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A CRITERION FOR TAYLOR SUMMABILITY OF FOURIER SERIES

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1. Let $\{a_{nk}\}$ be a matrix defined by

(1) $$\frac{(1-r)^{n+1}\theta^n}{(1-r\theta)^{n+1}}=\sum_{k=0}^{\infty} a_{nk}\theta^k \quad \text{for} \quad |r\theta|<1$$

and n taking only non-negative integer values.

Let $f(x)\in L[0,2\pi]$ and be periodic with period 2π outside this interval. Let the Fourier series associated with the function $f(x)$ be given by

$$\sum_{n=0}^{\infty}(a_n \cos nx+b_n \sin nx)=\sum_{n=0}^{\infty} A_n(x)$$

and let

$$\varphi(t)\equiv\tfrac{1}{2}\{f(x+t)+f(x-t)-2s\},$$

where s is a constant.

DEFINITION 1.1. Given a sequence $\{s_k\}_0^\infty$, we say that $\{s_k\}_0^\infty$ is Taylor summable, if

(2) $$\sigma_n^r=\sum_{k=0}^{\infty} a_{nk}s_k$$

tends to a limit as $n\to\infty$, where $0\le r<1$.

It is well known that Taylor summability is regular for $0\le r<1$ [2]. The asymptotic behaviour of the Lebesgue constant for Taylor summability of Fourier series has been studied by K. Ishiguro [4], L. Lorch and D. J. Newman [5] and R. L. Forbes [1].

Since the sequence of Lebesgue constants is not bounded, it is interesting to study sufficient criteria for Taylor summability of Fourier series. The main result of this paper is as follows.

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2. THEOREM. *With the previous notations, if:*

(3) (i) $\int_0^t |\varphi(u)|\, du = o(t)$ as $t \to 0^+$,

and

(4) (ii) $\lim_{n\to\infty} \int_{(1-r)\pi/n}^{\eta} \frac{|\varphi(t)-\varphi(t+(1-r)\pi/n)|}{t} \exp\left(-\frac{nrt^2}{2(1-r)^2}\right) dt = 0$

where η is a positive constant, then the Fourier series of f is Taylor summable to s at the point x. In order to prove the theorem we will need access to the following lemmas. Let us write $1 - re^{it} \equiv \rho e^{-i\theta}$.

LEMMA 2.1 [1]

(5) (i) $\left(\frac{1-r}{\rho}\right)^n \le e^{-Ant^2}$; $\quad A > 0, \quad 0 \le t \le \frac{\pi}{2}$

and

(6) (ii) $\left|\left(\frac{1-r}{\rho}\right)^n - \exp\left(-\frac{nrt^2}{2(1-r)^2}\right)\right| \le Bnt^4$; *B constant,* $t > 0$.

LEMMA 2.2 [6]

(7) $\left|\theta - \frac{rt}{1-r}\right| \le ct^3$; $\quad 0 \le t \le \frac{\pi}{2}$, *c constant.*

Proof of the theorem. If s_k is the kth partial sum of the Fourier series of f at the point x, then it is known [8], that

$$s_k - s = \frac{2}{\pi}\int_0^{\pi} \frac{\varphi(t)}{t} \sin kt\, dt + o(1). \tag{8}$$

The Taylor transform of $\{s_k - s\}$ is given by

$$\begin{aligned}\sigma_n^r &= \sum_{k=0}^{\infty} a_{nk}(s_k - s) = \sum_{k=0}^{\infty} a_{nk}\left\{\frac{2}{\pi}\int_0^{\pi} \frac{\varphi(t)}{t} \sin kt\, dt + o(1)\right\} \\ &= \frac{2}{\pi}\int_0^{\pi} \frac{\varphi(t)}{t} \operatorname{Im}\left\{\sum_{k=0}^{\infty} a_{nk} e^{ikt}\right\} dt + o(1),\end{aligned}$$

(since the series converges uniformly),

$$= p_1 + o(1), \text{ say.}$$

Now

$$p_1 = \frac{2}{\pi}\int_0^{\pi} \frac{\varphi(t)}{t} \operatorname{Im}\left\{\frac{(1-r)^{n+1} e^{int}}{(1-re^{it})^{n+1}}\right\} dt \tag{9}$$

and since we have written $1-re^{it}=\rho e^{-i\theta}$, then

$$p_1=\frac{2}{\pi}\int_0^{\pi}\frac{\varphi(t)}{t}\,\mathrm{Im}\left\{\frac{(1-r)^{n+1}e^{int}}{\rho^{n+1}e^{-i(n+1)\theta}}\right\}dt$$
$$=\frac{2}{\pi}\int_0^{\pi}\frac{\varphi(t)}{t}\left(\frac{1-r}{\rho}\right)^{n+1}\sin\{nt+(n+1)\theta\}\,dt.$$

Now let us denote

$$(10)\qquad q\equiv\frac{1}{1-r},\qquad a(n)\equiv\frac{\pi}{qn}\quad\text{and}\quad b(n)\equiv\left(\frac{\pi}{qn}\right)^{a},\quad\text{with}\quad \tfrac{1}{3}<a<\tfrac{1}{2}.$$

Thus we may write

$$(11)\qquad p_1=\frac{2}{\pi}\left[\int_0^{a(n)}+\int_{a(n)}^{b(n)}+\int_{b(n)}^{\pi}\right]\frac{\varphi(t)}{t}\left(\frac{1-r}{\rho}\right)^{n+1}\sin\{nt+(n+1)\theta\}\,dt$$
$$=\eta_1+\eta_2+\eta_3,\quad\text{say.}$$

Consider η_1. Since $1-r\leq\rho$, and using Lemma 2.2, we have

$$|\eta_1|\leq 0(1)\int_0^{a(n)}\frac{|\varphi(t)|}{t}\left\{nt+(n+1)\left(ct^3+\frac{rt}{1-r}\right)\right\}dt$$
$$\leq 0(n)\int_0^{a(n)}|\varphi(t)|\,dt$$
$$=0(n)o\left(\frac{\pi}{qn}\right),\quad\text{(by hypothesis)}$$
$$=o(1).$$

Consider η_3. Using Lemma 2.1, we have

$$|\eta_3|\leq 0(n^{\alpha})\exp\left(-A(n+1)\left(\frac{\pi}{qn}\right)^{2\alpha}\right)\int_{b(n)}^{\pi}|\varphi(t)|\,dt$$
$$=0(n^{\alpha})e^{-Dn^{1-2\alpha}},\quad D\text{ a positive constant.}$$

Since $\alpha<\frac{1}{2}$, then

$$|\eta_3|=o(1).$$

Bounds for $|\eta_2|$ will require a little more analysis and we proceed as follows:

$$\eta_2=\frac{2}{\pi}\int_{a(n)}^{b(n)}\frac{\varphi(t)}{t}\left(\frac{1-r}{\rho}\right)^{n+1}\sin\{nt+(n+1)\theta\}\,dt.$$

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Using Lemma 2.1 (ii), we have

$$\frac{2}{\pi}\int_{a(n)}^{b(n)} \frac{|\varphi(t)|}{t} B(n+1)t^4 \,|\sin\{nt+(n+1)\theta\}|\, dt$$

$$\le 0(n)0\left\{\left(\frac{\pi}{qn}\right)^{3\alpha}\right\}\int_{a(n)}^{b(n)} |\varphi(t)|\, dt$$

$$= 0(n^{1-3\alpha-\alpha})$$

$$= o(1) \quad \text{since} \quad \alpha > \tfrac{1}{3}.$$

Thus

$$\eta_2 = p_2 + o(1)$$

where

(12) $$p_2 = \frac{2}{\pi}\int_{a(n)}^{b(n)} \frac{\varphi(t)}{t} \exp\left(-\frac{(n+1)rt^2}{2(1-r)^2}\right)\sin\{nt+(n+1)\theta\}\, dt.$$

Now

$$\sin\{nt+(n+1)\theta\} = \sin(n+1)(t+\theta)\cos t - \cos(n+1)(t+\theta)\sin t$$
$$= \sin(n+1)(t+\theta) + 0(t^2) + 0(t)$$
$$= \sin(n+1)(t+\theta) + 0(t)$$

thus

$$\frac{2}{\pi}\int_{a(n)}^{b(n)} \frac{|\varphi(t)|}{t} 0(t)\, dt = o(1)$$

is the error if we write $(n+1)t$ for nt in (12). In a similar way we can replace in $\exp(-[(n+1)rt^2]/[2(1-r)^2])$, the $(n+1)$ by n. If

(13) $$p_3 = \frac{2}{\pi}\int_{a(n)}^{b(n)} \frac{\varphi(t)}{t} \exp\left(-\frac{nrt^2}{2(1-r)^2}\right)\sin n(t+\theta)\, dt$$

then

$$p_2 = p_3 + o(1).$$

Since

$$\left|\sin n(t+\theta) - \sin n\frac{t}{1-r}\right| \le n\left|t+\theta-\frac{t}{1-r}\right|$$

$$= n\left|\theta - \frac{rt}{1-r}\right|$$

$$\le cnt^3, \qquad 0 \le t \le \pi$$

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by Lemma 2.2, then

$$\frac{2}{\pi}\int_{a(n)}^{b(n)}\frac{|\varphi(t)|}{t}\exp\left(-\frac{nrt^2}{2(1-r)^2}\right)cnt^3\,dt=0(n)0\left\{\left(\frac{\pi}{n}\right)^{2\alpha}\right\}\int_{a(n)}^{b(n)}|\varphi(t)|\,dt$$
$$=0(n^{1-2\alpha-\alpha})$$
$$=o(1)\quad\text{since}\quad \alpha>\tfrac{1}{3}.$$

Consequently,

$$p_3=p_4+o(1)$$

where

$$p_4=\frac{2}{\pi}\int_{a(n)}^{b(n)}\frac{\varphi(t)}{t}\exp\left(-\frac{nrt^2}{2(1-r)^2}\right)\sin\left(nt\frac{1}{1-r}\right)dt. \tag{14}$$

We write as before, $1/(1-r)\equiv q$, then

$$\pi p_4=\int_{a(n)}^{b(n)}\frac{\varphi(t)}{t}\exp\left(-\frac{nrt^2}{2(1-r)^2}\right)\sin ntq\,dt$$
$$-\int_0^{b(n)-a(n)}\frac{\varphi\left(t+\frac{\pi}{qn}\right)}{t+\frac{\pi}{qn}}\exp\left(-\frac{nr}{2(1-r)^2}\left(t+\frac{\pi}{qn}\right)^2\right)\sin ntq\,dt$$
$$=\int_{a(n)}^{b(n)}\frac{\varphi(t)-\varphi\left(t+\frac{\pi}{qn}\right)}{t}\exp\left(-\frac{nrt^2}{2(1-r)^2}\right)\sin ntq\,dt$$
$$+\int_{a(n)}^{b(n)}\frac{\varphi\left(t+\frac{\pi}{qn}\right)}{t}\left[\exp\left(-\frac{nrt^2}{2(1-r)^2}\right)-\exp\left(-\frac{nr}{2(1-r)^2}\left(t+\frac{\pi}{qn}\right)^2\right)\right]\sin ntq\,dt$$
$$+\int_{a(n)}^{b(n)}\varphi\left(t+\frac{\pi}{qn}\right)\exp\left(-\frac{nr}{2(1-r)^2}\left(t+\frac{\pi}{qn}\right)^2\right)\left[\frac{1}{t}-\frac{1}{t+\frac{\pi}{qn}}\right]\sin ntq\,dt$$
$$-\int_0^{a(n)}\frac{\varphi\left(t+\frac{\pi}{qn}\right)}{t+\frac{\pi}{qn}}\exp\left(-\frac{nr}{2(1-r)^2}\left(t+\frac{\pi}{qn}\right)^2\right)\sin ntq\,dt$$
$$+\int_{b(n)-a(n)}^{b(n)}\frac{\varphi\left(t+\frac{\pi}{qn}\right)}{t+\frac{\pi}{qn}}\exp\left(-\frac{nr}{2(1-r)^2}\left(t+\frac{\pi}{qn}\right)^2\right)\sin ntq\,dt.$$

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By using the method similar to that of Sahney and Kathal [7] and Holland, Shaney and Tzimbalario [3], the five terms on the right hand side of the expression for πp_4 are all $o(1)$.

Thus η_1, η_2, and η_3 are all $o(1)$ and the theorem is proved.

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