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AN ENDOGENOUSLY DERIVED AK MODEL OF ECONOMIC GROWTH

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When the returns to scale of a production process vary with the intensity it is operated at, an AK model with constant returns to scale in production arises endogenously due to replication driven by profit maximization. If replication occurs at the efficiency-maximizing scale, as with perfect competition, the result applies also when the number of production processes must be discrete, thus, overcoming the so-called integer problem. When competition is imperfect, there is only convergence toward the AK model for large enough input use, so an economy is more prone to stalling in a steady state without growth, the smaller and less competitive it is.

Keywords: Endogenous Economic Growth, AK model, Replication, Returns to Scale, Integer Problem

1. INTRODUCTION

In economic growth theory, the returns to scale in the factors of production that an economy accumulates endogenously are crucial for its long-run evolution. In particular, when these are decreasing, growth comes to a halt in the absence of any external impetus, as illustrated by Solow (1956). Therefore, models with sustained endogenous growth assume increasing or constant returns to scale in the endogenously accumulated inputs, including those first proposed by Romer (1986) and Lucas (1988). Constant returns to scale are particularly popular, giving rise to the well-known AK model [Rebelo (1991)] class, which can endogenously sustain a strictly positive constant rate of growth, a balanced-growth path, consistent with the empirical stylized facts described by Kaldor (1963). Despite its crucial role, the nature of the returns to scale is always assumed, never derived, which is a significant deficiency of endogenous growth models [McCallum (1996)]. At most, authors argue that replication leads to non-decreasing returns to scale; since if a production process can be reproduced exactly, the copy should arguably yield the same output as the original [Koopmans (1957), Shell (1966), Romer (1990, 1994), Jones (1999, 2005)]. However, this is insufficient, as it says nothing about production levels that are not a multiple of what the original process yields, and can therefore not be generated by copying at the original scale; the so-called integer problem [Romer (1990)].

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If one assumes it is possible to maintain the same level of efficiency no matter the scale at which the production process is copied at, returns to scale are constant for all levels of production, but by assumption. Instead, we assume that the efficiency of a production process varies with the intensity it is operated at, an idea that goes back at least to Marshall (1890). Hence, exact replication, including the scale, yields the same output as the original, but copies that are scaled up or down are not equally efficient. Consequently, producers' scaling and replication determines the returns to scale in production. In order to maximize efficiency, producers exhaust increasing returns, and replicate the process as its returns become decreasing, thus, operating as close as possible to the intensity where returns to scale are constant. However, they cannot replicate the process exactly at this point, because the number of processes must be a positive integer, since otherwise the scale of the production process can be varied without affecting its efficiency, for example, by running half a process at the efficiency-maximizing scale, thus, violating our fundamental assumption. Such indivisibilities, inherently present in actual production, have been used to criticize the omnipresent assumption of constant return to scale [Scarf (1994)]. The present paper studies how these indivisibilities affect endogenous growth dynamics, and whether or not an AK model arises as a result of replication driven by profit maximization. It finds that total production only satisfies constant returns to scale when the production process is replicated at the efficiency-maximizing intensity, which requires that nothing distort the profit-maximizing intensity away from the efficiency-maximizing one, and that competition be perfect, or with imperfect competition, that the number of replications be large enough.

Because constant returns to scale, and the AK model, only arise for large enough input use when competition is imperfect, endogenous growth can come to a halt in poorer economies, even when more input-rich, but otherwise identical economies, keep growing forever. Moreover, the marginal product of inputs would be lower in the stalled economy, making it unable to attract inputs from others. The reason is that it is harder to achieve the efficiency-maximizing scale in smaller economies, which reduces the marginal products of inputs. As a result, an economy's starting point not only affects its growth rate, but could even determine whether it will stall, or keep growing endogenously forever. Consequently, policies usually considered to have only a short-term impact on the rate of growth, such as a temporary inflow of inputs, or a transitory increase in competition, can have permanent effects, by getting a stalled economy on to a path of never-ending endogenous growth.

There are many reasons why the efficiency of a production process can vary with the scale it is operated at. One is the physical nature of the process, for example, in mineral extraction returns to scale might be decreasing as a result of the most easily extractable resources being exploited first. Another is specialization, which can make efficiency increase with the scale, as each worker concentrates more on the task at which he has a comparative advantage, instead of having to do a little of everything.[1] Consequently, returns to scale might initially be increasing due to specialization and turn decreasing as fatigue or boredom kicks in. Additional factors that can contribute to increasing returns to scale are fixed costs, synergies, and learning-by-doing. Decreasing returns to scale can arise due to coordination and communication problems, which are more likely to emerge as the scale of operation becomes larger.[2] It can also be harder to supervise and motivate larger units, where the incentives to free-ride are greater.

Contrasting our results with empirical evidence is difficult. First of all, the degree of replication, the key variable, must be inferred from the number of producers or plants, or from the size of the economy or sector, which can be very imprecise. For example, a large-sized economy does not necessarily imply a greater degree of replication than a smaller economy since the production processes being replicated might differ due to differences in sectoral compositions. Comparisons over time are also subject to changes in the production processes and optimal scale. In addition, there may be frictions that prevent efficient scaling and convergence toward constant returns to scale. Furthermore, estimating returns to scale in production is difficult, see, for example, Bartelsman (1995), Basu and Fernald (1997), and Gorodnichenko (2008). Hence, although Romer (1986), Domowitz et al. (1988), Hall (1990), and Caballero and Lyons (1992) find that returns to scale are increasing, Burnside et al. (1995) and Basu and Fernald (1997) find they are decreasing or constant. Furthermore, Backus et al. (1992) and Jones (1995) find that there are no scale effects, so that larger countries do not generally grow faster, or have higher output per capita, than smaller ones. The empirical evidence on the relationship between competition and economic growth is also ambiguous, see, for example, Kahyarara (2004) and Sekkat (2009) versus Tybout (2000) and Singh (2002).

For simplicity, our results are derived assuming a single input, physical capital, and a logarithmic production process. However, replication leads to constant returns to scale in production with any number of inputs for any production process with returns to scale that vary with the scale it is operated at, and whose efficiency is maximized at an interior point characterized by the first-order conditions [Jensen (2014)]. To illustrate this, we first assume a generic two-input production process, and use it in the next section to derive constant returns to scale production function based on replication driven by producers' efforts to maximize efficiency. The subsequent section introduces the simpler single-input logarithmic production process studied in the remainder of the paper. The following two sections define the rest of the general equilibrium growth model and study its dynamics, respectively. Subsequently, we show that when the number of processes must be a positive integer, there is only convergence toward the AK model when replication occurs at the efficiency-maximizing intensity.

2. PRODUCTION WITH OPTIMAL REPLICATION

A production process is a producer's technology for converting inputs into output. It is the basic unit of production that producers replicate, and can be a single worker, a whole production line, or an entire plant. We assume that the output y of

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a production process depends on the quantity used of two inputs k and l through the function

$$y = f(k, l), \tag{1}$$

where f is strictly concave, twice continuously differentiable, and $f(0,0) < 0$, with $f(k,l) > 0$ for some $k, l > 0$.[3] Because the production process is concave, the optimal allocation among multiple identical processes is symmetrical, so the most total output Y that can be produced with N processes and K and L total input is

$$Y = N \times f\left(\frac{K}{N}, \frac{L}{N}\right) \equiv H(K, L, N), \tag{2}$$

where $Y = N \times y$, $K = N \times k$, and $L = N \times l$.[4] If N could be varied continuously, the first-order condition

$$\frac{\partial H(K, L, N)}{\partial N} = f\left(\frac{K}{\tilde{N}}, \frac{L}{\tilde{N}}\right) - f_1'\left(\frac{K}{\tilde{N}}, \frac{L}{\tilde{N}}\right) \times \frac{K}{\tilde{N}} - f_2'\left(\frac{K}{\tilde{N}}, \frac{L}{\tilde{N}}\right) \times \frac{L}{\tilde{N}} = 0 \tag{3}$$

would yield the optimal number of processes $\tilde{N}(K, L)$ since H is strictly concave in N.[5] The corresponding production function

$$Y = G(K, L) = H\left(K, L, \tilde{N}(K, L)\right) = \tilde{N} \times f\left(\frac{K}{\tilde{N}}, \frac{L}{\tilde{N}}\right) \tag{4}$$

which yields the most output that can be produced with any amount of total input K and L satisfies

$$G(K, L) = G_1'(K, L) \times K + G_2'(K, L) \times L \tag{5}$$

and has constant returns to scale. To see this, note that

$$G_i'(K, L) = f_i' + \left(f - f_1' \times \frac{K}{\tilde{N}} - f_2' \times \frac{L}{\tilde{N}}\right) \times \tilde{N}_i'(K, L) = f_i' \tag{6}$$

for $i = 1, 2$, due to the first-order condition (3) for $\tilde{N}$.

If N could be varied continuously, as above, it would be possible to change the scale of the production process without affecting its efficiency, for example, by running half a process at the efficiency maximizing scale, thus, violating our fundamental hypothesis, and making returns to scale constant by assumption. Instead, we follow Scarf (1994) and assume that the number of production processes must be varied in discrete units, whereas their input and output can be varied continuously. As an example, one can think of the number of workers and the hours that each works. It is not possible to hire half a worker, but it is possible to hire one to work part time. The distinction is relevant when, as we assume, a worker's productivity depends on the number of hours worked. As one can easily imagine, the joint output of two part-timers working four hours each can differ from that of a single employee working eight hours. Just as one can hire someone to work eight hours a day, one can do so for eight hours and five minutes. Hence, although the adjustment on the extensive margin is restricted to integers, that on the intensive margin is not. When the number of production processes must be a positive integer,

$$Y = \max\left\{0, f(K, L), 2f\left(\frac{K}{2}, \frac{L}{2}\right), 3f\left(\frac{K}{3}, \frac{L}{3}\right), \ldots\right\} \equiv F(K, L) \qquad \textbf{(7)}$$

is the most output that can be produced for a given amount of inputs K and L. Next we show that $F(K, L)$ converges toward $G(K, L)$, and hence constant returns to scale, as N grows.

For any K and L such that $\tilde{N}(K, L)$ from (3) is a positive integer, $F(K, L) = G(K, L)$ trivially. For any K and L such that $\tilde{N}(K, L)$ is not a positive integer, let

$$I(K, L) \equiv \tilde{N}(K, L) - \lambda(K, L), \qquad \textbf{(8)}$$

where $I \in \mathbb{N}$ is the natural number closest to $\tilde{N}$ such that $I < \tilde{N}$, implying that $\lambda \in (0, 1)$ and

$$F(K, L) = \max\left\{I \times f\left(\frac{K}{I}, \frac{L}{I}\right), (I+1) \times f\left(\frac{K}{I+1}, \frac{L}{I+1}\right)\right\}. \qquad \textbf{(9)}$$

Letting $\tilde{k} \equiv K/\tilde{N}(K, L)$ and $\tilde{l} \equiv L/\tilde{N}(K, L)$, we have

$$I \times f\left(\frac{K}{I}, \frac{L}{I}\right) = I \times f\left(\tilde{k} + \frac{\lambda}{I}\tilde{k}, \tilde{l} + \frac{\lambda}{I}\tilde{l}\right) \qquad \textbf{(10)}$$

and

$$(I+1) \times f\left(\frac{K}{I+1}, \frac{L}{I+1}\right) = (I+1) \times f\left(\tilde{k} + \frac{\lambda-1}{I+1}\tilde{k}, \tilde{l} + \frac{\lambda-1}{I+1}\tilde{l}\right), \qquad \textbf{(11)}$$

whereas

$$G(K, L) = (I + \lambda) \times f\left(\tilde{k}, \tilde{l}\right). \qquad \textbf{(12)}$$

According to Taylor's theorem [see, for example, Sydsæter et al. (1991)], for any function f that is twice continuously differentiable, there exists $\mu \in (0, 1)$ such that

$$f\left(\tilde{k} + \frac{\lambda}{I}\tilde{k}, \tilde{l} + \frac{\lambda}{I}\tilde{l}\right) = f\left(\tilde{k}, \tilde{l}\right) + f_1'\left(\tilde{k}, \tilde{l}\right)\frac{\lambda}{I}\tilde{k} + f_2'\left(\tilde{k}, \tilde{l}\right)\frac{\lambda}{I}\tilde{l} + \phi, \qquad \textbf{(13)}$$

where

$$\phi = \frac{\lambda^2}{2I^2} \times \begin{pmatrix}\tilde{k}\\ \tilde{l}\end{pmatrix}' \nabla^2 f\left(\tilde{k} + \mu\frac{\lambda}{I}\tilde{k}, \tilde{l} + \mu\frac{\lambda}{I}\tilde{l}\right)\begin{pmatrix}\tilde{k}\\ \tilde{l}\end{pmatrix} \qquad \textbf{(14)}$$

is the Lagrange error term for a first-order approximation. Multiplying by I yields

$$I \times f\left(\tilde{k} + \frac{\lambda}{I}\tilde{k}, \tilde{l} + \frac{\lambda}{I}\tilde{l}\right) = I \times f\left(\tilde{k}, \tilde{l}\right) + \lambda f_1'\left(\tilde{k}, \tilde{l}\right)\tilde{k} + \lambda f_2'\left(\tilde{k}, \tilde{l}\right)\tilde{l} + I \times \phi \qquad \textbf{(15)}$$

or

$$I \times f\left(\tilde{k} + \frac{\lambda}{I}\tilde{k}, \tilde{l} + \frac{\lambda}{I}\tilde{l}\right) = G(K, L) + I \times \phi \tag{16}$$

due to the first-order condition (3) for $\tilde{N}$. Similarly, Taylor's theorem implies that there exists some $\omega \in (0, 1)$ such that

$$(I+1) \times f\left(\tilde{k} + \frac{\lambda - 1}{I+1}\tilde{k}, \tilde{l} + \frac{\lambda - 1}{I+1}\tilde{l}\right) = G(K, L) + (I+1)\varphi, \tag{17}$$

where

$$\varphi = \frac{(\lambda - 1)^2}{2(I+1)^2} \times \begin{pmatrix} \tilde{k} \\ \tilde{l} \end{pmatrix}' \nabla^2 f\left(\tilde{k} + \omega\frac{\lambda - 1}{I+1}\tilde{k}, \tilde{l} + \omega\frac{\lambda - 1}{I+1}\tilde{l}\right)\begin{pmatrix} \tilde{k} \\ \tilde{l} \end{pmatrix}. \tag{18}$$

It follows that for large enough $\tilde{N}$, I will be large enough for $I \times \phi$ and $(I+1) \times \varphi$ to be arbitrary close to zero, making $F(K, L)$ arbitrarily close to $G(K, L)$. Hence, as replication increases, total output converges toward constant returns to scale, even if the number of production processes cannot be varied continuously.

Constant returns to scale do not guarantee never-ending endogenous growth. First of all, the economy might not endogenously accumulate all the inputs that enter the production function, making returns to scale decreasing in those it does accumulate. For example, in the Solow (1956) model, returns to scale are constant in capital and labor jointly, and therefore decreasing in capital alone, the only input it assumes the economy amasses endogenously. Of course, as Lucas (1988) shows, what matters is not just what the economy accumulates in quantity, but also in quality. If it does not amass more workers, but does accrue human capital in terms of improved skills, it can still grow endogenously, even in per-capita terms. The same is true if land, usually considered to be available in given amounts for the economy as a whole, is used more intensively or efficiently. Moreover, the economy might not accumulate inputs at a rate that yields balanced growth, which requires linear accumulation, as is known from the knife-edge assumption, or linearity, critique, discussed by Jones (2005) and Growiec (2007). In addition, balanced growth requires labor-augmenting technological improvements [Uzawa (1961)].

Convergence toward constant returns to scale might also be insufficient for never-ending endogenous growth if the economy does not grow large enough for this convergence to occur. This is the focus of the reminder of the paper, which shows that imperfect competition, or any other distortion that makes the profit-maximizing scale differ from the efficiency-maximizing one, can prevent never-ending endogenous growth by making the marginal product of inputs fall below what is required for their endogenous accumulation. We simplify by focusing on a single input called capital. This single input can alternatively be interpreted as a composite of all inputs, all assumed to be accumulated endogenously, though our specification of its accumulation is equivalent to what is commonly assumed for physical capital. Singling out different inputs would require specifying exactly how each is accumulated. We also assume a logarithmic production process, which further simplifies the presentation, and allows specifying the exact conditions under which endogenous growth would go on forever. These are obviously sensitive to how the marginal product of the input behaves away from the efficiency maximizing scale of the production process.

3. SINGLE-INPUT LOG PRODUCTION

Imagine a special case of that discussed above where the production process is

$$y = a \log (bk) \tag{19}$$

with given constants $a > 0$ and $b > 0$. Its returns to scale are increasing for $k \in (1/b, e/b)$, decreasing for $k > e/b$, and constant at $k = e/b$. Output is zero for $k = 1/b$, and strictly negative for $k < 1/b$. The most total output Y that can be produced with N processes and K total input is

$$Y = Na \log \left(b \frac{K}{N} \right) \equiv H (K, N), \tag{20}$$

and the optimal N is defined by

$$\frac{\partial H (K, N)}{\partial N} = a \left[\log \left(b \frac{K}{N} \right) - 1 \right] = 0, \tag{21}$$

which yields

$$\tilde{N} (K) = be^{-1} K. \tag{22}$$

The corresponding production function

$$Y = G (K) = H \left(K, \tilde{N} (K) \right) = abe^{-1} K \equiv AK. \tag{23}$$

When the number of production processes N must be a positive integer,

$$Y = \max \left\{ 0, a \log (bK), 2a \log \left(b \frac{K}{2} \right), 3a \log \left(b \frac{K}{3} \right), \ldots \right\} \equiv F (K) \tag{24}$$

is the most output that can be produced with a given amount of input K. Figure 1, which plots $F(K)$, AK and $H(K, N)$ for $N = 1, 2, 3, 4$ and $a = b = 1$, illustrates how $F(K)$ converges toward $G(K) = AK$ as K increases.[6]

Producers seek to maximize profits, and producing efficiently, getting the most output possible from the input, as determined above, is necessary for this. In addition, producers, who are assumed to be pricetakers in the input market, must decide how much input to rent from households. Letting output be numeraire, having a price of one, and assuming perfect competition also in the output market, they do so by maximizing profits

$$\pi (K) = F (K) - (r + \delta) K \tag{25}$$

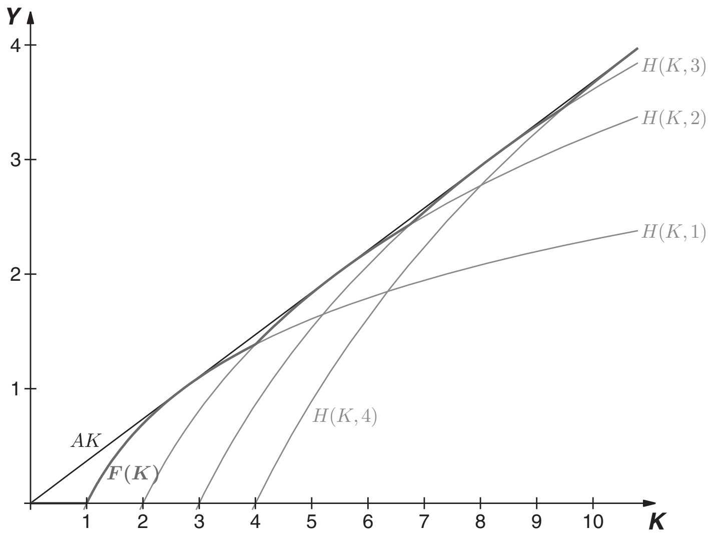

FIGURE 1. Convergence of $F(K)$ toward AK with efficient scaling.

for a given rental rate r and depreciation rate $\delta \in (0, 1)$, imagining input K is physical capital. In any competitive equilibrium with non-negative production, it must be that

$$r = A - \delta \tag{26}$$

holds.[7] If the interest rate were higher than this, profits would be negative for all input levels; since the average product is at most A, there would be no demand for the input and no production. If the interest rate were lower than $A - \delta$, each producer would demand an infinite amount of input, as this would make her average product equal A, which would be greater than her average cost $r + \delta$, making profits infinitely large. Inserting for the equilibrium interest rate (26) into the profit function (25) yields

$$\pi(K) = F(K) - AK \leq 0, \tag{27}$$

which is strictly negative whenever F deviates from AK. As a result, a competitive equilibrium is only feasible when all production occurs at points on the production function where returns to scale are exactly constant.[8] Hence, when the economy-wide output changes, the number of producers and production processes adjusts so that all end up with an average productivity of A, since otherwise their profits would be strictly negative.

With imperfect competition in production, we have

$$r = \frac{\epsilon - 1}{\epsilon} F'(K) - \delta, \tag{28}$$

where $\epsilon > 1$ is the elasticity of substitution between differentiated final goods (or the inverse of the elasticity of demand). This is the standard first-order condition for profit maximization with imperfect competition, after normalizing the price to unity, see, for example, Acemoglu (2008), and implies that producers apply a constant gross markup of $\epsilon/(\epsilon - 1) > 1$ to the marginal cost of production $(r + \delta)/F'(K)$.[9] Each of the

$$H(K, I) = Ia \log\left(b\frac{K}{I}\right) \tag{29}$$

functions that make up $F(K)$, were $I \in \mathbb{N}_+$, are strictly concave in K for all $I > 0$. Hence, unless profits are maximized at a point where the number of production processes changes, and we transition between $H(K, I)$ and $H(K, I + 1)$, or at $K = 0$, or for infinitely large K, the first-order condition (28) must hold. The latter two situations are not feasible in an equilibrium with production, as they would imply a zero or infinite demand for input. To see that profits cannot be maximized at any point K_I where F transitions between $H(K, I)$ and $H(K, I + 1)$, note that this would require

$$\frac{\epsilon - 1}{\epsilon} H_1'(K_I, I) \geq r + \delta \geq \frac{\epsilon - 1}{\epsilon} H_1'(K_I, I + 1), \tag{30}$$

the first inequality so that profits do not rise as K is reduced along $H(K, I)$, the second so that they do not rise as K is increased along $H(K, I + 1)$. The transition point K_I is given by $H(K, I) = H(K, I + 1)$, which yields

$$K_I = \frac{(I + 1)^{I+1}}{bI^I} \tag{31}$$

for $I = 1, 2, 3, \ldots$ Exploiting that $H_1'(K, I) = IaK^{-1}$, we have

$$H_1'(K_I, I) = ab\frac{(I + 1)^I}{I^I} < ab\frac{(I + 1)^{I+1}}{I^{I-1}} = H_1'(K_I, I + 1) \tag{32}$$

for finite I, contradicting the condition (30) necessary for profits to be maximized at K_I, $H_1'(K_I, I) \geq H_1'(K_I, I+1)$. The first-order condition (28) is also necessary for an equilibrium with production on the linear parts of F, which equal AK, since otherwise the marginal revenue $(\epsilon - 1)/\epsilon A$ of using an additional unit of input would always be greater, or smaller, than the marginal cost $r + \delta$, making input demand infinitely large, or zero, respectively. The condition with perfect competition (26) is a special case of that with imperfect competition (28) for $\epsilon \to \infty$, so the more general expression is used below. Inserted into the profit function (25) it yields equilibrium profits

$$\pi = F(K) - \frac{\epsilon - 1}{\epsilon} F'(K) K, \tag{33}$$

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which with imperfect competition (finite ϵ) can be non-negative at points where F has increasing, decreasing, or constant returns to scale.

The production function F assumes that the allocation among the underlying processes is optimal. Therefore, these must either be operated by the same producer, or be coordinated across producers in an effort to maximize efficiency and reduce costs. This can happen indirectly through outsourcing, or directly through collaborative arrangements such as code sharing among airlines. Alternatively, the coordination can arise as a result of all producers choosing their production levels so as to satisfy the same profit-maximizing first-order condition (28), assuming they have the same degree of market power. Hence, F can represent the production function of each individual producer, or that of the economy as a whole. With perfect competition, individual production satisfies constant returns to scale at all levels of replication, so the same applies to aggregate production.

4. EQUILIBRIUM

Households are assumed to be pricetakers and rent out their saved capital S for a rate of return r. In addition, they collect profits π generated by production. These resources are used to accumulate capital and acquire consumption goods C. Consumption and saving decisions are made so as to maximize the discounted lifetime utility

$$\int_0^\infty \frac{C(t)^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt \tag{34}$$

subject to the budget constraint

$$\dot{S}(t) = r(t) S(t) + \pi(t) - C(t) \tag{35}$$

with respect to the control C and the state S, given paths for the interest rate r and profits π, and values for the constant relative risk-aversion parameter $\theta > 0$, discount rate $\rho \in (0, 1)$, and initial capital stock $S_0 > 0$. The first-order condition

$$\frac{\dot{C}}{C} = \frac{r - \rho}{\theta} \tag{36}$$

is the usual requirement for the optimal consumption path.[10]

The market-clearing condition for the input

$$K = S \tag{37}$$

determines the equilibrium rental rate r. Combining the first-order conditions from maximizing profits (28) and utility (36) yields

$$\frac{\dot{C}}{C} = \frac{\frac{\epsilon-1}{\epsilon} F'(K) - \delta - \rho}{\theta}, \tag{38}$$

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whereas the budget constraint (35) becomes

$$\dot{K} = F(K) - \delta K - C \tag{39}$$

after substituting in for profits (25) and the market-clearing condition (37).[11]

The production function $F(K)$ is not concave, so the first-order condition for the corresponding consumer–producer or planner problem, $\dot{C}/C = [F'(K) - \delta - \rho]/\theta$, does not necessarily characterize the path that maximizes lifetime utility (34) subject to the resource constraint (39). However, for individual households, which take the real interest rate and profits as given, their budget constraints (35) are linear, making the first-order condition (36) necessary and sufficient for optimality. The two solutions can differ because the planner, or consumer–producer, might find it optimal to sacrifice current consumption in order to move to a point where the input is used more efficiently, thus, allowing for higher future consumption. For individual consumers that do not control production, the rate of return, and efficiency with which the input is used, is given, so they cannot consider such trade-offs. By separating the decision of how much input to accumulate from that of how much to use in production, the non-concave problem is isolated to the simpler non-dynamic profit maximization.

5. DYNAMICS

When competition is perfect, we have a standard AK model for all levels of input, with consumption, production, and capital always growing at the constant rate $\theta^{-1}(A - \delta - \rho)$ [see Rebelo (1991)]. With imperfect competition, the dynamics are more complicated. Whenever $K \leq 1/b$, production $F(K)$ is, and always will be, zero, so both consumption and the stock of input approach zero. For $K > 1/b$, a steady-state equilibrium with constant non-negative consumption and input use exists for any capital $\bar{K}$ and consumption $\bar{C}$ satisfying

$$F'\left(\bar{K}\right) = I^*\left(\bar{K}\right) a \bar{K}^{-1} = (\delta + \rho)\frac{\epsilon}{\epsilon - 1} \equiv \kappa \tag{40}$$

and

$$\bar{C} = F\left(\bar{K}\right) - \delta \bar{K} = I^*\left(\bar{K}\right) a \log\left[b \frac{\bar{K}}{I^*\left(\bar{K}\right)}\right] - \delta \bar{K} \tag{41}$$

where $I^*(K) \in \mathbb{N}$ denotes the optimal discrete number of processes associated with input level K.

Figure 1 shows that before $F(K)$ converges to AK, its slope varies with K. F goes from one H-function to the next, so its slope $F'(K)$ is decreasing in K while moving along any one H-function, but jumps up each time $I^*(K)$ increases and F moves to a new H-function. Inserting for the transition points K_I and K_{I-1} from (31) into $H_1'(K, I) = IaK^{-1}$, we find that while moving along $H(K, I)$,

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the slope

$$F'(K) \in \left(ab\left(\frac{I}{I+1}\right)^{I+1}, ab\left(\frac{I-1}{I}\right)^{I-1} \right) \qquad \textbf{(42)}$$

for $I = 2, 3, 4, \ldots$ Along $H(K, 1)$, $F'(K)$ falls from ab to $.25ab$. As I increases, the lower bound for $F'(K)$ rises, whereas the upper bound falls, both converging toward $A = abe^{-1} \approx .368ab$.[12] Since the optimal number of processes $I^*(K)$ is increasing in K, it follows that the smaller K is, the more prone an economy is to getting stuck at a constant steady state. That is, when $\kappa \in (.25ab, .368ab)$, it is possible for an economy that starts out with little input to stall completely, whereas one that starts out with just a little more input could grow endogenously forever, even if the two economies are identical in all other respects. Moreover, the rate of return of the input would be lower in the stalled economy, preventing it from attracting input from the other.

If $\kappa < .25ab$ (and $K > 1/b$), consumption growth is always strictly positive, and therefore production and input use must also increase over time (though not necessarily in every period). If $\kappa > ab$, consumption is always shrinking, which can only be optimal if the economy itself is shrinking. If $\kappa \in (.25ab, .368ab)$, consumption growth can be positive or negative, but if the economy does not stagnate in a constant steady state and accumulates enough input, the consumption growth rate converges toward a strictly positive number. The closer κ is to .368ab, the greater input stock an economy can have and still risk stalling. If $\kappa \in [.368ab, ab)$, consumption growth can take any sign, but there is a limit to how much the economy can grow, since consumption growth would become negative, or zero, if it ever accumulated enough input for F to converge to AK. Whenever K is large enough for F to be indistinguishable from AK, we have a standard AK model where consumption, input use and production are all growing at the constant rate $\theta^{-1}[A(\epsilon - 1)/\epsilon - \delta - \rho]$ [see Acemoglu (2008)].

The less competitive an economy is, the smaller is ϵ, and the higher is the threshold κ that the marginal product $F'(K)$ has to exceed in order to avoid stagnating in a steady state without growth. Hence, according to our model, less competitive economies are more prone to stalling.[13] Of course, the degree of competition could easily change over time, thereby, affecting the growth dynamics. For example, an economy stuck at a steady state could start growing again endogenously if the degree of competition increased sufficiently. Moreover, it could keep growing endogenously forever, even if the increase in competition was only temporary.

6. SUBOPTIMAL REPLICATION

When the number of processes must be a positive integer, so that returns to scale are not constant by assumption, replication only leads to constant returns to scale with the efficiency-maximizing scaling. There are many circumstances that can distort a producer's choice of how many processes to operate. Some examples are the time and costs incurred in setting up or dismantling a process, fixed costs associated with keeping it running, regulatory requirements that vary with the size of the operation, and credit constraints that inhibit producers from expanding at the efficiency-maximizing rate. For simplicity, we assume here that the distortion is due to a government imposed tax or subsidy of τ per production process (transferred lump sum to consumers). In addition, we imagine a perfectly competitive economy. As a result, a producer's profits are given by

$$Na \log\left(b\frac{K}{N}\right) - \tau N - (r+\delta)K \equiv \Pi(K,N), \quad (43)$$

which is strictly concave in N. Hence, the profit-maximizing number of processes is determined by the first-order condition

$$\frac{\partial \Pi(K,N)}{\partial N} = a\left[\log\left(b\frac{K}{N}\right) - 1\right] - \tau = 0, \quad (44)$$

assuming the maximum is non-negative ($\tau < -a\{1 + \log[(r+\delta)/(ab)]\}$). This yields the profit-maximizing number of processes

$$\hat{N}(K) = be^{-1-\frac{\tau}{a}}K, \quad (45)$$

and the production function

$$Y = H\left(K, \hat{N}(K)\right) = (a+\tau)\, be^{-1-\frac{\tau}{a}}K \equiv \hat{A}K, \quad (46)$$

which has constant returns to scale in K. For $\tau \neq 0$, we have $\hat{A} < A$, reflecting that the production is inefficient whenever the scaling is distorted.

When the number of processes must be a positive integer,

$$\max\left\{0, a\log(bK) - \tau - (r+\delta)K, 2a\log\left(b\frac{K}{2}\right) - 2\tau - (r+\delta)K, \ldots\right\} \quad (47)$$

yields the most profits that can be generated with a given amount of input K. The value of K at which one must switch from I to $I+1$ processes so as to maximize profits is now given by the point where $\Pi(K,I)$ and $\Pi(K,I+1)$ intersect, assuming profits are non-negative at such a point ($\tau < a\{\log[abI/(r+\delta)\log(1+1/I)] - I\log(1+1/I)\}$). This yields the transition points

$$\hat{K}_I = \frac{(I+1)^{I+1}}{bI^I}e^{\frac{\tau}{a}} \quad (48)$$

for $I = 1, 2, 3, \ldots$, which show that profit-maximizing producers use too few processes when $\tau > 0$, and too many when $\tau < 0$, compared to what maximizes output (31).

Figure 2 illustrates what happens when producers use too few processes. For $a = b = \tau = 1$, it plots $\hat{A}K$ and $H(K,I)$ for $I = 1, 2, 3, \ldots, 16$ (the latter are

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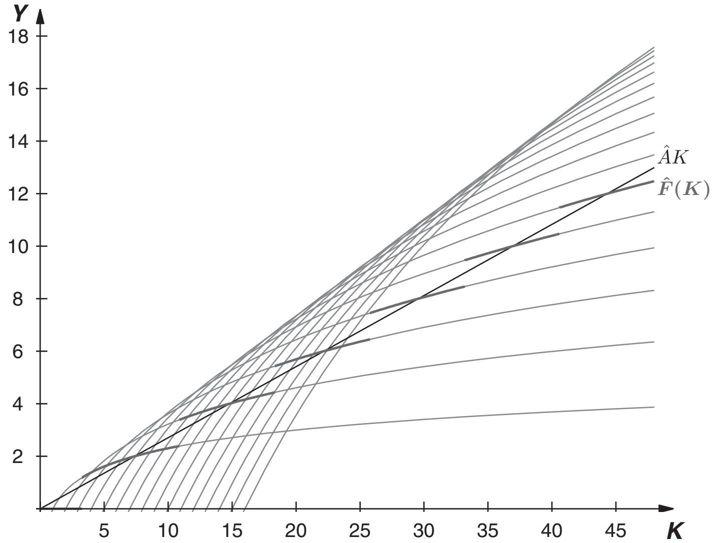

FIGURE 2. Lack of convergence toward AK with inefficient scaling.

not labeled in the figure), together with the total production, labeled $\hat{F}(K)$, that results with the profit-maximizing transition points (48). These make production jump up as we go from one H-function to the next, because the transitions are not where the H-functions intersect (but instead where the Π-functions intersect). One can easily show that

$$H\left(\hat{K}_{I+1}, I+1\right) - H\left(\hat{K}_I, I\right) = \tau, \tag{49}$$

implying that production jumps by τ units whenever the number of processes increases by one. Hence, when replication happens at a suboptimal scale, production with a discrete number of processes does not converge toward a linear production function. Instead, it converges toward a piecewise linear function that jumps when the profit-maximizing number of production processes changes. Because of these jumps, the production function does not satisfy constant returns to scale, even for large K.

Inserting for the transition points $\hat{K}_I$ and $\hat{K}_{I-1}$ into $H_1'(K, I) = IaK^{-1}$, we find that while moving along $H(K, I)$,

$$\hat{F}'(K) \in \left(ab\left(\frac{I}{I+1}\right)^{I+1} e^{-\frac{\tau}{a}}, ab\left(\frac{I-1}{I}\right)^{I-1} e^{-\frac{\tau}{a}}\right) \tag{50}$$

for $I = 2, 3, 4, \ldots$, which as I increases, converges towards $abe^{-1-\tau/a}$ $(\neq \hat{A})$. When $\tau > 0$, so that an inefficiently low number of processes is used, all operated at an inefficiently large scale, $\hat{F}'(K)$ is lower than it would be if efficiency were maximized for all I, thus, making the economy more prone to stalling in a steady state without growth, or even shrinking over time. If $\tau < 0$, the number of processes is inefficiently high, the scale inefficiently low, and $\hat{F}'(K)$ is higher than it would be if efficiency were maximized, but the economy shrinks whenever the number of processes increases.

7. CONCLUSIONS

We show how constant returns to scale and the AK model of economic growth can arise endogenously through the efficiency-maximizing replication of an underlying production process with returns to scale that vary with the intensity it is operated at. The result applies for a discrete number of replications, thus, overcoming the so-called integer problem. When competition is perfect, the AK model arises for all levels of input use and production. With imperfect competition, it only arises for a large enough number of replications, so it is possible for an economy to stagnate in a steady state without growth, whereas another that starts out with just a little more input, but is otherwise identical, could go on growing endogenously forever. Moreover, the marginal product of the input would be lower in the stalled economy, making it unable to attract inputs from the other. An economy is less prone to stalling the larger it is, and the higher the degree of competition among its producers. Our model suggests that even a temporary inflow of input, or transitory increase in competition, could start an everlasting growth spurt.

NOTES

1. In order for the degree of specialization to vary with the scale, there must be indivisibilities in production [Edgeworth (1911), Kaldor (1934), Wicksell (1934), Lerner (1944)].

2. The managerial input can lead to decreasing returns to scale [Marshall (1890), Kaldor (1934), Hicks (1939)] even when it increases proportionally with all other inputs, as it becomes overstretched due to the more than proportional complexity of the organization. The same applies for communication.

3. These assumptions are a simple way to make the returns to scale of the production process f go from being increasing to decreasing, and $f(0, 0) < 0$ plays a crucial role in that (since otherwise a concave f could have decreasing or constant returns to scale throughout). Of course, no optimizing producer would ever locate at a point where positive input use yields negative output. Other alternative assumptions are possible. As mentioned above, our results apply for any production process whose returns to scale vary with the scale it is operated at, and whose efficiency is maximized at an interior point characterized by the first-order conditions [Jensen (2014)].

4. Jensen's inequality implies that $N \times f(K/N, L/N) > \int_{n=0}^{N} f(k_n, l_n)$ and $N \times f(K/N, L/N) > \sum_{n=1}^{N} f(k_n, l_n)$ for any $k_n \neq K/N$ and $l_n \neq L/N$, for all $N > 1$ and strictly concave f.

5. Because f is strictly concave, its Hessian $\nabla^2 f(K/N, L/N)$ is negative definite, so $\partial^2 H(K, L, N)/\partial N^2 = N^{-1}[K/N, L/N]' \nabla^2 f(K/N, L/N)[K/N, L/N] < 0$, making H strictly concave in N.

6. If instead of being a constant, b were a non-rival input among production processes, so that $B = b$, the problem of choosing the optimal number of replications would remain unchanged and yield the production function $Y = ae^{-1}BK$, which has increasing returns to scale in inputs B and K jointly. Romer (1990 and 1994) and Jones (1999) argue that replication leads to increasing returns to scale, since technology, or ideas, are nonrival across production processes. Of course, not all innovations are nonrival, and even those that are nonrival are not always nonexcludable. Schumpeter (1934), Griliches

and Schmookler (1963), Schmookler (1966), Dasgupta and Stiglitz (1980), Grossman and Helpman (1991a, b), Rivera-Batiz and Romer (1991), Aghion and Howitt (1992), and Romer (1993) stress the importance of profits, and thus excludability, in driving innovation. Because externalities in production are non-rival inputs, they too can affect the returns to scale, as illustrated by Romer (1986) and Lucas (1988).

7. Alternatively, profits $H(K, N) - (r + \delta)K = Na\log(b\frac{K}{N}) - (r + \delta)K$ can be maximized with respect to K and N simultaneously which yields $\frac{Na}{K} - (r + \delta) = 0$, with respect to K, and the same condition as (22) for N. Combining these two yields $abe^{-1} - (r + \delta) = 0$, which is equivalent to condition (26).

8. This is in line with Romer's (1990, 1994) and Jones' (2005) point that perfect competition is incompatible with increasing returns to scale. In our setup it is also incompatible with K being at a point where $F(K)$ has decreasing returns, because a producer could then raise both her sales and profit margin with a large enough increase in input use. If the returns to scale of $F(K)$ were decreasing for all K, decreasing returns would be compatible with perfect competition.

9. For example, with an inverse demand function $p(Y)$ for the output Y of a particular producer, her profits are $p(F(K))F(K) - (r + \delta)K$. Assuming an interior solution, profit maximization is given by the first-order condition $(\epsilon - 1)/\epsilon \; p(F(K))F'(K) = r + \delta$, where $\epsilon = p'(F(K))F(K)/p(F(K))$. Alternatively, the condition (28) can be generated by assuming monopolistic competition a-la Dixit and Stiglitz (1977), see, for example, Zeng et al. (2014), Jensen (2016), and the appendix.

10. There is also a standard no-Ponzi game constraint, $\lim_{t\to\infty} S(t)\exp[-\int_0^t r(x)dx] \geq 0$, and transversality condition, $\lim_{t\to\infty} S(t)\exp[-\int_0^t r(x)dx] = 0$, see Acemoglu (2008).

11. Although convergence toward constant returns to scale, and thus a linear production function $F(K)$, makes this single-input model satisfy linear accumulation (39), and the necessary knife-edge assumption for balanced growth [Jones (2005), Growiec (2007)], this is not generally the case. In fact, the assumption of linear capital depreciation, which is standard in the literature, is also necessary to achieve linear accumulation in the present framework.

12. Convergence of $F'(K)$ toward A follows from $F(K)$ converging toward AK, but also from the fact that $\lim_{x\to\infty}(1 + m/x)^x = e^m$ for any constant m.

13. The same applies for high depreciation and discount rates, δ and ρ, respectively.

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APPENDIX

A.1. MODELING IMPERFECT COMPETITION

The present appendix models imperfect competition explicitly, assuming producers compete monopolistically a-la Dixit and Stiglitz (1977). Their differentiated goods are bundled together by identical households to effortlessly compose final goods. In particular, households combine the I differentiated goods in quantities X_i into Y units of final good through the aggregator

$$Y = \left(\sum_{i=1}^{I} X_i^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}}, \tag{A.1}$$

where $\epsilon \in (1, \infty)$ is the elasticity of substitution between any two differentiated goods. Households choose the mix of goods, $X_1, \ldots, X_I$ so as to minimize the cost of provisioning final goods, and hence minimize

$$\sum_{i=1}^{I} P_i X_i \tag{A.2}$$

subject to the aggregator (A.1), where P_i is the price of good i. The resulting demand for good i is

$$X_i = \left(\frac{P_i}{P} \right)^{-\epsilon} Y, \tag{A.3}$$

where the marginal cost of producing final goods is

$$P \equiv \left(\sum_{i=1}^{I} P_i^{1-\epsilon} \right)^{\frac{1}{1-\epsilon}}, \tag{A.4}$$

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which is obtained by inserting the demand for each good (A.3) into the production function for final goods (A.1). Because all households are identical, they compose identical final goods at identical cost, so its market price equals its marginal cost of production (A.4).

Given the cost of input use $(r+\delta)K_i$ and the production function $X_i = F(K_i)$ for each producer, the marginal cost of production equals $(r+\delta)/F'(K_i)$. Combined with the demand faced (A.3), producer i chooses her price P_i so as to maximize profits

$$\left(P_i - \frac{r+\delta}{F'(K_i)}\right)\left(\frac{P_i}{P}\right)^{-\epsilon} Y, \tag{A.5}$$

which yields

$$P_i = \frac{\epsilon}{\epsilon - 1}\frac{r+\delta}{F'(K_i)}, \tag{A.6}$$

the usual gross mark-up $\epsilon/(\epsilon-1)$ of marginal costs. Normalizing the price of final good to one, and exploiting that all good producers are identical, so that $P_i = 1$, yields the condition (28) used above.

A.2. SOCIAL OPTIMUM AND WELFARE

When the number of processes N can be varied continuously, the social optimum would only differ from the decentralized outcome with imperfect competition, or with a distortion that leads to replication at a suboptimal scale (Section 6), so the solution with perfect competition is the social optimum. The social planner problem is to maximize lifetime utility (34) subject to the resource constraint

$$\dot{K}(t) = N(t)a\log\left[b\frac{K(t)}{N(t)}\right] - C(t), \tag{A.7}$$

with respect to the controls N and C and the state K. The choice of N is not a dynamic decision, and deriving through the Hamiltonian with respect to it would yield the first-order condition (21). Exploiting this, the other condition would be

$$\frac{\dot{C}}{C} = \frac{abe^{-1} - \delta - \rho}{\theta} = \frac{A - \delta - \rho}{\theta}, \tag{A.8}$$

which is equivalent to (36), due to (23) and (26).

When N must be discrete, the optimization problem for the social optimum would require solving a dynamic program with a production function F that is not smooth or concave (everywhere). In this case, the decentralized and planner solution could differ. As mentioned in Section 4, the planner, or a household-producer, might find it optimal to sacrifice current consumption to move to a point where the input is used more efficiently, thus, allowing for higher future consumption. However, individual consumers, who have no impact on production and scaling, cannot make such considerations, and just take the efficiency with which the input is used, and the rate of return it earns, as given. Beyond what is mentioned above, steady-state welfare analysis is fairly standard. If the economy gets stuck, we are in the Solow model. If it does not get stuck, we will be in the AK model.