\varepsilon\|_{L^1((0,T)\times\mathbb{R}^n)} \leqslant C\|\phi_\varepsilon\|_{L^1}, \quad \varepsilon < 1.$$

We will use the estimate (see [7]):

$$\sup_{t>0}\|t^{m/2+n(1-1/r)/2}\partial_x^\alpha E_n(t,\cdot)\|_{L^r} < \infty, \quad |\alpha| \leqslant m,\ 1 \leqslant r \leqslant \infty. \tag{3.13}$$

Let $t \in (0,T)$, $\varepsilon < 1$, $i \in \{1,\dots,n\}$, $x \in \mathbb{R}^n$. Then taking $m = 1$ and $r = 1$, one has

$$\begin{aligned} t^{1/2}\partial_{x_i}U_\varepsilon(t,x) = {} & \int_{\mathbb{R}^n} t^{1/2}\partial_{y_i}E_n(t,y)U_{0\varepsilon}(x-y)\,\mathrm{d}y \\ & + \int_0^t t^{1/2}\int_{\mathbb{R}^n}(t-s)^{1/2}\partial_{y_i}E_n(t-s,y) \\ & \qquad \times \frac{V_\varepsilon(x-y)U_\varepsilon(s,x-y)}{(t-s)^{1/2}}\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

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Then

$$\|t^{1/2}\partial_{x_i}U_\varepsilon(t,\cdot)\|_{L^1} \leqslant \|t^{1/2}\partial_{x_i}E_n(t,\cdot)\|_{L^1}\|U_{0\varepsilon}\|_{L^1} + T\int_0^t \|(t-s)^{1/2}\partial_{x_i}E_n(t-s,\cdot)\|_{L^1}\frac{\|V_\varepsilon\|_{L^\infty}\|U_\varepsilon(s,\cdot)\|_{L^1}}{(t-s)^{1/2}}\,\mathrm{d}s$$

implies (with suitable $C>0$) that

$$\sup_{t\in(0,T)}\|t^{1/2}\partial_{x_i}U_\varepsilon(t,\cdot)\|_{L^1} \leqslant C, \quad \varepsilon<1.$$

Similar estimates with $m=2$, $r=1$, (3.13) and the previous step imply that

$$\begin{aligned} t\partial_{x_ix_j}U_\varepsilon(t,x) &= \int_{\mathbb{R}^n} t\frac{\partial}{\partial y_i\partial y_j}E_n(t,y)U_{0\varepsilon}(x-y)\,\mathrm{d}y \\ &\quad + t\int_0^t\int_{\mathbb{R}^n}(t-s)^{1/2}\frac{\partial}{\partial y_i}E_n(t-s,y)\frac{(\partial/\partial x_j)V_\varepsilon(x-y)U_\varepsilon(s,x-y)}{(t-s)^{1/2}}\,\mathrm{d}y\,\mathrm{d}s \\ &\quad + t\int_0^t\int_{\mathbb{R}^n}(t-s)^{1/2}\frac{\partial}{\partial y_i}E_n(t-s,y)\frac{V_\varepsilon(x-y)(\partial/\partial x_j)U_\varepsilon(s,x-y)}{(t-s)^{1/2}}\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

This gives

$$\sup_{t\in(0,T)}\|t\partial_{x_ix_j}U_\varepsilon(t,\cdot)\|_{L^1} \leqslant C, \quad \varepsilon<1.$$

Since

$$\frac{\partial}{\partial t}(tU_\varepsilon(t,x)) = U_\varepsilon(t,x) + (\Delta_x - V)(U_\varepsilon(t,x)),$$

it follows that

$$\sup_{t\in(0,T)}\left\|\frac{\partial}{\partial t}(tU_\varepsilon)\right\|_{L^1} \leqslant C, \quad \varepsilon<1.$$

Thus $(tU_\varepsilon)_\varepsilon$ is bounded in $W^{1,1}((0,T)\times\mathbb{R}^n)$. By the Rellich–Kondrachov compactness theorem (see [27, ch. 2, theorem 1.1]), there exists a sequence

$$h_\nu(t,x) = tU_{\varepsilon_\nu}(t,x), \quad (t,x)\in(0,T)\times\mathbb{R}^n,\ \nu\in\mathbb{N},$$

converging to $h(t,x)$ in $L^q_{\mathrm{loc}}((0,T)\times\mathbb{R}^n)$, $1\leqslant q<n/(n-1)$. It follows that

$$U_{\varepsilon_\nu}(t,x) \to \frac{h(t,x)}{t} = U(t,x) \quad \text{in } L^q_{\mathrm{loc}}((0,T)\times\mathbb{R}^n),\ \nu\to\infty,$$

and $U(t,x)\in L^q_{\mathrm{loc}}((0,T)\times\mathbb{R}^n)$. Then

$$\partial_t U = (\Delta - V)U \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^n).$$

3.2. The Lipschitz nonlinear case

LEMMA 3.9. *Let $f\in C^1([0,T)\times\mathbb{R})$ be real valued, $f(t,0)=0$, $t\in[0,T)$ and*

$$|f(t,y_1)-f(t,y_2)| \leqslant L_0(t)|y_1-y_2|, \quad y_1,y_2\in\mathbb{R},\ t\in[0,T), \tag{3.14}$$

for some positive bounded function $L_0:[0,T)\to\mathbb{R}$ and some $s>0$.

Then $(U_\varepsilon)_\varepsilon \mapsto (f(t,U_\varepsilon))_\varepsilon$ *defines the mappings*

$$\mathcal{E}_{C^0,W^1}([0,T):\mathbb{R}^n) \to \mathcal{E}_{C^0,W^1}([0,T):\mathbb{R}^n),$$
$$\mathcal{N}_{C^0,W^1}([0,T):\mathbb{R}^n) \to \mathcal{N}_{C^0,W^1}([0,T):\mathbb{R}^n)$$

and the corresponding mapping $[U_\varepsilon] \mapsto [f(t,U_\varepsilon)]$

$$\mathcal{G}_{C^0,W^1}([0,T):\mathbb{R}^n) \to \mathcal{G}_{C^0,W^1}([0,T):\mathbb{R}^n).$$

Proof. We will prove that $(U_\varepsilon)_\varepsilon \in \mathcal{E}_{C^0,W^1}([0,T):\mathbb{R}^n)$ implies that $(f(t,U_\varepsilon))_\varepsilon \in \mathcal{E}_{C^0,W^1}([0,T):\mathbb{R}^n)$. The other parts of the proof follow in a similar way.

One has to show that there exists $a \in \mathbb{R}$ such that

$$\sup_{t\in[0,T)} \|f(t,U_\varepsilon)\|_{W^1(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^a), \quad \varepsilon \to 0,$$

if there exists $b \in \mathbb{R}$ such that

$$\sup_{t\in[0,T)} \|U_\varepsilon\|_{W^1(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^b), \quad \varepsilon \to 0.$$

Relation (3.14) implies

$$\|f(t,U_\varepsilon)\|_{L^2} \leqslant L_0(t)\|U_\varepsilon\|_{L^2}, \quad t \in [0,T),\ \varepsilon < 1.$$

Thus, there exists $a \in \mathbb{R}$ such that

$$\sup_{t\in[0,T)} \|f(t,U_\varepsilon)\|_{L^2} = \mathcal{O}(\varepsilon^a), \quad \varepsilon \to 0.$$

Differentiation of $f(t,U_\varepsilon)$ with respect to some spatial variable x_i gives

$$\|\partial_y f(t,U_\varepsilon)\partial_{x_i} U_\varepsilon\|_{L^2} \leqslant \|\partial_y f(t,U_\varepsilon)\|_{L^\infty}\|\partial_{x_i} U_\varepsilon\|_{L^2} \leqslant L_0(t)\|\partial_{x_i} U_\varepsilon\|_{L^2},$$

where ∂_y denotes the differentiation with respect to the second variable.

Thus, the moderate bound of $\|U_\varepsilon\|_{W^1}$ implies a moderate bound for

$$\|\partial_y f(t,U_\varepsilon)\|_{L^2}, \quad \varepsilon < 1,$$

and the assertion follows. □

THEOREM 3.10. *Let* $T > 0$, $V \in \mathcal{G}_{W^{1,\infty}}(\mathbb{R}^n)$ *be of logarithmic type and* $U_0 \in \mathcal{G}_{W^2}(\mathbb{R}^n)$*. Suppose that* $f \in C^1([0,T) \times \mathbb{R})$ *is real valued,* $f(t,0) = 0$, $t \in [0,T)$ *and that (3.14) holds. Then there exists a net* $(U_\varepsilon)_\varepsilon$ *of solutions to*

$$\partial_t U_\varepsilon(t,x) = (\Delta - V_\varepsilon)U_\varepsilon(t,x) + f(t,U_\varepsilon(t,x)), U_\varepsilon(0,x) = U_{0,\varepsilon}(x), \quad \varepsilon < 1, \quad (3.15)$$

such that $U = [U_\varepsilon] \in \mathcal{G}_{C^1,W^2}([0,T):\mathbb{R}^n)$ *is the solution to*

$$\partial_t U = (\Delta - V)U + f(t,U) \quad (3.16)$$

in the sense of definition 3.1. Moreover, the solution to equation (3.16) is unique in the sense of the same definition.

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Proof. Recall that $U \in \mathcal{G}_{C^1,W^2}([0,T):\mathbb{R}^n)$ means that $U_\varepsilon \in C^0([0,T):W^2(\mathbb{R}^n)) \cap C^1((0,T):L^2(\mathbb{R}^n))$, $\varepsilon<1$, and there exists $a \in \mathbb{R}$ such that

$$\sup_{t\in[0,T)} \|U_\varepsilon(t)\|_{W^2}, \qquad \sup_{t\in(0,T)} \|\partial_t U_\varepsilon(t)\|_{L^2} = O(\varepsilon^a), \quad \varepsilon \to 0.$$

Let $\varepsilon<1$ be fixed. The classical solution to (3.15) satisfies

$$U_\varepsilon(t,x) = S_\varepsilon(t)U_{0\varepsilon}(x) + \int_0^t S_\varepsilon(t-s) f(s,U_\varepsilon(s,x))\,\mathrm{d}s, \quad t\in[0,T),\ x\in\mathbb{R}^n, \tag{3.17}$$

where S_ε is a representative of generalized C_0-semigroup generated by $A_\varepsilon = \Delta - V_\varepsilon$ (we refer to [24, ch. 6, theorem 1.5]).

We also have

$$\begin{aligned} U_\varepsilon(t,x) = {} & \int_{\mathbb{R}^n} E_n(t,x-y)U_{0\varepsilon}(y)\,\mathrm{d}y \\ & + \int_0^t\int_{\mathbb{R}^n} E_n(t-s,x-y)V_\varepsilon(y)U_\varepsilon(s,y)\,\mathrm{d}y\,\mathrm{d}s \\ & + \int_0^t\int_{\mathbb{R}^n} E_n(t-s,x-y) f(s,U_\varepsilon(s,y))\,\mathrm{d}s, \quad (t,x)\in(0,T)\times\mathbb{R}^n. \end{aligned} \tag{3.18}$$

The assumptions that $V \in \mathcal{G}_{W^{1,\infty}}(\mathbb{R}^n)$ is of log-type and (3.14) imply that there exists $C>0$ such that

$$\begin{aligned} \|U_\varepsilon(t,\cdot)\|_{L^2} &\leqslant \|E_n(t,\cdot)\|_{L^1}\|U_{0\varepsilon}\|_{L^2} \\ &\quad + \int_0^t \|E_n(t-s,\cdot)\|_{L^1}\|V_\varepsilon\|_{L^\infty}\|U_\varepsilon(s,\cdot)\|_{L^2}\,\mathrm{d}s \\ &\quad + \int_0^t \|E_n(t-s,\cdot)\|_{L^1}\|f(s,U_\varepsilon(s,\cdot))\|_{L^2}\,\mathrm{d}s \\ &\leqslant \|U_{0\varepsilon}\|_{L^2} + \int_0^t \|V_\varepsilon\|_{L^\infty}\|U_\varepsilon(s,\cdot)\|_{L^2}\,\mathrm{d}s \\ &\quad + \int_0^t L_0(s)\|U_\varepsilon(s,\cdot)\|_{L^2}\,\mathrm{d}s \\ &\leqslant \|U_{0\varepsilon}\|_{L^2} + C\log(1/\varepsilon)\int_0^t (1+L_0(s))\|U_\varepsilon(s,\cdot)\|_{L^2}\,\mathrm{d}s, \end{aligned}$$

for $t\in(0,T)$, i.e. there exists $a\in\mathbb{R}$ such that

$$\|U_\varepsilon(t,\cdot)\|_{L^2} \leqslant \|U_{0\varepsilon}\|_{L^2}\exp\biggl(C\log(1/\varepsilon)\int_0^t (1+L_0(s))\,\mathrm{d}s\biggr) = \mathcal{O}(\varepsilon^a), \quad \varepsilon\to 0, \tag{3.19}$$

uniformly for $t\in[0,T)$ (since $L_0(s)$ is bounded for $s\in[0,T)$). Here we use again the fact that $U_\varepsilon(t,\cdot)\to U_\varepsilon(0,\cdot)$ in $L^2(\mathbb{R}^n)$ as $t\to 0$, $\varepsilon<1$.

Differentiating (3.18) with respect to some spatial variable, we have

$$\begin{aligned}\partial_{x_i}U_\varepsilon(t,x) = \int_{\mathbb{R}^n} E_n(t,y)\partial_{x_i}U_{0\varepsilon}(x-y)\,\mathrm{d}y \\ + \int_0^t \int_{\mathbb{R}^n} \partial_{y_i}E_n(t-s,y)V_\varepsilon(x-y)U_\varepsilon(s,x-y)\,\mathrm{d}y\,\mathrm{d}s \\ + \int_0^t \int_{\mathbb{R}^n} \partial_{y_i}E_n(t-s,y)f(s,U_\varepsilon(s,x-y))\,\mathrm{d}y\,\mathrm{d}s,\end{aligned}$$

for $(t,x)\in(0,T)\times\mathbb{R}^n$. Young's inequality and the previously mentioned arguments for the estimate of $\|\partial_{x_i}U_\varepsilon(t,\cdot)\|_{L^2}$ imply the existence of $C>0$ such that

$$\begin{aligned}&\|\partial_{x_i}U_\varepsilon(t,\cdot)\|_{L^2} \\ &\quad\leqslant \|E_n(t,\cdot)\|_{L^1}\|\partial_{x_i}U_{0\varepsilon}\|_{L^2} \\ &\qquad+\int_0^t (t-s)^{-1/2}\|(t-s)^{1/2}\partial_{y_i}E_n(t-s,\cdot)\|_{L^1}\|V_\varepsilon\|_{L^\infty}\|U_\varepsilon(s,\cdot)\|_{L^2}\,\mathrm{d}s \\ &\qquad+\int_0^t (t-s)^{-1/2}\|(t-s)^{1/2}\partial_{y_i}E_n(t-s,\cdot)\|_{L^1}\|f(s,U_\varepsilon(s,\cdot))\|_{L^\infty}\,\mathrm{d}s \\ &\quad\leqslant \|\partial_{x_i}U_{0\varepsilon}\|_{L^2} + C\log(1/\varepsilon)\int_0^t (t-s)^{-1/2}(1+L_0(s))\|U_\varepsilon(s,\cdot)\|_{L^2}\,\mathrm{d}s,\end{aligned}$$

uniformly for $t\in(0,T)$. By the previous part of the proof it follows that there exists $a\in\mathbb{R}$ such that $\sup_{t\in(0,T)}\|\partial_{x_i}U_\varepsilon(t,\cdot)\|_{L^2}=\mathcal{O}(\varepsilon^a)$, $\varepsilon\to 0$. Moreover, using the same inequalities, we find, for every $\varepsilon<1$, that

$$\|\partial_{x_i}U_\varepsilon(t,\cdot)-\partial_{x_i}U_{0,\varepsilon}\|_{L^2}\to 0 \quad \text{as } t\to 0.$$

After another space-like differentiation one obtains (for $t\in(0,T)$)

$$\begin{aligned}\|\partial_{x_ix_j}U_\varepsilon(t,\cdot)\|_{L^2} &\leqslant \|E_n(t,\cdot)\|_{L^1}\|\partial_{x_ix_j}U_{0\varepsilon}\|_{L^2} \\ &\quad+\int_0^t (t-s)^{-1/2}\|(t-s)^{1/2}\partial_{y_i}E_n(t-s,\cdot)\|_{L^1} \\ &\qquad\times(\|\partial_{x_j}V\|_{L^\infty}\|U_\varepsilon(s,\cdot)\|_{L^2}+\|V\|_{L^\infty}\|\partial_{x_j}U_\varepsilon(s,\cdot)\|_{L^2})\,\mathrm{d}s \\ &\quad+\int_0^t (t-s)^{-1/2}\|(t-s)^{1/2}\partial_{y_i}E_n(t-s,\cdot)\|_{L^1} \\ &\qquad\times\|\partial_u f(s,U_\varepsilon(s,\cdot))\|_{L^\infty}\|\partial_{x_j}U_\varepsilon(s,\cdot)\|_{L^2}\,\mathrm{d}s.\end{aligned}$$

Now, by the use of the bounds on $\partial_u f$ and already obtained bounds for $\|\partial_{x_i}U_\varepsilon\|_{L^2}$, we obtain a moderate bound for all the second derivatives of U_ε.

The same inequalities also imply that, for every $\varepsilon<1$,

$$\|\partial_{x_ix_j}U_\varepsilon(t,\cdot)-\partial_{x_ix_j}U_{0,\varepsilon}\|_{L^2}\to 0 \quad \text{as } t\to 0.$$

Thus, $U_\varepsilon\in C^0([0,T):W^2(\mathbb{R}^n))$, $\varepsilon<1$, and the first part of (2.1) holds.

A moderate bound for the derivative with respect to the time variable $t\in(0,T)$ follows directly from the equation itself.

Thus $U_\varepsilon\in C^1([0,T):L^2(\mathbb{R}^n))$, $\varepsilon<1$, and the second part of (2.1) holds, i.e. $[U_\varepsilon]$ is a solution to (3.16).

Let us show the uniqueness of the above solution.

Let $U_{1\varepsilon}$ and $U_{2\varepsilon}$ be two solutions to equation (3.15) such that both satisfy (2.1). Denote $G_\varepsilon = U_{1\varepsilon} - U_{2\varepsilon}$, $\varepsilon < 1$ and $H_\varepsilon(t,\cdot) = f(t,U_{1\varepsilon}) - f(t,U_{2\varepsilon})$, $t \in [0,T)$, $\varepsilon < 1$.

Then G_ε is a solution to

$$\partial_t G_\varepsilon(t,x) = (\Delta - V_\varepsilon)G_\varepsilon(t,x) + H_\varepsilon(t,x) + N_\varepsilon(t,x),$$
$$(G_\varepsilon(0,x))_\varepsilon = (N_{0\varepsilon}(x))_\varepsilon \in \mathcal{N}_{W^2}(\mathbb{R}^n),$$

where $(N_\varepsilon)_\varepsilon \in \mathcal{N}_{C^0,L^2}([0,T) : \mathbb{R}^n)$. By (3.14), we have

$$\|H_\varepsilon(t,x)\| \leqslant L_0(t)G_\varepsilon(t,x), \quad t \in [0,T),\ x \in \mathbb{R}^n.$$

Let $\varepsilon < \varepsilon_0$. We then have

$$\begin{aligned} G_\varepsilon(t,x) = {} & \int_{\mathbb{R}^n} E_n(t,x-y)N_{0\varepsilon}(y)\,\mathrm{d}y \\ & + \int_0^t \int_{\mathbb{R}^n} E_n(t-s,x-y)V_\varepsilon(s,y)\,\mathrm{d}s \\ & + \int_0^t \int_{\mathbb{R}^n} E_n(t-s,x-y)(H_\varepsilon(s,y) + N_\varepsilon(s,y))\,\mathrm{d}s, \\ & \qquad\qquad (t,x) \in (0,T) \times \mathbb{R}^n. \end{aligned}$$

This implies (with suitable $C > 0$) that

$$\begin{aligned} \|G_\varepsilon(t,\cdot)\|_{L^2} \leqslant {} & \|E_n(t,\cdot)\|_{L^1}\|N_{0\varepsilon}\|_{L^2} \\ & + \int_0^t \|E_n(t-s,\cdot)\|_{L^1}\|V_\varepsilon\|_{L^\infty}\|G_\varepsilon(s,\cdot)\|_{L^2}\,\mathrm{d}s \\ & + \int_0^t L_0(s)\|E_n(t-s,\cdot)\|_{L^1}\|G_\varepsilon(s,\cdot)\|_{L^2}\,\mathrm{d}s \\ & + \int_0^t \|E_n(t-s,\cdot)\|_{L^1}\|N_\varepsilon(s,\cdot)\|_{L^2}\,\mathrm{d}s \\ \leqslant {} & \|N_{0\varepsilon}\|_{L^2} + C\log(1/\varepsilon)\int_0^t (1 + L_0(s))\|G_\varepsilon(s,\cdot)\|_{L^2}\,\mathrm{d}s \\ & + \int_0^t \|N_\varepsilon(s,\cdot)\|_{L^2}\,\mathrm{d}s, \quad t \in (0,T), \end{aligned}$$

and, for every $a \in \mathbb{R}$,

$$\begin{aligned} \|G_\varepsilon(t,\cdot)\|_{L^2} = {} & \Big(\|N_{0\varepsilon}\|_{L^2} + T\sup_{t\in(0,T)}\|N_\varepsilon(t,\cdot)\|_{L^2}\Big) \\ & \times \exp\bigg(C\log\bigg(\frac{1}{\varepsilon}\bigg)\int_0^t (1 + L_0(s))\,\mathrm{d}s\bigg) \\ = {} & \mathcal{O}(\varepsilon^a), \quad \varepsilon \to 0, \end{aligned}$$

uniformly for $t \in (0,T)$. Therefore, the solution is unique in $\mathcal{G}_{C^1,L^2}([0,T) : \mathbb{R}^n)$. □

EXAMPLE 3.11. Assume $n \geqslant 2$, $T > 0$, $V \in W^{1,\infty}(\mathbb{R}^n)$, and $f \in C^1([0,\infty) \times \mathbb{R}^n)$ satisfies $f(s,0) = 0$ and $|f(s,y_1) - f(s,y_2)| \leqslant C|y_1 - y_2|$ for $s \in [0,\infty)$, $y_1, y_2 \in \mathbb{R}^n$.

Let $U_0(x) = \delta(x)$, $x \in \mathbb{R}^n$, i.e. $U_{0\varepsilon} = \phi_\varepsilon(x)$, $\varepsilon < 1$ (cf. (3.2)). Then, for fixed $\varepsilon < 1$,

$$\partial_t U_\varepsilon(t,x) = (\Delta_x - V(x))U_\varepsilon(t,x) + f(t, U_\varepsilon(t,x)), U_\varepsilon(0,x) = \phi_\varepsilon$$

has a unique classical solution U_ε in $C^0([0,T), L^1(\mathbb{R}^n)) \cap C^1((0,T), L^1(\mathbb{R}^n))$ and $U_\varepsilon(t,\cdot) \in W^{2,1}(\mathbb{R}^n)$ for every $t > 0$. Again we have $U_\varepsilon(t,x) \in C^0((0,T) : W^2(\mathbb{R}^n))$, $\varepsilon < 1$.

We will show that there exists a sequence $(U_{\varepsilon_\nu})_{\nu \in \mathbb{N}}$ converging to

$$U \in L^q_{\text{loc}}((0,T), \mathbb{R}^n), \quad 1 \leqslant q < n/(n-1),$$

in $L^q_{\text{loc}}((0,T), \mathbb{R}^n)$ such that $\partial_t U = (\Delta - V)U$ in $\mathcal{D}'((0,T), \mathbb{R}^n)$.

There holds

$$\|U_\varepsilon(t,\cdot)\|_{L^1} \leqslant \|U_{0\varepsilon}\|_{L^1} + \int_0^t \|V\|_{L^\infty} \|U_\varepsilon(s,\cdot)\|_{L^1} \,\mathrm{d}s + \int_0^t C\|U_\varepsilon(s,\cdot)\|_{L^1} \,\mathrm{d}s, \quad 0 < t < T. \tag{3.20}$$

This implies $\|U_\varepsilon(t,\cdot)\|_{L^1} \leqslant C$, $t \in (0,T)$, $\varepsilon < 1$.

Let $i = 1,\dots,n$, $t \in (0,T)$, $x \in \mathbb{R}^n$ and $\varepsilon < 1$. Then

$$\begin{aligned} t^{1/2}\partial_{x_i} U_\varepsilon(t,x) &= \int_{\mathbb{R}^n} t^{1/2} \partial_{y_i} E_n(t,y) U_{0\varepsilon}(x-y) \,\mathrm{d}y \\ &\quad + t^{1/2} \int_0^t \int_{\mathbb{R}^n} \partial_{x_i} E_n(t-s, x-y) V_\varepsilon(y) U_\varepsilon(s,y) \,\mathrm{d}y \,\mathrm{d}s \\ &\quad + t^{1/2} \int_0^t \frac{1}{\sqrt{t-s}} \int_{\mathbb{R}^n} \sqrt{t-s}\,\partial_{x_i} E_n(t-s, x-y) V_\varepsilon(y) U_\varepsilon(s,y) \,\mathrm{d}y \,\mathrm{d}s \\ &\quad + t^{1/2} \int_0^t \frac{1}{\sqrt{t-s}} \int_{\mathbb{R}^n} \sqrt{t-s}\,\partial_{x_i} E_n(t-s, x-y) f(s, U_\varepsilon(s,y)) \,\mathrm{d}y \,\mathrm{d}s. \end{aligned}$$

This implies that

$$\begin{aligned} &\|t^{1/2}\partial_{x_i} U_\varepsilon(t,\cdot)\|_{L^1} \\ &\leqslant \|t^{1/2}\partial_{y_i} E_n(t,\cdot)\|_{L^1} \|U_{0\varepsilon}\|_{L^1} \\ &\quad + \sqrt{T} \int_0^t \frac{1}{\sqrt{t-s}} \|\sqrt{t-s}\,\partial_{x_i} E_n(t-s,\cdot)\|_{L^1} \|V_\varepsilon\|_{L^\infty} \|U_\varepsilon(s,\cdot)\|_{L^1} \,\mathrm{d}s \\ &\quad + \sqrt{T} \int_0^t \frac{1}{\sqrt{t-s}} \|\sqrt{t-s}\,\partial_{x_i} E_n(t-s,\cdot)\|_{L^1} \|V_\varepsilon\|_{L^\infty} \|U_\varepsilon(s,\cdot)\|_{L^1} \,\mathrm{d}s \\ &\quad + \sqrt{T} \int_0^t \frac{1}{\sqrt{t-s}} \|\sqrt{t-s}\,\partial_{x_i} E_n(t-s,\cdot)\|_{L^1} \|f(s, U_\varepsilon(s,\cdot))\|_{L^\infty} \|U_\varepsilon(s,\cdot)\|_{L^1} \,\mathrm{d}s. \end{aligned}$$

By (3.13) there exists a constant $C_T > 0$ such that

$$\|t^{1/2}\partial_{x_i} U_\varepsilon(t,\cdot)\|_{L^1} \leqslant C_T, \quad t \in (0,T),\ \varepsilon < 1. \tag{3.21}$$

For the second derivatives we have

$$\begin{aligned} t\partial_{x_i x_j} U_\varepsilon(t,x) &= \int_{\mathbb{R}^n} t\partial_{y_i y_j} E_n(t,y) U_{0\varepsilon}(x-y)\,\mathrm{d}y \\ &\quad + t\int_0^t \int_{\mathbb{R}^n} (t-s)^{1/2} \partial_{x_i} E_n(t-s,x-y) \frac{\partial_{x_j}(V_\varepsilon(y)U_\varepsilon(s,y))}{(t-s)^{1/2}}\,\mathrm{d}y\,\mathrm{d}s \\ &\quad + t\int_0^t \int_{\mathbb{R}^n} (t-s)^{1/2} \partial_{x_i} E_n(t-s,x-y) \frac{\partial_{x_j}(f(s,U_\varepsilon(s,y))U_\varepsilon(s,y))}{(t-s)^{1/2}}\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

Again by the previous step and (3.13) it follows that, for some $C > 0$,

$$\sup_{t\in(0,T)} \|t\partial_{x_i x_j} U_\varepsilon(t,\cdot)\|_{L^1} \leqslant C, \quad \varepsilon < 1.$$

Using equation (3.16), this implies

$$\sup_{t\in(0,T)} \left\| t\frac{\partial}{\partial t} U_\varepsilon(t,\cdot)\right\|_{L^1} \leqslant C, \quad \varepsilon < 1.$$

Thus $\{tU_\varepsilon(t,x) : \varepsilon < 1\}$ is bounded in $W^{1,1}((0,T)\times\mathbb{R}^n)$ and, as in example 3.8, there exist a sequence of functions $h_\nu(t,x) = tU_{\varepsilon_\nu}(t,x)$, $\nu \in \mathbb{N}$, and a function $h(t,x)$ in $L^q_{\mathrm{loc}}((0,T)\times\mathbb{R}^n)$, $1 \leqslant q < n/(n-1)$ such that $h_\nu \to h$ in $L^q_{\mathrm{loc}}((0,T)\times\mathbb{R}^n)$, $\nu\to\infty$.

We have

$$U_{\varepsilon_\nu}(t,x) \to \frac{h(t,x)}{t} = U(t,x) \quad \text{in } L^q_{\mathrm{loc}}((0,T)\times\mathbb{R}^n), \quad \nu\to\infty,$$

and $U(t,x) \in L^q_{\mathrm{loc}}((0,T)\times\mathbb{R}^n)$ satisfies

$$\partial_t U = (\Delta - V)U + f(t,U) \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^n).$$

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Appendix A. Generalized semigroups: proofs

First, we give a remark concerning definition 2.4. It is enough that (2.5) holds for all $x \in \Lambda$, where Λ is a dense subspace of E. This is a consequence of (2.4) and (2.6): fix $\varepsilon < \varepsilon_0$ and let $(x_n)_n$ be a sequence in Λ such that $x_n \to x \in E$. Then the estimate

$$\left\|\frac{N_\varepsilon(t)x}{t} - W_\varepsilon x\right\| \leqslant \left\|\frac{N_\varepsilon(t)}{t}\right\| \|x - x_n\| + \left\|\frac{N_\varepsilon(t)x_n}{t} - W_\varepsilon x_n\right\| + \|W_\varepsilon\|\|x_n - x\|,$$

implies (2.5) for every $x\in E$.

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Proof of proposition 2.5. Let

$$(S_\varepsilon(t))_\varepsilon \in \mathcal{S}E_M([0,\infty) : \mathcal{L}(E)) \quad \text{and} \quad (N_\varepsilon(t))_\varepsilon \in \mathcal{SN}([0,\infty) : \mathcal{L}(E)).$$

We will prove that

$$(S_\varepsilon(t)N_\varepsilon(t))_\varepsilon, (N_\varepsilon(t)S_\varepsilon(t))_\varepsilon \in \mathcal{SN}([0,\infty) : \mathcal{L}(E)),$$

where $S_\varepsilon(t)N_\varepsilon(t)$ denotes the composition of $S_\varepsilon(t)$ and $N_\varepsilon(t)$. The other parts of the assertions can be proved in a similar way.

Let $\varepsilon < \varepsilon_0$. By (2.2) and (2.3), for some $a \in \mathbb{R}$ and every $b \in \mathbb{R}$,

$$\|S_\varepsilon(t)N_\varepsilon(t)\| \leqslant \|S_\varepsilon(t)\| \cdot \|N_\varepsilon(t)\| = \mathcal{O}(\varepsilon^{a+b}), \quad \varepsilon \to 0.$$

The same holds for $\|N_\varepsilon(t)S_\varepsilon(t)\|$. Properties (2.2) and (2.5) yield

$$\sup_{t<t_0} \left\| \frac{S_\varepsilon(t)N_\varepsilon(t)}{t} \right\| \leqslant \sup_{t<t_0} \|S_\varepsilon(t)\| \sup_{t<t_0} \left\| \frac{N_\varepsilon(t)}{t} \right\| = \mathcal{O}(\varepsilon^a), \quad \varepsilon \to 0,$$

for some $t_0 > 0$ and $a \in \mathbb{R}$. Also,

$$\sup_{t<t_0} \left\| \frac{N_\varepsilon(t)S_\varepsilon(t)}{t} \right\| = \mathcal{O}(\varepsilon^a), \quad \varepsilon \to 0,$$

for some $t_0 > 0$ and $a \in \mathbb{R}$.

Let $\varepsilon < \varepsilon_0$ be fixed. Then

$$\begin{aligned} &\left\| \frac{S_\varepsilon(t)N_\varepsilon(t)}{t}x - S_\varepsilon(0)W_\varepsilon x \right\| \\ &\qquad = \left\| S_\varepsilon(t)\frac{N_\varepsilon(t)}{t}x - S_\varepsilon(t)W_\varepsilon x + S_\varepsilon(t)W_\varepsilon x - S_\varepsilon(0)W_\varepsilon x \right\| \\ &\qquad \leqslant \|S_\varepsilon(t)\| \left\| \frac{N_\varepsilon(t)}{t}x - W_\varepsilon x \right\| + \|S_\varepsilon(t)W_\varepsilon x - S_\varepsilon(0)W_\varepsilon x\|. \end{aligned}$$

By (2.2) and (2.5) as well as by the continuity of $t \mapsto S_\varepsilon(t)(W_\varepsilon x)$ at zero, it follows that the last expression tends to zero as $t \to 0$. Similarly, we have

$$\begin{aligned} &\left\| \frac{N_\varepsilon(t)S_\varepsilon(t)}{t}x - W_\varepsilon S_\varepsilon(0)x \right\| \\ &\qquad = \left\| \frac{N_\varepsilon(t)}{t}S_\varepsilon(t)x - \frac{N_\varepsilon(t)}{t}S_\varepsilon(0)x + \frac{N_\varepsilon(t)}{t}S_\varepsilon(0)x - W_\varepsilon S_\varepsilon(0)x \right\| \\ &\qquad \leqslant \left\| \frac{N_\varepsilon(t)}{t} \right\| \|S_\varepsilon(t)x - S_\varepsilon(0)x\| + \left\| \frac{N_\varepsilon(t)}{t}(S_\varepsilon(0)x) - W_\varepsilon(S_\varepsilon(0)x) \right\|. \end{aligned}$$

Assumptions (2.4), (2.5) and (2.2) imply that the last expression tends to zero as $t \to 0$. Thus, (2.5) is proved for $(N_\varepsilon(t)S_\varepsilon(t))_\varepsilon$ in both cases. □

Proof of proposition 2.7. Let $(N_\varepsilon)_\varepsilon = (S_\varepsilon - \tilde{S}_\varepsilon)_\varepsilon$. It is an element of $\mathcal{SN}([0,\infty) : \mathcal{L}(E))$. Let $\varepsilon < \bar{\varepsilon}_0$ be fixed and $x \in E$. Then

$$\frac{S_\varepsilon(t)x - x}{t} - \frac{\tilde{S}_\varepsilon(t)x - x}{t} = \frac{N_\varepsilon(t)}{t}x.$$

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Letting $t \to 0$, this implies that $D(A_\varepsilon) = D(\tilde{A}_\varepsilon)$. Thus

$$(A_\varepsilon - \tilde{A}_\varepsilon)x = \lim_{t\to 0} \frac{S_\varepsilon(t)x - x}{t} - \lim_{t\to 0} \frac{\tilde{S}_\varepsilon(t)x - x}{t}$$
$$= \lim_{t\to 0} \frac{N_\varepsilon(t)}{t} x = W_\varepsilon x, \quad x \in D(A_\varepsilon). \tag{A 1}$$

Since $D(A_\varepsilon)$ is dense in E, properties (2.5), (2.6) and (A 1) imply that, for every $a \in \mathbb{R}$, $\|A_\varepsilon - \tilde{A}_\varepsilon\| = \mathcal{O}(\varepsilon^a)$, as $\varepsilon \to 0$. □

We collect some obvious properties of semigroups (see [24]) in the following proposition.

PROPOSITION A.1. *Let $A \in \mathcal{A}/\sim$ and S be a generalized C_0-semigroup with the infinitesimal generator A. Then there exists $\varepsilon_0 \in (0,1)$ such that, for fixed $\varepsilon < \varepsilon_0$, the following hold:*

(a) *$t \mapsto S_\varepsilon(t)x : [0,\infty) \to E$ is continuous for every $x \in E$;*

(b) $\displaystyle\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} S_\varepsilon(s)\,\mathrm{d}s = S_\varepsilon(t)x, \quad x \in E;$

(c) $\displaystyle\int_0^t S_\varepsilon(s)x\,\mathrm{d}s \in D(A_\varepsilon), \quad x \in E;$

(d) *$S_\varepsilon(t)x \in D(A_\varepsilon)$ for every $x \in D(A_\varepsilon)$, $t \geqslant 0$, and*

$$\frac{\mathrm{d}}{\mathrm{d}t} S_\varepsilon(t)x = A_\varepsilon S_\varepsilon(t)x = S_\varepsilon(t)A_\varepsilon x;$$

(e) *for every $x \in D(A_\varepsilon)$ and every $t, s \geqslant 0$,*

$$S_\varepsilon(t)x - S_\varepsilon(s)x = \int_s^t S_\varepsilon(\tau)A_\varepsilon x\,\mathrm{d}\tau = \int_s^t A_\varepsilon S_\varepsilon(\tau)x\,\mathrm{d}\tau.$$

Let $(S_\varepsilon)_\varepsilon$ and $(\tilde{S}_\varepsilon)_\varepsilon$ be representatives S, with infinitesimal generators $(A_\varepsilon)_\varepsilon$ and $(\tilde{A}_\varepsilon)_\varepsilon$, respectively. Then the previous proposition implies that, for every $a \in \mathbb{R}$ and $t \geqslant 0$,

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t} S_\varepsilon(t) - \tilde{A}_\varepsilon S_\varepsilon(t) \right\| = \mathcal{O}(\varepsilon^a), \quad \varepsilon \to 0.$$

Proof of theorem 2.9. Let ε be small enough and $x \in D(A_\varepsilon) = D(\tilde{A}_\varepsilon)$. Proposition A.1(d) and the chain rule imply that, for every $t \geqslant 0$, the mapping

$$s \mapsto \tilde{S}_\varepsilon(t-s)S_\varepsilon(s)x, \quad s \in [0,t],$$

is differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}s}(\tilde{S}_\varepsilon(t-s)S_\varepsilon(s)x) = -\tilde{A}_\varepsilon \tilde{S}_\varepsilon(t-s)S_\varepsilon(s)x + \tilde{S}_\varepsilon(t-s)A_\varepsilon S_\varepsilon(s)x, \quad s \in [0,t].$$

Assumption $A = \tilde{A}$ implies $A_\varepsilon = \tilde{A}_\varepsilon + R_\varepsilon$, where $(R_\varepsilon)_\varepsilon$ is a zero operator net ($[R_\varepsilon] = 0$). Since $\tilde{A}_\varepsilon$ commutes with $\tilde{S}_\varepsilon$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}s}(\tilde{S}_\varepsilon(t-s)S_\varepsilon(s)x) = \tilde{S}_\varepsilon(t-s)R_\varepsilon S_\varepsilon(s)x, \quad t \geqslant s \geqslant 0,$$

for every $x \in D(A_\varepsilon)$. After integration, this implies that

$$\tilde{S}_\varepsilon(t-s)S_\varepsilon(s)x - \tilde{S}_\varepsilon(t)x = \int_0^s \tilde{S}_\varepsilon(t-u)R_\varepsilon S_\varepsilon(u)x\,\mathrm{d}u, \quad t \geqslant s \geqslant 0.$$

Now let $s = t$. Then

$$S_\varepsilon(t)x - \tilde{S}_\varepsilon(t)x = \int_0^t \tilde{S}_\varepsilon(t-u)R_\varepsilon S_\varepsilon(u)x\,\mathrm{d}u, \quad t \geqslant 0,\ x \in D(A_\varepsilon). \tag{A 2}$$

Since $D(A_\varepsilon)$ is dense in E and all the operators in the above inequality are continuous, we have

$$S_\varepsilon(t)y - \tilde{S}_\varepsilon(t)y = \int_0^t \tilde{S}_\varepsilon(t-u)R_\varepsilon S_\varepsilon(u)y\,\mathrm{d}u, \quad t \geqslant 0,\ y \in E.$$

Let us prove that $(N_\varepsilon)_\varepsilon = (S_\varepsilon - \tilde{S}_\varepsilon)_\varepsilon \in \mathcal{S}N([0,\infty) : \mathcal{L}(E))$. Equality (A 2) and definition 2.4 imply that, for some $C > 0$ and $a, \tilde{a} \in \mathbb{R}$,

$$\begin{aligned}\sup_{t\in[0,T)} \|N_\varepsilon(t)\| &\leqslant \sup_{t\in[0,T)} \int_0^t \|\tilde{S}_\varepsilon(t-u)\|\|R_\varepsilon\|\|S_\varepsilon(u)\|\,\mathrm{d}u \\ &\leqslant TC\varepsilon^{a+\tilde{a}}\|R_\varepsilon\|.\end{aligned}$$

Thus $(N_\varepsilon(t))_\varepsilon$ satisfies (2.3) since $\|R_\varepsilon\| = \mathcal{O}(\varepsilon^b)$, as $\varepsilon \to 0$, for every $b \in \mathbb{R}$. Condition (2.4) follows from the boundedness of $(\tilde{S}_\varepsilon)_\varepsilon$ and $(S_\varepsilon)_\varepsilon$ on each bounded domain $[0, t_0)$, the properties of $(R_\varepsilon)_\varepsilon$ and the expression

$$\begin{aligned}\left\|\frac{N_\varepsilon(t)}{t}\right\| &= \left\|\frac{1}{t}\int_0^t \tilde{S}_\varepsilon(t-u)R_\varepsilon S_\varepsilon(u)\,\mathrm{d}u\right\| \\ &\leqslant \|\tilde{S}_\varepsilon(t)\|\|R_\varepsilon\|\|S_\varepsilon(t)\| \leqslant \text{const.}, \quad t \leqslant t_0.\end{aligned}$$

Also,

$$\lim_{t\to 0}\frac{N_\varepsilon(t)}{t}x = \lim_{t\to 0}\frac{\tilde{S}_\varepsilon(t)x - x}{t} - \lim_{t\to 0}\frac{S_\varepsilon(t)x - x}{t} = R_\varepsilon x, \quad x \in D(A_\varepsilon).$$

Thus, (2.5) holds for a dense subset of E, and this concludes the proof. □

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