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On a non-local equation arising in population dynamics

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We study a one-dimensional non-local variant of Fisher's equation describing the spatial spread of a mutant in a given population, and its generalization to the so-called monostable nonlinearity. The dispersion of the genetic characters is assumed to follow a non-local diffusion law modelled by a convolution operator. We prove that, as in the classical (local) problem, there exist travelling-wave solutions of arbitrary speed beyond a critical value and also characterize the asymptotic behaviour of such solutions at infinity. Our proofs rely on an appropriate version of the maximum principle, qualitative properties of solutions and approximation schemes leading to singular limits.

1. Introduction

In 1930, Fisher [8] suggested modelling the spatial spread of a mutant in a given population by the reaction–diffusion equation

$$u_t - \Delta u = u(1-u), \tag{1.1}$$

where u represents the gene fraction of the mutant. Dispersion of the genetic characters is assumed to follow a diffusion law, while the logistic term $u(1-u)$ takes into account the saturation of this dispersion process.

Since then, much attention has been drawn to reaction–diffusion equations, as they have proven to give a robust and accurate description of a wide variety of phenomena, ranging from combustion to bacterial growth, nerve propagation or epidemiology. We point the interested reader to [7,10,12] and their many references.

In this work, we consider a variant of (1.1), in which diffusion is modelled by a convolution operator. Going back to the early work of Kolmogorov, Petrovsky and Piskounov [11], dispersion of the gene fraction at point $y \in \mathbb{R}^n$ should affect the gene fraction at $x \in \mathbb{R}^n$ by a factor $J(x,y)u(y)\,dy$, where $J(x,\cdot)$ is a probability

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density. Restricting our study to a one-dimensional setting and assuming that such a diffusion process depends only on the distance between two niches of the population, we end up with the equation

$$u_t - (J \star u - u) = f(u), \tag{1.2}$$

where $J : \mathbb{R} \to \mathbb{R}$ is a non-negative even function of mass 1 and for $x \in \mathbb{R}$,

$$J \star u(x) = \int_{\mathbb{R}} J(x-y)u(y)\,\mathrm{d}y.$$

More precisely, we assume in what follows that

$$J \in C^1(\mathbb{R}), \quad J \geqslant 0, \qquad J(x) = J(-x), \qquad \int_{\mathbb{R}} J = 1. \tag{H1}$$

We make the additional technical assumption

$$\exists \lambda > 0, \quad \int_{\mathbb{R}} J(x)\mathrm{e}^{\lambda x}\,\mathrm{d}x < +\infty. \tag{H2}$$

For example, (H2) is satisfied if J has compact support or if $J(x) = (1/2\lambda)\mathrm{e}^{-\lambda|x|}$ for some $\lambda > 0$.

The nonlinearity f in (1.2) can be chosen more generally than in equation (1.1). In the literature, three types of nonlinearities appear, according to the underlying application: we always assume that $f \in C^1(\mathbb{R})$, $f(0) = f(1) = 0$, $f'(1) < 0$ and

(i) we say that f is of bistable type if there exists $\theta \in (0,1)$ such that

$$f < 0 \in (0,\theta), \quad f(\theta) = 0 \quad \text{and} \quad f > 0 \in (\theta, 1),$$

(ii) f is of ignition type if there exists $\theta \in (0,1)$ such that

$$f|_{[0,\theta]} \equiv 0, \quad f|_{(\theta,1)} > 0 \quad \text{and} \quad f(1) = 0,$$

(iii) f is of monostable type if

$$f > 0 \in (0,1).$$

In this paper, we will focus on the monostable nonlinearity. Observe that equation (1.1) falls in this case.

Equation (1.1) can also be seen as a first-order approximation of (1.2). Indeed if any given niche of the species is assumed to interact mostly with close-by neighbours, the diffusion term is of the form

$$J_\varepsilon(x) := \frac{1}{\varepsilon} J\left(\frac{1}{\varepsilon}x\right),$$

where J is compactly supported and $\varepsilon > 0$ is small. We then have

$$\begin{aligned} J_\varepsilon \star u - u &= \frac{1}{\varepsilon}\int J\biggl(\frac{1}{\varepsilon}y\biggr)(u(x-y)-u(x))\,\mathrm{d}y \\ &= \int J(z)(u(x-\varepsilon z)-u(x))\,\mathrm{d}z \\ &= -\varepsilon\int J(z)u'(x)z\,\mathrm{d}z + \tfrac{1}{2}\varepsilon^2\int z^2 J(z)u''(x)\,\mathrm{d}z + o(\varepsilon^2) \\ &= c\varepsilon^2 u''(x) + o(\varepsilon^2), \end{aligned}$$

where we have used the fact that J is even in the last equality.

We observe that equation (1.2) can be related to a class of problems studied in [14, 15]. However, our approach differs in at least two ways. Firstly, from the technical point of view, inverting the operator $u \to u_t - (J \star u - u)$ in any reasonable space yields no *a priori* regularity property on the solution u and the compactness assumptions made in [14] no longer hold in our case.

Secondly, whereas Weinberger favoured discrete models over continuous ones to describe the dynamics of certain populations, we remain interested in the latter. In particular, we have in mind the following application to adaptive dynamics: in [9], the authors study a probabilistic model describing the microscopic behaviour of the evolution of genetic traits in a population subject to mutation and selection. Averaging over a large number of individuals in the initial state, they derive in the limit a deterministic equation, a special case of which can be written as

$$\partial_t u = [J \star u - u] + (1 - K \star u)u, \tag{1.3}$$

where $J(x)$ is a kernel taking into account mutation about trait x and $K(x)$ is a competition kernel, measuring the 'intensity' of the interaction between x and y. Taking $K(x) = \delta$, we recover equation (1.2) as a special case of (1.3).

The aim of this article is the study of so-called travelling-wave solutions of equation (1.2) i.e. solutions of the form

$$u(x,t) = U(x + ct),$$

where $c \in \mathbb{R}$ is the wave speed and U is the wave profile, which is required to solve the equation

$$\left.\begin{aligned} [J \star U - U] - cU' + f(U) = 0 \quad \text{in } \mathbb{R}, \\ U(-\infty) = 0, \\ U(+\infty) = 1, \end{aligned}\right\} \tag{1.4}$$

where $U(\pm\infty)$ denotes the limit of $U(x)$ as $x \to \pm\infty$.

Such solutions are expected to give the asymptotic behaviour in large time for solutions of (1.2) with, say, compactly supported initial data: in the Fisher equation, this is equivalent to saying that the mutant propagates (after some time) at constant speed and along the profile U. It is therefore of interest to prove the existence of such solutions.

The first results in this direction are due to Schumacher [13], who considered the monostable nonlinearity, under the extra assumption that $f(r) \geqslant h_0 r - K r^{1+\alpha}$, for some $h_0, K, \alpha > 0$ and all $r \in [0, 1]$. In this case, his results imply the existence of travelling waves with arbitrary speed $c \geqslant c^*$, where c^* is the smallest $c \in \mathbb{R}$ such that $\rho_c : \mathbb{R} \to \mathbb{R}$, defined by

$$\rho_c(\lambda) = -\lambda c + \int J(z) \mathrm{e}^{\lambda z} \, \mathrm{d}z - 1 + f'(0),$$

vanishes for some $\lambda > 0$. Observe from assumption (H1) that

$$\int J(z) \mathrm{e}^{\lambda z} = \int J(z) \mathrm{e}^{-\lambda z}.$$

So finding $\lambda > 0$ such that $\rho_c(\lambda) = 0$ amounts to looking for an explicit solution of the form $v(x) = \mathrm{e}^{\lambda x}$ of the equation

$$J \star v - v - cv' + f'(0)v = 0,$$

obtained by linearizing (1.4) near $x = -\infty$. v then yields the expected asymptotic decay near $x = -\infty$ of solutions of (1.4).

Finally, if $c > c^*$ and under some extra assumptions on f, Schumacher shows that the profile U of the associated travelling wave is unique up to translation.

Recently, Carr and Chmaj [3] completed the work of Schumacher. For the 'KPP' nonlinearity (i.e. if f is monostable and $f(r) \leqslant f'(0)r$ for all $r \in [0, 1]$) and if J has compact support, they show that the above uniqueness result can be extended to $c = c^*$.

Concerning the bistable nonlinearity, Bates *et al*. [1] and Chen [4] showed that there exists an increasing travelling wave U with speed c solving (1.4). Furthermore, if V is another non-decreasing travelling wave with speed c', then $c = c'$ and $V(x) = U(x + \tau)$ for some $\tau \in \mathbb{R}$.

Coville [5] then looked at the case of ignition nonlinearities and proved again the existence and uniqueness (up to translation) of an increasing travelling wave (U, c). Coville also obtained the existence of at least one travelling-wave solution in the monostable case.

Our first theorem extends some of the aforementioned results of Schumacher to the general monostable case.

THEOREM 1.1. *Assume that (H1) and (H2) hold and assume that f is of monostable type. There then exists a constant $c^* > 0$ (the minimal speed of the travelling wave) such that, for all $c \geqslant c^*$, there exists an increasing solution $U \in C^1(\mathbb{R})$ of (1.4), while no non-decreasing travelling wave of speed $c < c^*$ exists.*

Our second result extends previous work of Coville [5] regarding the behaviour of the travelling front U near $\pm\infty$.

PROPOSITION 1.2. *Assume that (H1) and (H2) hold. Then, given any travelling-wave solution (U, c) of (1.4) with f monostable, the following assertions hold.*

(i) *There exist positive constants A, B, M, λ_0 and δ_0 such that*

$$B\mathrm{e}^{-\delta_0 y} \leqslant 1 - U(y) \leqslant A\mathrm{e}^{-\lambda_0 y} \quad \text{for } y \geqslant M.$$

(ii) *If $f'(0) > 0$, then there exist positive constants K, N and λ_1 such that*

$$U(y) \leqslant K\mathrm{e}^{\lambda_1 y} \quad \text{for } y \leqslant -N.$$

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The first point is an easy consequence of a similar result when f is of bistable or ignition type, proved in [5].

Regarding theorem 1.1, our proof is based on the study of two auxiliary problems and the construction of adequate super and subsolutions. We work in three steps.

We start by showing existence and uniqueness of a solution for

$$\left.\begin{aligned} \mathcal{L}u + f(u) = -h_r(x) \quad \text{in } \Omega, \\ u(-r) = \theta, \\ u(+\infty) = 1, \end{aligned}\right\} \tag{1.5}$$

where, given $\varepsilon > 0$, $r \in \mathbb{R}$, $c \in \mathbb{R}$ and $\theta \in (0, 1)$,

$$\Omega = (-r, +\infty), \tag{1.6}$$

$$\mathcal{L}u = \mathcal{L}(\varepsilon, r, c)u = \varepsilon u'' + \left[\int_{-r}^{+\infty} J(x-y)u(y)\,\mathrm{d}y - u\right] - cu', \tag{1.7}$$

$$h_r(x) = \theta \int_{-\infty}^{-r} J(x-y)\,\mathrm{d}y. \tag{1.8}$$

The existence is obtained via an iterative scheme using a comparison principle and appropriate sub- and supersolutions.

In the second step, with a standard limiting procedure (as $r \to +\infty$), we prove theorem 1.1 for the problem

$$\left.\begin{aligned} \mathcal{M}u + f(u) = 0 \quad \text{in } \mathbb{R}, \\ u(-\infty) = 0, \\ u(+\infty) = 1, \end{aligned}\right\} \tag{1.9}$$

where, given $\varepsilon > 0$, $c \in \mathbb{R}$,

$$\mathcal{M}u = \mathcal{M}(\varepsilon, c)u = \varepsilon u'' + [J \star u - u] - cu'. \tag{1.10}$$

We stress the fact that, unlike (1.5), (1.9) does not have an (increasing and smooth) solution u for arbitrary values of $c \in \mathbb{R}$.

Finally, in the last step we send $\varepsilon \to 0$ and extract converging subsequences.

Though elementary in nature, the proofs require a number of lemmas, which we list and prove in the appendix. We construct sub- and supersolutions for (1.5) and (1.9) in § 2. After obtaining some useful *a priori* estimates in § 3, we prove existence and uniqueness of solutions of (1.5) in § 4. In § 5, we show the existence of a speed $c^*(\varepsilon) > 0$ such that (1.9) admits a solution for every $c \geqslant c^*(\varepsilon)$. We complete the proof of theorem 1.1 in § 6. Section 7 is devoted to the proof of proposition 1.2.

2. Existence of sub- and supersolutions

We start with the construction of a supersolution of (1.9) for speeds $c \geqslant \bar{\kappa}(\varepsilon)$ for some $\bar{\kappa}(\varepsilon) > 0$.

LEMMA 2.1. *Let $\varepsilon > 0$. There exists a real number $\bar\kappa(\varepsilon) > 0$ and an increasing function $\bar w \in C^2(\mathbb{R})$ such that, given any $c \geqslant \bar\kappa(\varepsilon)$,*

$$\mathcal{M}\bar w + f(\bar w) \leqslant 0 \quad \text{in } \mathbb{R},$$
$$\bar w(-\infty) = 0,$$
$$\bar w(+\infty) = 1,$$

where $\mathcal{M} = \mathcal{M}(\varepsilon, c)$ is defined by (1.10). Furthermore, $\bar w(0) = \frac{1}{2}$.

Proof. Fix positive constants N, λ, δ such that $\lambda > \delta$ and (H2) holds.

Let $\bar w \in C^2(\mathbb{R})$ be a positive increasing function satisfying

(i) $\bar w(x) = \mathrm{e}^{\lambda x}$ for $x \in (-\infty, -N]$,

(ii) $\bar w(x) \leqslant \mathrm{e}^{\lambda x}$ on $\mathbb{R}$,

(iii) $\bar w(x) = 1 - \mathrm{e}^{-\delta x}$ for $x \in [N, +\infty)$,

(iv) $\bar w(0) = \frac{1}{2}$.

Let $x_0 = \mathrm{e}^{-\lambda N}$ and $x_1 = 1 - \mathrm{e}^{-\delta N}$. We have $0 < x_0 < x_1 < 1$.

We now construct a positive function g defined on $(0, 1)$ which satisfies $g(\bar w) \geqslant f(\bar w)$. Since f is smooth near 0 and 1, for c large enough, say $c \geqslant \kappa_0$, we have

$$(c - \lambda)s \geqslant f(s) \quad \text{for } s \in [0, x_0] \tag{2.1}$$

and

$$\delta(c - \delta)(1 - s) \geqslant f(s) \quad \text{for } s \in [x_1, 1]. \tag{2.2}$$

Therefore, we can obtain $g(s) \geqslant f(s)$ for s in $[0, 1]$, with g defined by

$$g(s) = \begin{cases} \lambda(\kappa_0 - \lambda)s & \text{for } 0 \leqslant s \leqslant x_0, \\ l(s) & \text{for } x_0 < s < x_1, \\ \delta(\kappa_0 - \delta)(1 - s) & \text{for } x_1 \leqslant s \leqslant 1. \end{cases} \tag{2.3}$$

where l is any smooth positive function greater than f on $[x_0, x_1]$ such that g is of class C^1.

According to (2.3), for $x \leqslant -N$, i.e. for $w \leqslant \mathrm{e}^{-\lambda N}$, we have

$$\begin{aligned} \mathcal{M}\bar w + g(\bar w) &= \varepsilon \bar w'' + J \star \bar w - \bar w - c\bar w' + g(\bar w) \\ &= \varepsilon\lambda^2 \mathrm{e}^{\lambda x} + J \star \bar w - \mathrm{e}^{\lambda x} - \lambda c \mathrm{e}^{\lambda x} + \lambda(\kappa_0 - \lambda)\mathrm{e}^{\lambda x} \\ &\leqslant \varepsilon\lambda^2 \mathrm{e}^{\lambda x} + J \star \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} - \lambda c \mathrm{e}^{\lambda x} + \lambda(\kappa_0 - \lambda)\mathrm{e}^{\lambda x} \\ &\leqslant \mathrm{e}^{\lambda x}\biggl[\int_{\mathbb{R}} J(z)\mathrm{e}^{\lambda z}\,\mathrm{d}z - 1 - \lambda(c - \kappa_0) - \lambda^2(1 - \varepsilon)\biggr] \\ &\leqslant 0, \end{aligned}$$

for c large enough, say

$$c \geqslant \kappa_1 = \frac{1}{\lambda}\int_{\mathbb{R}} J(z)\mathrm{e}^{\lambda z}\,\mathrm{d}z - 1 + \lambda\kappa_0 - \lambda^2(1 - \varepsilon).$$

Furthermore, for $\bar{w} \geqslant 1 - \mathrm{e}^{-\delta N}$ we have

$$\begin{aligned}\mathcal{M}\bar{w} + g(\bar{w}) &= \varepsilon\bar{w}'' + J \star \bar{w} - \bar{w} - c\bar{w}' + g(\bar{w}) \\ &= \varepsilon\delta^2\mathrm{e}^{-\delta x} + J \star \bar{w} - (1 - \mathrm{e}^{-\delta x}) - \delta c\mathrm{e}^{-\delta x} + \delta(\kappa_0 - \delta)\mathrm{e}^{-\delta x} \\ &\leqslant \varepsilon\delta^2\mathrm{e}^{-\delta x} + 1 - 1 + \mathrm{e}^{-\delta x} - \delta c\mathrm{e}^{-\delta x} + \delta(\kappa_0 - \delta)\mathrm{e}^{-\delta x} \\ &\leqslant \mathrm{e}^{-\delta x}[1 - \delta(c - \kappa_0) - \delta^2(1 - \varepsilon)] \\ &\leqslant 0,\end{aligned}$$

for c large enough, say

$$c \geqslant \kappa_2 = \frac{1 + \delta\kappa_0 - \delta^2(1 - \varepsilon)}{\delta}.$$

Thus, by taking $c \geqslant \sup\{\kappa_0, \kappa_1, \kappa_2\}$, we obtain

$$g(\bar{w}) \geqslant f(\bar{w}), \quad J \star \bar{w} - \bar{w} - c\bar{w}' + g(\bar{w}) \leqslant 0 \quad \text{for } 0 \leqslant \bar{w} \leqslant \mathrm{e}^{-\lambda N}, \quad \bar{w} \geqslant 1 - \mathrm{e}^{-\delta N}.$$

For the remaining values of $\bar{w}$, i.e. for $x \in [-N, N]$, $\bar{w}' > 0$ and we may increase c further if necessary, to obtain

$$\varepsilon\bar{w}'' + J \star \bar{w} - \bar{w} - c\bar{w}' + g(\bar{w}) \leqslant 0 \quad \text{in } \mathbb{R}. \tag{2.4}$$

The result follows for

$$\bar{\kappa}(\varepsilon) := \sup\{\kappa_0, \kappa_1, \kappa_2, \kappa_3\},$$

where

$$\kappa_3 = \sup_{x \in [-N, N]} \left\{ \frac{\varepsilon|\bar{w}''| + |J \star \bar{w} - \bar{w}| + g(\bar{w})}{\bar{w}'} \right\}.$$

□

REMARK 2.2. $\bar{\kappa}(\varepsilon)$ is a non-decreasing function of ε.

REMARK 2.3. Observe that, given any $r \in \mathbb{R}$, for $c \geqslant \bar{\kappa}(\varepsilon)$, $\bar{w}$ also satisfies

$$\left.\begin{aligned} \mathcal{L}\bar{w} + f(\bar{w}) \leqslant 0 \quad \text{in } \Omega, \\ \bar{w}(-r) \geqslant 0, \\ \bar{w}(+\infty) = 1, \end{aligned}\right\} \tag{2.5}$$

where $\mathcal{L} = \mathcal{L}(\varepsilon, c, r)$ is defined by (1.7).

Next, we construct sub- and supersolutions of (1.5).

REMARK 2.4. Let $\varepsilon \geqslant 0$, $r \in \mathbb{R}$, $c \in \mathbb{R}$ and $\theta \in (0, 1)$. Then the constant functions $\underline{u} = \theta$ and $\bar{u} = 1$ are respectively a subsolution and a supersolution of problem (1.5), i.e.

$$\begin{aligned} &\mathcal{L}\underline{u} + f(\underline{u}) \geqslant -h_r(x) \quad \text{in } \Omega && (\text{respectively } \mathcal{L}\bar{u} + f(\bar{u}) \geqslant -h_r(x) \text{ in } \Omega), \\ &\underline{u}(-r) \leqslant \theta && (\text{respectively } \bar{u}(-r) \leqslant \theta), \\ &\underline{u}(+\infty) \leqslant 1 && (\text{respectively } \bar{u}(+\infty) \leqslant 1). \end{aligned}$$

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We now construct a subsolution of (1.5) satisfying stronger conditions on the boundary of Ω.

LEMMA 2.5. *Let $\varepsilon > 0$, $r \in \mathbb{R}$ and $\theta \in (0,1)$. There exists $\underline{\kappa}(\varepsilon) \in \mathbb{R}$ and an increasing function $\underline{w} \in C^2(\mathbb{R})$ such that, given any $c \leqslant \underline{\kappa}(\varepsilon)$,*

$$\begin{aligned}\mathcal{L}\underline{w} + f(\underline{w}) &\geqslant -h_r(x) \quad \text{in } \Omega,\\ \underline{w}(-r) &= \theta,\\ \underline{w}(+\infty) &= 1.\end{aligned}$$

Proof. Let f_{b} be a smooth bistable function (i.e. $f_{\mathrm{b}}(0) = f_{\mathrm{b}}(1) = 0$ and $\exists\theta \in (0,1)$ such that $f_{\mathrm{b}} < 0 \in (0,\theta)$, $f_{\mathrm{b}}(\theta) = 0$ and $f_{\mathrm{b}} > 0 \in (\theta,1)$) such that $f_{\mathrm{b}} \leqslant f$ and

$$\int_0^1 f_{\mathrm{b}}(s)\,\mathrm{d}s > 0.$$

Let $(u_{\mathrm{b}}, c_{\mathrm{b}})$ denote the unique (up to translation) increasing solution of (1.9) with f_{b} instead of f. Such a solution exists (see [1] for details). Moreover, $c_{\mathrm{b}} > 0$. Using the translation invariance of (1.9), one can easily show that, for any $c \leqslant c_{\mathrm{b}}$, $u_{\mathrm{b}}^{\tau} := u_{\mathrm{b}}(\,\cdot + \tau)$ is a subsolution of (1.5) for some $\tau \in \mathbb{R}$. Indeed, choose τ such that $u_{\mathrm{b}}^{\tau}(-r) = \theta$.

Since u_{b}^{τ} is increasing, we have

$$h_r(x) = \theta \int_{-\infty}^{-r} J(x-y)\,\mathrm{d}y \geqslant \int_{-\infty}^{-r} J(x-y)u_{\mathrm{b}}^{\tau}(y)\,\mathrm{d}y.$$

A simple computation shows that

$$\begin{aligned}\mathcal{L}u_{\mathrm{b}}^{\tau} + h_r(x) + f(u_{\mathrm{b}}^{\tau}) &\geqslant \mathcal{L}u_{\mathrm{b}}^{\tau} + \int_{-\infty}^{-r} J(x-y)u_{\mathrm{b}}^{\tau}(y)\,\mathrm{d}y + f_{\mathrm{b}}(u_{\mathrm{b}}^{\tau}) \quad \text{in } \Omega\\ &\geqslant \mathcal{M}u_{\mathrm{b}}^{\tau} + f_{\mathrm{b}}(u_{\mathrm{b}}^{\tau}) = (c_{\mathrm{b}} - c)(u_{\mathrm{b}}^{\tau})' \quad \text{in } \Omega.\end{aligned}$$

Hence, for $c \leqslant c_{\mathrm{b}}$,

$$\begin{aligned}\mathcal{L}u_{\mathrm{b}}^{\tau} + h_r(x) + f(u_{\mathrm{b}}^{\tau}) &\geqslant (c_{\mathrm{b}} - c)(u_{\mathrm{b}}^{\tau})' \geqslant 0 \quad \text{in } \Omega,\\ u_{\mathrm{b}}^{\tau}(-r) &= \theta,\\ u_{\mathrm{b}}^{\tau}(+\infty) &= 1.\end{aligned}$$

□

3. L^2 estimates

In this section, we obtain L^2 estimates for solutions u of the problems (1.5) and (1.9).

3.1. L^2 estimates for solutions of (1.9)

LEMMA 3.1. *Assume that $\varepsilon > 0$ and $c \in \mathbb{R}$ and let u be a smooth increasing solution of (1.9). Then*

(i) $u', u'' \in L^2(\mathbb{R})$,

(ii) $1 - u \in L^2(\mathbb{R}^+)$.

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Proof of lemma 3.1. Let u be a smooth increasing solution of (1.9). We start out by showing that u' and u'' vanish at infinity. We restrict our proof to the case when $u'(+\infty) = 0$, the other cases, $u'(-\infty) = 0$ and $u''(\pm\infty) = 0$, being similar.

Assume by contradiction that there exists an increasing sequence $(x_p)_{p\in\mathbb{N}}$ converging to $+\infty$ and $\alpha > 0$ such that

$$u'(x_p) \geqslant \alpha \quad \forall p \in \mathbb{N}. \tag{3.1}$$

Let $(u_p)_{p\in\mathbb{N}}$ be defined by

$$u_p(x) := u(x + x_p) \quad \text{for } x \in \mathbb{R}.$$

Clearly, u_p solves (1.9) and $0 \leqslant u_p \leqslant 1$. By definition of $(u_p)_{p\in\mathbb{N}}$, on every compact set we have

$$\lim_{p\to+\infty} u_p(x) \equiv 1.$$

Since u_p satisfies (1.9), using standard elliptic estimates, $u_p \to 1$ in $C^{2,\beta}_{\mathrm{loc}}$. In particular, $u_p \to 1$ in $C^{2,\beta}(-1,1)$ and $u_p'(0) \to 0$ as $p \to +\infty$. Now, using (3.1), we have

$$0 < \alpha \leqslant u'(x_p) = u_p'(0) \to 0,$$

which is our desired contradiction.

We show next that $f(u) \in L^1(\mathbb{R})$. Integrating (1.9) over $(-r, r)$ leads to

$$\varepsilon(u'(r) - u'(-r)) + \int_{-r}^{r} (J \star u - u)\,\mathrm{d}x - c(u(r) - u(-r)) = -\int_{-r}^{+r} f(u).$$

Assume for the moment that $J \star u - u \in L^1(\mathbb{R})$. Then we can pass to the limit as $r \to +\infty$ in the above expression. So we get

$$\int_{-\infty}^{+\infty} (J \star u - u)\,\mathrm{d}x - c = -\int_{-\infty}^{+\infty} f(u).$$

Therefore, $f(u) \in L^1(\mathbb{R})$, provided that $J \star u - u \in L^1(\mathbb{R})$.

Claim 3.2. $J \star u - u \in L^1(\mathbb{R})$. *Moreover,*

$$\|J \star u - u\|_{L^1} \leqslant \int_{\mathbb{R}} J(z)|z|\,\mathrm{d}z \quad \textit{and} \quad \int_{\mathbb{R}} (J \star u - u) = 0.$$

Proof of claim. Clearly,

$$\int_{-r}^{r} |(J \star u - u)| \leqslant \int_{-r}^{r} \int_{\mathbb{R}} J(x - y)|u(y) - u(x)|\,\mathrm{d}y\,\mathrm{d}x. \tag{3.2}$$

Using the change of variable in y, $z := y - x$, (3.2) becomes

$$\int_{-r}^{r} |(J \star u - u)| \leqslant \int_{-r}^{r} \int_{\mathbb{R}} J(z)|u(x + z) - u(x)|\,\mathrm{d}z\,\mathrm{d}x. \tag{3.3}$$

Since $u \in C^1(\mathbb{R})$,

$$|u(x + z) - u(x)| = |z| \int_0^1 u'(x + sz)\,\mathrm{d}s.$$

Plug this equality into (3.3) to obtain

$$\int_{-r}^{r}\int_{\mathbb{R}} J(z)|u(x+z)-u(x)|\,\mathrm{d}y\,\mathrm{d}x = \int_{-r}^{r}\int_{\mathbb{R}} J(z)|z|\int_0^1 u'(x+sz)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}x. \quad (3.4)$$

Since all terms are positive, using Tonnelli's theorem, we can permute the order of integration and obtain

$$\begin{aligned}\int_{-r}^{r}\int_{\mathbb{R}} J(z)|z|\int_0^1 u'(x+sz)\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}x \\ &= \int_{\mathbb{R}} J(z)|z|\int_{-r}^{r}\int_0^1 u'(x+sz)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}z \\ &= \int_0^1\int_{\mathbb{R}} J(z)|z|[u(r+sz)-u(-r+sz)]\,\mathrm{d}z\,\mathrm{d}s.\end{aligned}$$

Hence,

$$\int_{-r}^{r}\left|\int_{\mathbb{R}} J(x-y)(u(y)-u(x))\,\mathrm{d}y\right|\mathrm{d}x \leqslant \int_0^1\int_{\mathbb{R}} J(z)|z|[u(r+sz)-u(-r+sz)]\,\mathrm{d}z\,\mathrm{d}s.$$

Now, using Lebesgue dominated convergence, we can pass to the limit in the above expression to get

$$\|J\star u-u\|_{L^1} \leqslant \int_{\mathbb{R}} J(z)|z|\,\mathrm{d}z. \quad (3.5)$$

Let us now compute $\int_{\mathbb{R}}(J\star u-u)\,\mathrm{d}x$. Since J is symmetric, we have

$$\begin{aligned}\int_{\mathbb{R}}(J\star u-u)\,\mathrm{d}x &= \int_{\mathbb{R}^2} J(x-y)(u(y)-u(x))\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{\mathbb{R}^2} J(y-x)(u(y)-u(x))\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{\mathbb{R}^2} J(x-y)(u(x)-u(y))\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Hence,

$$2\int_{\mathbb{R}^2} J(x-y)(u(y)-u(x))\,\mathrm{d}y\,\mathrm{d}x = 0.$$

□

We now prove (i). Multiplying (1.9) by u and integrating over $\mathbb{R}$ yields

$$\varepsilon\int_{\mathbb{R}} u''u + \int_{\mathbb{R}}(J\star u-u)u - c\int_{\mathbb{R}} u'u = -\int_{\mathbb{R}} f(u)u.$$

Integrating the first term by parts yields

$$-\varepsilon\int_{\mathbb{R}}(u')^2 + \int_{\mathbb{R}}(J\star u-u)u - \tfrac{1}{2}c = -\int_{\mathbb{R}} f(u)u.$$

Since u is bounded and $f(u), J\star u-u \in L^1$, we conclude that $u' \in L^2$.

We obtain $u'' \in L^2$ similarly. Indeed, on multiplying (1.9) by u'' and integrating over $\mathbb{R}$ we get

$$\varepsilon \int_{\mathbb{R}} (u'')^2 + \int_{\mathbb{R}} (J \star u - u)u'' - c \int_{\mathbb{R}} u'u'' = \int_{\mathbb{R}} f(u)u''.$$

Integration by parts and uniform bounds yield

$$\varepsilon \int_{\mathbb{R}} (u'')^2 = - \int_{\mathbb{R}} (J \star u - u)u'' - \int_{\mathbb{R}} f(u)u'' \tag{3.6}$$

$$= \int_{\mathbb{R}} (J \star u' - u')u' + \int_{\mathbb{R}} f'(u)(u')^2 \tag{3.7}$$

$$\leqslant C_0 \int_{\mathbb{R}} u' + C_1 \|u'\|^2_{L^2(\mathbb{R})}, \tag{3.8}$$

where C_0 and C_1 are positive constants. This ends the proof of (i).

We can now show that $1 - u \in L^2(\mathbb{R}^+)$. Again multiplying (1.9) by $1 - u$ and integrating over $\mathbb{R}$ yields

$$\varepsilon \int_{\mathbb{R}} (u')^2 - \int_{\mathbb{R}} (J \star u - u)u - c/2 + \int_{\mathbb{R}} f(u)(1-u) = 0.$$

Now using claim 3.2 and choosing R so large that $f(u) \geqslant \frac{1}{2}|f'(1)|(1-u)$ on $[R, \infty)$, we obtain

$$\tfrac{1}{2}|f'(1)| \int_R^\infty (1-u)^2 \leqslant \int_{-\infty}^\infty f(u)(1-u) \leqslant C(\|u'\|^2_{L^2(\mathbb{R})} + 1) < \infty, \tag{3.9}$$

which proves (ii). □

REMARK 3.3. Note that these estimates easily extend to solutions of a bistable problem.

Finally, we obtain some useful L^2 estimates on $J \star u - u$. Namely, we have the following lemma.

LEMMA 3.4.

$$\|J \star u - u\|_{L^2} \leqslant C\|u'\|_{L^2}.$$

Proof. Using the fundamental theorem of calculus, we have

$$\int_{-\infty}^{+\infty} J(x-y)u(y)\,\mathrm{d}y - u(x) = \int_{-\infty}^{+\infty} J(x-y)(u(y) - u(x))\,\mathrm{d}y$$
$$= \int_{-\infty}^{+\infty} J(z)z\biggl(\int_0^1 u'(x+tz)\,\mathrm{d}t\biggr)\,\mathrm{d}z.$$

By the Cauchy–Schwarz inequality, it follows that

$$\biggl|\int_{-\infty}^{+\infty} J(x-y)u(y)\,\mathrm{d}y - u(x)\biggr|^2$$
$$= \biggl(\int_{-\infty}^{+\infty} J(z)z\biggl(\int_0^1 u'(x+tz)\,\mathrm{d}t\biggr)\,\mathrm{d}z\biggr)^2$$

$$\leqslant C\biggl[\int_{-\infty}^{+\infty}\int_0^1 J(z)|z|(u')^2(x+tz)\,\mathrm{d}t\,\mathrm{d}z\int_{-\infty}^{+\infty}J(z)|z|\,\mathrm{d}z\biggr]$$
$$\leqslant C'\biggl[\int_{-\infty}^{+\infty}\int_0^1 J(z)|z|(u')^2(x+tz)\,\mathrm{d}t\,\mathrm{d}z\biggr].$$

Hence, using Tonnelli's theorem and a standard change of variables,

$$\int_{-\infty}^{+\infty}\biggl|\int_{-\infty}^{+\infty}J(x-y)u(y)\,\mathrm{d}y-u(x)\biggr|^2\mathrm{d}x$$
$$\leqslant C'\biggl[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_0^1 J(z)|z|(u')^2(x+tz)\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}x\biggr]$$
$$\leqslant C''\int_{-\infty}^{+\infty}(u')^2(s)\,\mathrm{d}s.$$

□

REMARK 3.5. Lemmas 3.1 and 3.4 imply that $f(u)\in L^2(\mathbb{R})$.

3.2. L^2 estimates for solutions of (1.5)

LEMMA 3.6. *Let $\varepsilon>0$, $r\in\mathbb{R}$, $c\in\mathbb{R}$ and $\theta\in(0,1)$. Let u be a smooth non-decreasing solution of (1.5). Then*

(i) $u', u''\in L^2(\Omega)$,

(ii) $1-u\in L^2(\mathbb{R}^+\cap\Omega)$.

Proof. By following the lines of the proof of lemma 3.1, one can easily show that $u'(+\infty)=u''(+\infty)=0$.

Next we show that $f(u)\in L^1(\Omega)$. Integrating (1.5) over $(-r,R)$ leads to

$$\varepsilon(u'(R)-u'(-r))+\int_{-r}^{R}\biggl(\int_{-r}^{+\infty}J(x-y)u(y)\,\mathrm{d}y-u(x)\biggr)\mathrm{d}x-c(u(R)-u(-r))$$
$$=-\int_{-r}^{R}(f(u)-h_r(x))\,\mathrm{d}x.$$

Assume for the moment that

$$\int_{-r}^{+\infty}J(x-y)u(y)\,\mathrm{d}y-u$$

and $h_r(x)$ are in $L^1(\Omega)$. Then, passing to the limit as $R\to+\infty$, we deduce that $f(u)\in L^1(\Omega)$. It remains to prove the following claim.

CLAIM 3.7.

$$\int_{-r}^{+\infty}J(x-y)u(y)\,\mathrm{d}y-u$$

and $h_r(x)$ are in $L^1(\Omega)$.

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Proof. Start with $h_r(x)$. By definition of $h_r(x)$, one has

$$h_r(x) = \theta \int_{-\infty}^{-r-x} J(z)\,\mathrm{d}z =: \theta j(x).$$

Since $J \geqslant 0$ and satisfies (H2), a simple computation shows that, for some $\lambda > 0$,

$$|j(x)| = \int_{-\infty}^{-r-x} J(z)\,\mathrm{d}z \leqslant \mathrm{e}^{-\lambda(r+x)} \int_{\mathbb{R}} J(z)\mathrm{e}^{-\lambda z}\,\mathrm{d}z \leqslant K\mathrm{e}^{-\lambda(r+x)} \in L^1(\Omega). \quad (3.10)$$

Now, let us prove that

$$\int_{-r}^{+\infty} J(x-y)u(y)\,\mathrm{d}y - u \in L^1(\Omega).$$

Since u is smooth, using uniform bounds and the fundamental theorem of calculus, we have

$$\begin{aligned} \biggl|\int_{-r}^{+\infty} J(x-y)u(y)\,\mathrm{d}y - u(x)\biggr| &= \biggl|\int_{-r}^{+\infty} J(x-y)(u(y)-u(x))\,\mathrm{d}y - u(x)\int_{-\infty}^{-r} J(x-y)\,\mathrm{d}y\biggr| \\ &\leqslant \biggl|\int_{-r-x}^{+\infty} J(z)(u(x+z)-u(x))\,\mathrm{d}z\biggr| + u(x)\int_{-\infty}^{-r-x} J(z)\,\mathrm{d}z \\ &\leqslant \int_{-r-x}^{+\infty} J(z)|z|\biggl(\int_0^1 u'(x+tz)\,\mathrm{d}t\biggr)\mathrm{d}z + j(x). \end{aligned}$$

Since $j \in L^1(\Omega)$, we need to prove only that

$$\Gamma(x) := \int_{-r-x}^{+\infty} J(z)|z|\biggl(\int_0^1 u'(x+tz)\,\mathrm{d}t\biggr)\mathrm{d}z \in L^1(\Omega).$$

Integrating Γ over $(-r, R)$ yields

$$\begin{aligned} \int_{-r}^{R} \Gamma(x)\,\mathrm{d}x &= \int_{-r}^{R}\int_{-r-x}^{+\infty} J(z)|z|\int_0^1 u'(x+tz)\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}x \\ &= \int_{-r}^{R}\int_{0}^{+\infty} J(z)|z|\int_0^1 u'(x+tz)\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}x \\ &\quad + \int_{-r}^{R}\int_{-r-x}^{0} J(z)|z|\int_0^1 u'(x+tz)\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}x. \end{aligned}$$

Using Tonnelli's theorem, we end up with

$$\begin{aligned} \int_{-r}^{R} \Gamma(x)\,\mathrm{d}x = \int_0^1\int_0^{+\infty} J(z)|z|\biggl(\int_{-r}^{R} u'(x+tz)\,\mathrm{d}x\biggr)\mathrm{d}z\,\mathrm{d}t \\ + \int_0^1\int_{-r-R}^{0} J(z)|z|\biggl(\int_{-r-z}^{R} u'(x+tz)\,\mathrm{d}x\biggr)\mathrm{d}z\,\mathrm{d}t. \end{aligned}$$

Hence, we obtain

$$\int_{-r}^{R} \Gamma(x)\,\mathrm{d}x = \int_0^1 \int_0^{+\infty} J(z)|z|[u(R+tz) - u(-r+tz)]\,\mathrm{d}z\,\mathrm{d}t + \int_0^1 \int_{-r-R}^{0} J(z)|z|[u(R+tz) - u(-r+(t-1)z)]\,\mathrm{d}z\,\mathrm{d}t.$$

Since $0 \leqslant u \leqslant 1$, we end up with

$$\int_{-r}^{R} \Gamma(x)\,\mathrm{d}x \leqslant 2 \int_{-\infty}^{+\infty} J(z)|z|\,\mathrm{d}z,$$

which shows that $\Gamma \in L^1(\Omega)$. □

To obtain (iii) and (iv), we can then follow the proof of lemma 3.1. □

Finally, we obtain some useful L^2 estimates on

$$\int_{-r}^{+\infty} J(x-y)u(y)\,\mathrm{d}y - u.$$

More precisely we have the following lemma.

Lemma 3.8.

$$\int_{-r}^{+\infty} J(x-y)u(y)\,\mathrm{d}y - u \in L^2(\Omega).$$

Moreover,

$$\left\| \int_{-r}^{+\infty} J(x-y)u(y)\,\mathrm{d}y - u \right\|_{L^2(\Omega)} \leqslant C(\|u'\|_{L^2(\Omega)} + \|j\|_{L^2(\Omega)}),$$

where

$$j(x) := \int_{-\infty}^{-r-x} J(z)\,\mathrm{d}z.$$

Proof. Again, using the fundamental theorem of calculus, we have

$$\int_{-r}^{+\infty} J(x-y)u(y)\,\mathrm{d}y - u(x) = \int_{-r-x}^{+\infty} J(z)z\biggl(\int_0^1 u'(x+tz)\,\mathrm{d}t\biggr)\,\mathrm{d}z - u(x)j(x).$$

By the Young and the Cauchy–Schwarz inequalities, it follows that

$$\begin{aligned} &\left| \int_{-r}^{+\infty} J(x-y)u(y)\,\mathrm{d}y - u(x) \right|^2 \\ &\quad \leqslant 2\biggl[\biggl(\int_{-r-x}^{+\infty} J(z)z\biggl(\int_0^1 u'(x+tz)\,\mathrm{d}t\biggr)\,\mathrm{d}z\biggr)^2 + u^2 j^2\biggr] \\ &\quad \leqslant 2\biggl[\int_{-r-x}^{+\infty}\int_0^1 J(z)|z|(u')^2(x+tz)\,\mathrm{d}t\,\mathrm{d}z \cdot \int_{-r-x}^{+\infty} J(z)|z|\,\mathrm{d}z + u^2 j^2\biggr] \\ &\quad \leqslant C\biggl[\int_{-r-x}^{+\infty}\int_0^1 J(z)|z|(u')^2(x+tz)\,\mathrm{d}t\,\mathrm{d}z + u^2 j^2\biggr]. \end{aligned}$$

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Define

$$\Gamma_1(x) := \int_{-r-x}^{+\infty}\int_0^1 J(z)|z|(u')^2(x+tz)\,\mathrm{d}t\,\mathrm{d}z.$$

We then have

$$\left|\int_{-r}^{+\infty} J(x-y)u(y)\,\mathrm{d}y - u(x)\right|^2 \leqslant C[\Gamma_1(x)+j^2(x)].$$

By (3.10), $j \in L^2(\Omega)$. Therefore, to complete the proof, it remains to show that Γ_1 is in $L^1(\Omega)$ and satisfies

$$\|\Gamma_1\|_{L^1(\Omega)} \leqslant C\|u'\|^2_{L^2(\Omega)}. \tag{3.11}$$

Using Tonelli's theorem,

$$\int_{-r}^{R}\Gamma_1(x)\,\mathrm{d}x = \int_0^{+\infty} J(z)|z|\biggl(\int_{-r}^{R}\int_0^1 (u')^2(x+tz)\,\mathrm{d}t\,\mathrm{d}x\biggr)\mathrm{d}z + \int_{-r-R}^{0} J(z)|z|\biggl(\int_{-r-z}^{R}\int_0^1 (u')^2(x+tz)\,\mathrm{d}t\,\mathrm{d}x\biggr)\mathrm{d}z.$$

Using a standard change of variables we get

$$\int_{-r}^{R}\Gamma_1(x)\,\mathrm{d}x = \int_0^{+\infty} J(z)|z|\biggl(\int_0^1\int_{-r+tz}^{R+tz} (u')^2(s)\,\mathrm{d}s\,\mathrm{d}t\biggr)\mathrm{d}z + \int_{-r-R}^{0} J(z)|z|\biggl(\int_0^1\int_{-r+(t-1)z}^{R+tz} (u')^2(s)\,\mathrm{d}s\,\mathrm{d}t\biggr)\mathrm{d}z.$$

Since $u' \in L^2(\Omega)$, we then have

$$\int_{-r}^{R}\Gamma_1(x)\,\mathrm{d}x \leqslant \int_0^{+\infty} J(z)|z|\biggl(\int_0^1\int_{-r}^{+\infty} (u')^2(s)\,\mathrm{d}s\,\mathrm{d}t\biggr)\mathrm{d}z + \int_{-r-R}^{0} J(z)|z|\biggl(\int_0^1\int_{-r}^{+\infty} (u')^2(s)\,\mathrm{d}s\,\mathrm{d}t\biggr)\mathrm{d}z.$$

Hence,

$$\int_{-r}^{+\infty}\Gamma_1(x)\,\mathrm{d}x \leqslant \biggl(\int_{-\infty}^{+\infty} J(z)|z|\,\mathrm{d}z\biggr)\|u'\|^2_{L^2(\Omega)},$$

which is the desired conclusion. □

4. Construction of a solution of (1.5)

In this section, we show that for any fixed $r>0$, $c\in\mathbb{R}$, $\varepsilon>0$ and for any $\theta\in(0,1)$ there exists a unique increasing solution u_r of problem (1.5). More precisely, we show the following result.

THEOREM 4.1. *Let $\varepsilon>0$, $r>0$, $c\in\mathbb{R}$ and $\theta\in(0,1)$. There then exists a unique smooth increasing solution of (1.5).*

We prove only the existence; for the proof of uniqueness see [5].

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4.1. Preliminaries

Let G be a smooth non-decreasing function such that $G(-r) = \theta$, $\mathcal{L}G \in L^2(\Omega)$ and $1 - G \in L^2(\Omega)$. For $\lambda > 0$, define

$$T_{\lambda,r} : C_0(\Omega) \cap L^2(\Omega) \to C_0(\Omega) \cap L^2(\Omega),$$
$$v \mapsto z,$$

where z is the unique solution of

$$\left.\begin{aligned} \mathcal{L}z - \lambda z = F(v,x) \quad \text{in } \Omega, \\ z(-r) = 0, \\ z(+\infty) = 0, \end{aligned}\right\} \tag{4.1}$$

where $F(v,x) = -f(v+G) - \lambda v - \mathcal{L}G - h_r(x)$. Using lemma A.4, to prove that z is well-defined, it is sufficient to show that

$$v \in L^2(\Omega) \cap C_0(\Omega) \quad \Longrightarrow \quad F(v,x) \in L^2(\Omega) \cap C_0(\Omega).$$

By definition of G, $\mathcal{L}G \in L^2(\Omega)$. By (3.10), $h_r \in L^2(\Omega)$. So we are left to prove that $f(v+G) \in L^2(\Omega)$.

Given $v \in L^2(\Omega) \cap C_0(\Omega)$, since $f(1) = 0$ and $1 - G \in L^2(\Omega)$,

$$|f(v+G)| \leqslant \|f'\|_\infty |v + G - 1| \in L^2(\Omega) \quad \text{and} \quad \lim_{+\infty} f(v+G) = 0.$$

Hence, $f(v+G) \in L^2(\Omega) \cap C_0(\Omega)$.

4.2. Iteration procedure

We claim that there exists a sequence of functions $(u_n)_{n\in\mathbb{N}}$ satisfying

$$u_0 = G \text{ and, for } n \in \mathbb{N} \setminus \{0\},$$

$$\left.\begin{aligned} \mathcal{L}u_{n+1} - \lambda u_{n+1} = -f(u_n) - \lambda u_n - h_r(x) \quad \text{in } \Omega, \\ u_{n+1}(-r) = \theta, \\ u_{n+1}(+\infty) = 1. \end{aligned}\right\} \tag{4.2}$$

We proceed as follows. Using the substitution $v_n = u_n - G$, (4.2) reduces to

$$\left.\begin{aligned} \mathcal{L}v_{n+1} - \lambda v_{n+1} = F(v_n, x) \quad \text{in } \Omega, \\ v_{n+1}(-r) = 0, \\ v_{n+1}(+\infty) = 0, \end{aligned}\right\} \tag{4.3}$$

where $F(v,x) = -f(v+G) - \lambda v - \mathcal{L}G - h_r(x)$. Therefore, we want $v_{n+1} = T_{\lambda,r} v_n$. Using § 4.1 and induction, the sequence $(v_n)_{n\in\mathbb{N}}$ is well defined provided that $v_0 \in L^2(\Omega) \cap C_0(\Omega)$. This is trivial since $v_0 = 0$.

Remark 4.2. Observe that if u_0 is a supersolution (respectively, a subsolution) of (1.5) and if λ is chosen so large that $-f - \lambda$ is non-increasing, the maximum principle (theorem A.2) implies that $(u_n)_{n\in\mathbb{N}}$ is non-increasing (respectively, non-decreasing).

4.3. Passing to the limit as $n \to \infty$

Assume that u_0 is either a supersolution or a subsolution satisfying $\theta \leqslant u_0 \leqslant 1$. Recall that the constants θ and 1 are respectively a subsolution and a supersolution of (1.5).

It follows easily by induction and the maximum principle (theorem A.2) that, for all $n \in \mathbb{N}$,

$$\theta \leqslant u_n \leqslant 1. \tag{4.4}$$

Choosing $\lambda > 0$ so large that $-f - \lambda$ is non-increasing, we prove next by induction that, given $n \in \mathbb{N}$,

$$x \to u_n(x) \text{ is a non-decreasing function.} \tag{4.5}$$

First define

$$\tilde{u}_n(x) := \begin{cases} \theta & \text{if } x \in \mathbb{R} \setminus \varOmega, \\ u_n(x) & \text{if } x \in \varOmega. \end{cases}$$

We prove that $\tilde{u}_n$ is non-decreasing, which implies (4.5). Observe that $\tilde{u}_{n+1}$ solves

$$\left.\begin{aligned} \mathcal{M}\tilde{u}_{n+1} - \lambda\tilde{u}_{n+1} &= -(f+\lambda)(\tilde{u}_n(x)) \quad \text{in } \varOmega, \\ \tilde{u}_{n+1}(-r) &= \theta, \\ \tilde{u}_{n+1}(+\infty) &= 1. \end{aligned}\right\} \tag{4.6}$$

For $n = 0$, we already know that $\tilde{u}_0$ is non-decreasing. Now fix $n \geqslant 1$ and assume that $\tilde{u}_{n-1}$ is non-decreasing. Also, given any positive τ, let $w(x) = \tilde{u}_n(x+\tau) - \tilde{u}_n(x)$. It follows from (4.6) and the assumption that $\tilde{u}_{n-1}$ and $f + \lambda$ are non-decreasing that

$$\mathcal{M}w - \lambda w \leqslant 0 \quad \text{in } \varOmega, \tag{4.7}$$

$$w(x) \geqslant 0 \quad \text{for } x \in \mathbb{R} \setminus \varOmega, \tag{4.8}$$

$$w(+\infty) = 0, \tag{4.9}$$

whence by the maximum principle, $w \geqslant 0$. In particular, $\tilde{u}_n(x+\tau) - \tilde{u}_n(x) \geqslant 0$ for any positive τ. This shows that $\tilde{u}_n$ is non-decreasing.

Using remark 4.2 and the assumption on u_0, the sequence $(u_n)_{n\in\mathbb{N}}$ is monotone. Hence, using (4.4), (4.5) and Helly's lemma, it follows that $(u_n)_{n\in\mathbb{N}}$ converges pointwise to a non-decreasing function u satisfying

$$\theta \leqslant u \leqslant 1.$$

By the dominated convergence theorem, for all $x \in \varOmega$ we have

$$\int_{-r}^{+\infty} J(x-y)u_n(y)\,\mathrm{d}y - u_n(x) \to \int_{-r}^{+\infty} J(x-y)u(y)\,\mathrm{d}y - u(x) \quad \text{as } n \to \infty.$$

Rewriting (4.2) as

$$\varepsilon u''_{n+1} - cu'_{n+1} = u_{n+1} - \int_{-r}^{+\infty} J(x-y)u_{n+1}(y)\,\mathrm{d}y - \lambda(u_n - u_{n+1}) - f(u_n) - h_r(x), \tag{4.10}$$

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observing that the right-hand side of the above equation is uniformly bounded and using elliptic regularity, we conclude that $(u_n)_{n\in\mathbb{N}}$ is bounded e.g. in $C^{1,\alpha}(\omega)$, where $\alpha \in (0,1)$ and ω is an arbitrary bounded open subset of Ω. Bootstrapping the argument implies that $(u_n)_{n\in\mathbb{N}}$ is bounded in $C^{2,\alpha}(\omega)$. Hence, $u \in C^2(\Omega)$ and we can pass to the limit in the equation to obtain that u solves

$$\mathcal{L}u + f(u) + h_r(x) = 0 \quad \text{in } \Omega. \tag{4.11}$$

Observing that $u_n(-r) = \theta$ and that $(u_n)_{n\in\mathbb{N}}$ converges pointwise to u, we easily conclude that $u(-r) = \theta$.

To complete the construction of the solution, we prove that $u(+\infty) = 1$. Indeed, since u is uniformly bounded and non-decreasing, u achieves its limit at $+\infty$. Using lemma 3.6, $u'(+\infty) = u''(+\infty) = 0$. It follows from (4.11) that $f(u(+\infty)) = 0$. Hence, $u(+\infty) = 1$. We have thus constructed an increasing solution u of (1.5), provided we have an adequate sub- or supersolution u_0 of (1.5). □

Remark 4.3. In the case where u_0 is a subsolution of (1.5), one has

$$u_0 \leqslant u \leqslant 1.$$

Hence, $u(+\infty) = 1$ is a direct consequence of $u_0(+\infty) = 1$.

The construction of a non-decreasing solution of (1.5) is now reduced to finding a good sub- or supersolution u_0 satisfying $u_0(-r) = \theta$, $\mathcal{L}u_0 \in L^2(\Omega)$ and $1 - u_0 \in L^2(\Omega)$ for fixed $r > 0$, $\theta \in (0,1)$, $\varepsilon > 0$ and $c \in \mathbb{R}$.

4.4. Construction of a solution of (1.5) for $c \leqslant \underline{\kappa}(\varepsilon)$

Assume that $r > 0$, $\theta \in (0,1)$, $\varepsilon > 0$ are fixed and let $c \leqslant \underline{\kappa}(\varepsilon)$, where $\underline{\kappa}(\varepsilon)$ is given by lemma 2.5. Recall that $\underline{w}$ given by lemma 2.5 is a subsolution of (1.5), with $c \leqslant \underline{\kappa}(\varepsilon)$.

Using lemmas 3.1–3.4 and remark 3.3 yields

$$\underline{w}'', \underline{w}', (J \star \underline{w} - \underline{w}) \in L^2(\mathbb{R}) \quad \text{and} \quad 1 - \underline{w} \in L^2(\mathbb{R}^+).$$

Hence,

$$|\mathcal{L}\underline{w}| \leqslant \varepsilon|\underline{w}''| + |c\underline{w}'| + \left| \int_{-\infty}^{+\infty} J(x-y)\underline{w}(y)\,\mathrm{d}y - \underline{w} \right| \in L^2(\Omega).$$

We then apply the previous subsection with $u_0 = \underline{w}$ to obtain a non-decreasing solution of (1.5) for $c \leqslant \underline{\kappa}(\varepsilon)$.

4.5. Construction of a solution for $c > \underline{\kappa}(\varepsilon)$

To obtain solutions for $c > \underline{\kappa}(\varepsilon)$, we argue as follows. Assume, as in the previous subsection, that $r > 0$, $\theta \in (0,1)$, $\varepsilon > 0$ are fixed and choose $c > \underline{\kappa}(\varepsilon)$.

Let u_s be the smooth non-decreasing solution of (1.5) obtained with $c = \underline{\kappa}(\varepsilon)$. Since $c > \underline{\kappa}(\varepsilon)$ and u_s is increasing, u_s is a supersolution of (1.5) with speed c. By construction, we have $u_\mathrm{s} \geqslant \theta$ and θ is a subsolution of (1.5). Therefore, to obtain a solution of (1.5), it is sufficient to prove that $\mathcal{L}u_\mathrm{s} \in L^2(\Omega)$ and $1 - u_\mathrm{s} \in L^2(\Omega)$. The latter is easily obtained using the L^2 estimates (lemmas 3.6–3.8) obtained in the previous section.

5. Construction of solutions of (1.9) for all $c \geqslant c^*(\varepsilon)$

In this section, we study problem (1.9) and prove the following theorem.

THEOREM 5.1. *Let $\varepsilon > 0$. There then exists a positive real number $c^*(\varepsilon)$ such that for all $c \geqslant c^*(\varepsilon)$ there exists a positive smooth increasing solution u_ε of (1.9). Furthermore, if $c < c^*(\varepsilon)$, then problem (1.9) has no increasing solution.*

The proof of theorem 5.1 will be split in two parts. In the first part, § 5.1, we construct a solution of problem (1.9) for a specific value of the speed $c = \bar{\kappa}(\varepsilon)$, using solutions of approximate problems constructed in the previous section and a standard limiting procedure. Then, in the second part, § 5.2, we define the minimal speed $c^*(\varepsilon)$ and construct solutions of (1.9) for speeds $c \geqslant c^*(\varepsilon)$.

5.1. Construction of one solution of (1.9) for $c = \bar{\kappa}(\varepsilon)$

In this section, we consider problem (1.5) with $c = \bar{\kappa}(\varepsilon)$, where $\kappa(\varepsilon)$ is given by lemma 2.1.

By theorem 4.1, for any real number r and any $\theta \in (0, 1)$ there exists a unique solution of (1.5). For fixed $r > 0$, we claim that the solution of (1.5) satisfies the following normalization.

CLAIM 5.2. *Fix $\varepsilon > 0$ and $r > 0$. There exists $\theta_0 \in (0, 1)$ such that the corresponding solution $u_r^{\theta_0}$ of (1.5) with $\theta = \theta_0$ satisfies the normalization $u_r^{\theta_0}(0) = \frac{1}{2}$.*

Proof of claim 5.2. Define

$$\Theta = \{\theta \mid u_r^\theta(0) > \tfrac{1}{2}\}.$$

Choosing any $\theta \geqslant \frac{1}{2}$ and observing that u_r^θ is increasing, we have $[\frac{1}{2}, 1) \subset \Theta$. The uniqueness of the solution u_r^θ and standard *a priori* estimates imply that $\theta \to u_r^\theta(0)$ is continuous over $[0, 1]$. By continuity, we can therefore conclude that

(i) either there exists a positive θ_0 such that $u_r^{\theta_0}(0) = \frac{1}{2}$

(ii) or $(0, 1) \subset \Theta$.

We show that the latter case cannot occur, which proves the claim. For this, we argue by contradiction. Suppose that $(0, 1) \subset \Theta$. Let $(\theta_n)_{n\in\mathbb{N}}$ be a sequence such that $\theta_n \to 0$. Let $(u_n)_{n\in\mathbb{N}}$ be the corresponding sequence of solutions of (1.5) with $\theta = \theta_n$. Using Helly's lemma and standard *a priori* estimates, we can extract a subsequence, still denoted $(u_n)_{n\in\mathbb{N}}$, which converges to a non-decreasing function u. Clearly, $u(-r) = 0$. Since $u_n(0) > \frac{1}{2}$, we have $u(0) \geqslant \frac{1}{2}$. Hence, u is non-trivial and satisfies

$$\left.\begin{aligned} \mathcal{L}u + f(u) &= 0 \quad \text{for } x \in (-r, +\infty), \\ u(-r) &= 0, \\ u(+\infty) &= 1. \end{aligned}\right\} \tag{5.1}$$

The fact that $u(+\infty) = 1$ is obtained using lemma 3.6 and following the arguments of § 4.3.

Observe that $\bar{w}$ given by lemma 2.1 is a supersolution of (5.1).

One can show that $\bar{w} > u$ (see [5] for details), which provides a contradiction, since $\frac{1}{2} \leqslant u(0) < \bar{w}(0) = \frac{1}{2}$. □

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With the latter normalization, we are ready for the construction of a solution of (1.9). Let $(r_n)_{n\in\mathbb{N}} = (n)_{n\in\mathbb{N}}$ and $(u_n^{\theta_n})_{n\in\mathbb{N}}$ be the sequence of solutions of the corresponding approximate problem (1.5) with $r = r_n$ and $\theta = \theta_n$, where $(\theta_n)_{n\in\mathbb{N}}$ is such that $u_n^{\theta_n}(0) = \frac{1}{2}$. Define $(h_n)_{n\in\mathbb{N}}$ by

$$h_n(x) = \theta_n \int_{-\infty}^{-r_n} J(x-y)\,\mathrm{d}y. \tag{5.2}$$

By theorem 4.1 and claim 5.2 such sequences are well defined.

Clearly, $h_n \to 0$ pointwise, as $n \to \infty$. Observe now that $(u_n^{\theta_n})_{n\in\mathbb{N}}$ is a uniformly bounded sequence of increasing functions. Therefore, using Helly's lemma, there exists a subsequence which converges pointwise to a non-decreasing function u. Since $\varepsilon > 0$, using local $C^{2,\alpha}$ estimates, up to extraction, the subsequence converges in $C^{2,\alpha}_{\mathrm{loc}}$. Therefore, $u \in C^{2,\alpha}$ and satisfies

$$\mathcal{M}u + f(u) = 0 \quad \text{in } \mathbb{R}. \tag{5.3}$$

From the normalization and the fact that $f(\frac{1}{2}) \neq 0$, u is non-trivial. Since u is increasing and bounded, u achieves its limits $l^{\pm}$ at $\pm\infty$. A standard argument, using lemma 3.1, implies that $f(l^{\pm}) = 0$. Since $l^- \leqslant \frac{1}{2}$ and $l^+ \geqslant \frac{1}{2}$, we must have $u(-\infty) = 0$ and $u(+\infty) = 1$. Therefore, we have constructed a non-trivial solution of (1.9) for $c = \bar{\kappa}(\varepsilon)$.

Remark 5.3. Observe that the existence of a supersolution $\bar{w}$ is only needed in the normalization process. Therefore, the previous construction holds with any other supersolution ψ of (1.9) such that $\psi(0) = \frac{1}{2}$.

Let us now turn our attention to the second part of the proof.

5.2. Definition of $c^*(\varepsilon)$

Define

$$c^*(\varepsilon) := \inf\{c > 0 : (1.9) \text{ admits an increasing solution}\}. \tag{5.4}$$

By the previous section, $c^*(\varepsilon)$ is well defined. Obviously, from the definition of $c^*(\varepsilon)$, there is no increasing solution to (1.9) for speeds $c < c^*(\varepsilon)$. Our goal in this subsection is to provide a solution of (1.9) for all $c \geqslant c^*(\varepsilon)$.

First we observe that (1.9) has a solution for $c = c^*(\varepsilon)$. Let $(c_n)_{n\in\mathbb{N}}$ be a minimizing sequence for $c^*(\varepsilon)$. The corresponding solutions u_n of (1.9) are increasing (and uniformly bounded by 1) so that we may apply Helly's lemma and elliptic regularity as in the previous section to conclude that $(u_n)_{n\in\mathbb{N}}$ converges to an increasing solution of (1.9) for $c = c^*(\varepsilon)$, which we denote by u_ε. Boundary conditions for u_ε are obtained as in § 5.1 using the fact that $u_\varepsilon(0) = u_n(0) = \frac{1}{2}$.

Now fix $c > c^*(\varepsilon)$ and observe that $\bar{w} := u_\varepsilon$ is a smooth increasing supersolution of (1.9) with speed c. Since $u_\varepsilon(0) = \frac{1}{2}$, by remark 5.3, the construction of § 5.1 applies. Therefore, we get a solution of (1.9) for all $c \geqslant c^*(\varepsilon)$, which completes the proof of theorem 5.1.

6. Existence of a solution for $\varepsilon = 0$

In the previous section, we were able to prove that, for every positive ε, problem (1.9) admits a semi-infinite interval of solution, i.e. for $c \geqslant c^*(\varepsilon)$ there exists a positive increasing solution of (1.9). We will see that the same holds for (1.4). The idea is to let $\varepsilon \to 0$ in (1.9) and to extract a converging sequence of solutions. The main problem is to control $c^*(\varepsilon)$ when $\varepsilon \to 0$. We prove the following lemma.

LEMMA 6.1. *For every positive ε_0, there exists $\nu_0 > 0$ such that $c^*(\varepsilon) \leqslant \nu_0$ for all $\varepsilon \in [0, \varepsilon_0)$.*

Proof. According to remark 2.2, $\bar{\kappa}(\varepsilon)$ is an non-decreasing function of ε; therefore, $\bar{\kappa}(\varepsilon) \leqslant \bar{\kappa}(\varepsilon_0)$. The conclusion easily follows from the definition of $c^*(\varepsilon)$, i.e. $c^*(\varepsilon) \leqslant \bar{\kappa}(\varepsilon)$. □

We now derive existence of a solution of (1.4) for every speed c greater than ν_0. More precisely, we have the following result.

THEOREM 6.2. *There exists ν_0 such that, for every speed c greater than ν_0, there exists a solution u of (1.4) with speed c.*

Proof. According to lemma 6.1, for ε small, say $\varepsilon \leqslant \varepsilon_0$, equation (1.9) has a solution u_ε for every $c > \nu_0$ and $\varepsilon \leqslant \varepsilon_0$. Without loss of generality we assume that for all ε, $u_\varepsilon(0) = \frac{1}{2}$. From standard *a priori* estimates, u_ε is a bounded smooth increasing function. Let $\varepsilon \to 0$ along a sequence. As in the previous section, uniform *a priori* estimates and Helly's theorem applied to u_ε provide the existence of a monotone increasing solution u of

$$[J \star u - u] - cu' + f(u) = 0 \quad \text{in } \mathbb{R}. \tag{6.1}$$

The solution cannot be trivial, according to the normalization $\frac{1}{2} = u_\varepsilon(0) \to u(0)$. Boundary conditions are obtained as in § 5. □

We define another minimal speed

$$c^{**} = \inf\{c \mid \forall c' \geqslant c \text{ (1.4) has a positive increasing solution of speed } c'\}. \tag{6.2}$$

This minimal speed is well defined according to theorem 6.2.

REMARK 6.3. A quick computation shows that

$$c^{**} \leqslant \liminf_{\varepsilon \to 0} c^*(\varepsilon).$$

Nevertheless, to complete the characterization of the set of solutions of (1.4), we have to prove that there exist no travelling-wave solutions of speed $c < c^{**}$. In other words, if we define

$$c^* = \inf\{c \mid \text{(1.4) has a positive increasing solution of speed } c\}, \tag{6.3}$$

we have to show that $c^* = c^{**}$. Clearly, we have $c^{**} \geqslant c^*$; the main problem is to prove $c^{**} \leqslant c^*$. This will be done with the help of the monotony of the speed of truncated problems and its continuous behaviour at zero. More precisely, consider the equation

$$\left.\begin{aligned} \varepsilon u'' + [J \star u - u] - cu' + (f\chi_\theta)(u) = 0 \quad \text{in } \mathbb{R}, \\ u(-\infty) = 0, \\ u(+\infty) = 1, \end{aligned}\right\} \tag{6.4}$$

where $\varepsilon \geqslant 0$, $\theta > 0$ and χ_θ is such that

(i) $\chi_\theta \in C_0^\infty(\mathbb{R})$,

(ii) $0 \leqslant \chi_\theta \leqslant 1$,

(iii) $\chi_\theta(s) \equiv 0$ for $s \leqslant \theta$ and $\chi_\theta(s) \equiv 1$ for $s \geqslant 2\theta$.

We have the following existence and uniqueness theorem.

Theorem 6.4. *Let $\varepsilon > 0$ and $\theta > 0$. There exists a unique speed $c = c_\theta(\varepsilon)$ and, up to translation, a unique smooth increasing function u_θ such that (6.4) holds. Moreover, the speed $c_\theta(\varepsilon)$ is positive and satisfies*

$$c_\theta(\varepsilon) < c^*(\varepsilon), \tag{6.5}$$

$$\lim_{\theta \to 0} c_\theta(\varepsilon) = c^*(\varepsilon). \tag{6.6}$$

Remark 6.5. Theorem 6.4 still holds for $\varepsilon = 0$, with $c^*(0) := c^*$ (where c^* is given by (6.3)). We then designate the corresponding speed for (6.4) by $c_\theta := c_\theta(0)$.

A proof of theorem 6.4 and remark 6.5 can be found in [5,6], so we do not include it here. A natural corollary of this theorem is the continuity of the speed $c_\theta(\varepsilon)$ with respect to ε and θ. Namely, we have the following corollary.

Corollary 6.6. *Under the assumptions of theorem 6.4, the mapping*

$$\begin{aligned} (0,1) \times [0,1] &\to \mathbb{R}^+, \\ (\theta, \varepsilon) &\mapsto c_\theta(\varepsilon) \end{aligned}$$

is continuous.

Assume that corollary 6.6 holds. We then conclude that $c^* = c^{**}$. Indeed, assume by contradiction that $c^* < c^{**}$. Then choose c such that $c^* < c < c^{**}$. By theorem 6.4 and remark 6.5, since $c_\theta < c^*$ for every positive θ, we have $c_\theta < c^* < c$. Fix $\theta > 0$: since $c_\theta(\varepsilon)$ is a continuous function of ε, one has on the one hand $c_\theta(\varepsilon) < c$ for ε small, say $\varepsilon \in [0, \varepsilon_0]$. On the other hand, according to remark 6.3, we may obtain

$$c_\theta(\varepsilon) < c < c^*(\varepsilon) \quad \text{for all } \varepsilon \in [0, \varepsilon_0]. \tag{6.7}$$

From the above inequality, and according to (6.6), for each $\varepsilon \in (0, \varepsilon_0]$ there exists a positive $\theta(\varepsilon) \leqslant \theta$ such that $c = c_{\theta(\varepsilon)}(\varepsilon)$. Let $u_{\theta(\varepsilon)}$ be the associated solution, normalized by $u_{\theta(\varepsilon)}(0) = \frac{1}{2}$.

Now we take a sequence (θ_n) converging to 0. From the above construction, for each n there exists $\varepsilon_n \leqslant \theta_n$ and $\theta(\varepsilon_n) \leqslant \theta_n$ such that $c = c_{\theta(\varepsilon_n)}(\varepsilon_n)$ and $u_{\theta(\varepsilon_n)}$ is the corresponding normalized solution. By construction we have

$$\theta(\varepsilon_n) \to 0.$$

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Use now, as usual, uniform *a priori* estimates and Helly's theorem to get a solution $\bar{u}$ of (1.4) with speed c.

Since $c \in (c^*, c^{**})$ is arbitrary, there exists a non-trivial solution of (1.4) for any speed $c > c^*$, which contradicts the definition of c^{**}. We summarize the above proof in the following diagram:

$$\begin{array}{ccc} c_\theta & \xrightarrow{\theta \to 0} & c^* < c^{**} \leqslant \liminf_{\varepsilon \to 0} c^{**}(\varepsilon) \\ \uparrow{\scriptstyle \varepsilon \to 0} & & \uparrow{\scriptstyle \varepsilon \to 0} \\ c_\theta(\varepsilon) & \xrightarrow{\theta \to 0} & c^*(\varepsilon) = c^{**}(\varepsilon) \end{array}$$

□

We are left with establishing the following proof.

Proof of corollary 6.6. We know from theorem 6.4 and remark 6.5 that for every $\varepsilon \geqslant 0$ and $\theta > 0$ there exists a unique solution $(u_\theta^\varepsilon, c_\theta(\varepsilon))$ of (6.4).

Fix $\varepsilon_0 \geqslant 0$ and $\theta_0 > 0$. We want to show that, for any sequence $(\varepsilon_n, \theta_n) \to (\varepsilon_0, \theta_0)$, we have $c_{\theta_n}(\varepsilon_n) \to c_{\theta_0}(\varepsilon_0)$.

Let $u_{\theta_n}^{\varepsilon_n}$ be the normalized associated solution, i.e. $u_{\theta_n}^{\varepsilon_n}(0) = \frac{1}{2}$. Since $c_\theta(\varepsilon) > 0$ and since (6.5) holds, we have $(c_{\theta_n}(\varepsilon_n))$ bounded as $(\varepsilon_n, \theta_n) \to (\varepsilon_0, \theta_0)$. We can extract a sequence of speeds, which converges to some value γ. From the *a priori* estimates on $(u_{\theta_n}^{\varepsilon_n})_{n \in \mathbb{N}}$, there also exists a subsequence which converges to a smooth function u solving the following problem with speed γ:

$$\left.\begin{aligned} \varepsilon_0 u'' + [J \star u - u] - \gamma u' + f_{\theta_0}(u) = 0 \quad \text{in } \mathbb{R}, \\ u(-\infty) = 0, \\ u(+\infty) = 1. \end{aligned}\right\} \tag{6.8}$$

According to theorem 6.4, the speed and the profile are unique. Therefore, $\gamma = c_{\theta_0}(\varepsilon_0)$. Since $(c_{\theta_n}(\varepsilon_n))$ is precompact and has a unique accumulation point, the whole sequence $(c_{\theta_n}(\varepsilon_n))$ must converge to $c_{\theta_0}(\varepsilon_0)$. □

7. Asymptotic behaviour of solutions

In this section we establish the asymptotic behaviour of the solution u near $\pm\infty$, provided J satisfies (H2). The behaviour of the function near $+\infty$ was obtained in [5], therefore we deal only with the behaviour of u near $-\infty$.

Remark 7.1. The behaviour of u near $\pm\infty$ for bistable and ignition-type nonlinearities was also obtained in [5].

We use the same strategy as in [2] and start by proving the following lemma.

Lemma 7.2. *Assume that (H1) and (H2) hold. Also assume that f is monostable and that $f'(0) > 0$. Let u be an increasing solution of (1.4). There then exists $\beta > 0$ such that*

$$\int_{-\infty}^{\infty} u(x) \mathrm{e}^{-\beta x} \, \mathrm{d}x < \infty.$$

Proof. Let $\zeta \in C^\infty(\mathbb{R})$ be a non-negative, non-decreasing function such that $\zeta \equiv 0$ in $(-\infty, -2]$ and $\zeta \equiv 1$ in $[-1, \infty)$. For $N \in \mathbb{N}$, let $\zeta_N = \zeta(x/N)$. Multiplying (1.4) by $\mathrm{e}^{-\beta x}\zeta_N$ and integrating over $\mathbb{R}$, we get

$$\int (J \star u - u)(\mathrm{e}^{-\beta x}\zeta_N) - \int cu'(\mathrm{e}^{-\beta x}\zeta_N) + \int f(u)(\mathrm{e}^{-\beta x}\zeta_N) = 0. \tag{7.1}$$

Since J is even,

$$\begin{aligned}\int (J \star u - u)(\mathrm{e}^{-\beta x}\zeta_N) &= \int (J \star (\mathrm{e}^{-\beta x}\zeta_N) - \mathrm{e}^{-\beta x}\zeta_N)u \\ &= \int u(x)\mathrm{e}^{-\beta x}\biggl(\int J(y)\mathrm{e}^{\beta y}\zeta_N(x-y)\,\mathrm{d}y - \zeta_N(x)\biggr)\,\mathrm{d}x \\ &= \int u(x)\mathrm{e}^{-\beta x}\biggl(\int J(y)\mathrm{e}^{-\beta y}\zeta_N(x+y)\,\mathrm{d}y - \zeta_N(x)\biggr)\,\mathrm{d}x \\ &\geqslant \int u(x)\mathrm{e}^{-\beta x}\biggl(\int_{-R}^{\infty} J(y)\mathrm{e}^{-\beta y}\,\mathrm{d}y\zeta_N(x-R) - \zeta_N(x)\biggr)\,\mathrm{d}x,\end{aligned} \tag{7.2}$$

where we have used the monotone behaviour of ζ_N in the last inequality and where $R > 0$ is chosen as follows: first pick $0 < \alpha < f'(0)$ and $R > 0$ so large that

$$f(u)(x) \geqslant \alpha u(x) \quad \text{for } x \leqslant -R. \tag{7.3}$$

Next, one can increase R further if necessary so that

$$\int_{-R}^{\infty} J(y)\,\mathrm{d}y > (1 - \tfrac{1}{2}\alpha).$$

By continuity we obtain, for some $\beta_0 > 0$ and all $0 < \beta < \beta_0$,

$$\int_{-R}^{\infty} J(y)\mathrm{e}^{-\beta y}\,\mathrm{d}y \geqslant (1 - \tfrac{1}{2}\alpha)\mathrm{e}^{\beta R}. \tag{7.4}$$

Combining (7.2) and (7.4), we then obtain

$$\begin{aligned}&\int (J \star u - u)(\mathrm{e}^{-\beta x}\zeta_N) \\ &\qquad \geqslant \int u(x)\mathrm{e}^{-\beta x}((1 - \tfrac{1}{2}\alpha)\mathrm{e}^{\beta R}\zeta_N(x-R) - \zeta_N(x))\,\mathrm{d}x \\ &\qquad \geqslant (1 - \tfrac{1}{2}\alpha)\int u(x+R)\mathrm{e}^{-\beta x}\zeta_N(x)\,\mathrm{d}x - \int u(x)\mathrm{e}^{-\beta x}\zeta_N(x)\,\mathrm{d}x \\ &\qquad \geqslant -\tfrac{1}{2}\alpha\int u(x)\mathrm{e}^{-\beta x}\zeta_N(x)\,\mathrm{d}x,\end{aligned} \tag{7.5}$$

where we have used the monotone behaviour of u in the last inequality.

We now estimate the second term in (7.1):

$$\begin{aligned}\int u'\zeta_N\mathrm{e}^{-\beta x}\,\mathrm{d}x &= \beta\int u\zeta_N\mathrm{e}^{-\beta x} - \int u\zeta_n'\mathrm{e}^{-\beta x}\,\mathrm{d}x \\ &\leqslant \beta\int u\zeta_N\mathrm{e}^{-\beta x}.\end{aligned} \tag{7.6}$$

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Finally, using (7.3), the last term in (7.1) satisfies

$$\int f(u)\zeta_N \mathrm{e}^{-\beta x}\,\mathrm{d}x \geqslant \alpha \int_{-\infty}^{-R} u\zeta_N \mathrm{e}^{-\beta x}\,\mathrm{d}x - C. \tag{7.7}$$

By (7.1) and (7.5)–(7.7) we then obtain

$$(\tfrac{1}{2}\alpha - c\beta)\int_{-\infty}^{-R} u\zeta_N \mathrm{e}^{-\beta x}\,\mathrm{d}x \leqslant C.$$

Choosing $\beta < \alpha/2c$ and letting $N \to \infty$ proves the lemma. □

Using lemma 7.2 it is now easy to see that $u(x) \leqslant C\mathrm{e}^{\beta x}$ for all $x \in \mathbb{R}$. Suppose indeed this is not the case and let $x_n \in \mathbb{R}$ be such that $u(x_n) > n\mathrm{e}^{\beta x_n}$.

Since $0 \leqslant u \leqslant 1$, we may pick a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $x_{n_{k+1}} < x_{n_k} - 1$. But, since u is non-decreasing,

$$\begin{aligned}\int u(x)\mathrm{e}^{-\beta x}\,\mathrm{d}x &\geqslant \sum_{k\geqslant 1}\int_{x_{n_k}}^{x_{n_k-1}} u(x)\mathrm{e}^{-\beta x}\,\mathrm{d}x \\ &\geqslant \sum_{k\geqslant 1} n_k \int_{x_{n_k}}^{x_{n_k}-1} \mathrm{e}^{\beta(x_{n_k}-x)}\,\mathrm{d}x \\ &\geqslant \sum_{k\geqslant 1}\frac{n_k}{\beta}(1-\mathrm{e}^{-\beta}) = \infty.\end{aligned}$$

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Appendix A.

Here we prove some maximum principles and existence results for solutions of linear problems associated with the operator $\mathcal{L}$ defined by (1.7).

THEOREM A.1 (strong maximum principle for $\mathcal{L}$). *Let $\varepsilon \geqslant 0$, $r > 0$, $c \in \mathbb{R}$ and let $\mathcal{L}$ be defined by (1.7) on $\Omega = (-r, +\infty)$. Assume further that* $\mathrm{Int}(\mathrm{supp}\, J) \cap \Omega^- \neq \emptyset$, *where $\Omega^- = (-r, 0)$.*

Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy

$$\mathcal{L}u \geqslant 0 \in \Omega \quad (\textit{respectively, } \mathcal{L}u \leqslant 0 \in \Omega). \tag{A 1}$$

Then u may not achieve a positive maximum (respectively, negative minimum) inside Ω without being constant.

Similarly, we introduce the following theorem.

THEOREM A.2 (strong maximum principle for $\mathcal{L} + h_r(x)$). *Let $\varepsilon \geqslant 0$, $r > 0$, $c \in \mathbb{R}$, $\theta \in (0,1)$ and $\mathcal{L}$, $h_r(x)$ defined by (1.7) on $\Omega = (-r, +\infty)$.*

Assume further that $\mathrm{Int}(\mathrm{supp}\, J) \cap \Omega^- \neq \emptyset$*, where $\Omega^- = (-r, 0)$.*

Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy

$$\left.\begin{aligned} \mathcal{L}u &\geqslant -h_r(x) \in \Omega \qquad (\textit{respectively, } \mathcal{L}u \leqslant -h_r(x) \in \Omega),\\ u(-r) &= \theta,\\ u &\geqslant \theta \in \Omega \qquad (\textit{respectively, } u \leqslant \theta \in \Omega). \end{aligned}\right\} \tag{A 2}$$

Then u may not achieve a positive maximum (respectively, negative minimum) inside Ω without being constant.

Proof of theorem A.1. We argue by contradiction. Assume that u is non-constant and that it achieves a positive maximum at some point $x_0 \in \Omega$. Since

$$\int_{\mathbb{R}} J(z)\,\mathrm{d}z = 1,$$

we can rewrite (1.7) as

$$\mathcal{L}u = \varepsilon u'' + \int_{-r}^{+\infty} J(x-y)[u(y) - u(x)]\,\mathrm{d}y - cu' - d(x)u, \tag{A 3}$$

with

$$d(x) = \int_{-\infty}^{-r} J(x-y)\,\mathrm{d}y.$$

At the point x_0 of (positive) maximum, we have on the one hand

$$\varepsilon u''(x_0) \leqslant 0, \quad \int_{-r}^{+\infty} J(x_0 - y)[u(y) - u(x_0)]\,\mathrm{d}y \leqslant 0 \quad \text{and} \quad -d(x_0)u(x_0) \leqslant 0. \tag{A 4}$$

On the other hand, by (A 1),

$$\varepsilon u''(x_0) + \int_{-r}^{+\infty} J(x_0 - y)[u(y) - u(x_0)]\,\mathrm{d}y - \bar{d}(x_0)u(x_0) \geqslant 0. \tag{A 5}$$

Hence, $\varepsilon u''(x_0) = d(x_0)u(x_0) = 0$ and

$$\int_{-r}^{\infty} J(x_0 - y)[u(y) - u(x_0)]\,\mathrm{d}y = 0. \tag{A 6}$$

If $J > 0$ in $\mathbb{R}$, we conclude directly that $u(y) = u(x_0)$ for all $y \in \Omega$, contradicting our original assumption.

In general, J is a continuous non-negative even function with $\mathrm{Int}(\mathrm{supp}\, J) \cap \Omega^- \neq \emptyset$. In particular, there exist constants $0 < a < b$ such that $[-b, -a] \cup [a, b] \subset \mathrm{supp}(J)$ and $[a, b] \subset \Omega$. We deduce from (A 6) that

$$u(y) = u(x_0) \quad \text{for all } y \in (x_0 + [-b, -a] \cup [a, b]) \cap \Omega.$$

Let $z = x_0 + b$ and observe that $u(z) = u(x_0)$. We may thus argue as above and conclude that $u(y) = u(z)$ for all $y \in (z + [-b, -a] \cup [a, b]) \cap \Omega$. In particular,

$$u(y) = u(x_0) \quad \text{for all } y \in (x_0 + [0, b - a]) \cap \Omega.$$

Repeating the argument with $z = x_0 + a$, we obtain $u(y) = u(x_0)$ for all $y \in (x_0 + [-(b-a), 0]) \cap \Omega$. Thus,

$$u(y) = u(x_0) \quad \text{for all } y \in (x_0 + [-(b-a), b-a]) \cap \Omega.$$

Applying the above successively with $x_0 + b - a$ and $x_0 - (b-a)$ in place of x_0, we find that $u(y) = u(x_0)$ for all $y \in x_0 + [-2(b-a), 2(b-a)] \cap \Omega$. Working inductively, we conclude that $u \equiv u(x_0)$ in Ω, which contradicts our original assumption. □

Proof of theorem A.2. Define

$$\tilde{u}(x) := \begin{cases} u(x) & \text{in } \Omega, \\ \theta & \text{in } \mathbb{R} \setminus \Omega, \end{cases}$$

and observe that we can rewrite (A 2) as

$$\begin{aligned} \mathcal{M}\tilde{u} &\geqslant 0 \quad \text{in } \Omega, \\ \tilde{u}(x) &\geqslant \theta \quad \text{in } \Omega, \end{aligned}$$

where $\mathcal{M}\tilde{u} = \varepsilon\tilde{u}'' + [J \star \tilde{u} - \tilde{u}] - c\tilde{u}'$.

We argue by contradiction and assume that $\tilde{u}$ achieves a positive maximum at some point $x_0 \in \Omega$ and is non-constant. Since $u(x) \geqslant \theta$ in Ω we have $u(x_0) > \theta$. Working as in the proof of theorem A.1 we find that $u \equiv u(x_0)$ on $\bar{\Omega}$, which is a contradiction. □

Remark A.3. Theorems A.2 and A.1 remain valid when replacing $\mathcal{L}$ by $\mathcal{L} - d_0$, where d_0 is any positive constant.

Next, we provide an elementary lemma in order to construct solutions of Dirichlet problems associated with $\mathcal{L}$.

Lemma A.4. *Let $d_0 > 0$, $\varepsilon > 0$, $r > 0$, $c \in \mathbb{R}$ and let $\mathcal{L}$ be defined by (1.7) on $\Omega = (-r, +\infty)$.*

Assume further that $\mathrm{Int}(\mathrm{supp}\, J) \cap \Omega^- \neq \emptyset$, *where $\Omega^- = (-r, 0)$. Given $f \in C_0(\Omega) \cap L^2(\Omega)$, there exists a unique solution $u \in C^2(\Omega) \cap L^2(\Omega)$ of*

$$\left.\begin{aligned} \mathcal{L}u - d_0 u = f \quad &\text{in } \Omega, \\ u(-r) = 0&, \\ u(+\infty) = 0&. \end{aligned}\right\} \tag{A 7}$$

Proof. Uniqueness follows from the maximum principle. Let $X = H_0^1(\Omega)$ and define the bilinear form $\mathcal{A}(u, v)$ for $u, v \in X$ by

$$\begin{aligned} \mathcal{A}(u, v) = \varepsilon \int_\Omega u'v' + \frac{1}{2} \int_\Omega \int_\Omega J(x-y)(u(y) - u(x))(v(y) - v(x)) \,\mathrm{d}y \,\mathrm{d}x \\ - c \int_\Omega u'v + \int_\Omega d(x)uv, \end{aligned}$$

where

$$d(x) = \int_{-\infty}^{-r} J(x-y) \,\mathrm{d}y + d_0.$$

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To solve (A 7), we just need to find $u \in X$ such that $\mathcal{A}(u,v) = \int_\Omega uv$ for all $v \in X$. We will show that $\mathcal{A}$ is coercive and continuous in X. Existence will then be given by the Lax–Milgram lemma. Clearly,

$$\mathcal{A}(u,u) \geqslant \varepsilon \int_\Omega (u')^2 - c\int_\Omega u'u + d_0 \int_\Omega u^2 = \varepsilon \int_\Omega (u')^2 + d_0 \int_\Omega u^2.$$

Thus, $\mathcal{A}$ is coercive in X. It remains to prove the continuity of $\mathcal{A}$. Let ϕ and ψ be two smooth functions with compact support in Ω:

$$|\mathcal{A}(\phi,\psi)| \leqslant \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} J(x-y)|\phi(y)-\phi(x)|\,|\psi(y)-\psi(x)|\,\mathrm{d}y\,\mathrm{d}x.$$

By the fundamental theorem of calculus and the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
|\mathcal{A}(\phi,\psi)| &\leqslant \int_{\mathbb{R}^2}\int_0^1\int_0^1 J(z)z^2|\phi'(x+tz)|\,|\psi'(x+sz)|\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s \\
&\leqslant \int_{\mathbb{R}}\int_{[0,1]^2} J(z)z^2 \int_{\mathbb{R}} |\phi'(h)|\,|\psi'(h+(s-t)z)|\,\mathrm{d}h\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}t \\
&\leqslant \int_{\mathbb{R}}\int_{[0,1]^2} J(z)z^2\,\mathrm{d}z\,\mathrm{d}t\,\mathrm{d}s \|\phi'\|_{L^2(\mathbb{R})}\|\psi'\|_{L^2(\mathbb{R})} \\
&\leqslant \biggl(\int_{\mathbb{R}} J(z)z^2\,\mathrm{d}z\biggr)\|\phi'\|_{L^2(\mathbb{R})}\|\psi'\|_{L^2(\mathbb{R})},
\end{aligned}$$

which shows the continuity of $\mathcal{A}$. □

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