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Dynamic analysis and numerical simulation of a discrete model of a bistable system†

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Numerical simulation is the generally used method for studying stochastic resonance (SR), which is a kind of non-linear phenomenon that usually occurs in non-linear bistable systems. It has been found that the input signal needs to be over-sampled during the numerical simulation of SR. In this paper we provide an explanation of this phenomenon based on a stability analysis of the bistable system. We begin by studying the stability of a discrete model of a bistable system in numerical simulations. We then give a theoretical derivation of the stability conditions for the simulation model with different parameters, and carry out numerical experiments to show that the results coincide with the predictions of the theory. We explain why the input signal needs to be over-sampled in the simulation and provides guidelines for the choice of system parameters for the bistable system and the sampling time step in the numerical simulation of SR. Finally, we present the results of simulations showing an example of SR occurring in a bistable system and an example of weak periodic signal detection when it is processed by a bistable system.

1. Introduction

In the last twenty years, stochastic resonance (SR) in bistable systems has been extensively exploited both theoretically and experimentally. SR is a non-linear phenomenon of bistable systems arising in conjunction with a periodic stimulus and noise (Benzi *et al.* 1981; Hu *et al.* 2003; Leng and Wang 2003; Jung and Hanggi 1991; Yang and Hu 2004; Gammaitoni 1968; McNamara and Wiesenfeld 1998). When the non-linearity of the bistable system, periodic input signal and noise satisfy a sort of matching condition, the response of the system to the weak input becomes enhanced and the plot of the output signal-to-noise ratio against the intensity of the noise has a peak. As a result of this property, SR has been shown to have potential for applications in signal processing.

Numerical simulation is the most commonly used method for studying SR in bistable systems (Asdi and Tewfik 1995; Gang *et al.* 2007; Min and Ying 2010). A suitable system for study is a discrete model of a bistable system, which can demonstrate SR. So the first problem we must address is the stability of the discrete model of a bistable system, which is of great importance for the simulation of SR. However, there are no

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existing research reports describing the quantitative relationship between the sampling time (sampling frequency), system parameters and the stability of the discrete model. In fact, the stability of the discrete system is dominated by the parameters of the bistable system and the sampling time. The sampling time is usually determined by experiment. It is well known that SR is suitable for low frequency signals, and that the signal should be over-sampled in numerical simulations, since otherwise the output of the system will diverge (Leng and Wang 2003; Jung and Hanggi 1991; Yang and Hu 2004). The reason for the divergence of the output is related to the stability of the discrete model of the bistable system.

1.1. *Organisation of the paper*

In Section 2, we present a stability analysis of the discrete model of a bistable systems using a theoretical derivation. In Section 3, we give the numerical simulation results for the output of the bistable for different sampling times – these results show the validity of the stability analysis and why the low-frequency input periodic signal should be over-sampled. We conclude the section by showing that SR can occur in the bistable using a numerical method and giving an example of the detection of a weak periodic signal. Finally, we give a brief summary in Section 4.

2. Stability analysis of the discrete model of a bistable system

At present, the most commonly studied SR system is the bistable system. The dynamic equation of a bistable system driven by weak sinusoidal signal and noise can be described by the following Langevin equation (Gammaitoni 1968):

$$\frac{dx}{dt} = ax - bx^3 + A\sin(2\pi f t) + n(t) \tag{1}$$

where $a > 0$, $b > 0$ and $n(t)$ is Gaussian white noise with

$$\langle n(t) \rangle = 0$$
$$\langle n(t)N(t') \rangle = \sigma^2 \delta(t - t'),$$

where $\langle\cdot\rangle$ is the ensemble average. The system's two stable states are located at $\pm\sqrt{a/b}$, and σ^2 is the variance of the noise. In a numerical simulation, the continuous system should be translated to a discrete model. Here, we use the first-order Euler numerical method to discretise equation (1). Assuming the sample frequency is f_s, the sampling time step $h = 1/f_s$. Letting

$$u(t) = A\sin(2\pi f t) + n(t),$$

the discrete form of equation (1) becomes

$$x_{n+1} - x_n = h(ax_n - bx_n^2 + u_n). \tag{2}$$

We first assume the input $u(t)$ to be zero and the initial value $x(0) = x_0$ to be a real number not equal to zero. Equation (2) then turns into the iterative equation

$$x_{n+1} = (1 + ah - bhx_n^2). \tag{3}$$

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We will now analyse the stability of equation (3). We will first consider the case where x_n satisfies

$$1 + ah - bhx_n^2 < -1. \tag{4}$$

This means that x_{n+1} and x_n will have opposite signs, and $|x_{n+1}| > |x_n|$. Therefore,

$$bhx_{n+1}^2 > bhx_n^2,$$

and

$$1 + ah - bhx_{n+1}^2 < 1 + ah - bhx_n^2 < -1.$$

Iterating equation (3), we have

$$|x_{n+2}| > |x_{n+1}|,$$

and x_{n+2} and x_{n+1} have opposite signs. In a similar way, we can derive

$$|x_{n+2}| < |x_{n+3}| < |x_{n+4}| \cdots ,$$

and the output of the discrete system will tend to infinity. So the necessary condition for equation (3) to be stable is

$$1 + ah - bhx_n^2 \geqslant -1,$$

which can be solved for the inequality to give

$$|x_n| < \sqrt{\frac{ah+2}{bh}}. \tag{5}$$

In the case of equality, the output of the system is

$$|x_n| \equiv \sqrt{(ah+2)/(bh)},$$

and consecutive outputs are opposite in sign. So the system produces a persistent oscillation.

If we let

$$|x_{lim}| \equiv \sqrt{(ah+2)/(bh)}$$

and assume that

$$|x_n| < |x_{lim}|,$$

there is also the possibility of the output of the discrete model tending to infinity – we will discuss this case further.

Let

$$f(x) = (1 + ah)x - bhx^3.$$

$f(x)$ is odd and has a maximum value at

$$x_{max} = \sqrt{(1+ah)/(3bh)}$$

in the interval ($-x_{lim}$, x_{lim}), and

$$f(x_{max}) = \tfrac{2}{3}(1+ah)\sqrt{(1+ah)/(3bh)}.$$

According to the property of odd functions, $f(x)$ has a minimum value at $x = -x_{max}$, so $f(x)$ is an increasing function in the interval ($-x_{max}$, x_{max}). When x is $\pm\sqrt{a/b}$, we have

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$f(x) = x$; when $|x| < \sqrt{a/b}$ with $x \neq 0$, we have $|f(x)| > |x|$. If we let

$$x_{max} = \sqrt{(1 + ah)/(3bh)},$$

we have $f(x_z) = 0$ and $f(-x_z) = 0$. We will now discuss the stability of the model (3) for four cases:

— Case $f(x_{max}) > x_{min}$

The solution of the inequality for this case is

$$ah > 2. \tag{6}$$

Figure 1 shows the curve of $f(x)$ for the interval considered in this case. As $f(x)$ is an odd function, the numeric area of x_0 can be reduced; any real number in the interval $(0, x_{max})$ is taken as the initial value x_0. Because

$$f(x_{max}) > x_{lim},$$

there must be a left neighbourhood of x_{max}, denoted by K, such that $\forall x \in K$, there is $f(x) > x_{lim}$. According to the necessary stability condition given by equation (3), if a real number in the neighbourhood K is taken as the initial value, the output of the discrete model of equation (3) must tend to infinity. Now assuming that $x_n \in K$, according to the property of $f(x)$, we have $f(x_n) > x_n$, and we can write

$$x_n = f(x_n - 1),$$

so we have

$$f(xn) > f(xn - 1).$$

Because $f(x)$ is an increasing function in the interval, we must have $x_n > x_{n-1}$. In a similar way, we can show

$$x_n > x_{n-1} > x_{n-2} > \cdots,$$

and the series tend to zero. This shows that there must exist a countless number of initial points $x_0 \in (0, \varepsilon)$, so after a limited number of iterations, the output of equation (3) will tend to infinity.

In fact, there also exist initial points in the interval $(0, x_{max})$ such that after the nth iteration, the output of equation (3) will give

$$f(x_{n+1}) = x_n$$

or

$$f(x_{n+1}) = -x_n.$$

But if the influence of the input signal $u(n)$ is considered, the system output will not maintain the equilibrium states and a tiny perturbation will make the output depart from these states and tend towards infinity. So, if a real number in the interval $(0, x_{max})$ is taken as the initial value, the output of equation (2) will diverge.

Because the numeric area of $f(x)$ in the interval is the same as that in the interval $(0, x_{max})$, if a real number in the interval (x_{max}, x_z) is taken as the initial value, we get similar results as for taking a point in $(0, x_{max})$ as the initial value, and the output of

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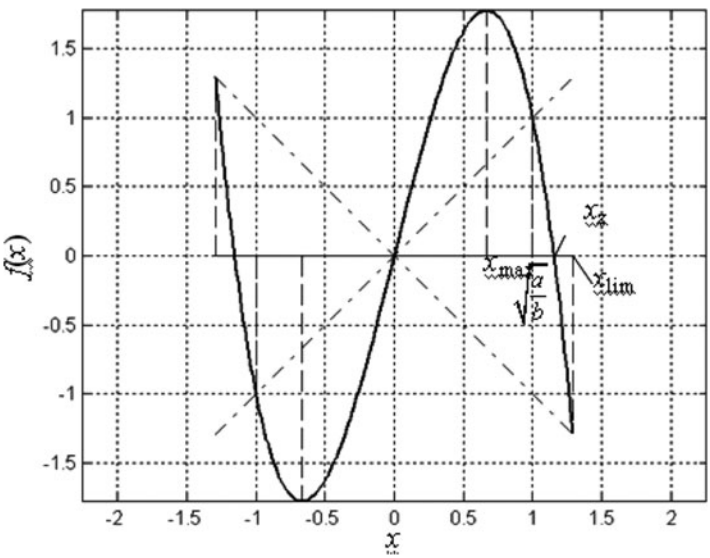

Fig. 1. Curve of $f(x) = 4x - 3x^3$, with $a = 1$, $b = 1$, $h = 3$

the system defined by equation (2) will diverge. It can be proved that if any point in (x_z, x_{lim}) is taken as the initial value, after n iterations we have

$$f(x_n) \in (-x_z, x_z).$$

And because $f(x)$ is odd, the output of equation (2) will diverge if a real number in $(-x_z, 0)$ is taken as the initial value. To summarise, the output of the discrete model (2) will tend to infinity if any real number in the interval $(-x_{lim}, x_{lim})$ is taken as the initial value, and it is thus unsuitable for numerical simulation of SR in the bistable system.

— Case $x_z < f(x_{max}) < x_{lim}$
In this case, the solution of the inequality is

$$\tfrac{3\sqrt{3}}{2} - 1 < ah < 2. \tag{7}$$

Figure 2 shows the curve of $f(x)$ for the interval $[-x_{lim}, x_{lim}]$.
It can be proved that if a real number in the interval $(-x_{lim}, x_{lim})$ is taken as the initial value, the output of equation (4) will oscillate between a range of positive and negative values. If the influence of the input signal is taken into account, the output of system (4) will tend to diverge, and it is thus unsuitable for numerical simulation of SR in the bistable system.

— Case $f(x_{max}) < x_z$ and $x_{max} < \sqrt{a/b}$
In this case, the solution of the inequality is

$$\tfrac{1}{2} < ah < \tfrac{3\sqrt{3}}{2} - 1. \tag{8}$$

Figure 3 shows the curve of $f(x)$ for the interval $[-x_{lim}, x_{lim}]$.
In this case, if a real number in the interval $(-x_{lim}, x_{lim})$ is taken as the initial value, the output of system (3) will oscillate locally within a range of positive or negative values.

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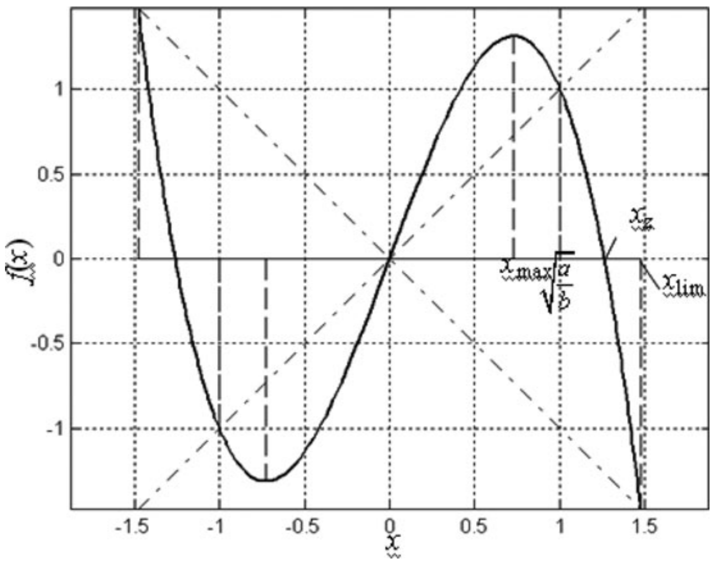

Fig. 2. Curve of $f(x) = 2.7x - 1.7x^3$, with $a = 1$, $b = 1$, $h = 1.7$

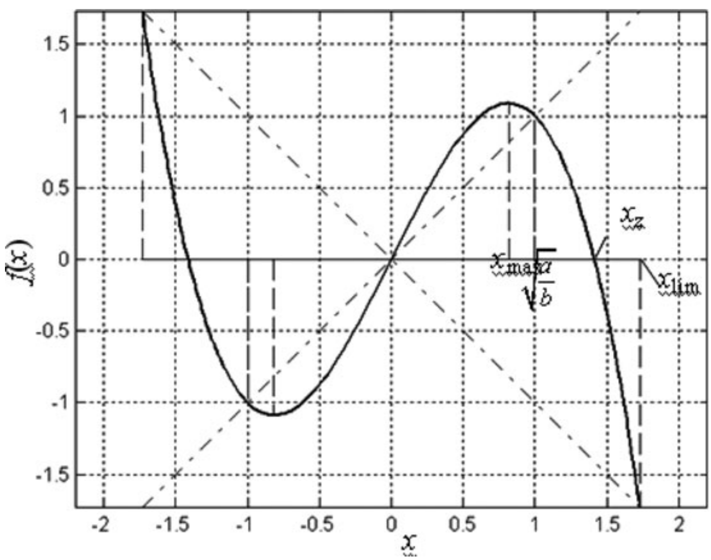

Fig. 3. Curve of $f(x) = 2x - x^3$, with $a = 1$, $b = 1$, $h = 1$

Hence, the output will always have the same sign, the actual sign in a particular case being determined by the initial value.

— Case $f(x_{max}) \geqslant \sqrt{a/b}$

In this case, the solution of the inequality is

$$ah < \tfrac{1}{2}. \tag{9}$$

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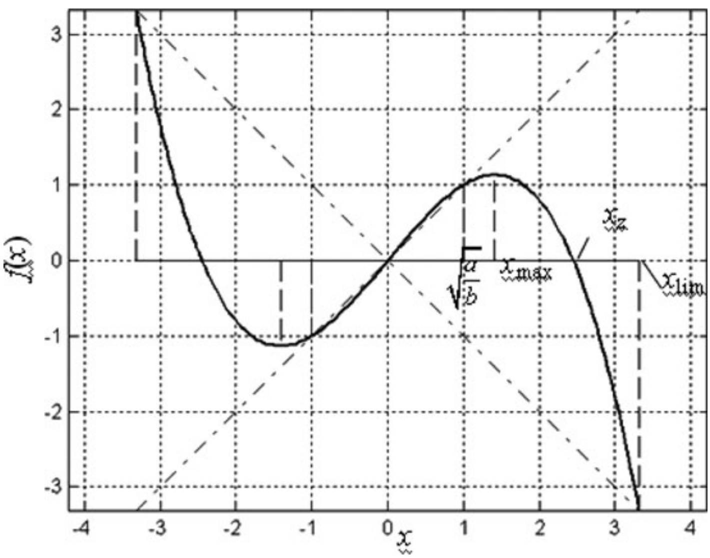

Fig. 4. Curve of $f(x) = 1.2x - 0.2x^3$, with $a = 1$, $b = 1$, $h = 0.2$

Figure 4 shows the curve of $f(x)$ for the interval $[-x_{lim}, x_{lim}]$. In this case, if a real number in the interval $(-x_{lim}, x_{lim})$ is taken as the initial value, the output of system (4) will tend to stable points $\pm\sqrt{a/b}$.

This completes our analysis of the stability of the discrete model of the bistable system. It is obvious that when the output diverges ($ah > 2$) or oscillates between a range of negative and positive values ($3\sqrt{3}/2 - 1 < ah < 2$), the model is unsuitable for numerical simulation of SR. However, it is suitable for numerical simulation of SR when the output converges to the stable points ($ah \leqslant 1/2$). But we need to investigate whether it is suitable for SR simulation when the output oscillates locally ($1/2 < ah < 3\sqrt{3}/2 - 1$). Through simulation experiments, it has been found that when ($1/2 < ah \leqslant 1$), the amplitude of the local oscillation of the system output is very small, and attenuates gradually, so the oscillation has no influence on the accuracy of the simulation of SR. To summarise, the condition required for a suitable numerical simulation of SR in the bistable system is

$$\begin{aligned} &ah \leqslant 1 \\ &|x_0| \leqslant x_{lim}. \end{aligned} \tag{10}$$

When the influence of the input signal and noise is taken into account, the output of the system (3) will tend to infinity when the noise intensity is too large. The reason for this is that when the noise intensity is too large, after the nth iteration we have

$$|x_n + u_n| > x_{lim}.$$

So, in order to avoid divergence of the output, we need to reduce the sampling time step to increase the value of x_{lim}.

In the next section, we will discuss numerical experiments on the discrete model of the bistable system to validate the analysis given in the current section.

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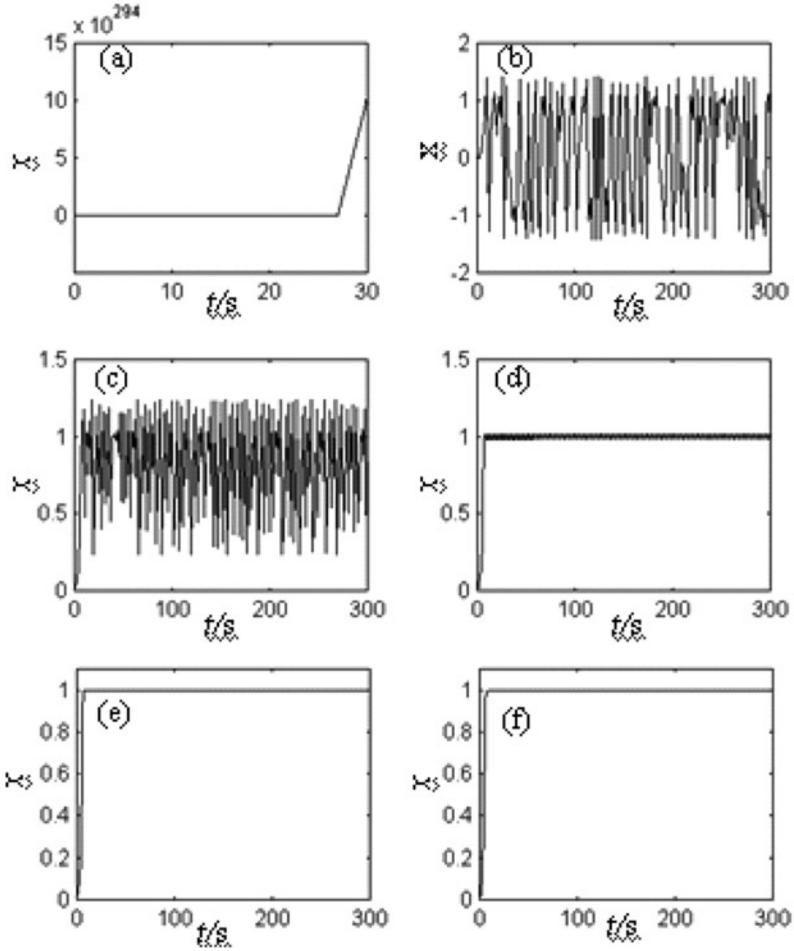

Fig. 5. Output of system (2) with $a = 1$ and $b = 1$ for different time steps (a) $h = 3$, (b) h=1.9, (c) h=1.5, (d) h=1, (e) h=0.5, (f) h=0.2

3. Numerical simulation experiments on the discrete model of the bistable system

3.1. *Stability of the numerical simulation of the bistable system*

In order to verify the stability condition presented in the previous section, we used different sampling time steps in solving equation (4). Using values for the system parameters $a = b = 1$, the output results for equation (4) obtained using Euler's numerical integral method are shown in Figure 5 for different time steps and an initial value for x_0 of 0.01.

Figure 5(a) shows that when $ah = 3$, the output diverges. Figure 5(b) shows that when $ah = 1.9$, the output oscillates between a range of negative and positive values. Figure 5(c) shows that when $ah = 1.5$, there is oscillation within a range of positive values. Figure 5(d) shows that when $ah = 1$, the oscillation amplitude around the stable state $x = 1$ is very small. Figures 5(e) and 5(f) show that when $ah \leqslant 0.5$, the output tends rapidly

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to the stable state $x = 1$. These numerical simulation results are in accordance with the theoretical analysis.

According to adiabatic approximation theory and the linear response theory of SR (Jung and Hanggi 1991; McNamara and Wiesenfeld 1998), the frequency of the periodic signal should satisfy $f \ll 1$. Here we let $f = 0.01$Hz and $A = 0.01$, so according to the sampling theorem, the sample frequency should be larger than $2f$. But even if the sample frequency $f_s = 40f$ (that is, $h = 2.5s$), the output of the system shown in equation (1) still diverges because we have $a_h > 2$. In fact, the frequency must be larger than $100f$ for the system to exhibit the SR phenomenon in the bistable system. This kind of phenomenon is called over-sampling of the input signals in SR. The basic reason for the over-sampling is that during the numerical simulation, the most important condition is that the discrete model of the bistable should be stable. Hence, the stability condition in equation (9) leads directly to the necessity of over-sampling of the input signals.

3.2. *Stochastic resonance in the bistable system*

We carried out a numerical simulation to show stochastic resonance in the bistable system. Figure 6 shows the outputs of system (1) driven by a signal with noise of different noise intensity with signal parameters $A = 0.3$ and $f = 0.01$Hz, sample frequency $f_s = 5$Hz ($h = 0.2s$), and system parameters $a = 1$ and $b = 1$. This is suitable for simulation of SR because we have $ah = 0.2$. Figure 6(a) shows that the output of the bistable system oscillates locally within a single stable state in accordance with the signal frequency. Figure 6(b) shows occasional hopping between two stable states. Figure 6(c) shows that with increasing noise, the hopping between the two stable states becomes synchronised with the periodic signal, and SR is exhibited. Figure 6(d) shows that when the noise intensity is too large, the hopping is mainly decided by the noise. Summarising, Figure 6, shows that the system (2a) exhibits a typical SR phenomenon.

The bistable system can be used to extract a weak periodic signal from strong background noise using the SR mechanism. To show this, we passed the mixed signal through the model of the bistable system and analysed the spectrum of the output signal of the system. The system parameters are as follows: $a = b = 1$, $A = 0.5$ and $f = 0.1$Hz. The over-sampling frequency f_s is 200 times that of the periodic signal, so $f_s = 20$Hz (that is, $h = 0.05$s). Figure 7 shows the mixed input signal and the output signal of the bistable system:

— Figure 7(a) shows the waveform of the mixed signal;
— Figure 7(b) shows the spectrum of the mixed signal;
— Figure 7(c) shows the waveform of the output signal;
— Figure 7(d) shows the spectrum of the output signal.

It can be seen from Figure 7(d), that there is a clear frequency component at $f = 0.1$Hz, and the noise fades obviously, whereas Figure 7(b) shows the frequency component of the periodic signal is submerged in noise and is difficult to distinguish from the mixed signal spectrum. During the process, some energy of the noise is transferred to the energy of the periodic signal, so the weak periodic signal is enhanced through the bistable system using the SR mechanism.

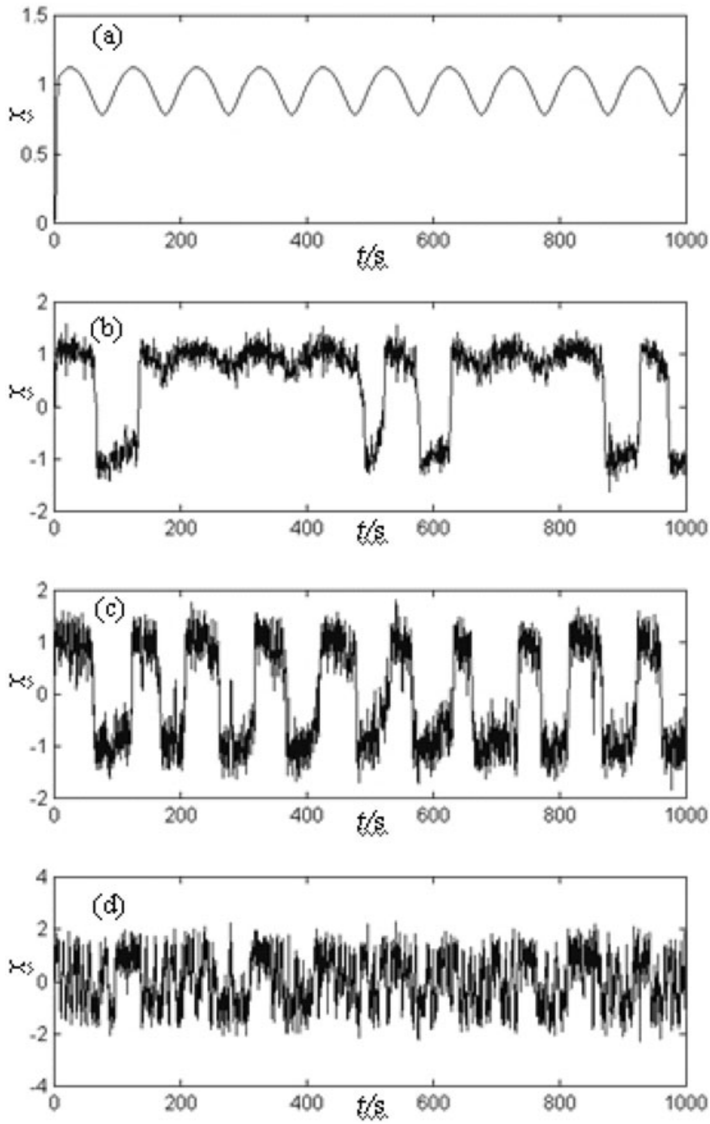

Fig. 6. Output of system (2a) for different time steps (a) $\sigma = 0$, (b) $\sigma = 0.3$, (c) $\sigma = 1.0$, (d) $\sigma = 3$

4. Conclusions

In the paper, we have analysed the stability of the discrete model of a bistable system, which is of great importance for the simulation of SR. We have derived a stability condition for the system and carried out numerical experiments whose results coincide with the theoretical predictions. We also explained why a low frequency input periodic signal should be over-sampled during numerical simulation of SR.

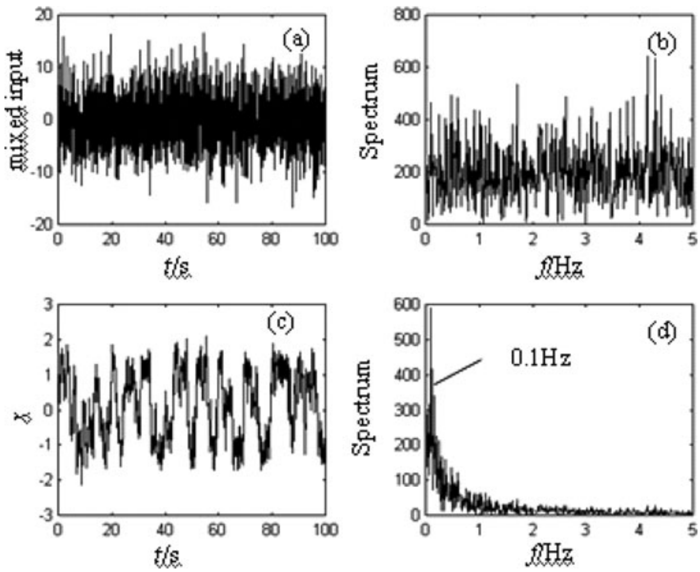

Fig. 7. Simulation of detection of weak perioidic signal with $f = 0.1Hz$

In the current paper, we used a first-order Euler method for discrimination; if we had used a Runge–Kutta method, the accuracy of the output would have increased, though it would have had little effect on the stability condition for the discrete model.

These research results provide guidelines for the choice of system parameters and sampling time step in numerical simulations of SR in bistable system. SR has been proposed as a means for improving periodic signal detection in a wide variety of systems, and our numerical simulation of weak periodic signal detection showed the benefit of SR.

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