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APPROXIMATING LOCAL VOLATILITY FUNCTIONS OF STOCHASTIC VOLATILITY MODELS: A CLOSED-FORM EXPANSION APPROACH

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We propose a method for approximating equivalent local volatility functions of stochastic volatility models. Enlightened by the theory of generalized Wiener functionals proposed by Watanabe and Yoshida (1987, 1992), our key technique is to propose a closed-form expansion of conditional expectations involving marginal distributions generated by stochastic differential equations. A numerical test and an illustration of application are provided to demonstrate the efficiency of our approach.

1. INTRODUCTION

Since the debut in Hull and White [25], Scott [39] and Heston [24] among others, stochastic volatility has become an important feature for modeling the real world dynamics of asset prices. Even after carefully trading off between empirical features and mathematical convenience in model design, efficient calibration poses significant challenges. Among many other approaches, the Markovian projection approach proposed by Piterbarg [37] hinges on equivalent local volatility functions; see, for example, Dupire [10], Derman and Kani [8] and a comprehensive survey in Gatheral [14]. Based on the theoretical result from Gyöngy [20], the equivalent local volatility function can be constructed by a conditional expectation of the spot variance given the underlying asset price; see, for example, Dupire [11] and Piterbarg [34–36,38]. Consequently, the stochastic volatility model can be calibrated to market via the calibration of local volatility function using European option trading data; see, for example, Dupire [10].

In spite of the convenience of the framework proposed in Piterbarg [37], the central and most challenging issue amounts to the valuation of the aforementioned conditional expectation form of equivalent local volatility functions. A most commonly-used method is the Gaussian approximations discussed in Piterbarg [37], Antonov and Misirpashaev [2] and Antonov, Misirpashaev, and Piterbarg [3]; related applications can be seen in Dupire

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[11], Piterbarg [34–36,38], Xu and Zheng [45,46], Antonov and Misirpashaev [1], and Kienitz and Wittke [28]. Another method is to employ heat-kernel approximation; see, for example, Avellaneda et al. [4] and Henry-Labordere [22]. Motivated by a broader range of important applications in applied probability and stochastic modeling, the valuation of conditional expectations involving marginal distributions generated by stochastic differential equations (SDE hereafter) attracts attentions owing to the significant challenge even from Monte Carlo simulation based methods; see, for example, Fournie et al. [12,13], Benhamou [5] and Bouchard, Ekeland, and Touzi [7].

In this paper, we will proposed a widely applicable and efficient closed-form expansion approach, which serves as a complementary choice of other methods. Once the expansion formulas are calculated and shortened using a symbolic computation platform, for example, Mathematica, the computing time in the calibration can be significantly saved from heavy and repeated numerical procedures. From a technical perspective, our expansion approach roots in the method for analyzing generalized random variables initiated by Watanabe [44] and Yoshida [47] as well as its substantial development for statistical inference and option valuation in, for example, Yoshida [47], Takahashi [40,41], Kunitomo and Takahashi [29,30], Uchida and Yoshida [43], Gobet, Benhamou, and Miri [18] and Li [31,33].

The rest of this paper is organized as follows. In Section 2, we discussed about our motivation and introduce the model with basic setup. In Section 3, we propose our method of closed-form expansion approximation. In Section 4, we test the performance of our expansion and illustrate its applications. In Section 5, we conclude the paper. The proofs are included in Appendices 5.

2. MOTIVATION AND THE MODEL

To articulate our motivation, we start from the following local stochastic volatility model with a general specification of the volatility process, that is,

$$\frac{dS(t)}{S(t)} = rdt + v(t,S(t))\sqrt{V(t)}\left(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)\right), \quad S(0) = s_0, \tag{2.1a}$$

$$dV(t) = a(V(t))dt + b(V(t))dW_1(t), \quad V(0) = v_0, \tag{2.1b}$$

for some functions v, a and b, where (W_1, W_2) is a standard two-dimensional Brownian motion.

An application of Theorem 4.6 in Gyöngy [20] leads to the following distributional equivalence:

$$\{S(t); t \geq 0\} \overset{\text{in law}}{=} \{\widehat{S}(t) + s_0; t \geq 0\},$$

where $\widehat{S}$ is governed by the following local volatility model

$$\frac{d\widehat{S}(t)}{\widehat{S}(t)} = rdt + v(t,\widehat{S}(t))\sqrt{u(t,\widehat{S}(t))}dW(t), \quad \widehat{S}(0) = 0,$$

with the volatility function

$$u(t,s) := E(V(t)|S(t) = s). \tag{2.2}$$

According to Piterbarg [37], this observation provides a convenient method for calibrating stochastic volatility models through the local volatility functions, which can be directly implied by cross-sectional data of option prices; see, for example, Gatheral et al. [16]. However, given the complex structure of the underlying model, efficient valuation of the conditional expectation (2.2) poses a challenge to the practice of this method. We will propose a flexible approximation method via closed-form expansion.

For illustration purpose and without loss of generality, we consider the conditional expectation $E(X_2(T)|X_1(T)=x_1)$ for some $T>0$, where the bivariate process $X=(X_1,X_2)$ is governed by the following two-dimensional SDE

$$dX(t)=\mu(X(t))dt+\sigma(X(t))dW(t), X(0)=x_0=(x_{10},x_{20}). \tag{2.3}$$

Here, $W=(W_1,W_2)$ is a two-dimensional standard Brownian Motion; $\mu=(\mu_1,\mu_2):\mathbb{R}^2\to\mathbb{R}^2$ is the drift function and $\sigma=(\sigma_{ij})_{2\times 2}:\mathbb{R}^2\to\mathbb{R}^{2\times 2}$ is the diffusion matrix. Under some standard conditions on μ and σ, we assume that (2.3) admits a unique solution in distribution.

Our approximation hinges on the following expression of the conditional expectation $E(X_2(T)|X_1(T)=x_1)$:

$$E(X_2(T)|X_1(T)=x_1)=\frac{E\left[X_2(T)\delta(X_1(T)-x_1)\right]}{E\left[\delta(X_1(T)-x_1)\right]}. \tag{2.4}$$

From a theoretical perspective, the random variable $\delta(X_1(T)-x_1)$ and the expectations $E\left[\delta(X_1(T)-x_1)\right]$ and $E\left[X_2(T)\delta(X_1(T)-x_1)\right]$ can be interpreted in a generalized sense as studied in Watanabe [44]. More detailed explanations can be found in Section 7.2 of Li [32] and references therein. Formula (2.4) can be obtained from the definition of Dirac Delta function; see, for example, p. 4 in Kanwal [26]. Indeed, denote by $p_1(x)$ the probability density function of $X_1(T)$. By conditioning on $X_1(T)$, it follows that

$$\begin{aligned}E\left[\delta(X_1(T)-x_1)\right]&=\int_{-\infty}^{\infty}E\left[\delta(X_1(T)-x_1)|X_1(T)=x\right]P(X_1(T)\in dx)\\&=\int_{-\infty}^{\infty}\delta(x-x_1)p_1(x)dx=p_1(x_1).\end{aligned}$$

Similarly, we have

$$\begin{aligned}E\left[X_2(T)\delta(X_1(T)-x_1)\right]&=\int_{-\infty}^{\infty}E\left[X_2(T)\delta(X_1(T)-x_1)|X_1(T)=x\right]P(X_1(T)\in dx)\\&=\int_{-\infty}^{\infty}\delta(x-x_1)E\left[X_2(T)|X_1(T)=x\right]p_1(x)dx\\&=E\left[X_2(T)|X_1(T)=x_1\right]p_1(x_1),\end{aligned}$$

which results in (2.4). Using Malliavin calculus, Monte Carlo simulation methods for approximating (2.4) are discussed in, for example, Fournie et al. [12,13], Benhamou [5] and Bouchard et al. [7]. In what follows, we will propose a closed-form expansion based on the expression (2.4).

3. A CLOSED-FORM EXPANSION

To apply the method proposed in Li [31,33], we introduce the following parameterized model $X^{\epsilon}(t)$ governed by the following SDE:

$$dX^{\epsilon}(t) = \epsilon \left[\mu^T(X^{\epsilon}(t))dt + \sigma^T(X^{\epsilon}(t))dW_T(t)\right], \quad X^{\epsilon}(0) = x_0, \tag{3.1}$$

with the functions

$$\mu^T(x) = T\mu(x) \text{ and } \sigma^T(x) = \sqrt{T}\sigma(x), \tag{3.2}$$

where $\{W_T(t)\}$ is a standard two-dimensional Brownian motion constructed by

$$W_T(t) = \frac{1}{\sqrt{T}} W(Tt) \tag{3.3}$$

via the Brownian scaling property. We note that $X^{\epsilon}(1)|_{\epsilon=1}$ and $X(T)$ are equivalent in distribution. To verify this useful property, we first notice that $\{X^{\epsilon}(t/T)|_{\epsilon=1}\}$ solves the SDE (2.3). Indeed, by change-of-variable and the definitions of (3.2) and (3.3), we have

$$\begin{aligned} X^{\epsilon}\left(\frac{t}{T}\right)\bigg|_{\epsilon=1} &= x_0 + \int_0^{\frac{t}{T}} \mu^T(X^{\epsilon}(s)|_{\epsilon=1})ds + \int_0^{\frac{t}{T}} \sigma^T(X^{\epsilon}(s)|_{\epsilon=1})dW_T(s) \\ &= x_0 + \int_0^{t} \mu^T\left(X^{\epsilon}\left(\frac{s}{T}\right)\bigg|_{\epsilon=1}\right) d\left(\frac{s}{T}\right) + \int_0^{t} \sigma^T\left(X^{\epsilon}\left(\frac{s}{T}\right)\bigg|_{\epsilon=1}\right) dW_T\left(\frac{s}{T}\right) \\ &= x_0 + \int_0^{t} \mu\left(X^{\epsilon}\left(\frac{s}{T}\right)\bigg|_{\epsilon=1}\right) ds + \int_0^{t} \sigma\left(X^{\epsilon}\left(\frac{s}{T}\right)\bigg|_{\epsilon=1}\right) dW(s). \end{aligned}$$

Hence, $\{X^{\epsilon}(t/T)|_{\epsilon=1}\}$ is a solution to SDE (2.3). Thus, our assumption on the uniqueness of the solution to SDE (2.3) guarantees that $X^{\epsilon}(1)|_{\epsilon=1} \equiv X^{\epsilon}(T/T)|_{\epsilon=1}$ and $X(T)$ are equivalent in distribution. In the following expositions, we will abbreviate $W_T(t)$ as $W(t)$.

Thus, we obtain a parameterized version of (2.4) as

$$E(X_2(T)|X_1(T) = x_1) = \frac{E\left[X_2^{\epsilon}(1)\delta(X_1^{\epsilon}(1) - x_1)\right]}{E\left[\delta(X_1^{\epsilon}(1) - x_1)\right]}\bigg|_{\epsilon=1},$$

which will serve as a starting point for our expansion. Denote by $p^{\epsilon} = E\left[\delta(X_1^{\epsilon}(1) - x_1)\right]$ and $q^{\epsilon} = E\left[X_2^{\epsilon}(1)\delta(X_1^{\epsilon}(1) - x_1)\right]$. We will expand p^{ϵ} and q^{ϵ} as

$$p^{\epsilon} := E\left[\delta(X_1^{\epsilon}(1) - x_1)\right] = \frac{D_1(x_0)}{\epsilon} \sum_{k=0}^{J} \Omega_k \left[D_1(x_0)\left(\frac{x_1 - x_{10}}{\epsilon} - \mu_1^T(x_0)\right)\right] \epsilon^k + \mathcal{O}(\epsilon^{J+1}), \tag{3.4}$$

and

$$\begin{aligned} q^{\epsilon} &:= E\left[X_2^{\epsilon}(1)\delta(X_1^{\epsilon}(1) - x_1)\right] \\ &= \frac{D_1(x_0)}{\epsilon} \sum_{k=0}^{J} \Psi_k \left[D_1(x_0)\left(\frac{x_1 - x_{10}}{\epsilon} - \mu_1^T(x_0)\right)\right] \epsilon^k + \mathcal{O}(\epsilon^{J+1}), \end{aligned} \tag{3.5}$$

for some functions $D_1(x_0)$, Ω_k and Ψ_k to be determined in closed-form, respectively. Then, by plugging in $\epsilon = 1$, we define a Jth order approximation to the conditional expectation

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$E(X_2(T)|X_1(T) = x_1)$ as

$$CE^{(J)} := \frac{q^{(J)}}{p^{(J)}}, \tag{3.6}$$

where

$$p^{(J)} = \sum_{k=0}^{J} \Omega_k \left[D_1(x_0) \left(x_1 - x_{10} - \mu_1^T(x_0) \right) \right]$$

and

$$q^{(J)} = \sum_{k=0}^{J} \Psi_k \left[D_1(x_0) \left(x_1 - x_{10} - \mu_1^T(x_0) \right) \right].$$

3.1. Pathwise Expansions

Similar to Lemma 1 in Li [31], we begin by introducing a pathwise expansion of $X^\epsilon(t)$ as a Taylor-like power series of the variable ϵ with closed-form random coefficients. For such a purpose, we define, for $i = 1, 2$,

$$b_i^T(x) = -\frac{1}{2} \sum_{k=1}^{2} \sum_{j=1}^{2} \sigma_{kj}^T(x) \frac{\partial}{\partial x_k} \sigma_{ij}^T(x), \tag{3.7}$$

and introduce the following differential operators:

$$\mathcal{A}_0 := \sum_{i=1}^{2} b_i^T(x_0) \frac{\partial}{\partial x_i}, \quad \mathcal{A}_j := \sum_{i=1}^{2} \sigma_{ij}^T(x_0) \frac{\partial}{\partial x_i}, \text{ for } j = 1, 2, \quad \mathcal{A}_3 := \sum_{i=1}^{2} \mu_i^T(x_0) \frac{\partial}{\partial x_i},$$

which map $\mathbb{R}^2$-valued functions to $\mathbb{R}^2$-valued functions. More precisely, for any integer ν and a ν-dimensional vector-valued function $\varphi(x) = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_\nu(x))^\top$ with $\top$ denoting vector/matrix transpose, we have

$$(\mathcal{A}_k(\varphi))(x) = ((\mathcal{A}_k(\varphi_1))(x), (\mathcal{A}_k(\varphi_2))(x), \ldots, (\mathcal{A}_k(\varphi_\nu))(x))^\top, \text{ for } k = 0, 1, 2, 3.$$

For an index $\mathbf{i} = (i_1, \ldots, i_n) \in \{0, 1, 2, 3\}^n$ and a right-continuous stochastic process $\{f(t)\}$, we define an iterated Stratonovich integral with integrand f as

$$J_{\mathbf{i}}[f](t) := \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(t_n) \circ dW_{i_n}(t_n) \cdots \circ dW_{i_2}(t_2) \circ dW_{i_1}(t_1). \tag{3.8}$$

To lighten the notation, for $f \equiv 1$, the integral $J_{\mathbf{i}}[1](t)$ is abbreviated as $J_{\mathbf{i}}(t)$. By convention, we assume $W_0(t) := t$ and $W_3(t) := t$. We define

$$\|\mathbf{i}\| := \sum_{k=1}^{n} \left[2 \cdot 1_{\{i_k = 0\}} + 1_{\{i_k \neq 0\}} \right] \tag{3.9}$$

as the norm of index $\mathbf{i}$, in which the occurrence of 0 is counted twice in this norm.

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Lemma 1: *$X^\epsilon(1)$ admits the following pathwise expansion:*

$$X^\epsilon(1)=\sum_{k=0}^{J}F_k\epsilon^k+\mathcal{O}(\epsilon^{J+1}),\ \text{for } i=1,2, \tag{3.10}$$

with $F_0=x_0$ and higher order correction terms represented by linear combinations of iterated Stratonovich integrals (3.8), that is,

$$F_k=\sum_{\boldsymbol{i}=(i_1,\ldots,i_n),\|\boldsymbol{i}\|=k}C_{\boldsymbol{i}}(x_0)J_{\boldsymbol{i}}(1),\ \text{for } k=1,2,3,\ldots. \tag{3.11}$$

Here, the coefficients are given by

$$C_{\boldsymbol{i}}(x_0)=\mathcal{A}_{i_n}\left(\cdots\left(\mathcal{A}_{i_3}\left(\mathcal{A}_{i_2}\left(\sigma^T_{\cdot i_1}\right)\right)\right)\cdots\right)(x_0). \tag{3.12}$$

For $i_1\in\{1,2\}$, the vector $\sigma^T_{\cdot i_1}(x)$ denotes the i_1th column vector of the dispersion matrix $\sigma^T(x)$, that is, $\sigma^T_{\cdot i_1}(x)=(\sigma^T_{1i_1}(x),\sigma^T_{2i_1}(x))^\top$; for $i_1=0$, the vector $\sigma^T_{\cdot 0}(x)$ refers to the vector $b^T(x)$; for $i_1=3$, the vector $\sigma^T_{\cdot 3}(x)$ refers to the vector $\mu^T(x)$.

Proof: See Appendix 5. ■

Under some standard sufficient conditions, the validity of this expansion can be justified in the sense of Malliavin calculus; see Theorem 3.3 in Watanabe [44] for a similar case. In the subsequent subsections, all the expansions of generalized random variables and their expectations can be guaranteed by Theorems 2.2 and 2.3 of Watanabe [44]. A survey of these theoretical issues can be found in Sections 7.2 and 7.3 of Li [32]. To focus on computational issues and applications in this paper, we omit similar discussions.

In particular, for an arbitrary $r=1,2$, we have an elementwise representation of $X^\epsilon(1)$ as $X^\epsilon_r(1)=\sum_{k=0}^{\infty}F_{k,r}\epsilon^k$, where $F_{k,r}=\sum_{\mathbf{i}=(i_1,\ldots,i_n),\|\mathbf{i}\|=k}C_{\mathbf{i},r}(x_0)J_{\mathbf{i}}(1)$ for any $k=1,2,3,\ldots$, and $C_{\mathbf{i},r}(x_0)=\mathcal{A}_{i_n}\left(\cdots\left(\mathcal{A}_{i_3}\left(\mathcal{A}_{i_2}\left(\sigma^T_{ri_1}\right)\right)\right)\cdots\right)(x_0)$. For example, the first two terms can be calculated explicitly as

$$F_{1,r}=\mu^T_r(x_0)+\sum_{j=1}^{2}\sigma^T_{rj}(x_0)W_j(1),$$

$$F_{2,r}=b^T_r(x_0)+\sum_{i_1,i_2=1}^{2}\mathcal{A}_{i_2}\left(\sigma^T_{ri_1}\right)(x_0)J_{(i_1,i_2)}(1)+\sum_{i_2=1}^{2}\mathcal{A}_3\left(\sigma^T_{ri_1}\right)(x_0)J_{(i_1,3)}(1)$$
$$+\sum_{i_2=1}^{2}\mathcal{A}_{i_2}\left(\sigma^T_{r3}\right)(x_0)J_{(3,i_2)}(1)+\mathcal{A}_3\left(\sigma^T_{r3}\right)(x_0)J_{(3,3)}(1).$$

For ease of exposition, we introduce a two-dimensional correlated Brownian motion $B(t)=(B_1(t),B_2(t))$, where $B(t)=D(x_0)\sigma^T(x_0)W(t)$ with the diagonal matrix $D(x)$ defined by

$$D(x):=\text{diag }(D_1(x),D_2(x))\ \text{and}\ D_i(x)=\left(\sum_{j=1}^{2}\sigma^T_{ij}(x)^2\right)^{-1/2},\ \text{for } i=1,2. \tag{3.13}$$

So, the covariance matrix of $B(1)$ can be written as $\Sigma(x_0)=D(x_0)\sigma^T(x_0)\sigma^T(x_0)^\top D(x_0)$. Employing these notations, we have, for example, $F_1=\mu^T(x_0)+D(x_0)^{-1}B(1)$.

3.2. Expansion for the Denominator $p^{\epsilon} = E\left[\delta(X_1^{\epsilon}(1) - x_1)\right]$

In this subsection, we derive a closed-form expansion for denominator $p^{\epsilon} = E\left[\delta(X_1^{\epsilon}(1) - x_1)\right]$ in the form of (3.4). To guarantee the convergence based on Watanabe [44], we standardize $X^{\epsilon}(1)$ into $Y^{\epsilon}(1)$

$$Y^{\epsilon}(1) := D(x_0)\left(\frac{X^{\epsilon}(1) - x_0}{\epsilon} - \mu^T(x_0)\right), \tag{3.14}$$

which converges to the bivariate normal variable $(B_1(1), B_2(1))$ as $\epsilon \to 0$. Similar setup can be found in Yoshida [47,48], Kunitomo and Takahashi [29,30], Takahashi, Takehara, and Toda [40], Takahashi, Takehara, and Toda [42], and Li [31,33]. Assume that we have an asymptotic expansion for $Y^{\epsilon}(1)$ as

$$Y^{\epsilon}(1) = \sum_{k=0}^{J} Y_k \epsilon^k + \mathcal{O}(\epsilon^{J+1}). \tag{3.15}$$

According to (3.10) and (3.14), it is easy to find the coefficients as $Y_i = D(x_0)F_{i+1}$, for $i = 1, 2, \ldots$.

Now, we observe that

$$p^{\epsilon} = E\left[\delta(X_1^{\epsilon}(1) - x_1)\right] = \frac{D_1(x_0)}{\epsilon} E\left[\delta(Y_1^{\epsilon}(1) - y_1) | X(0) = x_0\right],$$

where

$$y_1 = D_1(x_0)\left(\frac{x_1 - x_{10}}{\epsilon} - \mu_1^T(x_0)\right), \tag{3.16}$$

and x_{01} is the first element of x_0. By regarding $D_1(x_0)/\epsilon$ as a constant, our expansion will be carried out with $\mathbb{E}\left[\delta(Y_1^{\epsilon}(1) - y_1) | X(0) = x_0\right]$. Without any confusion, the initial condition $X(0) = x_0$ will be omitted in what follows.

Based on the expansion (3.15) and the classical rule for Taylor expansion, we obtain an expansion of $\delta(Y_1^{\epsilon}(1) - y_1)$ as

$$\delta(Y_1^{\epsilon}(1) - y_1) = \sum_{k=0}^{J} \Phi_k(y_1)\epsilon^k + \mathcal{O}(\epsilon^{J+1}), \tag{3.17}$$

where $\Phi_k(y_1)$ represents the kth order expansion term. Taking expectations, we obtain that

$$\mathbb{E}\left[\delta(Y_1^{\epsilon}(1) - y_1)\right] := \sum_{k=0}^{J} \Omega_k(y_1)\epsilon^k + \mathcal{O}(\epsilon^{J+1}),$$

where $\Omega_k(y_1) := \mathbb{E}\Phi_k(y_1)$ can be explicitly calculated. Thus, the Jth order expansion approximation of p^{ϵ} can be defined as

$$p^{\epsilon,(J)} := \frac{D_1(x_0)}{\epsilon} \sum_{k=0}^{J} \Omega_k\left(D_1(x_0)\left(\frac{x_1 - x_{10}}{\epsilon} - \mu_1^T(x_0)\right)\right)\epsilon^k.$$

By plugging in $\epsilon = 1$, we obtain the following Jth order approximation of the marginal density of $X_1(T)$:

$$p^{(J)} := D_1(x_0) \sum_{k=0}^{J} \Omega_k\left(D_1(x_0)\left(x_1 - x_{10} - \mu_1^T(x_0)\right)\right). \tag{3.18}$$

To give a closed-form formula for $\Omega_k(y)$ with $k \geq 1$, we define the following differential operator $\mathcal{D}$ via

$$\mathcal{D}(f)(x) := \frac{\partial f(x)}{\partial x} - xf(x), \text{ for any differentiable function } f(x). \tag{3.19}$$

Denote by $\phi(\cdot)$ the probability density function of a standard normal variable. Note that, for any function $g(x)$ and $\phi(x)$, the derivative of the multiplication $g(x)\phi(x)$ can be simply expressed using (3.19) as follows:

$$\frac{\partial}{\partial x}\left[g(x)\phi(x)\right] = \left[\frac{\partial}{\partial x}g(x) - xg(x)\right]\phi(x) = \mathcal{D}(g)(x)\phi(x).$$

Denote by

$$P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}(z) := E\left(\prod_{v=1}^{l} J_{\mathbf{i}_v}(1)|B_1(1) = z\right), \tag{3.20}$$

for some indices $\mathbf{i}_1, \mathbf{i}_2, \ldots, \mathbf{i}_l$.

In the following proposition, we give a closed-form formula for the correction term Ω_k.

Proposition 1: *For any $k \in \mathbb{N}$, the correction term $\Omega_k(y_1)$ in (3.18) admits the following explicit expression:*

$$\Omega_k(y_1) = \sum_{1\leq l\leq k}(-1)^l \frac{D_{11}(x_0)^l}{l!} \sum_{(j_1,j_2,\ldots,j_l)\in S(k,l)} N(j_1,j_2,\ldots,j_l) \sum_{\{(i_1,i_2,\ldots,i_l)|\|i_v\|=j_v+1, v=1,2,\ldots,l\}} \left(\prod_{v=1}^{l} C_{i_v,1}(x_0)\right)\mathcal{D}^{(l)}\left(P_{(i_1,i_2,\ldots,i_l)}\right)(y_1)\phi(y_1), \tag{3.21}$$

where the index set $S(k,l)$ is defined as

$$S(k,l) := \{(j_1,j_2,\ldots,j_l)|1 \leq j_1 \leq j_2 \leq \cdots \leq j_l, \quad j_1 + j_2 + \cdots + j_l = k\}. \tag{3.22}$$

And $N(j_1,j_2,\ldots,j_l)$ denotes the total number of different permutations for $j_1,j_2,\ldots,j_l$. For example, if $(j_1,j_2,\ldots,j_8) = (2,2,2,3,3,6,6,6)$, we have $N(j_1,j_2,\ldots,j_8) = (8!/(3!2!3!)) = 560$. And $C_{i_v,1}$, $D_{11}(x_0)$, $\mathcal{D}$, $P_{(i_1,i_2,\ldots,i_l)}$ are defined in (3.12), (3.13), (3.19) and (3.20), respectively.

Proof: See Appendix 5. ■

The conditional expectation (3.20) plays an important role in expressing expansion terms. An efficient algorithm for calculating (3.20) as a closed-form polynomial in z can be found in Section 4 of Li [33].

3.3. Expansion for the Numerator $q^\epsilon = E\left[X_2^\epsilon(1)\delta(X_1^\epsilon(1)-x_1)\right]$

To obtain the expansion (3.5) for numerator $q^\epsilon = E\left[X_2^\epsilon(1)\delta(X_1^\epsilon(1)-x_1)\right]$, we note that it involves a multiplication of a classical random variable $X_2^\epsilon(1)$ and a generalized random variable $\delta(X_1^\epsilon(1)-x_1)$. For $\delta(X_1^\epsilon(1)-x_1)$, as in the previous section, we standardize $X_1^\epsilon(1)$ into $Y_1^\epsilon(1)$ by (3.14), and deduce that

$$E\left[X_2^\epsilon(1)\delta(X_1^\epsilon(1)-x_1)\right] = \frac{D_1(x_0)}{\epsilon}E\left[X_2^\epsilon(1)\delta\left(Y_1^\epsilon(1)-y_1\right)\right],$$

where y_1 is given by (3.16).

By multiplying the expansions (3.10) and (3.17), we obtain the expansion of the product $X_2^\epsilon(1)\delta\left(Y_1^\epsilon(1)-y_1\right)$ as

$$X_2^\epsilon(1)\delta\left(Y_1^\epsilon(1)-y_1\right) = \sum_{k=0}^{J}\Theta_k(y_1)\epsilon^k + \mathcal{O}(\epsilon^{J+1}), \tag{3.23}$$

where

$$\Theta_k(y_1) := \sum_{r=0}^{k}F_{r,2}\Phi_{k-r}(y_1), \text{ with } F_{k,2} = \sum_{\mathbf{i}=(i_1,\ldots,i_n),\|\mathbf{i}\|=k} C_{\mathbf{i},2}(x_0)J_{\mathbf{i}}(1). \tag{3.24}$$

Taking expectation on the both sides of (3.23), we obtain that

$$E\left[X_2^\epsilon(1)\delta\left(Y_1^\epsilon(1)-y_1\right)\right] = \sum_{k=0}^{J}\Psi_k(y_1)\epsilon^k + \mathcal{O}(\epsilon^{J+1}),$$

where the correction term $\Psi_k(y_1)$ follows from

$$\Psi_k(y_1) = E\Theta_k(y_1) = \sum_{r=0}^{k}E\left[F_{r,2}\Phi_{k-r}(y_1)\right]. \tag{3.25}$$

Thus, the Jth order expansion approximation of $E\left[X_2^\epsilon(1)\delta(X_1^\epsilon(1)-x_1)\right]$ can be defined as

$$q^{\epsilon,(J)} := \frac{D_1(x_0)}{\epsilon}\sum_{k=0}^{J}\Psi_k\left[D_1(x_0)\left(\frac{x_1-x_{10}}{\epsilon}-\mu_1^T(x_0)\right)\right]\epsilon^k.$$

By plugging in $\epsilon=1$, we obtain the following Jth order approximation of $E\left[X_2(T)\delta(X_1(T)-x_1)\right]$ as

$$q^{(J)} := D_1(x_0)\sum_{k=0}^{J}\Psi_k\left[D_1(x_0)\left(x_1-x_{10}-\mu_1^T(x_0)\right)\right]. \tag{3.26}$$

Similar to Proposition 1, we give a closed-form formula for $\Psi_k(y_1)$ in the following proposition.

Proposition 2: *For any $k \in \mathbb{N}$, the correction term $\Psi_k(y_1)$ in (3.25) admits the following explicit expression:*

$$\Psi_k(y_1) = \sum_{r=0}^{k} \sum_{\mathbf{i}=(i_1,\ldots,i_n),\|\mathbf{i}\|=r} C_{\mathbf{i},2}(x_0) \sum_{l\le k-r} \frac{1}{l!} D_{11}(x_0)^l \sum_{(j_1,j_2,\ldots,j_l)\in S(k-r,l)} N(j_1,j_2,\ldots,j_l) \tag{3.27}$$

$$\sum_{\{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)|\|\mathbf{i}_v\|=j_v+1,v=1,2,\ldots,l\}} \left(\prod_{v=1}^{l} C_{\mathbf{i}_v,1}(x_0)\right) \mathcal{D}^{(l)}\left(P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l,\mathbf{i})}\right)(y_1)\phi(y_1),$$

with $C_{\mathbf{i}_v,1}$, $D_{11}(x_0)$, $\mathcal{D}$, $P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}$, $S(k,l)$ defined in (3.12), (3.13), (3.19), (3.20) and (3.22), respectively, as well as $N(j_1,j_2,\ldots,j_l)$ defined in Proposition 1.

Proof: See Appendix 5. ■

4. COMPUTATIONAL RESULTS

In this section, we will test the performance of our expansion and provide an application following the motivating example discussed in Section 2. We will first test our expansion method by approximating a conditional expectation involving a two-dimensional geometric Brownian motion, which renders analytical tractability for explicitly exhibiting the approximation errors. Next, for illustration of application, we will employ our expansion method in approximating local volatility functions of the celebrated SABR local-stochastic volatility model (see, e.g., Hagan et al. [21]).

4.1. A Test of Performance

We consider the following two-dimensional geometric Brownian motion model.

Model 1: Two-dimensional geometric Brownian motion (GBM):

$$dX_1(t) = \sigma_1 X_1(t) dW_1(t), \quad X_1(0) = x_{10},$$
$$dX_2(t) = \sigma_2 X_2(t) dW_2(t), \quad X_2(0) = x_{20}.$$

for some $x_{10} > 0$, $x_{20} > 0$, $\sigma_1 > 0$, $\sigma_2 > 0$, where (W_1, W_2) is a correlated two-dimensional Brownian motion with $dW_1(t)dW_2(t) = \rho dt$ with $-1 \le \rho \le 1$.

Because of

$$X_i(t) = x_{i0} \exp\left(\sigma_i W_i(t) - \frac{1}{2}\sigma_i^2 t\right) \text{ for } i = 1, 2,$$

it is straightforward to obtain the following closed-form formula for the conditional expectation, which will serve as a benchmark for testing the performance of our expansion method:

$$E(X_2(T)|X_1(T) = x_1) = x_{20}\left(\frac{x_1}{x_{10}}\right)^{(\rho\sigma_2/\sigma_1)} \exp\left(\frac{1}{2}T\rho\sigma_2(\sigma_1 - \rho\sigma_2)\right). \tag{4.1}$$

We compare the third order and sixth order ($CE^{(3)}$ and $CE^{(6)}$ as defined in (3.6)) approximations of $E(X_2(T)|X_1(T) = x_1)$ with the benchmark values calculated from

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(4.1) for different time horizons ($T = 1/2, 1, 3, 10$) and different conditioning values ($x_1 = 70, 80, 90, 100, 110, 120, 130$). In Table 1, True value corresponds to the benchmark values $E(X_2(T)|X_1(T) = x_1)$; Asymptotic value corresponds to the values calculated from our expansions, say, the Jth order expansion $CE^{(J)}$; Error refers to the relative error calculated via

$$RE^{(J)}(x_1, T) = \frac{CE^{(J)} - E(X_2(T)|X_1(T) = x_1)}{E(X_2(T)|X_1(T) = x_1)}. \tag{4.2}$$

From Table 1, it is obvious that the performance of our expansion is systematically enhanced as the order is increased.

In Figure 1, we further illustrate the efficiency of our algorithm by plotting the relationship between expansion orders and uniform relative error for different maturities $T = 1/2, 1, 3, 10$. For each maturity, we define the uniform relative error as the maximum of the absolute relative errors across all conditioning values of x_1 in a certain region, that

Table 1. Numerical results for the performance test in Section 4.1

Parameters		Benchmark	Third-order expansion		Sixth-order expansion	
T	x_1	True value	Asymptotic	Error	Asymptotic	Error
0.5	70	69.00496527	68.50524690	-7.2418×10^{-3}	69.02096103	2.3181×10^{-4}
	80	76.27361000	76.23893890	-4.5456×10^{-4}	76.27312958	-6.2986×10^{-6}
	90	83.31797262	83.32496958	8.3979×10^{-5}	83.31797148	-1.3637×10^{-8}
	100	90.16890830	90.16917293	2.9348×10^{-6}	90.16890509	-3.5616×10^{-8}
	110	96.85037920	96.85903796	8.9403×10^{-5}	96.85036290	-1.6829×10^{-7}
	120	103.38148022	103.33924103	-4.0858×10^{-4}	103.38197502	4.7861×10^{-6}
	130	109.77775014	109.52183464	-2.3312×10^{-3}	109.76899208	-7.9780×10^{-5}
1	70	69.13447095	68.87098550	-3.8112×10^{-3}	69.12696300	-1.0860×10^{-4}
	80	76.41675717	76.47256098	7.3026×10^{-4}	76.41669262	-8.4472×10^{-7}
	90	83.47434036	83.48140138	8.4589×10^{-5}	83.47430935	-3.7156×10^{-7}
	100	90.33813360	90.33919598	1.1760×10^{-5}	90.33810785	-2.8510×10^{-7}
	110	97.03214401	97.04118886	9.3215×10^{-5}	97.03208242	-6.3475×10^{-7}
	120	103.57550234	103.65030364	7.2219×10^{-4}	103.57537520	-1.2274×10^{-6}
	130	109.98377651	109.67287138	-2.8268×10^{-3}	109.99142705	6.9561×10^{-5}
3	70	69.65492876	70.02561944	5.3218×10^{-3}	69.65850846	5.1392×10^{-5}
	80	76.99203746	77.09768717	1.3722×10^{-3}	76.99187144	-2.1562×10^{-6}
	90	84.10275151	84.10586834	3.7060×10^{-5}	84.10200083	-8.9258×10^{-6}
	100	91.01821673	91.02791878	1.0659×10^{-4}	91.01751449	-7.7154×10^{-6}
	110	97.76262096	97.76634832	3.8127×10^{-5}	97.76160868	-1.0354×10^{-5}
	120	104.35523896	104.52509708	1.6277×10^{-3}	104.35204925	-3.0566×10^{-5}
	130	110.81175588	111.45461822	5.8014×10^{-3}	110.80027165	-1.0364×10^{-4}
10	70	71.50758031	71.95153547	6.2085×10^{-3}	71.49378585	-1.9291×10^{-4}
	80	79.03983824	79.09443042	6.9069×10^{-4}	79.01495140	-3.1486×10^{-4}
	90	86.33967998	86.38669494	5.4453×10^{-4}	86.31481091	-2.8804×10^{-4}
	100	93.43907974	93.55263158	1.2152×10^{-3}	93.41216203	-2.8808×10^{-4}
	110	100.36286870	100.41303853	4.9988×10^{-4}	100.32967244	-3.3076×10^{-4}
	120	107.13083429	107.21821509	8.1565×10^{-4}	107.08434540	-4.3394×10^{-4}
	130	113.75907885	114.72103904	8.4561×10^{-3}	113.66448109	-8.3156×10^{-4}

Note: Parameters: $x_{10} = 100$, $x_{20} = 90$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$ and $\rho = 0.5$.

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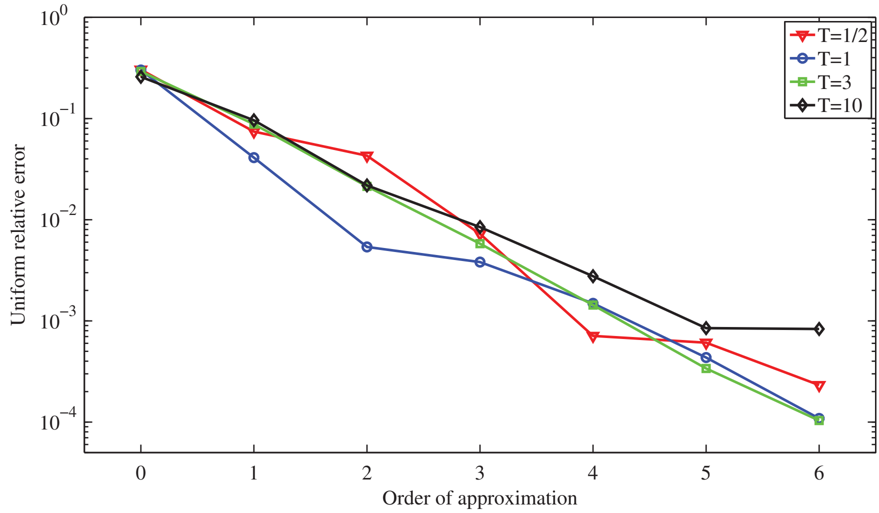

FIGURE 1. Uniform relative errors.
Note: Parameters: $x_{10} = 100$, $x_{20} = 90$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$ and $\rho = 0.5$.

is, the Jth order uniform relative error is calculated as

$$e^{(J)} := \max_{x_1 \in \{70,71,\ldots,129,130\}} |RE^{(J)}(x_1, T)|,$$

where $RE^{(J)}(x_1, T)$ is the relative error defined via (4.2). It is evident from Figure 1 that the uniform relative error decreases rapidly as the expansion order increases. This suggests that our method is relatively robust to wide range of parameters.

4.2. An Application in Approximating Local Volatility Functions

In this section, we will apply our asymptotic expansion method to approximate local volatility functions of the celebrated SABR local-stochastic volatility model, which is famous for its extensive applications in derivatives pricing; see, for example, Hagan et al. [21].

Model 2: The SABR model:

$$dS(t) = \sigma(t)S(t)^{\beta}dW_1(t),\ S(0) = s_0,$$
$$d\sigma(t) = \alpha\sigma(t)[\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)], \sigma(0) = \sigma_0,$$

where (W_1, W_2) is standard two-dimensional Brownian motion. Here, we assume $-1 \leq \rho \leq 1$, $0 \leq \beta \leq 1$ and $\alpha \geq 0$.

We denote by $V(t) = \sigma^2(t)$ the stochastic variance. Thus, an application of the Itô formula yields that

$$dS(t) = \sqrt{V(t)}S(t)^{\beta}dW_1(t), \quad S(0) = s_0,$$
$$dV(t) = \alpha^2 V(t)dt + 2\alpha V(t)[\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)], \quad V(0) = \sigma_0^2.$$

We let $(X_1(t), X_2(t)) = (S(t), V(t))$ and directly apply our expansion proposed in Section 3 to approximate the local volatility function (2.2). Taking a set of typical market parameters, we plot the sixth order approximation of the local variance

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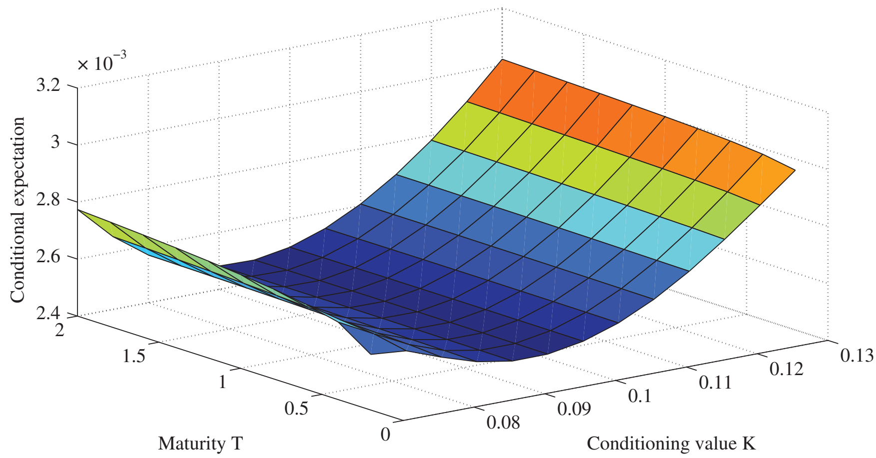

FIGURE **2.** Local variance surface of SABR model.
Note: Parameters: $s_0 = 0.1$, $\sigma_0 = 0.05$, $\alpha = 0.2$, $\beta = 0.5$ and $\rho = 0.1$.

surface $\{(T, K, CE^{(6)}(T, K))\}$ over the grids of maturity $T = \{0.2, 0.4, \ldots, 1.8, 2.0\}$ and strike $K = \{0.07, 0.075, \ldots, 0.125, 0.13\}$ in Figure 2, where $CE^{(6)}(T, K)$ is the sixth order approximation of the local volatility function

$$u(T, K) := E(V(T)|S(T) = K) \equiv E(X_2(T)|X_1(T) = K).$$

5. CONCLUDING REMARKS

In this paper, we proposed a method for approximating local volatility functions of stochastic volatility models via a closed-form expansion for conditional expectations involving marginal distributions generated by SDEs. A numerical test and an illustration of application are provided to demonstrate the efficiency of our approach. In spite of our exposition on a two-dimensional model, our method is flexible and can be generalized to various multidimensional diffusion models and further applied to various multi-factor stochastic volatility models; see, for example, Duffie, Pan, and Singleton [9] and Gatheral [15]. This and its subsequent applications can be set as a future research direction. Among many others, one can also employ our expansion of local volatility functions as input to obtain expansions of implied volatility following, for example, Berestycki et al. [6], Henry-Labordère [23], Guyon and Henry-Labord'ere Gatheral et al. [19], Gatheral et al. [16] and Gatheral and Wang [17].

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APPENDIX

PROOFS

Proof of Lemma 1

PROOF: Based on the following equivalent Stratonovich form (see, e.g., Section 3.3 in Karatzas and Shreve [27]) of the SDE (3.1)

$$dX^{\epsilon}(t) = \epsilon\mu^{T}(X^{\epsilon}(t))dt + \epsilon^{2}b^{T}(X^{\epsilon}(t))dt + \epsilon\sigma^{T}(X^{\epsilon}(t)) \circ dW(t),$$

where $b^{T}(x) = (b_1^{T}(x), b_2^{T}(x))^{\top}$ defined in (3.7), the proof follows from successive applications of the Itô–Stratonovich formula; see, for example, Section 3.3 in Karatzas and Shreve [27]. Thus, we omit the details. ∎

Proof of Proposition 1

PROOF: The kth order correction term for the pathwise expansion (3.17) follows from the chain rule, that is,

$$\Phi_0(y_1) = \delta(B_1(1) - y_1),$$

and, for $k \geq 1$,

$$\begin{aligned}\Phi_k(y_1) &= \sum_{(l,\mathbf{j})\in\mathcal{R}(k)} \frac{1}{l!}\partial^{(l)}\delta(B_1(1) - y)Y_{j_1,1}Y_{j_2,1}\cdots Y_{j_l,1} \\ &= \sum_{(l,\mathbf{j})\in\mathcal{R}(k)} \frac{D_{11}(x_0)^{l}}{l!}\partial^{(l)}\delta(B_1(1) - y_1)\left(\prod_{v=1}^{l} F_{j_v+1,1}\right), \qquad \textbf{(A.1)}\end{aligned}$$

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where the index set $\mathcal{R}(k)$ is defined as

$$\mathcal{R}(k) := \{(l, (j_1, j_2, \ldots, j_l)) \,|\, j_1, j_2, \ldots, j_l \geq 1, j_1 + j_2 + \cdots + j_l = k\}. \tag{A.2}$$

By plugging in the closed-form expansion (3.11), we deduce that

$$\Phi_k(y_1) = \sum_{(l,\mathbf{j})\in\mathcal{R}(k)} \frac{1}{l!} D_{11}(x_0)^l \partial^{(l)} \delta(B_1(1) - y_1) \left(\prod_{v=1}^{l} \sum_{\|\mathbf{i}\|=j_v+1} C_{\mathbf{i},1}(x_0) J_{\mathbf{i}}(1) \right),$$

where we note that

$$\prod_{v=1}^{l} \sum_{\|\mathbf{i}\|=j_v+1} C_{\mathbf{i},1}(x_0) J_{\mathbf{i}}(1) = \sum_{\{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l) | \|\mathbf{i}_v\|=j_v+1, v=1,2,\ldots,l\}} \left(\prod_{v=1}^{l} C_{\mathbf{i}_v,1}(x_0) \right) \left(\prod_{v=1}^{l} J_{\mathbf{i}_v}(1) \right).$$

By changing the order of summation and collecting same terms, we obtain that

$$\begin{aligned} \Phi_k(y_1) = &\sum_{1\leq l\leq k} \frac{D_{11}(x_0)^l}{l!} \sum_{(j_1,j_2,\ldots,j_l)\in S(k,l)} N(j_1, j_2, \ldots, j_l) \\ &\sum_{\{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l) | \|\mathbf{i}_v\|=j_v+1, v=1,2,\ldots,l\}} \left(\prod_{v=1}^{l} C_{\mathbf{i}_v,1}(x_0) \right) \partial^{(l)} \delta(B_1(1) - y_1) \left(\prod_{v=1}^{l} J_{\mathbf{i}_v}(1) \right), \end{aligned} \tag{A.3}$$

with the index set $S(k, l)$ defined in (3.22) and $N(j_1, j_2, \ldots, j_l)$ defined in Proposition 1. By taking expectation, the calculation of $E\Phi_k(y_1)$ boils down to

$$\begin{aligned} &E\left[\partial^{(l)} \delta(B_1(1) - y_1) \left(\prod_{v=1}^{l} J_{\mathbf{i}_v}(1) \right) \right] \\ &= \int_{b_1\in\mathbb{R}} \partial^{(l)} \delta(b_1 - y_1) \left(E\left[\left(\prod_{v=1}^{l} J_{\mathbf{i}_v}(1) \right) | B_1(1) = b_1 \right] \phi(b_1) \right) db_1 \\ &= (-1)^l \frac{\partial^{(l)}}{\partial y_1^{(l)}} \left(E\left[\left(\prod_{v=1}^{l} J_{\mathbf{i}_v}(1) \right) | B_1(1) = y_1 \right] \phi(y_1) \right), \end{aligned}$$

where we used the integration-by-parts formula of Dirac Delta function (see, e.g., section 2.6 of Kanwal [26]). Hence, the formula follows from the definition of the differential operator (3.19). ■

Proof of Proposition 2

Proof: By plugging in (3.24) and (A.1) and collecting similar terms, we obtain that

$$\begin{aligned} &F_{r,2}\Phi_{k-r}(y_1) \\ &= \left(\sum_{\mathbf{i}=(i_1,\ldots,i_n),\|\mathbf{i}\|=r} C_{\mathbf{i},2}(x_0) J_{\mathbf{i}}(1) \right) \left(\sum_{l\leq k-r} \frac{1}{l!} D_{11}(x_0)^l \sum_{(j_1,j_2,\ldots,j_l)\in S(k-r,l)} N(j_1, j_2, \ldots, j_l) \right. \end{aligned}$$

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$$\sum_{\{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)|\|\mathbf{i}_v\|=j_v+1,v=1,2,\ldots,l\}} \left(\left(\prod_{v=1}^{l} C_{\mathbf{i}_v,1}(x_0)\right) \partial^{(l)}\delta(B_1(1)-y_1)\left(\prod_{v=1}^{l} J_{\mathbf{i}_v}(1)\right)\right)$$

$$= \sum_{\mathbf{i}=(i_1,\ldots,i_n),\|\mathbf{i}\|=r} C_{\mathbf{i},2}(x_0) \sum_{l\le k-r} \frac{1}{l!} D_{11}(x_0)^l \sum_{(j_1,j_2,\ldots,j_l)\in S(k-r,l)} N(j_1,j_2,\ldots,j_l)$$

$$\sum_{\{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)|\|\mathbf{i}_v\|=j_v+1,v=1,2,\ldots,l\}} \left(\prod_{v=1}^{l} C_{\mathbf{i}_v,1}(x_0)\right) \partial^{(l)}\delta(B_1(1)-y_1)\left(\prod_{v=1}^{l} J_{\mathbf{i}_v}(1)\right) J_{\mathbf{i}}(1).$$

Thus, by taking expectation, we have

$$E\left[F_{r,2}\Phi_{k-r}(y_1)\right] = \sum_{\mathbf{i}=(i_1,\ldots,i_n),\|\mathbf{i}\|=r} C_{\mathbf{i},2}(x_0) \sum_{l\le k-r} \frac{1}{l!} D_{11}(x_0)^l \sum_{(j_1,j_2,\ldots,j_l)\in S(k-r,l)} N(j_1,j_2,\ldots,j_l)$$

$$\sum_{\{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)|\|\mathbf{i}_v\|=j_v+1,v=1,2,\ldots,l\}} \left(\prod_{v=1}^{l} C_{\mathbf{i}_v,1}(x_0)\right) E\left[\partial^{(l)}\delta(B_1(1)-y_1)\left(\prod_{v=1}^{l} J_{\mathbf{i}_v}(1)\right) J_{\mathbf{i}}(1)\right].$$

Similar to the proof of Proposition 1, we obtain that

$$E\left[\partial^{(l)}\delta(B_1(1)-y_1)\left(\prod_{v=1}^{l} J_{\mathbf{i}_v}(1)\right) J_{\mathbf{i}}(1)\right] = \mathcal{D}^{(l)}\left(P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l,\mathbf{i})}\right)(y_1)\phi(y_1).$$

■

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