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On Safety Metrics related to Aircraft Separation

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ICAO safety standards require low probabilities of collision of the order of 5×10^{-9} per hour. Such low probabilities are virtually impossible to confirm through simulations, implying the need for alternative and related safety metrics, which are easier to use. Several such alternative safety metrics are discussed in this paper. For example, it is shown that if the r.m.s. position error σ is less than about one-tenth $L/\sigma \sim 10$ of the minimum separation distance L, then the probabilities of collision will be less than the ICAO standard, in the case of aircraft flying in opposite directions on parallel tracks. This result is derived from an analytical formula based on the statistical application of a Gaussian probability distribution to a worst-case collision scenario and a rather different result $L/\sigma \sim 20$ arises for a Laplace probability distribution. The implication is that the ICAO standard of low probability of collision can be satisfied, by checking that the r.m.s. position error does not exceed a given value, though that value is dependent on the shape of the 'tail' of the probability distribution. Whether the condition is met can be checked in simulations, by measuring the drift between the intended and actual trajectory, due to all factors (navigation inaccuracy, atmospheric disturbances, trajectory drift between position updates, etc.). Thus the r.m.s. deviation from the intended trajectory can be used as a safety metric, as an alternative to the probability of collision, using the formulas and tables provided. The formulas concern a variety of possible safety metrics, including the maximum and cumulative probabilities of coincidence, probabilities of overlap, collision rates and collision probabilities; the tables apply to horizontal and vertical separation in controlled and transoceanic airspace. The sensitivity of the results to the probability distribution assumed (Gaussian or Laplace) suggests the introduction of a parametric family exponential of probability distributions, of which these are particular cases. A choice of the parameter is given that could lead to a probability distribution with a 'tail' shape more suited to typical Air Traffic Management (ATM) scenarios than the Gaussian and Laplace distributions, while being simpler than multi-parameter distributions.

KEY WORDS

1. Air. 2. Safety. 3. Modelling.

1. INTRODUCTION. One of the important aspects of future ATM scenarios[1] is to increase capacity without reducing safety.[2] This requires consideration of smaller separations[3] together with measures ensuring that the risk of collision is reduced and the need for collision avoidance manoeuvres[4] is not increased. The methods of calculation of probability of collision have been developed in considerable detail[5–7] involving both collection[8–10] and analysis[11–13] of traffic data. A good example is the Reduced Vertical Separation Minima (RVSM), halving the vertical separation in controlled air space from 2000 ft to 1000 ft, based on a careful study of collision

probabilities.[14] The latter was based on flight data on aircraft altitude deviations fitted by appropriate probability distributions,[15–23] generally non-Gaussian.[24–26] For the purpose of establishing a safety metric, the probability of collision is the most obvious choice, but it is not the only one. Other related parameters may be used as safety metrics,[27] which may be advantageous if they are easier to measure.

In order to identify potential safety metrics, the starting point is to review the methods of calculation collision probability, which identify the probability of deviation as an upper bound for some simple ATM scenarios; alternatively, the probability of deviation leads to a probability of coincidence and, through the aircraft size, to a probability of overlap. Use of the general probability distributions from the theory of statistics, identifies the r.m.s. position error as the essential safety parameter; it applies to double or multiple collisions and specifies the probability of deviation. The safety assessment should consider not only the probability of coincidence, but also the probability of overlap; the sensitivity of results to the case of different probability distributions should also be assessed. The preceding relations involve the minimum separation distance, which is chosen according to the ATM scenario, e.g. horizontal separation in controlled and transoceanic airspace, and vertical separation. The preceding one-dimensional assessments can be combined in three-dimensional estimates for comparison with the ICAO Target Level of Safety (TLS) standard. Given the sensitivity of the results to the probability distribution assumed, a parametric family is introduced, and its parameters chosen to match aircraft altitude deviation data.

2. SAFETY REQUIREMENTS FOR AIRCRAFT COLLISION AVOIDANCE.

A natural safety metric is the probability of collision, for which the probability of deviation may serve as an upper bound in some conditions. An alternative is to use the probability of deviation to calculate the probability of coincidence, leading to the probability of overlap via appropriate integrations.

2.1. *Safety volume and collisions due to penetration.* To each aircraft may be associated a 'safety volume', so that a collision between aircraft occurs when their safety volumes first touch. Thus the probability of collision depends on the intended flight paths of the aircraft, and the deviations from them, which could lead to their safety volumes overlapping. A simple approximation to the safety volume of an aircraft is a rectangle with sides equal to the length R_x, span R_y and height R_z of the aircraft. The exact safety volume would depend on the shape of the aircraft and its angular position (heading and bank angle) relative to other aircraft; the rectangular safety volume is a simple approximation that over-estimates slightly the collision risk (an even simpler but more crude single aircraft size parameter is the radius R of a sphere containing the aircraft).

The collision rate between two aircraft is given by the probability that the safety volume of one of the aircraft is penetrated on any side,[5] viz.:

$$P_r = P_x P_y N_z + P_y P_z N_x + P_x P_z N_y \equiv \sum_{ijk}^{xyz} P_i P_j N_k, \tag{1}$$

where: the (P_x, P_y, P_z) are the probabilities of separations of less than (R_x, R_y, R_z) respectively along track, across track and in altitude, and (N_x, N_y, N_z) the frequency with these separation reduce to less than (R_x, R_y, R_z); Σ_{ijk}^{xyz} means the sum of the three cyclic permutations of $(i, j, k) \equiv (x, y, z)$. The frequencies of penetration (N_x, N_y, N_z) are the probabilities of deviation (P_x, P_y, P_z) divided by the time periods (t_x, t_y, t_z) when the deviations exceed (R_x, R_y, R_z), viz.:

$$i \equiv x, y, z: \qquad N_i = P_i/t_i, \tag{2}$$

and thus:

$$P_r = P_x P_y P_z \left(\frac{1}{t_x} + \frac{1}{t_y} + \frac{1}{t_z} \right) = \prod_{j=1}^{3} P_j \sum_{i=1}^{3} \frac{1}{t_i}. \tag{3}$$

In order to obtain a collision rate per aircraft pair, this must be summed over the safety volume of the aircraft:

$$P_a = \int_0^{R_x} \mathrm{d}x \int_0^{R_y} \mathrm{d}y \int_0^{R_z} \mathrm{d}z \, P_r \left(x, y, z \right). \tag{4}$$

Note that the probabilities of deviation (P_x, P_y, P_z) have the dimensions of the inverse of length L^{-1}, the frequency of penetration (N_x, N_y, N_z) has dimensions of the inverse of length and time $L^{-1}T^{-1}$, the collision rate (3) has dimensions $L^{-3}T^{-1}$, and the collision probability (4) has the dimensions of the inverse time T^{-1}, and thus can be compared directly to the ICAO TLS standard.

If the collision rate (3) varies slowly over the aircraft size, the collision rate per aircraft pair (4) simplifies to:

$$P_a = R_x P_x R_y P_y R_z P_z \left(\frac{1}{t_x} + \frac{1}{t_y} + \frac{1}{t_z} \right). \tag{5}$$

The frequency of penetration is[5] approximately:

$$N_i = P_i \bar{V}_i / 2R_i, \tag{6}$$

where: $\bar{V}_i$ is the average rate of change of relative position between aircraft, and relates to the time spent at separation larger than R_i by (2) viz.:

$$t_i = 2R_i/\bar{V}_i. \tag{7}$$

Substituting (7) in (5) yields for the collision rate per aircraft pair is given by:

$$P_a = \tfrac{1}{2} P_x P_y P_z (\bar{V}_x R_y R_z + \bar{V}_y R_x R_z + \bar{V}_z R_x R_y) = \tfrac{1}{2} \prod_{l=1}^{3} P_l \sum_{ijk}^{xyz} \bar{V}_i R_j R_k. \tag{8}$$

The collision probability is the collision rate per pair multiplied by the time the aircraft spend in close proximity:

$$P_c = \sum_{prox} P_a T_a, \tag{9}$$

summed for all cases where aircraft fly by each other. This sum is dimensionless and will depend on the ATM scenario, viz. the geometry of flight paths and traffic flows along them. These general formulas will be illustrated in some simple cases in the sequel.

2.2. *Probability of deviation as an upper bound.* If it is assumed that along track, across track and altitude flight data deviations are statistically independent, then the collision probability in three dimensions is the product of three one-dimensional collision probabilities. The one-dimensional case of (1)≡(3), e.g. for along track deviations is that in which there are no across track or altitude penetrations, viz.:

$$N_y = 0 = N_z: \quad P_r = N_x P_y P_z = P_x P_y P_z / t_x. \tag{10}$$

The integration in (4) now applies only in dx, viz.:

$$P_a = R_y R_z \int_0^{R_x} P_r \, \mathrm{d}x, \tag{11}$$

and assuming that the collision rate varies slowly over the length of the aircraft, this is approximated by:

$$P_a = R_x R_y R_z P_r = P_x P_y P_z R_x R_y R_z / t_x. \tag{12}$$

Bearing in mind that the probabilities of deviation cannot exceed unity:

$$P_y R_y,\, P_z R_z \leqslant 1: \quad P_a \leqslant P_x R_x / t_x \tag{13}$$

the collision rate has the upper bound (13). Using (7) specifies the upper bound to the collision rate as:

$$P_a \leqslant \tfrac{1}{2} P_x \bar{V}_x. \tag{14}$$

For aircraft flying in parallel paths in:

(a) opposite directions, $\bar{V}_x^- = 2V$ is twice the airspeed:

$$\bar{V}_x^- = 2V: \quad P_a^- \leqslant P_x V; \tag{15}$$

(b) in the same direction $\bar{V}_x^+ = 2\Delta V$ is twice the speed error:

$$\bar{V}_x^+ = 2\Delta V: \quad P_a^+ \leqslant P_x \Delta V. \tag{16}$$

Note that the collision rate is much larger for aircraft flying in the opposite direction than for aircraft flying in the same direction:

$$P_a^+ / P_a^- \sim \Delta V / V \ll 1; \tag{17}$$

this is partly compensated by the aircraft spending much longer in the proximity of each other when the total collision probability is calculated, as will be shown in (24).

The result (15) shows that a bound for the collision rate for aircraft flying in opposite directions along parallel tracks is specified by the probability of deviation times the airspeed. Thus the collision probability per unit distance:

$$P_d \equiv P_a^- / V \leqslant P_x, \tag{18}$$

has for upper bound the probability of deviation. This is a very simple result, which supplies an estimate of the order of magnitude of collision probability[27], and will be used later. In the case of two aircraft on parallel tracks, the probability of collision (9) is the collision rate per pair P_a multiplied by the time the aircraft spend in proximity:

$$P_c = P_a T_a. \tag{19}$$

For the case of aircraft flying in opposite directions the proximity time is short:

$$T_a^- = R_x / 2V, \tag{20}$$

but the collision rate is high (15), leading to a collision probability:

$$P_c^- = P_a^- T_a^- = \tfrac{1}{2} P_x R_x \tag{21}$$

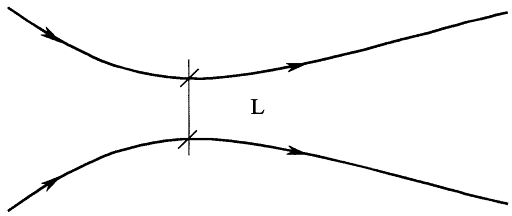

Figure 1. In the derivation of the equivalent safety metrics it may be assumed that, for the minimum separation distance L, the probability of collision is higher in 2-D than in 3-D.

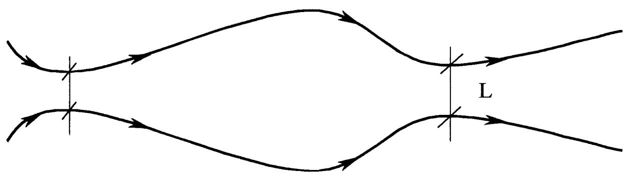

Figure 2. For an encounter geometry in 2-D, i.e. flight paths in the same plane, it may be assumed that the collision probability is higher the more often the two aircraft are at minimum separation distance, i.e. higher in Figure 2 than in Figure 1.

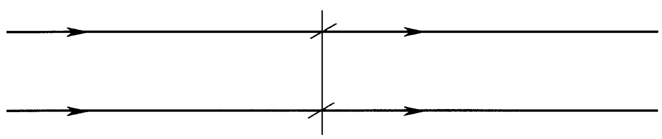

Figure 3. It follows from maximum probability of collision will be for two aircraft flying always at minimum separation distance, e.g. parallel paths, and it will be higher than the probability of triple or multiple collisions.

one-half the probability of deviation times aircraft size; for the case of aircraft flying in the same direction, the time spent in proximity is long, viz. the length l of the flight path divided by the speed:

$$T_a^+ = l/V, \tag{22}$$

but the collision rate is low (16), leading to a probability of collision:

$$P_c^+ = P_a^+ T_a^+ = P_x l \, \Delta V/V, \tag{23}$$

which equals the probability of deviation times the length of the flight path, times the velocity error divided by the speed. The probabilities of collision for aircraft flying in the same and opposite directions are in the ratio

$$P_c^+/P_c^- = 2(l/R_x)(\Delta V/V) \tag{24}$$

of twice the length of flight path divided by aircraft size, times the speed error divided by the speed, showing a competition of opposite effects: (i) the first factor is larger than unity for aircraft flying in the same direction, a longer distance than the aircraft size; (ii) the second factor is smaller then unity, with the speed error being a fraction of airspeed, smaller for more accurate flying.

2.3. *Probability of overlap and aircraft size*. The one-dimensional collision probability may be considered, taking as an example across track deviations, for aircraft flying in the same plane, with minimum lateral deviation L (Figure 1). The probability of collision is larger (Figure 2) the more often the aircraft are at minimum separation distance L. The worst case (Figure 3) is when both aircraft are always at the minimum separation distance L, which implies that they fly in the same direction at the same speed. If the probability of lateral deviation x is $P_1(x)$ for the first aircraft and $P_2(L-x)$ for the lateral deviation $L-x$ of the second aircraft, the probability of coincidence is:

$$P_{12}(x) = P_1(x)\,P_2(L-x). \tag{25}$$

The cumulative probability of coincidence for all possible lateral positions:

$$\bar{P}(L) = \int_{-\infty}^{+\infty} P_1(x) P_2(L-x)\,\mathrm{d}x \tag{26}$$

is the convolution of the probabilities of deviations of each aircraft. The probability of overlap is obtained approximately[14] by integrating over the aircraft size:

$$P_b \sim \int_0^R \bar{P}_x(y)\,\mathrm{d}y. \tag{27}$$

Substitution of (26) into (27) gives as approximate formula for the probability of overlap:

$$P_b \sim \int_0^R \mathrm{d}y \int_{-\infty}^{+\infty} \mathrm{d}x\; P_1(x)\,P_2(L-x), \tag{28}$$

to be compared with[14] the exact value:

$$P_b = \int_{-\infty}^{+\infty} \mathrm{d}x\; P_1(x) \int_{x-R}^{x+R} \mathrm{d}y\; P_2(y-x). \tag{29}$$

Note that the probability of deviation P_1, P_2 has the dimensions of inverse of length L^{-1}, the probability of coincidence (25) has the dimensions of inverse square of length L^{-2}, the cumulative probability of coincidence (26) has the dimensions inverse of length L^{-1}, and the probability of overlap (27) has no dimensions. The cumulative probability of coincidence (26) has the same dimensions as (18), allowing a comparison with the ICAO TLS standard. In order to consider the various probabilities associated with collisions in more detail, it is necessary to assume a probability distribution for the deviations.

3. USE OF GENERAL PROBABILITY DISTRIBUTIONS AND STATISTICS.

The central limit theorem of the theory of statistics suggests a Gaussian probability distribution for position error, although there are counter-arguments that make it advisable to consider alternative probability distributions. The Gaussian distribution can serve as an example that double collisions are more likely than multiple collisions; it also serves as example to calculate the maximum probability for coincidence between two aircraft.

3.1. *The effect of r.m.s. position error.* Consider next, two aircraft flying on parallel paths at a constant distance L (Figure 3) equal to the minimum separation distance. In this collision scenario, the aircraft can at all times drift into positions less than a minimum separation distance apart. The minimum separation distance is in general smaller in the vertical direction than in the lateral or longitudinal direction; a similar analysis would apply to aircraft on the same flight path with a given longitudinal separation. In general, it will be assumed that along track, across track and altitude errors are statistically independent. Thus the three-dimensional collision

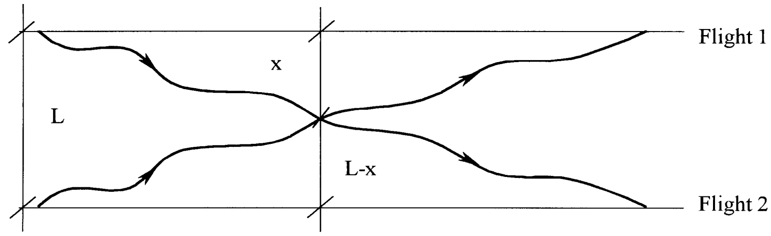

Figure 4. In the case of aircraft flying on parallel paths a minimum separation distance L, a coincidence will occur if the lateral position errors are x and $L-x$, for any value of x.

problem is decoupled into three one-dimensional collision problems. Each may have different parameters, e.g. separation distances but the basic analysis is the same.

A convenient assumption would be that the position error satisfies Gaussian statistics for both aircraft. Note that the central limit theorem of the theory of probability [25] indicates that a long sequence N of statistically independent events, in this case position errors, tend to a Gaussian distribution with an accuracy of order $1/\sqrt{N}$, if the Lindeberg condition[26] is met, that events with large separation make a small contribution to the total variance. These two conditions, viz. (i) Lindeberg and (ii) large number of events can be questioned: aircraft collisions are extremely rare events, involving large deviations from the mean. Thus the number of statistically independent events may not be enough to justify a law of large numbers. Also, collisions correspond to the 'tails' of the probability distribution, i.e. the large deviations, which the Lindeberg condition assumes to make a small contribution to the variance. The theoretical counter-arguments to the Gaussian distribution seem to be supported by observations of navigation errors,[14] which suggest[15–23] that some form of generalized exponential distribution could be more appropriate. In order to assess the sensitivity of results to the assumed probability distribution, the Gaussian is considered first, then the Laplace, and then an exponential parametric family including both is considered. Starting with the Gaussian case, the probability of the first aircraft having a lateral position error x is:

$$P_1(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{x^2}{2\sigma^2} \right], \tag{30}$$

where: σ is the r.m.s. position error.

3.2. *Case of double vs. multiple coincidences.* The Gaussian probability distribution (30) is taken with zero mean, because the average position error should be zero, i.e. the mean flight path is a straight line. The Gaussian probability distribution with zero mean (30) involves a single parameter σ, which is the r.m.s. position error due to all factors affecting the aircraft flight path, e.g. atmospheric disturbances, inaccuracies of the navigation system, pilot or controller distraction, aircraft drift between position fixes, etc. The same Gaussian statistics are assumed to apply to the second aircraft, so

$$P_2(x) = P_1(L-x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(L-x)^2}{2\sigma^2} \right], \tag{31}$$

is the probability it has a lateral position error $L-x$, leading (Figure 4) to a coincidence with the first aircraft at position x, if both aircraft were represented as

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mass points. Taking into account finite aircraft size, e.g. R_1 for the first and R_2 for the second, means that a collision occurs when the centres of mass are at a distance of not more than $R \equiv (R_1 + R_2)/2$; this approach will be followed subsequently with the Laplace probability distribution. In the present case of the Gaussian distribution, it will be assumed that the effect of aircraft size is to reduce the minimum separation distance from L to $L - R$ (the conditions of validity of this assumption will be discussed later); in this case, the aircraft size is more important if it is not negligible relative to the separation, and less important otherwise.

Using the preceding assumptions, the probability of coincidence of the two aircraft at position x is thus the product of the two probabilities.

$$P_{12}(x) = P_1(x)\,P_2(x) = P_1(x)\,P_1(L-x). \tag{32}$$

The probability of coincidence of three aircraft

$$P_{123}(x) = P_1(x)\,P_2(x)\,P_3(x) \leqslant P_{12}(x), \tag{33}$$

could never exceed the probability of coincidence of two aircraft, since $P_3(x) \leqslant 1$. Thus the highest coincidence probability remains that specified by (32) for two aircraft.

3.3. *Position for the highest probability of coincidence.* From (30), (31) and (32), it follows that the probability that two aircraft, flying at minimum separation distance L, with Gaussian statistics with the same r.m.s. position error σ, coincide at position x is given by:

$$P_{12}(x) = \frac{1}{2\pi\sigma^3} \exp\left[-\frac{x^2 + (L-x)^2}{2\sigma^2} \right]. \tag{34}$$

Note that the coincidence is possible for any lateral position x from $\pm\infty$ i.e. $-\infty < x < +\infty$. The extremum in the probability of collision occurs if the first derivative is zero:

$$0 = \frac{\mathrm{d}P_{12}}{\mathrm{d}x}(x) = -\frac{1}{2\pi\sigma^4}(2x - L)\exp\left[-\frac{x^2 + (L-x)^2}{2\sigma^2} \right], \tag{35}$$

i.e. the extremum in the coincidence probability occurs for $2x - L = 0$, i.e. $x = L/2$ halfway between the two parallel paths (Figures 5 and 6).

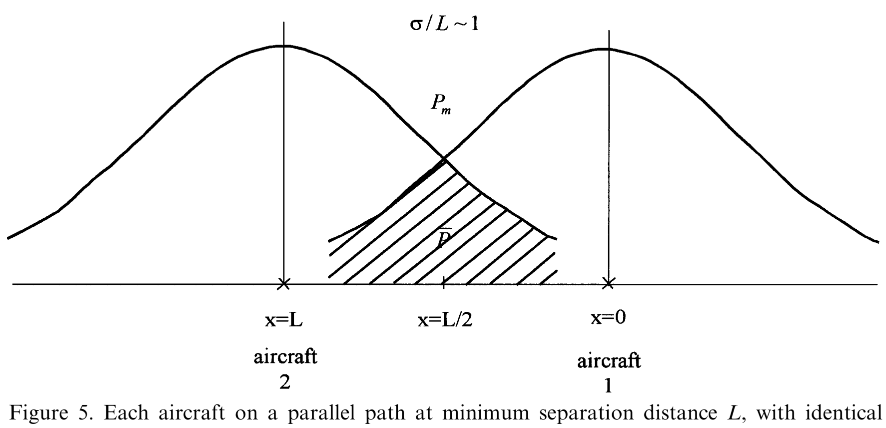

Figure 5. Each aircraft on a parallel path at minimum separation distance L, with identical Gaussian distributions of position error, leads to a maximum probability of coincidence P_m at mid-position, and cumulative coincidence probability $\bar{P}$ which depends on the region of overlap.

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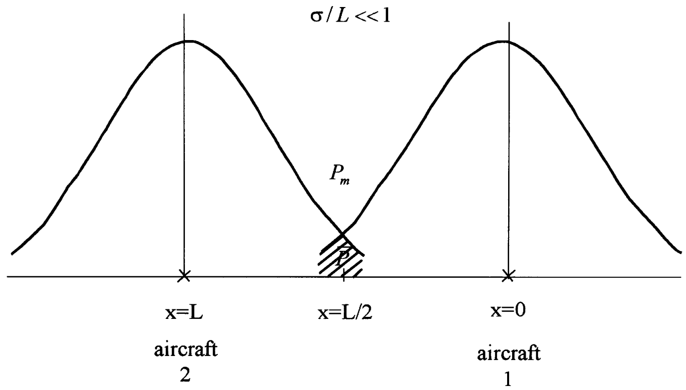

Figure 6. Both the maximum probability of coincidence per square mile P_m, and the cumulative probability of coincidence per mile $\bar{P}$, reduce as the r.m.s. position error σ becomes smaller relative to minimum separation distance L, ensuring that the ICAO Target Level of Safety is met for values of less than about one-tenth $\sigma/L \leqslant 0{\cdot}1$, if Gaussian statistics are used.

Thus the position for extreme probability of coincidence is $x = L/2$, and is given by:

$$x = L/2: \quad P_m = P_{12}(L/2) = \frac{1}{2\pi\sigma^2} \exp\left[-\left(\frac{L}{2\sigma} \right)^2 \right], \tag{36}$$

which is the extremum for the probability of coincidence. This extremum is actually a maximum, because the second derivative:

$$\frac{d^2 P_{12}}{dx^2}(x) = -\frac{1}{\pi\sigma^4}\left[1 - \left(\frac{2x-L}{2\sigma^2} \right)^2 \right] \exp\left[-\frac{x^2+(L-x)^2}{2\sigma^2} \right], \tag{37}$$

is negative at a mid-position:

$$\frac{d^2 P_{12}}{dx^2}\left(\frac{L}{2} \right) = -\frac{1}{\pi\sigma^4} \exp\left[-\frac{L^2}{2\sigma^2} \right] < 0, \tag{38}$$

Thus the maximum probability of coincidence (36) occurs at the mid-position between the two flight paths, as should be expected for two aircraft with identical r.m.s. position errors σ.

4. MAXIMUM AND CUMULATIVE PROBABILITIES OF COINCIDENCE OR OVERLAP. Since coincidences can occur at all positions, i.e. not only at the position of maximum probability, it is appropriate to calculate also a cumulative probability of coincidence over all possible positions of coincidence. The latter leads to a formula relating the probability of coincidence to the r.m.s. position error, for a given minimum separation distance. In order to assess the sensitivity of the coincidence probabilities to the choice of distributions, an exponential is considered instead of Gaussian, as well as integration leading to the probability of overlap.

4.1. *Cumulative probability of coincidence for all positions.* The highest probability of coincidence (36) is not the only relevant result, because coincidences can also occur for other positions $x \neq L/2$, even if they are less likely; in fact, the probability of coincidence (34) decays rapidly for x greater than L, but remains close to the maximum for x close to $L/2$. One way to address this aspect is to consider the total of cumulative probability of coincidence, summed or integrated over all possible positions $-\infty < x < +\infty$ transverse to the flight paths and in the same plane, viz.:

$$\bar{P} \equiv \int_{-\infty}^{+\infty} P_{12}(x)\,\mathrm{d}x, \tag{39}$$

which is specified (34) by:

$$\bar{P} = \frac{1}{2\pi\sigma^2}\int_{-\infty}^{+\infty} \exp\left[-\frac{2x^2+L^2-2xL}{2\sigma^2}\right]\mathrm{d}x \tag{40}$$

and thus depends only on separation distance L and r.m.s. position error σ. In order to determine explicitly this relation $\bar{P}(L, \sigma)$, the integral over x in (40) must be evaluated.

The change of variable:

$$y \equiv (x-L/2)/\sigma, \qquad \mathrm{d}x = \sigma\mathrm{d}y, \tag{41a, b}$$

leads to:

$$y^2 = (2x^2-2xL+L^2/2)/(2\sigma^2) = (2x^2-2xL+L^2)/(2\sigma^2)-L^2/4\sigma^2, \tag{42}$$

and transforms (41) into:

$$\bar{P} = \frac{1}{2\pi\sigma}\exp\left[-\frac{L^2}{4\sigma^2}\right]\int_{-\infty}^{+\infty}\exp[-y^2]\mathrm{d}y, \tag{43}$$

where the last factor:

$$\int_{-\infty}^{+\infty}\exp[-y^2]\mathrm{d}y = \sqrt{\pi}, \tag{44}$$

is the well-known Gaussian integral.

4.2. *Probability of coincidence versus separation distance and r.m.s. position error.* Substituting (44) into (43) leads to:

$$\bar{P} = \frac{1}{2\sigma\sqrt{\pi}}\exp\left[-\left(\frac{L}{2\sigma}\right)^2\right], \tag{45}$$

namely, the cumulative probability of coincidence $\bar{P}$ as a function of separation distance L and r.m.s. position error σ due to all causes. For a fixed separation distance L, it is possible to use as a safety metric, instead of the cumulative probability of coincidence $\bar{P}$, the r.m.s. position error σ.

Before proceeding to evaluate the result (45), it may be worthwhile to recall the eight assumptions related to its derivation, namely that:

(a) the probability of coincidence in 2-D is higher than in 3-D;
(b) the probability of coincidence in 2-D is highest for paths spaced by the minimum separation distance L at all times;
(c) the probability of coincidence between two aircraft is higher than that between three or more aircraft;
(d) the probability of lateral position error is specified by Gaussian statistics with zero mean and variance σ^2;

(e) that along track, across track and altitude errors are statistically independent, so that each can be treated as a separate one-dimensional problem, with different parameters, e.g. separation distances;
(f) that the effect of finite aircraft sizes R_1 and R_2, is to reduce the minimum separation distance from L to $L-R$, with $R \equiv (R_1+R_2)/2$ specifying the distance of closest approach of the two aircraft;
(g) that the aircraft fly in unbounded airspace without altitude or terrain limitations, or restricted areas, which may be true in cruise, but not in other flight phases;
(h) the aircraft dynamics is not taken into account, allowing arbitrary displacements, and thus leading to an over-estimate of the probability of coincidence, i.e. to upper bounds.

Since several of the assumptions have already been discussed, one may proceed to consider (d) and (g); they are re-considered next, to assess the extent to which they affect coincidence probabilities.

4.3. *Effect of aircraft size and Laplace distribution on probability of overlap*. There are both theoretical and observational[14, 24] counter-arguments to the use of a Gaussian probability distribution for position errors, and a Laplace distribution has often been used[5–7] instead:

$$P_0(x) = \frac{1}{\sigma\sqrt{2}} \exp\left[-\sqrt{2}\frac{|x|}{\sigma} \right], \tag{46}$$

where: σ is again the r.m.s. position error. The coincidence probabilities are mainly affected by large deviations $x \gg \sigma$, i.e. the 'tails' of the probability distribution; since the Laplace distribution (46) decays more slowly than the Gaussian (30), it should lead to larger probabilities of coincidence. Besides this difference from the preceding calculation, the joint probability that aircraft one is at position x and aircraft two is at position y will be considered:

$$P_0(x, y) = P_0(x)\,P_0(y) = \frac{1}{2\sigma^2} \exp\left[-\frac{\sqrt{2}}{\sigma}(|x|+|y|) \right], \tag{47}$$

so that an overlap occurs if the separation $L-x-y$ is less than half the sum $R \equiv (R_1+R_2)/2$ of aircraft sizes R_1 and R_2:

$$\tilde{P} = \int_{|L-y-x| \leqslant R} P_0(x, y)\,\mathrm{d}y; \tag{48}$$

substitution of (47) in (48), for the case $x, y > 0$:

$$\tilde{P} = \frac{e^{\frac{-\sqrt{2}x}{\sigma}}}{2\sigma^2} \int_{L-x-R}^{L-x+R} e^{\frac{-\sqrt{2}y}{\sigma}}\mathrm{d}y, \tag{49}$$

leads to a result:

$$\tilde{P} = \frac{e^{\frac{-\sqrt{2}x}{\sigma}}}{2\sqrt{2}\sigma} \left\{ e^{-\sqrt{2}\frac{L-x-R}{\sigma}} - e^{-\sqrt{2}\frac{L-x+R}{\sigma}} \right\}, \tag{50}$$

which is independent of x:

$$\tilde{P} = \frac{1}{\sigma\sqrt{2}} e^{\frac{-\sqrt{2}L}{\sigma}} \sinh\left[\frac{\sqrt{2}R}{\sigma}\right]. \tag{51}$$

This result may be simplified for $\xi \equiv \sqrt{2}L/\sigma$ large or small, noting that $\sinh\xi \sim \xi$ for $\xi \ll 1$ and $\sinh\xi \sim e^{\xi}/2$ for $\xi \gg 1$, so that:

(a) if the aircraft size is small relative to the r.m.s. position error, which is the usual case, the probability of overlap:

$$R\sqrt{2} \ll \sigma: \quad \tilde{P} \sim \frac{R}{\sigma^2} e^{-\sqrt{2}\frac{L}{\sigma}}, \tag{52a}$$

gains a small factor $R/\sigma \ll 1$ and is proportional to aircraft size;

(b) in the opposite case, which is less likely, of aircraft size large relative to the r.m.s. position error, in the probability of overlap:

$$R\sqrt{2} \gg \sigma: \quad \tilde{P} \sim \frac{1}{2\sqrt{2}\sigma} e^{\frac{-\sqrt{2}(L-R)}{\sigma}}, \tag{52b}$$

the aircraft size subtracts from the separation.

If the aircraft size is much smaller than the separation $L \gg R$, then the cumulative joint probability becomes independent of aircraft size. Since the cumulative joint probability (51) is independent of position, the product by the aircraft size specifies (27) the probability of overlap:

$$P_b = R\tilde{P} = \frac{R}{\sigma\sqrt{2}} e^{-\sqrt{2}\frac{L}{\sigma}} \sinh\left(\frac{\sqrt{2}R}{\sigma}\right), \tag{53}$$

which is dimensionless. The Laplace distribution (46) leads to the cumulative joint probability (51), which is generally larger than the cumulative probability of coincidence (45) associated with the Gaussian distribution (30), viz.:

(a) the main difference is the argument of the exponential, viz. $\sqrt{2}L/\sigma$ in (51) and one-eighth the square $(L/2\sigma)^2 = (\sqrt{2}L/\sigma)^2/8$ in (45), the latter being much larger for $L/\sigma \gg 1$, and leading to smaller probabilities;
(b) the multiplying factor is also larger $1/(\sigma\sqrt{2}) \approx 0{\cdot}707/\sigma$ for the Laplace (51) than for the Gaussian $1/(2\sigma\sqrt{\pi}) \approx 0{\cdot}282/\sigma$ distribution (45).

5. APPLICATION TO SEPARATION IN CONTROLLED AND UNCONTROLLED AIRSPACE.

The difference between the Gaussian and Laplace distributions will become apparent in the application to vertical separation and horizontal separation in controlled and uncontrolled airspace.

5.1. *Application to horizontal separation in controlled airspace.* Gaussian cumulative probability of coincidence (45) is most convenient use in logarithmic form:

$$\log \bar{P} = -[L/(2\sigma)]^2 - \log\sigma - \log 2 - \tfrac{1}{2}\log\pi, \tag{54}$$

where: the constant values can be inserted:

$$\log \bar{P} = -0{\cdot}25(L/\sigma)^2 - \log\sigma - 1{\cdot}2655. \tag{55}$$

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Taking $L_h = 5$ nm for the lateral separation distance specified by ICAO in controlled airspace leads to:

$$\log \bar{P}_h = -6{\cdot}25/\sigma_h^2 - \log \sigma_h - 1{\cdot}2655, \tag{56}$$

as the relation between cumulative probability of coincidence $\bar{P}_h$ and r.m.s. position error σ_h in Table 1. It can be seen from the table that large r.m.s. horizontal position

Table 1. Equivalent safety metrics for five nautical mile horizontal separation.

Probability distribution	Gaussian		Laplace	
r.m.s. position error	Cumulative probability of coincidence	Maximum probability of coincidence	Exponential factor	Cumulative joint probability
σ_h (nautical miles)	$\bar{P}_h$ (per-nautical mile)	P_{mh} (per-square nautical mile)	$\tilde{P}_{fh}$ (per nautical mile)	$\tilde{P}_h$ (per nautical mile)
3	$4{\cdot}70 \times 10^{-2}$	$8{\cdot}83 \times 10^{-3}$	$4{\cdot}46 \times 10^{-2}$	$6{\cdot}92 \times 10^{-4}$
2	$2{\cdot}96 \times 10^{-2}$	$8{\cdot}34 \times 10^{-3}$	$2{\cdot}06 \times 10^{-2}$	$4{\cdot}79 \times 10^{-4}$
1	$5{\cdot}45 \times 10^{-4}$	$3{\cdot}07 \times 10^{-4}$	$1{\cdot}20 \times 10^{-3}$	$5{\cdot}58 \times 10^{-5}$
0·9	$1{\cdot}40 \times 10^{-4}$	$8{\cdot}76 \times 10^{-5}$	$6{\cdot}08 \times 10^{-4}$	$3{\cdot}14 \times 10^{-5}$
0·8	$2{\cdot}02 \times 10^{-5}$	$1{\cdot}43 \times 10^{-5}$	$2{\cdot}56 \times 10^{-4}$	$1{\cdot}49 \times 10^{-5}$
0·7	$1{\cdot}16 \times 10^{-6}$	$9{\cdot}38 \times 10^{-7}$	$8{\cdot}29 \times 10^{-5}$	$5{\cdot}51 \times 10^{-6}$
0·6	$1{\cdot}36 \times 10^{-8}$	$1{\cdot}28 \times 10^{-8}$	$1{\cdot}80 \times 10^{-5}$	$1{\cdot}40 \times 10^{-6}$
0·5	$7{\cdot}83 \times 10^{-12}$	$8{\cdot}84 \times 10^{-12}$	$2{\cdot}04 \times 10^{-6}$	$1{\cdot}90 \times 10^{-7}$
0·4	$7{\cdot}65 \times 10^{-18}$	$1{\cdot}08 \times 10^{-17}$	$7{\cdot}43 \times 10^{-8}$	$8{\cdot}66 \times 10^{-9}$
0·3	$6{\cdot}51 \times 10^{-31}$	$1{\cdot}23 \times 10^{-30}$	$2{\cdot}73 \times 10^{-10}$	$4{\cdot}96 \times 10^{-11}$
0·2	$1{\cdot}95 \times 10^{-68}$	$5{\cdot}51 \times 10^{-68}$	$3{\cdot}12 \times 10^{-15}$	$7{\cdot}32 \times 10^{-30}$
0·1	$1{\cdot}04 \times 10^{-133}$	$5{\cdot}85 \times 10^{-129}$	$2{\cdot}76 \times 10^{-30}$	$1{\cdot}33 \times 10^{-30}$

errors σ_h of 2 to 3 nm, still below the minimum separation distance $L_h = 5$ nm, give high cumulative probabilities of coincidence. A smaller r.m.s. position error of the order 0·7 to 1 nm would lead to lower probabilities of coincidence (10^{-4} to 10^{-6}), which could be tested in simulations; the aim of these simulations would be to check that the formula (56) for the cumulative probability of coincidence. The main use of the formula (56) is for smaller r.m.s. position errors, as it shows that a value of $\sigma \leqslant 0{\cdot}5$ nm leads to a cumulative probability of coincidence of less than $\bar{P}_h \leqslant 7{\cdot}83 \times 10^{-12}$ per nautical mile; for an aircraft cruising at a speed not exceeding $V = 600$ kt, this leads to a cumulative probability of collision $\bar{P}_h V \leqslant 4{\cdot}7 \times 10^{-9}$ per hour, which meets the ICAO requirement of less than 5×10^{-9} in the conditions for which (18) holds.

Specifying a larger r.m.s. position error quickly increases the probability of collision, e.g. to $\bar{P}_h \leqslant 1{\cdot}36 \times 10^{-8}$ for $\sigma_h = 0{\cdot}6$ nm. A smaller r.m.s. position error decreases coincidence probability to minute levels, e.g. $\bar{P}_h \leqslant 6{\cdot}51 \times 10^{-31}$ for $\sigma = 0{\cdot}3$ nm, but this places an unnecessarily severe demand on position accuracy, which could be costly to meet in terms of aircraft on-board equipment and hard to comply with by the ground-based ATM system. The r.m.s. position error $\sigma_h = 0{\cdot}5$ nm of one-tenth the separation distance $L_h = 5 \text{ nm} = 10\sigma_h$, is a fairly robust result for meeting the ICAO target safety level (TLS), since:

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(a) a larger r.m.s. position error will rapidly increase the cumulative probability of coincidence to unacceptable levels; and
(b) a smaller r.m.s. position error reduces the cumulative probability of coincidence, but is unnecessary.

The ICAO target level of safety (TLS) of low probability of collision (5×10^{-9} per hour) can be obtained, with a five nautical mile minimum separation distance $L_h = 5$ nm, by requiring a $\sigma_h = 0{\cdot}5$ nm r.m.s. position error; the latter leads to a cumulative probability of coincidence not exceeding $\bar{P}_h \leqslant 7{\cdot}83\times10^{-12}$ per nautical mile flown, or $\bar{P}_h D \leqslant 1{\cdot}69\times10^{-7}$ for a $D = 40000$ km $= 21587$ nm flight around the earth on a great circle. This low upper bound for the cumulative probability of coincidence makes it unnecessary to demand a higher r.m.s. position accuracy. To degrade the r.m.s. position accuracy would quickly lead to much higher probabilities of coincidence. The safety standard of 0·5 nm r.m.s. position error should include all causes of position error, e.g. inaccuracy of the navigation system, effects of atmospheric disturbances, trajectory drift between updates of position fixes, etc. The 0·5 nm r.m.s. position error is easy to use as a safety metric in simulations: it just requires calculation of the r.m.s. deviation from the desired flight path. The preceding discussion has been based on the Gaussian cumulative probability of coincidence (56) per nautical mile flown by one aircraft. It is also possible to use the Gaussian maximum probability of coincidence (36) per square nautical mile, i.e. per nautical mile flown by each aircraft:

$$P_m = (0{\cdot}1592/\sigma^2)\exp[-0{\cdot}25(L/\sigma)^2], \tag{57}$$

or for five nautical mile minimum separation distance:

$$P_{mh} = (0{\cdot}1592/\sigma_h^2)\exp(-6{\cdot}25/\sigma_h^2). \tag{58}$$

The values of the Gaussian maximum probability of coincidence (58) are indicated in Table 1, for the same values of r.m.s. position error σ_h as used for the cumulative probability of coincidence (56). For the previously recommended r.m.s. position error not exceeding $\sigma \leqslant 0{\cdot}5$ nm, the maximum probability of coincidence would be $P_{mh} \leqslant 8{\cdot}84\times10^{-12}$ per nautical mile flown by each aircraft. For a $D = 21587$ nm flight around the earth on a great circle, the maximum probability of coincidence would be $P_{mh}\mathrm{D}^2 \leqslant 4{\cdot}12\times10^{-3}$.

Table 1 also includes in the fourth column the factor in the exponential joint probability (51) that is independent of aircraft size:

$$\tilde{P}_f = \frac{1{\cdot}4142}{\sigma}\exp\left[-1{\cdot}4142\frac{L}{\sigma}\right], \tag{59}$$

in the case of five nautical mile lateral separation:

$$\tilde{P}_{fh} = \frac{1{\cdot}4142}{\sigma}\exp[-7{\cdot}0711/\sigma]. \tag{60}$$

The aircraft size appears in the exponential joint probability (51):

$$\tilde{P} = \tilde{P}_f \sinh[1{\cdot}4142R/\sigma], \tag{61}$$

indicated in the fifth column of Table 1 for an aircraft span $R_y = 200$ ft $= 60{\cdot}8$ m $= 3{\cdot}29\times10^{-2}$nm:

$$\tilde{P}_h = \tilde{P}_{fh}\sinh[4{\cdot}6525\times10^{-2}/\sigma]. \tag{62}$$

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Table 2. Cumulative and maximum probabilities of deviation for vertical separation.

Probability distribution	Gaussian		Laplace	
Vertical separation error	Cumulative probability of coincidence	Maximum probability of coincidence	Exponential factor	Cumulative joint probability
σ_v (*ft*)	$\bar{P}_v$ (per-nautical mile)	P_{mv} (per-square nautical mile)	$\tilde{P}_{fv}$ (per nautical mile)	$\tilde{P}_v$ (per nautical mile)
500	1·26	8·66	1·02	$1·44 \times 10^{-1}$
400	$8·99 \times 10^{-1}$	7·70	$6·26 \times 10^{-1}$	$1·11 \times 10^{-1}$
300	$3·55 \times 10^{-1}$	4·06	$2·57 \times 10^{-1}$	$6·80 \times 10^{-2}$
200	$1·65 \times 10^{-2}$	$2·84 \times 10^{-1}$	$3·65 \times 10^{-2}$	$1·32 \times 10^{-2}$
100	$2·38 \times 10^{-10}$	$8·17 \times 10^{-9}$	$6·20 \times 10^{-5}$	$4·76 \times 10^{-5}$
90	$7·51 \times 10^{-13}$	$2·86 \times 10^{-11}$	$1·43 \times 10^{-5}$	$1·24 \times 10^{-5}$
80	$2·33 \times 10^{-16}$	$9·97 \times 10^{-15}$	$2·26 \times 10^{-6}$	$2·27 \times 10^{-6}$
70	$1·70 \times 10^{-21}$	$8·35 \times 10^{-20}$	$2·07 \times 10^{-7}$	$2·47 \times 10^{-7}$
60	$1·98 \times 10^{-29}$	$1·13 \times 10^{-27}$	$8·32 \times 10^{-9}$	$1·22 \times 10^{-8}$
50	$1·28 \times 10^{-42}$	$8·75 \times 10^{-41}$	$8·95 \times 10^{-11}$	$1·73 \times 10^{-10}$
40	$1·28 \times 10^{-67}$	$4·66 \times 10^{-66}$	$9·50 \times 10^{-14}$	$2·70 \times 10^{-13}$
30	$1·32 \times 10^{-119}$	$1·98 \times 10^{-117}$	$9·65 \times 10^{-19}$	$5·05 \times 10^{-18}$

It can be seen in Table 1 that the correction for aircraft size (62) is more important for large r.m.s. position error and not very significant for small σ. Thus the Laplace joint probability is dominated, for small r.m.s. position error σ, by the exponential factor (60), which is much larger than the Gaussian. The r.m.s. position error was taken as low as $\sigma = 0·1$ nm, because a small Laplace joint probability requires a smaller value of σ than a comparable Gaussian cumulative probability. A r.m.s. position error of $\sigma = 0·25$ nm or $L/\sigma = 20$ is needed for a Laplace exponential factor $\tilde{P}_{fh} = 2·94 \times 10^{-12}$ per nautical mile, and a cumulative joint probability $\tilde{P}_h = 5·50 \times 10^{-13}$ per nautical mile; this meets the ICAO TLS standard of $S = 5 \times 10^{-9}$ per hour, for speeds up to $V = S/\tilde{P}_h = 9·09 \times 10^3$ kt, which substantially exceeds the speed capability of current airliners.

5.2. *Application to reduced vertical separation minima.* The values indicated in Table 1 are calculated from (56), (58), (60) and (62), and apply to a five nautical mile minimum horizontal separation distance in controlled airspace; the formulae (55), (57), (59) and (61) could also be applied to other minimum separation distances, e.g. to the vertical instead of the horizontal separation distance. The horizontal separation distance $L_h = 5$ nm used in (56, 58, 60, 62) is that which applies to flight in controlled air space, for which the vertical separation is $L_v = 1000$ ft $= 0·1645$ nm at lower flight levels (below FL 290); the same vertical separation distance is being applied for higher flight levels[14] where the earlier value of 2000 ft is being replaced by the RVSM (Reduced Vertical Separation Minima) of $L_v = 1000$ ft $= 305$ m $= 0·1645$ nm. Using this value in (55) specifies the cumulative probability of coincidence due to error in vertical position σ_v:

$$\log \bar{P}_v = -6·7631 \times 10^{-3}/\sigma_v^2 - \log \sigma_v - 1·2655, \quad (63)$$

which is indicated in Table 2. The cumulative probability of coincidence is relatively large for vertical separation error of more than 200 ft and can even exceed unity, since

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the values indicated are upper bounds. For a vertical separation error of $\sigma = 100$ ft, the cumulative probability of coincidence does not exceed $\bar{P}_v \leqslant 2{\cdot}38 \times 10^{-10}$ per nautical mile or $P_v D \leqslant 5{\cdot}14 \times 10^{-6}$ for a great circle tour of the earth. The recommended r.m.s. error for vertical separation is smaller $\sigma_v = 90$ ft, i.e. about one-eleventh of the minimum vertical separation $L_v = 1000$ ft, i.e. $\sigma_v/L_v = 0{\cdot}09$. Note that the recommended r.m.s. horizontal position error $\sigma_h = 0{\cdot}5$ nm was one-tenth $\sigma_h/L_h = 0{\cdot}1$ of the minimum horizontal separation $L_h = 5$ nm. The reason for a smaller relative value here is that a r.m.s. vertical position error $\sigma_v = 90$ ft leads to an upper bound for the cumulative probability of coincidence $\bar{P}_v \leqslant 7{\cdot}51 \times 10^{-13}$per nautical mile; for the fastest commercial aircraft (Concorde), which cruises at a speed of $V = 1166$ kt, the cumulative probability of collision does not exceed $\bar{P}_v V \leqslant 8{\cdot}76 \times 10^{-10}$ per hour, which meets the ICAO TLS standard of less than 5×10^{-9} per hour, in the condition for which (18) holds.

The Gaussian maximum probability of coincidence per square mile (57), can also be calculated for the reduced vertical separation minima of $L_v = 1000$ ft $= 0{\cdot}1645$ nm:

$$P_{mv} = (0{\cdot}1592/\sigma_v^2)\exp(-6{\cdot}7631 \times 10^{-3}/\sigma_v^2), \tag{64}$$

and is also indicated in Table 2. For the recommended r.m.s. vertical position error of $\sigma_v = 90$ ft, the maximum probability of coincidence is $P_{mv} \leqslant 2 \times 10^{-11}$ per nautical mile flown by each aircraft, or $P_{mv} D^2 \leqslant 1{\cdot}33 \times 10^{-2}$ for a great circle tour of the earth suggesting that a smaller r.m.s. position error be considered. For two aircraft with a cruise speed not exceeding $V \leqslant 547$ kt, the maximum probability of coincidence $P_{mv} \leqslant 9{\cdot}97 \times 10^{-15}$ per square nautical mile, for a $\sigma_v = 80$ ft vertical r.m.s. position error, leads to an upper bound for the maximum probability of coincidence of $P_{mv} V^2 \leqslant 2{\cdot}98 \times 10^{-9}$ per flight hour squared, below the ICAO modified TLS value of 5×10^{-9}per hour squared.

The Laplace exponential factor (59) for the same vertical separation is given by:

$$\tilde{P}_{fv} = \frac{1{\cdot}4142}{\sigma} \exp\left[-\frac{0{\cdot}2326}{\sigma} \right], \tag{65}$$

per nautical mile, and is not affected by aircraft size; the values in Table 2 show that the Gaussian cumulative probability of coincidence is much smaller than for the Laplace exponential factor. The aircraft size enters through the factor (61) in the cumulative joint probability:

$$\tilde{P}_v = \tilde{P}_{fv} \sinh\,[1{\cdot}1631 \times 10^{-2}/\sigma], \tag{66}$$

where the aircraft size was taken to be the height $R_z = 50$ ft $= 15{\cdot}2$ m $= 8{\cdot}2245 \times 10^{-3}$ nm. The correction for aircraft size is less than unity for small r.m.s. altitude error $\sigma < 90$ ft and becomes larger than unity for $\sigma > 90$ ft. A rather smaller r.m.s. altitude error $\sigma_h = 40$ ft, which is very small compared to the separation $L_v/\sigma_v = 25$ leads to a Laplace cumulative joint probability $\tilde{P}_h = 2{\cdot}70 \times 10^{-13}$ per nautical mile, which is consistent with the ICAO TLS standard $S = 5 \times 10^{-9}$ per hour for speeds up to $V = S/\tilde{P}_h = 1{\cdot}85 \times 10^4$ kt.

5.3. *Application to horizontal separation for transoceanic routes.* The vertical separation $L_v = 1000$ ft applies both to flight in controlled air space and to transoceanic routes at lower flight levels. The lateral separation for the latter is

Table 3. Application to horizontal separation in transoceanic routes.

Probability distribution	Gaussian		Laplace	
Vertical separation error	Cumulative probability of coincidence	Maximum probability of coincidence	Exponential factor	Cumulative joint probability
σ_t (nautical miles)	$\bar{P}_t$ (per nautical mile)	P_{mt} (per-square nautical mile)	$\tilde{P}_{ft}$ (per-square nautical mile)	$\tilde{P}_t$ (per-square nautical mile)
40	$1{\cdot}22 \times 10^{-2}$	$5{\cdot}67 \times 10^{-5}$	$4{\cdot}24 \times 10^{-3}$	$3{\cdot}70 \times 10^{-6}$
30	$3{\cdot}46 \times 10^{-3}$	$6{\cdot}51 \times 10^{-5}$	$2{\cdot}79 \times 10^{-3}$	$3{\cdot}24 \times 10^{-6}$
20	$1{\cdot}99 \times 10^{-3}$	$4{\cdot}19 \times 10^{-5}$	$1{\cdot}02 \times 10^{-3}$	$1{\cdot}78 \times 10^{-6}$
10	$3{\cdot}48 \times 10^{-6}$	$1{\cdot}96 \times 10^{-6}$	$2{\cdot}92 \times 10^{-5}$	$1{\cdot}02 \times 10^{-7}$
9	$4{\cdot}68 \times 10^{-7}$	$2{\cdot}94 \times 10^{-8}$	$1{\cdot}27 \times 10^{-5}$	$4{\cdot}92 \times 10^{-8}$
8	$2{\cdot}75 \times 10^{-8}$	$1{\cdot}94 \times 10^{-9}$	$4{\cdot}38 \times 10^{-6}$	$1{\cdot}91 \times 10^{-8}$
7	$4{\cdot}25 \times 10^{-10}$	$3{\cdot}43 \times 10^{-11}$	$1{\cdot}10 \times 10^{-6}$	$5{\cdot}48 \times 10^{-9}$
6	$6{\cdot}53 \times 10^{-13}$	$6{\cdot}14 \times 10^{-14}$	$1{\cdot}70 \times 10^{-7}$	$9{\cdot}89 \times 10^{-10}$
5	$1{\cdot}31 \times 10^{-17}$	$1{\cdot}48 \times 10^{-18}$	$1{\cdot}21 \times 10^{-8}$	$8{\cdot}44 \times 10^{-11}$
4	$2{\cdot}63 \times 10^{-25}$	$3{\cdot}70 \times 10^{-27}$	$2{\cdot}17 \times 10^{-9}$	$1{\cdot}89 \times 10^{-11}$
3	$3{\cdot}50 \times 10^{-45}$	$6{\cdot}58 \times 10^{-46}$	$2{\cdot}45 \times 10^{-13}$	$2{\cdot}85 \times 10^{-15}$
2	$2{\cdot}71 \times 10^{-99}$	$7{\cdot}65 \times 10^{-100}$	$2{\cdot}65 \times 10^{-19}$	$4{\cdot}62 \times 10^{-21}$

$L_t = 60$ nm rather than $L_h = 5$ nm for the former. Thus the Gaussian cumulative probability of coincidence (55) is given by:

$$\log \bar{P}_t = -900/\sigma_t^2 - \log \sigma_t - 1{\cdot}2655, \quad (67)$$

and is indicated in Table 3 for several values of the r.m.s. horizontal position error. For the case of interest, i.e. σ_t much smaller than $L_t = 60$ nm, the cumulative Gaussian probability of coincidence decreases with σ_t, rapidly for σ_t less than 20 nm. For a recommended r.m.s. horizontal position error not exceeding $\sigma_t = 6$ nm, the cumulative probability of coincidence does not exceed $\bar{P}_t \leqslant 6{\cdot}53 \times 10^{-13}$ per nautical mile; for a great circle tour of the earth $D = 40\,000$ km $= 21\,587$ nm, the cumulative probability of coincidence would be less than $\bar{P}_t D \leqslant 1{\cdot}41 \times 10^{-8}$. For Concorde cruising at twice the speed of sound in the stratosphere $V = 1166$ kt, the cumulative probability of coincidence $\bar{P}_t V \leqslant 7{\cdot}61 \times 10^{-10}$ per flight hour, meets the ICAO TLS standard of no more than 5×10^{-9} per hour, in the conditions for which (18) holds. Note that the recommended r.m.s. horizontal separation error $\sigma_t = 6$ nm is about one-tenth $\sigma_t/L_t = 0{\cdot}1$ of the horizontal separation $L_t = 60$ nm for transoceanic routes, as it was ten percent for the horizontal separation $\sigma_h/L_h = 0{\cdot}1$ in controlled airspace, and a little less (nine percent) $\sigma_v/L_v = 0{\cdot}09$ for the vertical separation in both cases.

Table 3 also indicates the maximum probability of coincidence (57) for the horizontal separation $L_t = 60$ nm, calculated from:

$$P_{mt} = \frac{0{\cdot}1592}{\sigma_t^2} \exp\left[-\frac{900}{\sigma_t^2}\right]. \quad (68)$$

The maximum probability of coincidence is relatively small and reduces slowly as the r.m.s. horizontal position error reduces from $\sigma_t = 30$ nm to 10 nm; this result is of

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possible interest for fast time simulations. For the recommended r.m.s. horizontal position error $\sigma_t = 6$ nm, the maximum probability of coincidence is $P_{mt} \leqslant 6{\cdot}19 \times 10^{-14}$ per square nautical mile. For two aircraft on a great circle tour of the earth $D = 21587$ nm, the maximum probability of coincidence would be $P_{mt} D^2 \leqslant 2{\cdot}86 \times 10^{-5}$, if they flew in parallel all the time at minimum separation distance. For two aircraft cruising at $V \leqslant 547$ kt, the maximum probability of coincidence would not exceed $P_{mt} V^2 \leqslant 1{\cdot}83 \times 10^{-8}$ per flight hour squared, somewhat in excess of the modified ICAO TLS standard 5×10^{-9} per flight hour squared.

The Laplace exponential factor probability (59) for lateral separation $L_t = 60$ nm in transoceanic airspace is given by:

$$\tilde{P}_{ft} = \frac{1{\cdot}4142}{\sigma} \exp\left[-\frac{84{\cdot}853}{\sigma} \right], \tag{69}$$

per nautical mile, and is again higher than the Gaussian cumulative coincidence probability, as seen in Table 3. The correction (61) for aircraft size is:

$$\tilde{P}_t = \tilde{P}_{ft} \sinh[3{\cdot}4894 \times 10^{-2}/\sigma], \tag{70}$$

for an aircraft length $R_x = 150$ ft $= 45{\cdot}7$ m $= 2{\cdot}4674 \times 10^2$ nm. For these large r.m.s. position errors $\sigma \geqslant 2$ nm aircraft size is important. A r.m.s. position error of $\sigma_t = 3$ nm, very small compared with the separation $L_t/\sigma_t = 20$ nm leads to a Laplace cumulative joint probability $\tilde{P}_t = 2{\cdot}85 \times 10^{-15}$ per nautical mile, which is much smaller than the exponential factor $\tilde{P}_{ft} = 2{\cdot}45 \times 10^{-13}$ per nautical mile. Both would meet the ICAO TLS standard for all current aircraft speeds $V \leqslant S/\tilde{P}_{ft} = 2{\cdot}04 \times 10^4$ kt and $V \leqslant S/\tilde{P}_t = 1{\cdot}75 \times 10^6$ kt, the latter by a wide margin.

6. THREE-DIMENSIONAL SEPARATION WITH GAUSSIAN OR LAPLACIAN STATISTICS.

The preceding results on one-dimensional separation can be extended to three-dimensions, and show again much sensitivity to the tail shape of probability distributions, as can be confirmed from simple order-of-magnitude estimations.

6.1. *Separation in altitude, and along and across track.* The preceding analysis of one-dimensional separation, can be combined for two and three-dimensional separation, for example; for aircraft staggered along parallel tracks. Consider:

(a) a $L_y = 60$ nm lateral separation in transoceanic airspace, with $\sigma_y = 5$ nm r.m.s. position error, leading by Table 3 to a Gaussian cumulative probability of coincidence $P_y = 1{\cdot}31 \times 10^{-17}$ (Laplace joint cumulative probability $8{\cdot}44 \times 10^{-11}$) per nautical mile;

(b) a $L_x = 5$ nm along track stagger, with a $\sigma_x = 0{\cdot}7$ nm longitudinal position error, leading by Table 1 to a Gaussian cumulative probability of coincidence $P_x = 1{\cdot}16 \times 10^{-6}$(Laplace joint cumulative probability $5{\cdot}52 \times 10^{-6}$) per nautical mile.

Then the combined two-dimensional probability of coincidence is $P_x P_y = 1{\cdot}52 \times 10^{-23}(4{\cdot}65 \times 10^{-16})$ per nautical mile squared. If there is an altitude difference $L_z = 1000$ ft and vertical r.m.s. position error $\sigma_z = 90$ ft, Table 2 gives the corresponding Gaussian cumulative probability of coincidence $P_z = 7{\cdot}51 \times 10^{-13}$(Laplace joint cumulative probability $1{\cdot}24 \times 10^{-5}$) per nautical mile; hence, the combined three-dimensional probability of collision is $P_3 = P_x P_y P_z = 1{\cdot}14\times10^{-35}(5{\cdot}77\times10^{-21})$ per nautical mile cubed. This shows that stagger and altitude difference combined with lateral separation lead to very low probabilities of coincidence. Taking for the aircraft 'size' a span $R_y = 200$ ft $= 60{\cdot}8$ m $= 3{\cdot}29\times10^{-2}$ nm, a length $R_x = 150$ ft $= 45{\cdot}7$ m $= 2{\cdot}47\times10^{-2}$ nm and height $R_z = 50$ ft $= 15{\cdot}2$ m $= 8{\cdot}22\times10^{-3}$ nm, the aircraft volume is $R_3 = R_x R_y R_z = 6{\cdot}68\times10^{-6}(\text{nm})^3$, and the Gaussian (Laplace) cumulative probability of overlap $\bar{P} = P_3 R_3 = 7{\cdot}62\times10^{-41}(3{\cdot}85\times10^{-26})$ is dimensionless. It has been assumed that the aircraft remain always at the minimum separation distance, but if this happened say only a fraction of the time $(40\%, f = 0{\cdot}4)$, the probability of collision would be further reduced to $f\bar{P} = 3{\cdot}04\times10^{-41}(1{\cdot}54\times10^{-26})$. The significant difference between Gaussian and Laplace probabilities will be discussed next.

6.2. *Relation with the ICAO Target Level of Safety*. The main conclusion of the preceding analysis is that an ICAO target level of safety (TLS) of the probability of collision of less than 5×10^{-9} per flight hour can be ensured, in the conditions for which (18) is valid, by requiring a r.m.s. position error of about one-tenth to one-eleventh of the minimum separation distance if a Gaussian probability distribution is assumed. As typical examples:

(a) a r.m.s. horizontal position error of $\sigma_h = 0{\cdot}5$ nautical mile for flight in controlled air space, with a minimum horizontal separation distance of $L_h = 5$ nautical miles, hence $L_h/\sigma_h = 10$;
(b) a r.m.s horizontal position error $\sigma_t = 6$ nm for flight in transoceanic airspace with a minimum horizontal separation distance of $L_t = 60$ nm, hence $L_t/\sigma_t = 10$;
(c) a r.m.s. altitude (or vertical separation) error of $\sigma_h = 90$ ft for flight in controlled or uncontrolled airspace with minimum altitude separation $L_t = 1000$ ft, hence $L_h/\sigma_h = 11$.

These r.m.s. position errors $(\sigma_h, \sigma_v, \sigma_t)$ are easy-to-use safety metrics, since they can be assessed in simulations, by measuring the deviation of the actual from the intended trajectory, respecting the minimum separation distances (L_h, L_v, L_t). The metrics $(\sigma_h, \sigma_v, \sigma_t)$ are comprehensive in that they include all causes of position error, viz. instrumental, environmental and man-made, due to airborne sensing and ground control.

The apparently universal result that the r.m.s. position error σ should one order-of-magnitude less $\sigma/L \leqslant 0{\cdot}1$ than the minimum separation distance L, emerges from the preceding detailed analysis, and can be justified roughly as follows. The probability of coincidence being considered is for two aircraft flying at the minimum separation distance L, e.g. on parallel paths (Figure 3). Viewed head-on (Figures 5 and 6) each aircraft carries a Gaussian probability distribution for position error, for which the same variance σ^2 is assumed for both aircraft. If the position error is large, i.e. σ/L not far from unity, the aircraft carry wide probability distributions, with significant overlap, and high-probability of collision (Figure 5); if the r.m.s. position error is small, in the sense $\sigma/L \ll 1$, then the probability distributions are narrowly concentrated on each aircraft, and decay rapidly away from them, leading to small overlap and low probability of collision. Note that, in Figures 5 and 6:

(a) the maximum probability of coincidence is the product of probabilities of collision at the 'crossing point' of the two curves;

(b) the cumulative probability of coincidence is not the shaded area, but rather the integral (26), although the former may serve as rough qualitative indication for comparison of the cases of 'high' (Figure 5) and 'low' (Figure 6) probability of coincidence.

6.3. *Order of magnitude estimate of coincidence probabilities.* The probability of coincidence is dominated by exponential in the Gaussian bivariate distribution:

$$P(x) \sim \exp\left[-\frac{x^2+(L-x)^2}{2\sigma^2} \right], \tag{71}$$

so that:

$$P(L/2) \sim \exp\left[-\frac{L^2}{4\sigma^2} \right], \tag{72}$$

for a separation $x = L/2$. Taking $L/\sigma = 10$ leads to $\log \bar{P} \sim -25$ or $\bar{P} \sim 1{\cdot}4 \times 10^{-11}$, which is a low value for either:

(a) the cumulative probability of coincidence per mile $\bar{P}$, or
(b) the maximum probability of coincidence per square mile P_m.

For flights not exceeding the perimeter of the earth $D = 4 \times 10^4$ km $=$ $2{\cdot}158 \times 10^4$ nm, the probabilities of coincidence $\bar{P}D \sim 3{\cdot}0 \times 10^{-7}$ or $P_m D^2 \sim 6{\cdot}5 \times 10^{-3}$ are small. For the fastest subsonic airliners in cruise $V \leqslant 547$ kt, the probability of coincidence per flight hour $\bar{P}V \leqslant 7{\cdot}6 \times 10^{-9}$ is of the order of the ICAO target level of safety of 5×10^{-9}. This brief order-of-magnitude justification is no replacement for the detailed analysis in the paper, but it may serve as a simple indication of order of magnitude of the results.

The same kind of order of magnitude assessment can be applied to the Laplace cumulative joint probability distribution (51), where the dominant term is still the exponential $\log P \sim -\sqrt{2}L/\sigma$. The same low value of probability of coincidence $\log \tilde{P} \sim -25$ needed to satisfy the ICAO TLS standard, now leads to $L/\sigma \sim 25/\sqrt{2} \approx 18$, i.e. a separation about 18 times larger than the r.m.s. position error. This is confirmed by the more detailed calculations, showing:

(a) a r.m.s. position error $\sigma_h = 0{\cdot}25$ nm for horizontal separation $L_h = 5$ nm in controlled airspace, hence $L_h/\sigma_h = 20$;
(b) a r.m.s. position error $\sigma_t = 3$ nm for transoceanic lateral separation $L_t = 60$ nm, hence $L_t/\sigma_t = 20$;
(c) a r.m.s. altitude error $\sigma_v = 40$ ft for a RVSM of $L_h = 1000$ ft, hence $L_v/\sigma_v = 25$.

The significant differences found between the Gaussian and Laplace probability distributions, confirms that the probabilities of coincidence are very sensitive to tail shape. This is also suggested by Figures 5 and 6, where it is seen that the shaded area depends very much on the small probabilities of large deviations. Thus it is important to choose the 'right' probability distribution, and the Gaussian and Laplace forms offer only a limited choice, with rather different results. This suggests the introduction of a continuous family of probability distributions, of which the Gaussian and the Laplace distributions are particular cases. This family would allow a more precise fit to collected flight data, and a more accurate calculation of coincidence collision probabilities.

7. THE GENERALIZED EXPONENTIAL FAMILY OF PROBABILITY DISTRIBUTIONS.

The Gaussian (30) and Laplace (46) probability distributions are respectively the particular cases $k = 2$ and $k = 1$ of the generalized probability distribution:

$$F_k(x) = A \exp(-a|x|^k), \tag{73}$$

where: A, a are two constants, viz.: (para 7.1.) the normalization constant A is determined by the condition of unit total probability:

$$1 = \int_{-\infty}^{+\infty} F_k(x)\,\mathrm{d}x; \tag{74}$$

(para 7.2.) the constant a relates to the variance σ^2 r.m.s. error:

$$\sigma^2 = \int_{-\infty}^{+\infty} x^2 F_k(x)\,\mathrm{d}x. \tag{75}$$

Each value of k specifies one member of the family (73) of probability distributions, viz.:

(a) $k < 0$ may be excluded because it would lead to zero probability at the centre $F_k(0) = 0$ and constant probability at infinity $F_k(\pm\infty) = A$;
(b) $k = 0$ is an uniform distribution;
(c) $0 < k < 1$ leads to 'tails' decaying more slowly than for the Laplace distribution, $k = 1$;
(d) $1 < k < 2$ leads to tail decay between the Laplace and the Gaussian case $k = 2$;
(e) $k > 2$ would certainly lead to collision probabilities smaller than Gaussian, which would be misleading in the ATM application.

Comparison with flight data suggests (para 7.3.) a value $k = 0{\cdot}5$.

7.1. *Normalization constant for unit total probability*. In order to calculate the normalization constant, (73) is substituted into (74):

$$\frac{1}{A} = 2\int_0^{+\infty} \exp(-ax^k)\,\mathrm{d}x. \tag{76}$$

The change of variable:

$$\xi = ax^k: \quad \frac{1}{2A} = \frac{a^{-1/k}}{k}\int_0^{+\infty} \xi^{1/k-1} e^{-\xi}\mathrm{d}\xi, \tag{77}$$

reduces the integral to a Gamma function:

$$\Gamma(\alpha) \equiv \int_0^{+\infty} \xi^{\alpha-1} e^{-\xi}\mathrm{d}\xi, \tag{78}$$

of argument $\alpha = 1/k$, viz.:

$$ka^{1/k}/(2A) = \Gamma(1/k). \tag{79}$$

Substituting (79) in (73) leads to:

$$F_k(x) = \frac{ka^{1/k}}{2\Gamma(1/k)}\exp(-a|x|^k), \tag{80}$$

as the normalized probability distribution.

7.2. *Variance or r.m.s. position error as parameter*. The parameter a in (80) can be related to the r.m.s. deviation σ in (75), viz.:

$$\sigma^2 = \frac{ka^{1/k}}{\Gamma(1/k)}\int_0^{+\infty} x^2 \exp(-ax^k)\mathrm{d}x. \tag{81}$$

The same change of variable as in (77), viz.:

$$\xi = ax^k: \quad \sigma^2 = \frac{a^{-2/k}}{\Gamma(1/k)}\int_0^{+\infty} \xi^{3/k-1} e^{-\xi}\mathrm{d}\xi, \tag{82}$$

leads again to a Gamma function (78), this time of argument $\alpha = 3/k$:

$$a^{2/k}\sigma^2 = \Gamma(3/k)/\Gamma(1/k). \tag{83}$$

Substituting of a from (83) into (80) leads to the family of probability distributions:

$$F_k(x) = \frac{1}{2\sigma}\frac{k}{\Gamma(1/k)}\sqrt{\frac{\Gamma(3/k)}{\Gamma(1/k)}}\exp\left\{-\left[\frac{\Gamma(3/k)}{\Gamma(1/k)}\right]^{k/2}\left(\frac{|x|}{\sigma}\right)^k\right\}, \tag{84}$$

with the r.m.s. deviation σ as parameter.

7.3. *Comparison with altitude deviations from flight data*. In (73) and (84) a zero mean $\mu = 0$ was assumed. If the mean is not zero $\mu \neq 0$, then x is replaced by $x-\mu$, leading to:

$$F_k(x) = \frac{1}{2\sigma\Gamma(1+1/k)}\frac{1}{}\sqrt{\frac{\Gamma(3/k)}{\Gamma(1/k)}}\exp\left\{-\left[\frac{\Gamma(3/k)}{\Gamma(1/k)}\right]^{k/2}\left(\frac{|x-\mu|}{\sigma}\right)^k\right\}, \tag{85}$$

as the generalized exponential distribution with mean μ and variance σ^2, where was used the property of the Gamma function:

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha). \tag{86}$$

It can be checked that the case $k = 1$ is the Laplace distribution, viz.:

$$F_1(x) = \frac{1}{\sigma\sqrt{2}}\exp\left(-\sqrt{2}\frac{|x-\mu|}{\sigma}\right), \tag{87}$$

agrees with (46) for $\mu = 0$. Using the property (86) of the Gamma function together with

$$\Gamma(1/2) = \sqrt{\pi}, \tag{88}$$

shows that the case $k = 2$ coincides with the Gaussian distribution, viz.:

$$F_2(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{|x-\mu|^2}{2\sigma^2}\right), \tag{89}$$

agrees with (30) for $\mu = 0$. The more interesting instances of the new family of probability distributions (83), for ATM applications, should be $1 < k < 2$ and $0 < k < 1$.

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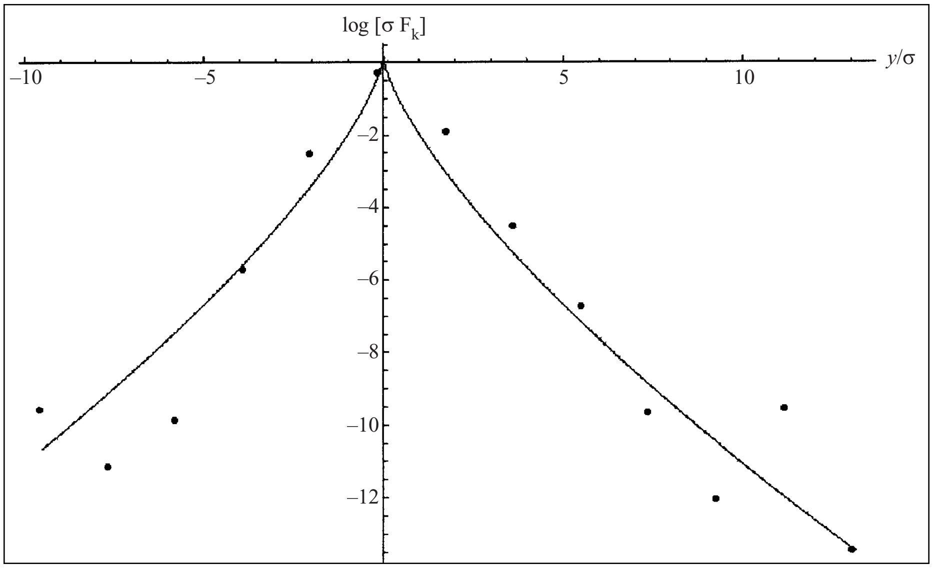

Figure 7. The exponential probability distribution (85) with weight $k = 0{\cdot}53$ close to one-half (90) approximates the altitude deviations measured for aircraft in flight.[14]

In Figure 7, the distributions of large altitude errors in real flight[14] is shown to be consistent with the extended exponential probability distribution (83) with $k = 0{\cdot}5$, viz.

$$F_{1/2}(x) = \frac{1}{\sigma}\sqrt{\frac{15}{2}}\exp\left\{-\sqrt[4]{120}\sqrt{\frac{|x-\mu|}{\sigma}}\right\}, \tag{90}$$

is a simple and relatively accurate probability distribution for position errors.

8. CONCLUSION. It has been pointed out[5–7] that the Laplace distribution $k = 1$ underestimates the 'tails' of the probability distribution, and the uniform distribution $k = 0$ overestimates, so that a more accurate assessment of collision risk lies somewhere in between $0 < k < 1$. The value $k = 0{\cdot}5$ is consistent with these observations, and arises out a comparison with altitude deviations of aircraft measured from flight data.[14] This data has been closely fitted [15–23] using double exponential or Gaussian probability distributions, with five parameters, allowing a close match both to the 'body' and 'tails' of the probability distribution. The choice of a generalized exponential distribution with weight $k = 0{\cdot}5$ is much simpler, in that it involves a single parameter (besides the mean), viz. the r.m.s. deviation σ, which is readily estimated from the data. Given the various sources of error involved in estimation of collision risk, this simple one-parameter probability distribution may do nearly as well as more complex multi-parameter models.

The probability distribution for large rare deviations is the key input is assessing collision risk. The actual calculation, for a simple or complex ATM scenario, involves several other probabilities, all related to the probability of deviation of a single aircraft from its flight path. The difference between simple and complex ATM scenarios depends on the number of aircraft involved and their relative paths, which determine how many proximities have to be considered; the calculations become more complex for higher traffic densities and crossings from many different

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directions. Based on the (i) probability of deviation from the flight path of a single aircraft, the calculation of collision rates or assessment of collision risks, involves several others probabilities which could serve as intermediate safety metrics; (ii) the probability of coincidence of two aircraft at the same position; (iii) the maximum probability of coincidence, at the most likely position of coincidence; (iv) the cumulative probability of coincidence, at all possible positions; (v) the probability of overlap, taking into account finite aircraft size. These can be used to calculate (vi) collision rates, which can be compared to the ICAO TLS standard per unit time or (vii) per unit distance. The dimensionless (viii) collision probability for a given traffic system over a given time is the final safety metric, which depends on many parameters, since most of the preceding are used as building blocks.

REFERENCES

[1] Eurocontrol. (1998). *Air Traffic Management Strategy for 2000+*.

[2] CARE-INTEGRA. (2000). *Objective measures of ATM system safety: safety metrics*. Report to Eurocontrol. 01.03.2000.

[3] Ballin, M. G., Wing, D. J., Hughes, M. F. and Conway, S. R. (1999). Airborne separation assurance and traffic management research concepts and techniques. *AIAA Paper*, 99–3989.

[4] Tomlin, C., Pappas, J. and Sastry, S. (1998). Conflict resolution in air traffic management: a study in multi-agent hybrid systems. *IEEE Trans. Aunt. Control*, **43** (1998), 509–521.

[5, 6&7] Reich, P. G. (1996). Analysis of long-range air traffic systems: separation standards. This *Journal*, Vol **19**: I, 88–98; II, 169–186; III, 328–347.

[8] IATA. (1963). *Report on vertical separation study North Atlantic region – 15 July to 30 September, 1963*. Doc-Gen. 1951.

[9] NASA. (1963). *Random deviations from stabilized cruise altitude of commercial transports at altitude up to 4000 ft with autopilot in altitude hold*. Tech. Note 1950.

[10] FAA. (1964). *Operation Accordion-navigational accuracies of civil jet aircraft over the North Atlantic Feb. 1962 to Sept. 1963*. Rep. RD-64-52, Vol. I.

[11] Hampton, D. E. and Mills, J. R. (1964). Long-range navigation of civil aircraft. This *Journal*, Vol **17,** 167–174.

[12] Treweek, K. H. (1965). An approach to the problem of estimating safe separation standards for air traffic. This *Journal*, Vol. **18**, 185–195.

[13] Attwool, V. W. (1965). Costing air traffic control deviations. This *Journal*, Vol **19**, 99–108.

[14] Eurocontrol. (1988). *European studies of vertical separation above FL 290- Summary Report*. Doc. 88/20/10.

[15] Moek, G. (1987). *Fitting generalized Laplace densities to the results of the main data collection*. NLR.

[16] Harrison, D. (1987). *Height-keeping distributions based on the single aircraft approach*. CAA, UK.

[17] Harrison, D. (1987). Further analysis to obtain Pz1000 based on the single aircraft approach. CAA, UK.

[18] Mock, G. (1987). *Additional results of fitting double generalized Laplace densities to data*. NLR.

[19] Mock, G. (1987). *Estimating the probability of vertical overlap Pz(1000) based on the results of the European vertical radar collection*. NLR.

[20] Mock, G. and Mimoun, S. (1987). *Some preliminary results of fitting double generalized Laplace densities to grouped data by means to the maximum likelihood method*.

[21] Harrison, D. (1987). *Some preliminary results of estimating the probability of vertical overlap from the distribution of single aircraft deviations from North Atlantic traffic*.

[22] Harrison, D. (1987). *Initial analysis via the single aircraft approach, of North Atlantic traffic*. CAA, UK.

[23] Campos, L. M. B. C. and Marques, J. M. G. *On the generalized exponential distribution with application to Air Traffic Management problems*. (in preparation).

[24] Anderson, E. W. *The principles of navigation*.

[25] Mises, R. V. (1960). *Theory of Probability and Statistics*. Academic Press.

[26] Lindeberg, J. W. (1922). Ueber der Exponentialgezetses in der Wahrlicheinkalkulus. *Zeits. Math.* **15**, 211–225.

[27] Campos, L. M. B. C. On the probability of collision of aircraft with dissimilar position errors. *A.I.A.A. Journ. of Aircraft* **38**, 593–599.

LIST OF SYMBOLS

l	length of the flight path (22)
t_i	time period during which deviation exceed R_i, with $i \equiv x, y, z$ (2)
L	separation distance between aircraft (25)
N_i	frequency of separation less than R_i, with $i \equiv x, y, z$ (2)
F_k	generalized exponential probability distribution (70, 80, 81)
F_1	Laplace probability distribution (83)
F_2	Gaussian probability distribution (85)
$F_{1/2}$	probability distribution for ATM scenarios (86)
P	probability distribution
P_a	collision rate (4)
$P_a^{\pm}$	collision rate for aircraft flying on parallel tracks in the same/opposite directions (15, 16)
P_b	probability of overlap (27)
P_c	probability of collision (9)
$P_c^{\pm}$	probability of collision for aircraft flying in the same/opposite directions (21, 23)
P_d	probability per unit distance (18)
P_i	probability of separation less than R_i, with $i \equiv x, y, z$ (1)
P_m	maximum probability of collision between aircraft
P_r	probability of penetration safety volume (1)
P_0	Laplace probability of an aircraft having a given position error (46)
P_1	Gaussian probability of the first aircraft having a given position error (30)
P_2	Gaussian probability of the second aircraft having a given position error (31)
P_{12}	probability of coincidence of the two aircraft (25)
P_{123}	probability of coincidence of three aircraft (33)
$\bar{P}$	cumulative probability of coincidence (26)
$\tilde{P}$	cumulative joint probability (48)
R_x	aircraft length (5)
R_y	aircraft span (5)
R_z	aircraft height (5)
R	aircraft size (27)
$T_a^{\pm}$	time spent in proximity for aircraft flying on parallel tracks in the same/opposite directions (20, 21)
V	airspeed (15)
$\bar{V}_i$	average rate of change of relative position between aircraft (6)
ΔV	airspeed error (16)
σ	root mean square position error (30)
$\sum_{prox}$	proximity sum (9)
$\sum_{i=1}^{3}$	sum (3) over $i \equiv x, y, z$
$\sum_{ijk}^{xyz}$	sum (1) of the three cyclic permutations of $(i, j, k) \equiv (x, y, z)$
$\prod_{j=1}^{3}$	product (3) over $j \equiv x, y, z$

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