SOLUBLE GROUPS WITH COMPLEMENTED SUBNORMAL SUBGROUPS

R. KOCHENDÖRFFER

(Received 19 March 1969)

To Bernhard Hermann Neumann on his 60th birthday

Communicated by G. B. Preston

A subgroup A of a group G is said to be *complemented* in G if G contains a subgroup C such that

$$G = AC, \ A \cap C = 1.$$

Every subgroup C with this property is called a *complement* of A in G. Various results have been obtained about groups in which each member of a given set of subgroups is complemented. Some of these results state, roughly speaking, that the existence of complements of all members of a given set of subgroups implies that all members of a larger set are also complemented. In this paper we derive another theorem of this kind.

Adopting a notation introduced by C. Christensen [1] we define a group G to be an aC-group (sC-group, nC-group) if every subgroup (every subnormal subgroup, every normal subgroup) of G is complemented in G. We are concerned with conditions under which an sC-group is an aC-group. This is certainly not always the case since every aC-group is soluble (cf. [2] for finite groups and [3], [4] for arbitrary groups). On the other hand, not all sC-groups are soluble as any non-abelian simple group shows. Therefore it is natural to restrict oneself to soluble groups. We shall prove the following:

THEOREM. *Every soluble sC-group is an aC-group.*

We shall need the following lemmas:

LEMMA 1. *Every subnormal subgroup of an sC-group is an sC-group.*

Since subnormality is transitive the proof is simple and will be omitted.

LEMMA 2 (cf. [5], Lemma 4). *If $B \leqq A \leqq G$, if A and B are complemented in G, and if N is a complement of B in G, then A has a complement in G which is contained in N.*

For the proof we refer to [5].

Lemma 3. *Let G be an sC-group and A a normal subgroup of G. Then every complement of A in G is an sC-group.*

Proof. Let B be any complement of A in G. For a subnormal subgroup S of B the product AS is subnormal in G. Therefore there exists a subgroup C of G such that $ASC = G, AS \cap C = 1$. By Lemma 2 we may assume that $C \leqq B$. This gives $SC = B, S \cap C = 1$ which completes the proof.

Lemma 4 (cf. [4], Theorem 7, Corollary 2). *Let A be an abelian normal subgroup of G and let A be a (restricted) direct product of subgroups whose orders are prime numbers. If every subgroup of A is complemented in G, then there exists a direct decomposition of A into groups of prime orders which are all normal in G.*

Proof. By Zorn's Lemma there exists a maximal set of independent subgroups of prime orders of A which are normal in G. Let M be the direct product of all members of such a maximal set. Let us assume that M is a proper subgroup of A.

Then A has a direct decomposition $A = M \times A_1$. Since $A_1 \neq 1$ there is a subgroup P of prime order such that $A_1 = P \times A_2$ where the case $A_2 = 1$ is not excluded. We write $A = N \times P$ with $N = M \times A_2$. By the condition on G there exists a subgroup H of G such that

$$G = NH, \; N \cap H = 1. \tag{1}$$

This implies $A = N \times Q$ where $Q = A \cap H$. Since A is normal in G it follows that Q is normal in H, and since A is abelian, Q and N commute elementwise. Making use of (1) we conclude that Q is normal in G. Moreover $P \cong A/N \cong Q$. Thus Q is a subgroup of A of prime order which is independent of M and normal in G. This contradicts the assumption on M so that the Lemma is proved.

In view of Lemma 4 a direct decomposition of A will be said to be *normal* in G if the direct factors have prime orders and are normal in G.

Proof of the theorem. Let G be a soluble sC-group and let

$$G = G^{(0)} > G^{(1)} > G^{(2)} > \cdots > G^{(n)} = 1$$

be its derived series. Since G is in particular an nC-group we can apply Lemma (4.3) of [1] to obtain a factorization

$$G = A_1 A_2 \cdots A_n \tag{2}$$

where $A_1, A_2, \cdots, A_n$ are abelian subgroups of G which satisfy the following conditions

(i) $A_i \cap A_{i+1} A_{i+2} \cdots A_n = 1 \quad (i = 1, \cdots, n-1)$

(ii) $A_{i+r} \leqq N_G(A_i) \quad (i = 1, \cdots, n-1; r = 1, \cdots, n-i)$

(iii) $G^{(n-i)} = A_1 \cdots A_i \quad (i = 1, \cdots, n)$.

By Lemma 1, every group $G^{(n-i)}$ is an sC-group and hence it follows from Lemma 3 that every A_i is an sC-group. This means that every subgroup of the abelian group A_i is a direct factor. As is well-known A_i is therefore a direct product of subgroups whose orders are prime numbers.

For every i, $1 \leqq i \leqq n-1$,

$$G_i = A_i \cdots A_n$$

is an sC-group by Lemma 3. Every subgroup of A_i is subnormal in G_i and is therefore complemented in G_i. Hence, by Lemma 4, every A_i has a direct decomposition which is normal in G_i.

We choose arbitrary elements u, v, w, x of G. Corresponding to (2) we have

$$u = u_1 u_2 \cdots u_n, \quad u_i \in A_i.$$

For each u_i, $i = 1, \cdots, n$, we form the direct product U_i of all those simple direct factors of a normal decomposition of A_i which contain components $\neq 1$ of u_i. In an analogous way we define the subgroups V_i, W_i, X_i of A_i. It follows that for every i, $1 \leqq i \leqq n$,

$$R_i = U_i V_i W_i X_i$$

is a finite subgroup of A_i which is normal in G_i and hence, in particular, $R_i R_j = R_j R_i$. Therefore

$$R = R_1 R_2 \cdots R_n$$

is a finite subgroup of G which contains u, v, w, x. Since each R_i is a product of factors of a normal decomposition of A_i it is easy to see that R is supersoluble. Hence, by a well-known theorem (cf. [6], Theorem 7.2.13) the derived group R' is nilpotent.

Let S be a Sylow p-subgroup of R. From the construction of R it is evident that S is obtained by forming consecutive splitting extensions of finite elementary abelian p-groups by finite elementary abelian p-groups. Moreover each Sylow p-subgroup of R_i is a product of factors of a normal decomposition of A_i. Hence S is abelian. It follows that the Sylow subgroups of the nilpotent group R' are abelian so that R' itself is abelian, i.e. $R'' = 1$. This gives $[[u, v], [w, x]] = 1$ and since u, v, w, x are arbitrary elements, we obtain the result that $G'' = 1$.

Thus the structure of G can be described as follows. G is a splitting extension of an abelian group $A (= G')$ by an abelian group B where both A and B are direct products of groups whose orders are prime numbers. Moreover, A has a direct decomposition which is normal in G.

From results of N. V. Baeva [3] and S. N. Černikov [4] it now follows that G is an aC-group. We repeat the simple proof. For any subgroup T of G let $A_0 = A \cap T$ and let B_0 be the subgroup of B generated by the

B-components of the elements of T. We can find a complement $\bar{A}_0$ of A_0 in A which is the product of suitable direct factors of a normal decomposition of A so that $\bar{A}_0 \lhd G$. Moreover let $\bar{B}_0$ be any complement of B_0 in B. Then $\bar{A}_0 \bar{B}_0$ is a complement of T in G.

References

[1] C. Christensen, 'Complementation in Groups', *Math. Z.* 84 (1964), 52—69.
[2] P. Hall, 'Complemented Groups', *J. London Math. Soc.* 12 (1937), 201—204.
[3] N. V. Baeva, 'Completely Factorisable Groups', *Doklady Akad. Nauk S.S.S.R.* (*N.S.*) 93 (1953), 877—880.
[4] S. N. Černikov, 'Groups with Systems of Complemented Subgroups', *Mat. Sbornik* 35 (77) (1954), 93—128.
[5] N. T. Dinerstein, 'Finiteness Conditions in Groups with Systems of Complemented Subgroups', *Math. Z.* 106 (1968), 321—326.
[6] W. Scott, *Group Theory*, Englewood Cliffs, New Jersey: Prentice Hall Inc. 1964.

Department of Mathematics
University of Tasmania
Hobart, Tasmania

https://doi.org/10.1017/S1446788700007722 Published online by Cambridge University Press