Proceedings of the Edinburgh Mathematical Society (2016) **59**, 65–76
DOI:10.1017/S0013091515000590

ON FRACTIONAL HEAT EQUATIONS WITH NON-LOCAL INITIAL CONDITIONS

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(Received 16 August 2013)

Abstract In this paper we consider the problem of existence of mild solutions to semilinear fractional heat equations with non-local initial conditions. We provide sufficient conditions for existence and regularity of such solutions.

Keywords: fractional heat equations; non-local initial conditions; Mittag-Leffler functions

2010 *Mathematics subject classification:* Primary 35K05; 35K55
Secondary 33E12

1. Introduction and statement of the main results

In this paper we are interested in ensuring the existence of solutions to problems that arise in the theory of heat conduction with non-local initial conditions. Specifically, we will study the problem

$$\left.\begin{aligned} D_t^\gamma u &= \Delta u + f(t,u), \quad [0,\infty)\times\Omega, \\ u &= 0, \qquad\qquad\quad\; [0,\infty)\times\partial\Omega, \\ u(0,x) &= u_0(x) + \sum_{i=1}^{k}\beta_i(x)u(T_i,x), \quad x\in\Omega, \end{aligned}\right\} \tag{1.1}$$

where D_t^γ is Caputo's derivative of order $\gamma\in(0,1]$, $\Omega\subset\mathbb{R}^n$ is a bounded smooth domain, $T_i\in(0,\infty)$ are fixed real numbers, $\beta_i\colon\Omega\to\mathbb{R}$, $i=1,\dots,k$, and $f\colon[0,\infty)\times\mathbb{R}\to\mathbb{R}$ are continuous functions with f satisfying, for some $\rho>1$,

$$|f(t,s)-f(t,r)|\leqslant c(1+|s|^{\rho-1}+|r|^{\rho-1})|s-r| \tag{1.2}$$

and

$$|f(t,s)| \leqslant c(1+|s|^{\rho}) \tag{1.3}$$

for all $t \in [0,\infty)$ and $r, s \in \mathbb{R}$. In particular, if $\gamma = 1$, the above problem becomes the well-known nonlinear heat equation.

The model problem that we have in mind is

$$\left.\begin{aligned} D_t^{\gamma} u(t,x) = \Delta u(t,x) + \kappa u(t,x)|u(t,x)|^{\rho-1}, \quad [0,\infty)\times\Omega, \\ u(t,x) = 0, \quad [0,\infty)\times\partial\Omega, \end{aligned}\right\} \tag{1.4}$$

with initial condition

$$u(0,x) = u_0(x) + \sum_{i=1}^{k} \beta_i u(T_i, x), \tag{1.5}$$

where $\kappa > 0$, $\beta_i \in \mathbb{R}$, $T_i \in (0,\infty)$ and $x \in \Omega$.

Physically, (1.5) says that some initial measurements were made at the times 0 and $T_i \in (0,\infty)$, $i = 1, \dots, k$, and the observer uses this previous information in their model. This type of situation can lead us to a better description of the phenomenon. For example, consider the phenomenon of diffusion of a small amount of gas in a tube and assume that the diffusion is observed via the surface of the tube (see [**4**]). If there is too small an amount of gas at the initial time, then the measurement $u(0,x)$ of the amount of the gas in this instant may be less precise than the measurement $u(0,x) + \sum_{i=1}^{k} \beta_i u(T_i,x)$. We remark that initial conditions of type (1.5) were previously considered in [**2**–**4**, **6**, **17**, **19**, **20**].

We will treat (1.1) as an abstract evolution equation in the L^q setting with $1 < q < \infty$. To this end, let $A_q = \Delta$ with Dirichlet boundary conditions in Ω. Then A_q can be seen as an unbounded operator in $X_q^0 := L^q(\Omega)$ with domain

$$X_q^1 := W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega).$$

It is well known that the scale of fractional power spaces $\{X_q^{\alpha}\}_{\alpha\in\mathbb{R}}$ associated with A_q verifies (see [**1**, **5**])

$$\begin{aligned} X_q^{\alpha} &\hookrightarrow H_q^{2\alpha}(\Omega), \quad \alpha \geqslant 0,\ 1<q<\infty, \\ X_q^{-\alpha} &\hookrightarrow (E_{q'}^{\alpha})', \qquad \alpha \geqslant 0,\ 1<q<\infty,\ q' = \frac{q}{q-1}. \end{aligned}$$

Therefore, we have

$$\begin{gathered} X_q^{\alpha} \hookrightarrow L^r(\Omega) \quad \text{for } r \leqslant \frac{Nq}{N-2q\alpha},\ 0 \leqslant \alpha < \frac{N}{2q}, \\ X_q^0 = L^q(\Omega), \\ X_q^{\alpha} \hookleftarrow L^s(\Omega) \quad \text{for } s \geqslant \frac{N}{N-2q\alpha},\ -\frac{N}{2q'} < \alpha \leqslant 0, \end{gathered}$$

with continuous embeddings.

https://doi.org/10.1017/S0013091515000590 Published online by Cambridge University Press

It is important to observe that A_q is a sectorial operator, that is, there exist positive constants C and $\phi \in (\pi/2, \pi)$ such that

$$\Sigma_\phi := \{\lambda \in \mathbb{C} \setminus \{0\} \colon |\arg(\lambda)| < \phi\} \subset \rho(A_q)$$

and

$$\|(\lambda - A_q)^{-1}\|_{\mathcal{L}(X_q^0, X_q^0)} \leqslant \frac{C}{|\lambda|} \quad \forall \lambda \in \Sigma_\phi. \tag{1.6}$$

With the above considerations we can rewrite (1.1) on the Banach space X_q^0 in the abstract form

$$\left.\begin{aligned} D_t^\gamma u(t) = A_q u(t) + f(t, u(t)), \quad t \in [0, \infty),\\ u(0) = g(u), \end{aligned}\right\} \tag{1.7}$$

where $\gamma \in (0, 1]$ for $x \in \Omega$ and $T_i > 0$, $u(t)(x) = u(t, x)$, $f(t, u(t))(x) = f(t, u(t, x))$ and

$$g(u)(x) = u_0(x) + \sum_{i=1}^{k} \beta_i u(T_i, x). \tag{1.8}$$

Consider $\alpha \in [0, 1)$ such that the function f in (1.7) is well defined from $[0, \infty) \times X_q^\alpha$ into X_q^0. By (1.2), (1.3) and the fact that if $\alpha \in (0, 1)$, then $X_q^\alpha \hookrightarrow X_q^0$ has continuous inclusion, we can see that

$$\|f(t, x) - f(t, y)\|_{X_q^0} \leqslant c(1 + \|x\|_{X_q^\alpha}^{\rho-1} + \|y\|_{X_q^\alpha}^{\rho-1})\|x - y\|_{X_q^\alpha} \tag{1.9}$$

and

$$\|f(t, x)\|_{X_q^0} \leqslant c(1 + \|x\|_{X_q^\alpha}^{\rho}) \tag{1.10}$$

for all $x, y \in X_q^\alpha$, all $t \geqslant 0$ and some $c > 0$.

Going further, let us make some considerations. Firstly, for $\gamma, \theta \in (0, \infty)$, consider the function

$$E_{\gamma,\theta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\gamma m + \theta)}, \quad z \in \mathbb{C}.$$

Clearly, $E_{\gamma,\theta}$ is an entire function for all $\gamma, \theta \in (0, \infty)$. Furthermore, since $\Gamma(m+1) = m!$ for all $m \in \mathbb{N}$, we have

$$E_{1,1}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(m + 1)} = \mathrm{e}^z, \quad z \in \mathbb{C},$$

where $z \mapsto \mathrm{e}^z$ is the exponential function. Actually, we observe that other notable elementary functions can be recovered by $E_{\gamma,\theta}$. For example, if $\gamma = 2$ and $\theta = 1$, we have

$$E_{2,1}(z^2) = \cosh(z) \quad \text{and} \quad E_{2,1}(-z^2) = \cos(z), \quad z \in \mathbb{C}.$$

A very useful property of the function $E_{\gamma,\theta}$ is its integral representation. Indeed, we have

$$E_{\gamma,\theta}(z) = \frac{1}{2\pi \mathrm{i}} \int_{Ha'} \mathrm{e}^\lambda \lambda^{\gamma-\theta} (\lambda^\gamma - z)^{-1} \, \mathrm{d}\lambda, \quad z \in \mathbb{C}, \tag{1.11}$$

where Ha' is a contour that starts and ends at $-\infty$ and encircles the origin once anticlockwise. We will fix $E_\gamma := E_{\gamma,1}$. The function E_γ is called a *Mittag-Leffler function* in honour of Mittag-Leffler, who introduced and investigated it at the beginning of the twentieth century (see [**9**–**11**]). Many studies on Mittag-Leffler functions and their applications to several branches of knowledge can be found in the literature. We particularly recommend to the reader the books [**8**, **12**, **14**] and the papers [**7**, **13**, **15**, **16**, **18**].

On the other hand, let us suppose for a moment that $u\colon [0,\infty) \to X_q^0$ verifies (1.7). Formally, applying the Laplace transform in (1.7) we have

$$\lambda^\gamma \hat{u}(\lambda) - \lambda^{\gamma-1} g(u) = A_q \hat{u}(\lambda) + F(\lambda) \quad \Longrightarrow \quad (\lambda^\gamma - A_q)\hat{u}(\lambda) = \lambda^{\gamma-1} g(u) + F(\lambda),$$

where $F(\lambda)$ is the Laplace transform of the function $t \mapsto f(t, u(t))$. Taking $\lambda^\gamma \in \rho(A_q)$, it follows that

$$\hat{u}(\lambda) = \lambda^{\gamma-1}(\lambda^\gamma - A_q)^{-1} g(u) + (\lambda^\gamma - A_q)^{-1} F(\lambda).$$

Using the inverse Laplace transform we deduce that

$$u(t) = E_\gamma(tA_q) g(u) + \int_0^t (t-s)^{\gamma-1} E_{\gamma,\gamma}((t-s)A_q) f(s, u(s))\, \mathrm{d}s, \quad t \geqslant 0,$$

where $E_\gamma(tA_q)$ and $t^{\gamma-1}E_{\gamma,\gamma}(tA_q)$ are, formally at least, the inverse Laplace transform of

$$\lambda \mapsto \lambda^{\gamma-1}(\lambda^\gamma - A_q)^{-1}$$

and

$$\lambda \mapsto (\lambda^\gamma - A_q)^{-1},$$

respectively. In the next section we prove that these functions are well defined for all $\gamma \in (0,1]$. Indeed, we start the next section with the proof of the following result.

Proposition 1.1. *Consider $\gamma \in (0,1]$. The functions*

$$E_\gamma(tA_q) := \frac{1}{2\pi \mathrm{i}} \int_{Ha} \mathrm{e}^{\lambda t} \lambda^{\gamma-1}(\lambda^\gamma - A_q)^{-1}\, \mathrm{d}\lambda, \quad t \geqslant 0,$$

and

$$E_{\gamma,\gamma}(A_q t) := \frac{t^{1-\alpha}}{2\pi \mathrm{i}} \int_{Ha} \mathrm{e}^{\lambda t}(\lambda^\gamma - A_q)^{-1}\, \mathrm{d}\lambda, \qquad t \geqslant 0,$$

where Ha is a suitable path, are well defined and there exists a constant $M > 0$ such that

$$\|E_\gamma(tA_q)x\|_{X_q^0} \leqslant M\|x\|_{X_q^0} \quad \text{and} \quad \|E_{\gamma,\gamma}(A_q t)x\|_{X_q^0} \leqslant M\|x\|_{X_q^0}$$

for every $t \geqslant 0$ and $x \in X_q^0$.

The strategy to prove Proposition 1.1, and later Proposition 1.4, basically consists of considering the path

$$Ha = \{s\mathrm{e}^{\mathrm{i}\eta} \colon r \leqslant s < \infty\} \cup \{r\mathrm{e}^{\mathrm{i}s} \colon |s| \leqslant \eta\} \cup \{s\mathrm{e}^{-\mathrm{i}\eta} \colon r \leqslant s < \infty\}$$

for $r > 0$ and suitable $\eta \in (\pi/2, \pi)$, both independent of $\gamma \in (0,1]$, and to use the estimate of the resolved operator (1.6). (See the next section for more details.)

Motivated by this discussion and by the previous related literature, we adopt the following concept for a solution to (1.7).

Definition 1.2. Let $\tau > 0$. A function $u\colon [0,\tau] \to X$ is said to be a local mild solution to (1.7) in $[0,\tau]$ if $u \in C([0,\tau]; X_q^0)$ and

$$u(t) = E_\gamma(tA_q)u_0 + \sum_{i=1}^{k} \beta_i E_\gamma(tA_q)u(T_i) + \int_0^t (t-s)^{\gamma-1} E_{\gamma,\gamma}((t-s)A_q) f(s, u(s))\,\mathrm{d}s$$

for all $t \geqslant 0$.

Our main result ensures existence of a local mild solution to (1.7), or equivalently to (1.1), in X_q^α, $\alpha \in [0,1)$. Furthermore, we prove that these solutions regularize immediately. Precisely, we prove the following existence result.

Theorem 1.3. *Consider $\gamma \in (0,1]$ and $\tau \in (0,\infty)$ such that $T_i \leqslant \tau < \infty$, $i = 1,\dots,k$. Suppose that $\alpha \in [0,1)$ is such that the function $f\colon [0,\tau] \times X_q^\alpha \to X_q^0$ is well defined. Consider $u_0 \in X_q^\alpha$ and $\beta_i\colon \Omega \to \mathbb{R}$ bounded functions. If $c > 0$ and $\|\beta_i\|_\infty$ are small enough, where c is given in (1.9) and (1.10), then (1.7) has at least one mild solution $u \in C([0,\tau], X_q^\alpha)$. If u is such a mild solution, then, for all $t > 0$,*

$$u(t) \in X_q^{\alpha+\theta} \quad \forall \theta \in [0, 1-\alpha). \tag{1.12}$$

Furthermore, if $n > q$, then the set

$$\{u(t)\colon 0 < t \leqslant \tau\} \subset X_q^\alpha$$

is a compact set.

It is interesting to note that the regularity result (1.12) is independent of the order of derivation $\gamma \in (0,1]$.

To prove the above result we just need to understand the behaviour of the Mittag-Leffler families on the scale of fractional power spaces associated with the linear operator A_q. For this reason, in the next result we give some information on these families in the spaces X_q^α associated with the sectorial operator A_q.

Proposition 1.4. *Consider $\gamma \in (0,1]$ and $0 \leqslant \alpha \leqslant 1$. There then exists a constant $M > 0$ such that*

$$\|E_\gamma(tA_q)x\|_{X_q^\alpha} \leqslant Mt^{-\gamma\alpha}\|x\|_{X_q^0} \quad \text{and} \quad \|t^{\gamma-1}E_{\gamma,\gamma}(tA_q)x\|_{X_q^\alpha} \leqslant Mt^{\gamma(1-\alpha)-1}\|x\|_{X_q^0}$$

for all $t > 0$.

Remark 1.5. For the special situation in which $\gamma = 1$, the above result recovers the well-known estimates for the semigroup generated by A_q.

Having obtained the above result, our strategy will be to define a nonlinear operator T on $C([0,\tau];X_q^\alpha)$, $\alpha\in[0,1)$, by

$$Tu(t)=E_\gamma(tA_q)u_0+\sum_{i=1}^{k}\beta_iE_\gamma(tA_q)u(T_i)+\int_0^t(t-s)^{\gamma-1}E_{\gamma,\gamma}((t-s)A_q)f(s,u(s))\,\mathrm{d}s,\quad t\geqslant 0,$$

and to confirm the existence of a fixed point to it.

As an immediate application of Theorem 1.3 we consider, for example, the problem

$$\left.\begin{aligned}D_t^\gamma u(t,x)&=\Delta u(t,x)+\kappa u(t,x)|u(t,x)|,\quad [0,\infty)\times\Omega,\\ u(t,x)&=0,\quad [0,\infty)\times\partial\Omega,\\ u(0,x)&=u_0(x)+\sum_{i=1}^{k}\beta_iu(T_i,x),\quad x\in\Omega,\end{aligned}\right\}\tag{1.13}$$

where $\kappa>0$, $\gamma\in(0,1]$, $\Omega\subset\mathbb{R}^3$ is a bounded smooth domain, $T_i\in(0,\infty)$ and $\beta_i\in\mathbb{R}$ are fixed real numbers, $i=1,\dots,k$. We will consider problem (1.13) with $u_0\in H_0^1(\Omega)$. To this end, set $X_2^0=L^2(\Omega)$ and

$$D(\Delta):=X_2^1=H^2(\Omega)\cap H_0^1(\Omega).$$

The nonlinear term in (1.13) is given by

$$f(t,s)=\kappa s|s|,\quad t\geqslant 0,\ s\in\mathbb{R},$$

which verifies (1.2) and (1.3) with $\rho=2$ and $c=\kappa$. To use Theorem 1.3 we just need to prove that the abstract nonlinear term associated with (1.13) is well defined from $[0,\infty)\times H_0^1$ into $L^2(\Omega)$, that is, the function $f\colon[0,\infty)\times X_2^{1/2}\to X_2^0$ given by

$$f(t,\phi)(x):=\kappa\phi(x)|\phi(x)|,\quad x\in\Omega,$$

is well defined. But the above follows immediately by Hölder's inequality and by the fact that $H_0^1(\Omega)\hookrightarrow L^2(\Omega)$ is a compact inclusion. Hence, if κ and $\max\{\beta_i\}$ are small enough, problem (1.13) has at least one mild solution. Furthermore, if

$$u\colon[0,\tau]\to X_2^{1/2}(\Omega)$$

is such a solution, then

$$u(t)\in X_2^{1/2+\theta}\quad\forall\theta\in[0,1/2)$$

and the set

$$\{u(t)\colon 0<t\leqslant\tau\}\subset H_0^1(\Omega)$$

is a compact set. Particularly, if $\beta_i=0$, $i=1,\dots,k$, we have that the problem

$$\begin{aligned}D_t^\gamma u(t,x)&=\Delta u(t,x)+\kappa u(t,x)|u(t,x)|, && [0,\infty)\times\Omega,\\ u(t,x)&=0, && [0,\infty)\times\partial\Omega,\\ u(0,x)&=u_0(x), && x\in\Omega,\end{aligned}$$

has at least one mild solution with the above properties.

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2. Proof of the main results

In this section we will prove our main results. We start with the proof of Proposition 1.1.

Proof of Proposition 1.1. Consider $\phi \in (\pi/2, \pi)$ and let Σ_ϕ be the sector associated with the sectorial operator A_q and choose arbitrary values $\varepsilon > 0$ and $\theta \in (\pi/2, \phi]$. Let $Ha = Ha(\varepsilon, \theta)$ be the Hankel path given by $Ha = Ha_1 + Ha_2 - Ha_3$, where the Ha_i are such that

$$\left.\begin{aligned} Ha_1 &:= \{t\mathrm{e}^{\mathrm{i}\theta} \colon t \in [\varepsilon, \infty)\},\\ Ha_2 &:= \{\varepsilon\mathrm{e}^{\mathrm{i}t} \colon t \in [-\theta, \theta]\},\\ Ha_3 &:= \{t\mathrm{e}^{-\mathrm{i}\theta} \colon t \in [\varepsilon, \infty)\}.\end{aligned}\right\} \tag{2.1}$$

We will estimate the function $\|E_\gamma(tA_q)\|$ on each Ha_i, according to definition (2.1), for any $t > 0$. Just observe that for each fixed $t \neq 0$, if we assume that $\varepsilon = 1/t$, then the following hold.

- On Ha_1, it holds that

$$\left\|\frac{1}{2\pi\mathrm{i}}\int_{Ha_1} \mathrm{e}^{\lambda t}\lambda^{\gamma-1}(\lambda^\gamma - A_q)^{-1}\,\mathrm{d}\lambda\right\| \leqslant \frac{1}{2\pi}\left\|\int_\varepsilon^\infty \mathrm{e}^{ts\mathrm{e}^{\mathrm{i}\theta}}(s\mathrm{e}^{\mathrm{i}\theta})^{\gamma-1}((s\mathrm{e}^{\mathrm{i}\theta})^\gamma - A_q)^{-1}\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}s\right\|$$

and, using that if $\lambda = s\mathrm{e}^{\mathrm{i}\theta} \in Ha(\varepsilon, \theta) \subset \Sigma_\phi$, then $\lambda^\gamma \in \Sigma_\phi$, we obtain by the sectorial property

$$\begin{aligned}\left\|\frac{1}{2\pi\mathrm{i}}\int_{Ha_1} \mathrm{e}^{\lambda t}\lambda^{\gamma-1}(\lambda^\gamma - A_q)^{-1}\,\mathrm{d}\lambda\right\| &\leqslant \frac{N}{2\pi}\int_\varepsilon^\infty \mathrm{e}^{ts\cos(\theta)}|(se^{i\theta})|^{-1}\,\mathrm{d}s\\ &\leqslant \frac{N}{2\pi\varepsilon}\int_\varepsilon^\infty \mathrm{e}^{ts\cos(\theta)}\,\mathrm{d}s\\ &= \frac{N\mathrm{e}^{\cos(\theta)}}{-2\pi\cos(\theta)}.\end{aligned}$$

- On Ha_2, we have

$$\begin{aligned}\left\|\frac{1}{2\pi\mathrm{i}}\int_{Ha_2} \mathrm{e}^{\lambda t}\lambda^{\gamma-1}(\lambda^\gamma - A_q)^{-1}\,\mathrm{d}\lambda\right\| &= \frac{1}{2\pi}\left\|\int_{-\theta}^\theta \mathrm{e}^{t\varepsilon\mathrm{e}^{\mathrm{i}s}}(\varepsilon\mathrm{e}^{\mathrm{i}s})^{\gamma-1}((\varepsilon\mathrm{e}^{\mathrm{i}s})^\gamma - A_q)^{-1}\mathrm{i}\varepsilon\mathrm{e}^{\mathrm{i}s}\,\mathrm{d}s\right\|\\ &\leqslant \frac{N}{2\pi}\int_{-\theta}^\theta \mathrm{e}^{t\varepsilon\cos(s)}\,\mathrm{d}s\\ &\leqslant \frac{\theta N\mathrm{e}}{\pi}.\end{aligned}$$

- On Ha_3 we proceed in the same way as in Ha_1.

Taking M as the maximum over all the bounds obtained above, we deduce that $E_\gamma(tA_q)$ is well defined for each $t > 0$ and $\|E_\gamma(tA_q)\| \leqslant M$.

We shall now seek a uniform bound for the function $\|E_\gamma(tA_q)\|$ for all $t>0$. For this, just observe that if $\varepsilon<\varepsilon'$ and $\pi/2<\theta'<\theta<\phi$ and we take $Ha=Ha(\varepsilon,\theta)$ and $Ha'=Ha(\varepsilon',\theta')$, we obtain the equality

$$\frac{1}{2\pi \mathrm{i}}\int_{Ha}\mathrm{e}^{\lambda t}\lambda^{\gamma-1}(\lambda^\gamma-A_q)^{-1}\,\mathrm{d}\lambda=\frac{1}{2\pi \mathrm{i}}\int_{Ha'}\mathrm{e}^{\lambda t}\lambda^{\gamma-1}(\lambda^\gamma-A_q)^{-1}\,\mathrm{d}\lambda\quad\forall t\geqslant 0.$$

This is enough to justify that the estimate obtained before is uniform on $t>0$.

Finally, we observe that, making a change of variables,

$$E_\gamma(tA_q)=\frac{1}{2\pi \mathrm{i}}\int_{Ha}\mathrm{e}^{\lambda}\lambda^{\gamma-1}(\lambda^\gamma-A_qt^\gamma)^{-1}\,\mathrm{d}\lambda,\quad t\geqslant 0,$$

and therefore, using the dominated convergence theorem, we conclude that $E_\gamma(0)$ is the identity operator.

A similar procedure proves that $\|E_{\gamma,\gamma}(tA_q)\|\leqslant M$ for all $t\in[0,\infty)$. □

Proof of Proposition 1.4. We have that

$$\begin{aligned}\|E_\gamma(tA_q)x\|_{X^\alpha}&=\left\|\frac{1}{2\pi \mathrm{i}}\int_{Ha}\mathrm{e}^{\lambda t}\lambda^{\gamma-1}A_q^\alpha(\lambda^\gamma-A_q)^{-1}x\,\mathrm{d}\lambda\right\|_{X_q^0}\\&\leqslant\frac{t^{-\gamma}}{2\pi}\left\|\int_{Ha}\mathrm{e}^{\lambda}\lambda^{\gamma-1}A_q^\alpha((\lambda/t)^\gamma-A_q)^{-1}x\,\mathrm{d}\lambda\right\|_{X_q^0}\\&\leqslant t^{-\gamma\alpha}\left(\frac{C}{2\pi}\int_{Ha}|\mathrm{e}^{\lambda}\lambda^{\gamma\alpha-1}|\,\mathrm{d}|\lambda|\right)\|x\|_{X_q^0}.\end{aligned}$$

Hence, it is sufficient to choose $M>1$ such that

$$\frac{C}{2\pi}\int_{Ha}|\mathrm{e}^{\lambda}\lambda^{\gamma\alpha-1}|\,\mathrm{d}|\lambda|\leqslant M.$$

By a similar procedure one may prove the second estimate. □

Proof of Theorem 1.3. Consider $r>0$ such that

$$\|u_0\|_{X_q^\alpha}<\min\left\{\frac{r}{4M},\frac{r}{4}\right\}.$$

Let B_r^α be the space

$$B_r^\alpha:=\left\{v\in C([0,\tau];X_q^\alpha)\colon \sup_{0\leqslant s\leqslant\tau}\|v(s)\|_{X_q^\alpha}\leqslant r\right\}$$

with norm $\|v\|_{B_r^\alpha}:=\sup_{0\leqslant s\leqslant\tau}\|v(s)\|_{X_q^\alpha}$. On B_r^α define the operator

$$Tu(t)=E_\gamma(tA_q)g(u)+\int_0^t(t-s)^{\gamma-1}E_{\gamma,\gamma}((t-s)A_q)f(s,u(s))\,\mathrm{d}s. \tag{2.2}$$

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Let us first prove that T is a well-defined map and $T(B_r^\alpha) \subset B_r^\alpha$. Initially, fix $t_2 \in [0,\tau)$ and let $t_2 < t_1 \leqslant \tau$. We then have

$$\begin{aligned}\|Tu(t_1) - Tu(t_2)\|_{X_q^\alpha} \leqslant{}& \|(E_\gamma(t_1A_q) - E_\gamma(t_2A_q))g(u)\|_{X_q^\alpha} \\ &+ \left\| \int_0^{t_2} [(t_1-s)^{\gamma-1}E_{\gamma,\gamma}((t_1-s)A_q) \right. \\ &\qquad \left. - (t_2-s)^{\gamma-1}E_{\gamma,\gamma}((t_2-s)A_q)]f(s,u(s))\,\mathrm{d}s \right\|_{X_q^\alpha} \\ &+ \left\| \int_{t_2}^{t_1} (t_1-s)^{\gamma-1}E_{\gamma,\gamma}((t_1-s)A_q)f(s,u(s))\,\mathrm{d}s \right\|_{X_q^\alpha}.\end{aligned}$$

In the above, it is not hard to see that the first term goes to zero as $t_1 \to t_2^+$. Using Lebesgue's dominated convergence theorem, we can prove that the second term also goes to zero as $t_1 \to t_2^+$. Let us consider the third term. For it we have

$$\begin{aligned}&\left\| \int_{t_2}^{t_1} (t_1-s)^{\gamma-1}E_{\gamma,\gamma}((t_1-s)A_q)f(s,u(s))\,\mathrm{d}s \right\|_{X_q^\alpha} \\ &\qquad\leqslant M\int_{t_2}^{t_1} (t_1-s)^{\gamma(1-\alpha)-1}\|f(s,u(s))\|_{X_q^0}\,\mathrm{d}s \\ &\qquad\leqslant Mc\int_{t_2}^{t_1} (t_1-s)^{\gamma(1-\alpha)-1}(1+\|u(s)\|_{X_q^\alpha}^\rho)\,\mathrm{d}s \\ &\qquad\leqslant Mc(1+r^\rho)\int_{t_2}^{t_1} (t_1-s)^{\gamma(1-\alpha)-1}\,\mathrm{d}s \\ &\qquad\leqslant Mc(1+r^\rho)\int_0^{t_1-t_2} s^{\gamma(1-\alpha)-1}\,\mathrm{d}s,\end{aligned}$$

which goes to zero as $t_1 \to t_2^+$. The case in which $t_1 < t_2$ is similar. Then, for all $u \in B_r^\alpha$, $Tu \in C([0,\tau]; X_q^\alpha)$.

On the other hand, if $u \in B_r^\alpha$, we have

$$\begin{aligned}\|Tu(t)\|_{X_q^\alpha} &\leqslant \|E_\gamma(tA_q)g(u)\|_{X_q^\alpha} + M\int_0^t (t-s)^{\gamma(1-\alpha)-1}\|f(s,u(s))\|_{X_q^0}\,\mathrm{d}s \\ &\leqslant M\|u_0\|_{X_q^\alpha} + M\sum_{i=1}^k \|\beta_i\|_\infty \|u(T_i)\|_{X_q^\alpha} \\ &\quad + Mc\int_0^t (t-s)^{\gamma(1-\alpha)-1}(1+\|u(s)\|_{X_q^\alpha}^\rho)\,\mathrm{d}s \\ &\leqslant M\|u_0\|_{X_q^\alpha} + Mr\sum_{i=1}^k \|\beta_i\|_\infty + Mc\left(\frac{(1+r^\rho)\tau^{\gamma(1-\alpha)}}{\gamma(1-\alpha)}\right).\end{aligned}$$

Then, if

$$\sum_{i=1}^k \|\beta_i\|_\infty < \frac{1}{4M} \tag{2.3}$$

https://doi.org/10.1017/S0013091515000590 Published online by Cambridge University Press

and

$$0 < c < \left(\frac{\gamma(1-\alpha)}{2M\tau^{\gamma(1-\alpha)}}\right)\frac{r}{1+r^{\rho}},$$

we have that T takes B_r^α into itself. From now we fix $\beta_i > 0$, $i = 1, \dots, k$, with the above property.

The next step is to show that T has a fixed point in B_r^α. We claim that if $c > 0$ is small enough, the operator T is a contraction in B_r^α. In fact, consider $u, v \in B_r^\alpha$. Then,

$$\begin{aligned}
&\|Tu(t) - Tv(t)\|_{X_q^\alpha} \\
&\quad\leqslant \sum_{i=1}^{k} \|\beta_i\|_\infty \|E_\gamma(tA_q)(u(T_i) - v(T_i))\|_{X_q^\alpha} \\
&\qquad + M\int_0^t (t-s)^{\gamma(1-\alpha)-1}\|f(s,u(s)) - f(s,v(s))\|_{X_q^0}\,\mathrm{d}s \\
&\quad\leqslant M\sum_{i=1}^{k}\|\beta_i\|_\infty \sup_{0\leqslant t\leqslant\tau}\|u(t)-v(t)\|_{X_q^\alpha} \\
&\qquad + Mc\int_0^t (t-s)^{\gamma(1-\alpha)-1}(1 + \|u(s)\|_{X_q^\alpha}^{\rho-1} + \|v(s)\|_{X_q^\alpha}^{\rho-1})\|u(s)-v(s)\|_{X_q^\alpha}\,\mathrm{d}s \\
&\quad\leqslant \left[M\sum_{i=1}^{k}\|\beta_i\|_\infty + Mc\left(\frac{(1+2r^{\rho-1})\tau^{\gamma(1-\alpha)}}{\gamma(1-\alpha)}\right)\right]\sup_{0\leqslant t\leqslant\tau}\|u(t)-v(t)\|_{X_q^\alpha}.
\end{aligned}$$

It is then sufficient to consider

$$0 < c < \left(\frac{\gamma(1-\alpha)}{2M\tau^{\gamma(1-\alpha)}}\right)\frac{1}{1+2r^{\rho-1}}$$

and we have that T is a $\frac{3}{4}$-contraction.

Finally, considering β_i given by (2.3) and

$$0 < c < \left(\frac{\gamma(1-\alpha)}{2M\tau^{\gamma(1-\alpha)}}\right)L(r),$$

where

$$L(r) = \min\left\{\frac{r}{1+r^{\rho}}, \frac{1}{1+2r^{\rho-1}}\right\},$$

by the Banach fixed point theorem it follows that T has a fixed point $u \in B_r^\alpha$.

From now on, consider $u \in C([0,\tau]; X_q^\alpha)$ a mild solution to (1.7) and $\theta \in [0, 1-\alpha)$. If $t > 0$, we have

$$\begin{aligned}
\|u(t)\|_{X_q^{\alpha+\theta}} \leqslant \|E_\gamma(tA_q)u_0\|_{X_q^{\alpha+\theta}} + \sum_{i=1}^{k}\|\beta_i\|_\infty\|E_\gamma(tA_q)u(T_i)\|_{X_q^{\alpha+\theta}} \\
+ \int_0^t \|(t-s)^{\gamma-1}E_{\gamma,\gamma}((t-s)A_q)f(s,u(s))\|_{X_q^{\alpha+\theta}}\,\mathrm{d}s
\end{aligned}$$

$$\leqslant Mt^{-\gamma\theta}\|u_0\|_{X_q^\alpha} + Mt^{-\gamma\theta}\sum_{i=1}^{k}\|\beta_i\|_\infty\|u(T_i)\|_{X_q^\alpha} + Mc\int_0^t (t-s)^{\gamma(1-\alpha-\theta)-1}(1+\|u(s)\|_{X_q^\alpha}^\rho)\,\mathrm{d}s$$

$$\leqslant Mt^{-\gamma\theta}\|u_0\|_{X_q^\alpha} + Mt^{-\gamma\theta}\sum_{i=1}^{k}\|\beta_i\|_\infty\|u(T_i)\|_{X_q^\alpha} + \frac{Mc(1+\sup_{0\leqslant t\leqslant\tau}\|u(t)\|_{X_q^\alpha}^\rho)t^{\gamma(1-\alpha-\theta)}}{\gamma(1-\alpha-\theta)},$$

and therefore $u(t) \in X_q^{\alpha+\theta}$, proving the second part of the theorem.

Finally, to conclude the proof, consider $\alpha' \in (\alpha, 1)$ and $\tau_0 > 0$. For all $t \in [\tau_0, \tau]$ we have

$$\|u(t)\|_{X_q^{\alpha'}} \leqslant Mt^{\alpha-\alpha'}\|u_0\|_{X_q^\alpha} + Mt^{\alpha-\alpha'}\sum_{i=1}^{k}\|\beta_i\|_\infty\|u(T_i)\|_{X_q^\alpha} + \frac{Mc(1+\sup_{0\leqslant t\leqslant\tau}\|u(t)\|_{X_q^\alpha}^\rho)t^{\gamma(1-\alpha')}}{\gamma(1-\alpha')},$$

which proves that $\{u(t)\colon t \in [\tau_0, \tau]\} \subset X_q^{\alpha'}$ is a bounded set. Since $X_q^{\alpha'} \hookrightarrow X_q^\alpha$ is a compact inclusion, we conclude the proof. $\square$

Acknowledgements. The results in this work were partly obtained during a visit in August 2012 of B.A. to the Departamento de Matematica y Estadstica, Universidad de La Frontera, Temuco, Chile: he thanks Professor Herme Soto for his kind hospitality. B.A. is partly supported by CNPQ-Brazil (Grants 460800/2014-0, 100994/2011-3 and 478053/2013-4) and CAPES-Brazil (Grant BEX 5549/11-6). C.C. is partly supported by CNPQ-Brazil (Grant 478053/2013-4). H.S. is sponsored partly by DIUFRO DI15-0050.

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