DISAPPOINTMENT AVERSION PREMIUM PRINCIPLE

BY

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ABSTRACT

In recent years, the determination of premium principle under various non-expected utility frameworks has become popular, such as the pioneer works by Tsanakas and Desli (2003) and Kaluszka and Krzeszowiec (2012). We here revisit the problem under another prevalent behavioral economic theory, namely the Disappointment Aversion (DA) Theory proposed by Gul (1991). In this article, we define and study the properties of the *DA premium principle*, which builds on the equivalent utility premium principle. We derive various properties of this premium principle, such as non-negative and no unjustified risk loading, translation invariance, monotonicity, convexity, positive (non-)homogeneity, independent (non-)additivity, comonotonic (non-)additivity and monotonicity with respect to the extent of disappointment. A generalized Arrow–Pratt approximation is also established. Explicit representations of the premium principle are obtained for linear and exponential utilities, and they reveal that the premium principle proposed echoes the capital reserve regulatory requirement in practice.

KEYWORDS

Disappointment aversion theory, equivalent utility premium principle, generalized Arrow–Pratt approximation, explicit representations, capital reserve regulatory requirement.

1. INTRODUCTION

In their celebrated work, von Neumann and Morgenstern (2007) were the first to develop the Expected Utility Theory (EUT). The theory stated that if decision makers behave in accordance with the four axioms of (1) completeness, (2) transitivity, (3) independence, and (4) continuity on the preference ordering, they will essentially make a choice that gives the "maximum" possible expected utility. Despite its elegance, their theory has constantly been challenged against

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its failure to explain the actual behavior of decision makers. For example, the Allais Paradox by Allais (1953) could be one of the most illuminating. Therefore, it is crucial to revisit some fundamental problems such as the determination of insurance pricing principle under the non-EUT framework.

Under an insurance agreement, the insured pays a premium in advance for a protection $I(\cdot)$ from the insurer against an insurable risk X. The premium set by the insurer is used to fund the cost of economic capital to compensate the future contingent liabilities. This plays the same role of the recently popular risk measure[1], being a functional of loss variables, in finance. As depicted by Young (2006), three methods were employed by actuaries to establish the premium principle, namely: (I) the ad-hoc approach; (II) the characterization method; and (III) the economic method. For the ad-hoc approach, it assumes firstly that the premium principle takes a particular functional form, for example, net premium principle, expected value premium principle, variance premium principle, and exponential premium principle. The principles are then examined through their properties, for instance, no unjustified risk loading, translation invariance, monotonicity, and (sub-)additivity. Also see Gerber (1979) and Goovaerts *et al.* (1984) for comprehensive studies on various existing premium principles including those mentioned above. For the characterization method, it is the converse of the ad-hoc approach. In other words, this method identifies premium principles that satisfy certain reasonable axioms. For example, Wang *et al.* (1997) proved that if the premium principle satisfies properties, such as comonotonic additivity, the premium takes the form, now known as distortion premium principle, similar to the net premium principle but with a concave distortion on the survival function. For the economic method, which was originated by Bühlmann (1970), the premium is priced not only according to the nature of the loss, but also with the consideration on the risk aversion and surplus of the insurer or insured; in particular, Bühlmann (1970) proposed the zero-utility premium principle H^S (resp. H^B) for the insurer (resp. insured) which is defined as the unique solution of the following equation. For *insurer*:

$$u(w^S) = \mathbb{E}[u(w^S + H^S(I(X)) - I(X))]; \tag{1}$$

and respectively, for *insured*:

$$\mathbb{E}\left[v(w^B - X)\right] = \mathbb{E}[v(w^B - H^B(I(X)) - X + I(X))], \tag{2}$$

where u, v are the respective utility functions of the insurer and insured; w^S, w^B are the respective initial wealths.

Due to the deficiencies of the EUT, Bühlmann (1970)'s model has recently been revisited, via the concept of indifference pricing (see Carmona (2009), Denuit *et al.* (2006) and Tsanakas and Desli (2003) for an overview), through the following indifference pricing equations. For *insurer*:

$$U^S(w^S) = U^S\left(w^S + H^S(I(X)) - I(X)\right); \tag{3}$$

and respectively, for *insured*:

$$U^B\left(w^B - X\right) = U^B\left(w^B - H^B(I(X)) - X + I(X)\right), \tag{4}$$

where U^S and U^B are the respective general preference operators of the insurer and insured. The premium principle determined by (3) or (4) is also named as the equivalent utility premium principle, in which the "utility" stands for the general decision making criterion instead of the measurement of individual satisfaction.

In our present work, we revisit the insurance pricing problem, by using the third method via the equivalent utility premium principle, under one of the most popular non-EUT models. The advantage of using economic method is to provide a convenient link with the contemporary prevalent theory of risk measures through suitable choices of preference operators. For example, the linear and exponential utilities in zero-utility premium principle give rise to the net premium principle and the exponential premium principle, or entropic risk measure, respectively. Furthermore, with a small enough variance of the loss, the exponential premium principle can be connected to the variance premium principle as a reasonably good approximation. This perspective has been echoed in the works such as Denuit *et al.* (2006): setting of risk measures in practice should take into account the utility concepts in order to avoid any deficiencies. Besides, the economic method can result in a direct connection with behavioral economics; indeed, the premium should be calibrated not only by the nature of the loss itself, but also by the risk aversion and other potential behavioral preference of the insurers. For example, with the choice of preference functionals as specified by the Dual Theory by Yaari (1987), the distortion premium principle can be obtained. Tsanakas and Desli (2003) derived a premium principle called the generalized expected utility premium principle based on the rank-dependent EUT developed by Quiggin (1993); in particular, their proposed premium principle is a combination of the utility function of X and the distortion on the loss survival distribution. Besides, Kaluszka and Krzeszowiec (2012) studied the equivalent utility premium principle based on the Cumulative Prospect Theory developed by Kahneman and Tversky (1992).

A popular and representative non-EUT is the *Disappointment Theory*, which was originated by Bell (1985) and further studied by Loomes and Sugden (1986). This theory deploys that decision makers experience disappointment if, prior to resolving of the lottery, they took a certain level of expectation on the value of the lottery, which eventually turned out to be greater than the actual outcome. The theory further asserts that the decision maker takes his/her potential disappointment feeling into account for ordering the preference. The models proposed by Bell (1985) and Loomes and Sugden (1986) regard the mathematical expectation as the reference point. As an alternative to their models, *DA Theory* as developed by Gul (1991) provides an intuitive explanation of the Allais Paradox by introducing the weaker independence axiom; in particular, he decomposed lotteries into elation and disappointment parts with respect to a certainty equivalent instead of the mathematical expectation as considered by

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Bell (1985) and Loomes and Sugden (1986). Gul (1991) also showed that, under his newly proposed axiom, the preference ordering of a decision maker can be induced by the corresponding DA utility (see Section 2 for details).

There are a number of recent advances implementing the psychological disappointment factor into a variety of applied problems in economics and finance. Gollier and Muermann (2010) considered the portfolio choice and insurance problem using the Loomes and Sugden disappointment theory. Under the DA theory, treatments on the optimal portfolio selection problem are provided in the literature. See applications in Ang *et al.* (2005) to explain Equity Premium Puzzle and the analytical solution for small risks in Saltari and Travaglini (2010). Recently, Cheung *et al.* (2015) solved classical optimal insurance problems under various disappointment theories. Nevertheless, to the best of our knowledge, a comprehensive study of the determination of the insurance pricing principle under the disappointment framework is still absent in the literature.

In this article, we first introduce the DA premium principle from the insurer's perspective under the DA Theory by Gul (1991). A comprehensive study on its properties and explicit representations with linear and exponential utilities will be discussed. We observe that our proposed premium principle H can capture the important behavioral factor: the DA coefficient of the insurer. Generalization of the Arrow–Pratt approximation under the present DA framework will also be provided. Our results demonstrate that equivalent utility premium principle under the DA model is more realistic than that under the EUT, in the sense that the premium principle representations under the linear and exponential utilities unveil an interesting connection with the capital reserve regulatory requirement in practice. Our work complements Tsanakas and Desli (2003) and Kaluszka and Krzeszowiec (2012), who are the first laying down the zero-utility premium principle under behavioral frameworks.

The organization of our article is as follows. In Section 2, the DA utility under the framework in Gul (1991) together with its existence and elementary properties will be introduced. Under this DA setting, the corresponding premium principle H for the insurer, together with its equivalent formulations, properties, Arrow–Pratt type approximation, and convex risk measure representation, will be established in Section 3. When the underlying utility u is either linear or exponential, the explicit representations of H, with a concrete identification of the underlying family of probability measures, will be provided in Sections 4.1 and 4.2. Finally, further numerical examples will be illustrated.

2. Disappointment aversion utility

In this section, we recall a particular preference operator, namely, the DA utility proposed by Gul (1991), which will be used to define our premium principle in Section 3. We shall also study some basic properties of the DA utility in this section.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ be the convex space of all essentially bounded $\mathcal{F}$-measurable random variables. Let $u : \mathbb{R} \to \mathbb{R}$ be an underlying utility function which is strictly increasing, concave (not necessarily strictly concave), and vanishing at 0, i.e. $u(0) = 0$.

Definition 2.1. *Given the underlying utility function u and a constant $D \in [0, 1]$, the* DA utility *of the payoff $Y \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is the real number $\mathcal{U}(Y)$ such that:*

i. for $D = 0$,

$$\mathcal{U}(Y) := \operatorname{essinf} u\,(Y)\,;$$

ii. for $D \in (0, 1]$, $\mathcal{U}(Y)$ is implicitly defined through the equation:

$$\mathcal{U}(Y) = \mathbb{E}^{\mathbb{P}}\left[u(Y) + \left(\frac{1}{D} - 1\right)(u(Y) - \mathcal{U}(Y))\,\mathbb{1}_{\{u(Y)<\mathcal{U}(Y)\}}\right]. \tag{5}$$

In fact, the DA utility is the composite function $u \circ \mu$ of the underlying utility function u and the DA certainty equivalent μ introduced by Gul (1991). When $D = 1$, which corresponds to the scenario that the disappointment effect does not affect the insurer's decision, Equation (5) reduces to

$$u(\mu(Y)) = \mathbb{E}^{\mathbb{P}}[u(Y)]\,.$$

This is used to define the usual certainty equivalent under the EUT. Note that Equation (5) can be written as

$$\mathcal{U}(Y) = \mathbb{E}^{\mathbb{P}}\left[u(Y) - \left(\frac{1}{D} - 1\right)(\mathcal{U}(Y) - u(Y))_+\right], \tag{6}$$

from which we interpret the DA utility $\mathcal{U}(Y)$ as the expected penalized utility of the payoff Y, where the penalization is imposed on the downside risk with respect to $\mathcal{U}(Y)$ itself through the scaling factor $\frac{1}{D} - 1$. We also remark that when D decreases, the insurer is more disappointment averse, and vice versa. Therefore, the parameter D captures the degree of DA of the insurer and thus it is called the *DA coefficient*.

Proposition 2.2. *For any $Y \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{U}(Y)$ has the following properties:*

i. $\mathcal{U}(Y)$ uniquely exists in $[\operatorname{essinf} u\,(Y)\,, \operatorname{esssup} u\,(Y)]$.
ii. If $Y = c$ where $c \in \mathbb{R}$, $\mathcal{U}(Y) = u(c)$.
iii. $\mathcal{U}(Y)$ preserves stochastic order[2].

Proof. Statement (i) follows from a simple application of the Intermediate Value Theorem. Statement (ii) follows from the definition. For statement (iii), assume $Y \preceq_{\text{st}} Z$ but $\mathcal{U}(Y) > \mathcal{U}(Z)$. Since

$$f(y) := u(y) - \left(\frac{1}{D} - 1\right)(\mathcal{U}(Y) - u(y))_+$$

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is increasing in y, $Y \preceq_{\text{st}} Z$ implies that

$$\mathcal{U}(Y) = \mathbb{E}^{\mathbb{P}}[f(Y)] \leq \mathbb{E}^{\mathbb{P}}[f(Z)] < \mathbb{E}^{\mathbb{P}}\left[u(Z) - \left(\frac{1}{D} - 1\right)(\mathcal{U}(Z) - u(Z))_+\right]$$
$$= \mathcal{U}(Z).$$

which leads to a contradiction. ■

3. Disappointment aversion premium principle

From now on, we fix a non-negative bounded random variable $X \in L^\infty_+(\Omega, \mathcal{F}, \mathbb{P})$, which models the insurable loss of the insured. Its distribution and survival functions are denoted by F_X and S_X respectively. Let $I(X) \in L^\infty_+(\Omega, \mathcal{F}, \mathbb{P})$ be the insurance indemnity chosen by the insured or offered by the insurer. We now formally recall the definition of the equivalent utility premium principle for the insurer.

Definition 3.1. *Let $w \geq 0$ be the initial wealth and $U : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be a general preference functional of the insurer. The* equivalent utility premium $H : L^\infty_+(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$*, to be charged by the insurer for the indemnity I is defined as the solution of*

$$U(w) = U\left(w + H(I(X)) - I(X)\right). \tag{7}$$

Without loss of generality, we take X in place of $I(X)$ in the study of the properties of H in the rest of the article.

Remark 3.2. *The functional H defined in Definition 3.1 bears different terminologies. In the insurance and actuarial science context, H is coined as the principle of equivalent utility; while, in the finance literature, H is referred as the indifference price or ask-price. The underlying philosophy of Equation (7) is that H is the minimal possible price to be charged so that the pricing agent with an initial wealth w is not worse off by undertaking the financial obligation. This actuarial indifference determination for the insurance premium has been thoroughly studied in the recent decades, see, for instance, Denuit et al. (2006), Heilpern (2003), Kaluszka and Krzeszowiec (2012), Laeven and Goovaerts (2008), Tsanakas and Desli (2003, 2005) and Tsanakas (2008). In the field of finance, the indifference price is defined through a more general equation, which allows the pricing agent to invest the initial fortune and the reward arising from transaction if any, in order to fulfill the future financial obligation. For studies of indifference pricing in financial markets, see Henderson and Hobson (2009) in Carmona (2009), Elliott and Van Der Hoek (2004) and Elliott and Siu (2010, 2011).*

Definition 3.3. *Given the underlying utility function u, the DA coefficient $D \in$* $[0, 1]$*, and the initial wealth w of the insurer, the* DA premium principle *H is defined through Equation (7) with the decision criterion U replaced by the DA utility $\mathcal{U}$ in Definition 2.1. More precisely, given $X \in L^\infty_+(\Omega, \mathcal{F}, \mathbb{P})$, $H(X)$ is the*

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real number that satisfies the following equation:

$$\mathcal{U}(w) = \mathcal{U}\left(w + H(X) - X\right). \tag{8}$$

Notice that $H(X) = \operatorname{esssup} X$ when $D = 0$. In the following, we focus only on the nontrivial case of $D \in (0, 1]$ when considering properties of H. The next proposition gives two equivalent formulations for the DA premium principle. Both formulations will be useful in the discussion afterwards.

Proposition 3.4. *For any $X \in L^\infty_+(\Omega, \mathcal{F}, \mathbb{P})$, the DA premium principle $H(X)$ is implicitly defined by:*

i.

$$v_{u,D,w}(w) = \mathbb{E}^{\mathbb{P}}\left[v_{u,D,w}(w + H(X) - X)\right], \tag{9}$$

where $v_{u,D,w}(x) := u(x) - \left(\frac{1}{D} - 1\right)(u(w) - u(x))_+$; or equivalently,

ii.

$$u(w) = \mathbb{E}^{\mathbb{Q}_D}\left[u\left(w + H(X) - X\right)\right], \tag{10}$$

where $\mathbb{Q}_D$ is the $\mathbb{P}$-equivalent probability measure on $(\Omega, \mathcal{F})$ defined by

$$\frac{d\mathbb{Q}_D}{d\mathbb{P}} := \frac{D\mathbb{1}_{\{X \le H(X)\}} + \mathbb{1}_{\{X > H(X)\}}}{D\mathbb{P}(X \le H(X)) + \mathbb{P}(X > H(X))}. \tag{11}$$

Proof. Replace Y by $w + H(X) - X$ in Equations (5) or (6) and apply Equation (8) and Property 2 in Proposition 2.2. Notice that $v_{u,D,w}(w) = u(w)$ for any $D \in (0, 1]$. ■

In Equation (9), given the underlying utility function u, $D \in (0, 1]$ and initial wealth w, the function $v_{u,D,w}$ behaves like a utility function as it is strictly increasing and concave. The function $v_{u,D,w}$ is obtained from the underlying utility function u where the portion to the left of the initial wealth w is distorted downwards, resulting in a non-differentiable point at w. Figure 1 shows the comparison of the underlying utility u and the function $v_{u,D,w}$ when u is linear or exponential. Proposition 3.4 depicts that the DA premium principle is indeed a zero-utility premium principle with either replacing

i. the utility function u by the new utility function $v_{u,D,w}$; or,
ii. the real-world probability measure $\mathbb{P}$ by the $\mathbb{P}$-equivalent probability measure $\mathbb{Q}_D$.

Hence, properties of the DA premium can be derived from that of the zero-utility premium principle. In the sequel, for any $X \in L^\infty_+(\Omega, \mathcal{F}, \mathbb{P})$, if the dependence of $H(X)$ on D is emphasized, $H(X)$ will be written as $H_D(X)$.

Theorem 3.5. *For any $X \in L^\infty_+(\Omega, \mathcal{F}, \mathbb{P})$,*

i. (Unique existence) $H(X)$ uniquely exists;
ii. (Law invariance) $H(X)$ is law-invariant;
iii. (No unjustified risk loading) if $X = c$ where $c \in \mathbb{R}^+$, $H(c) = c$;

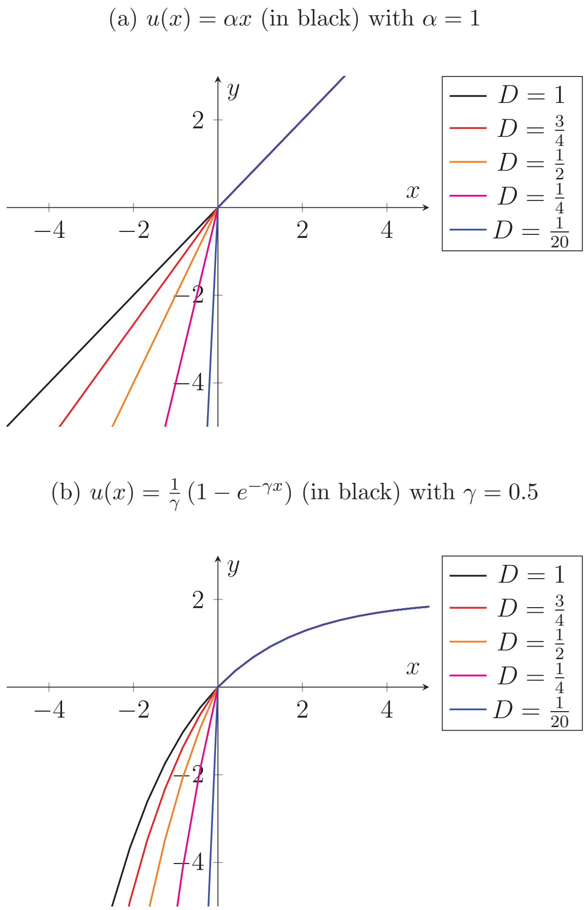

FIGURE 1: Distorted utility function $v_{u,D,w}(x)$ with (i) $w = 0$ and (ii) $D = 1, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{20}$. (a) $u(x) = \alpha x$ (in black) with $\alpha = 1$. (b) $u(x) = \frac{1}{\gamma}\left(1 - e^{-\gamma x}\right)$ (in black) with $\gamma = 0.5$. (Color online)

iv. *(Risk loading and no rip-off)* $\mathbb{E}^{\mathbb{P}}[X] \leq H(X) \leq \operatorname{esssup} X$;

v. *$H(X)$ is convex, increasing and translation invariant, i.e.*
   a. *(Translation invariance) for any* $c \in \mathbb{R}^+$, $H(X + c) = H(X) + c$;
   b. *(Monotonicity in stochastic order sense)* $X \preceq_{st} Y \Rightarrow H(X) \leq H(Y)$;
   c. *(Convexity) for any* $\lambda \in [0, 1]$, $H(\lambda X + (1 - \lambda)Y) \leq \lambda H(X) + (1 - \lambda)H(Y)$;

vi. *(Fatou property) for any* $X_n \in L^{\infty}_{+}(\Omega, \mathcal{F}, \mathbb{P})$ *such that* $\sup_n \|X_n\|_\infty < \infty$ *and* $X_n \to X$ *in probability*, $H(X) \leq \liminf H(X_n)$.

Proof. The arguments are straightforward by (I) using the definition, (II) applying Proposition 2.2, and (III) using the standard argument commonly found in the literature, for instance, Gerber (1979), Goovaerts *et al.* (1984) and Jouini *et al.* (2006). ■

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Theorem 3.6. *For any* $X \in L^{\infty}_{+}(\Omega, \mathcal{F}, \mathbb{P})$,

i. *(Positive (non-)homogeneity)* $H(X)$ *is positively homogeneous if and only if* u *is (I) linear, or (II) piecewise linear with only one non-differentiable point at* $x = w$.
ii. *(Independent (non-)additivity)* $H(X)$ *is independent additive if and only if (I)* $D = 1$*, and (II)* u *is linear or exponential.*
iii. *(Comonotonic (non-)additivity)* $H(X)$ *is comonotonic additive if and only if (I)* $D = 1$*, and (II)* u *is linear.*
iv. *(Monotonicity with respect to DA coefficient) If* $0 \leq D_1 \leq D_2 \leq 1$, $H_{D_1}(X) \geq H_{D_2}(X)$.

Proof.

i. By Theorem A.1 and the fact that v is concave, $H(X)$ is positively homogeneous if and only if

$$v_{u,D,w}(x) = \begin{cases} u(w) + \beta(x - w), & \text{if } x \geq w \\ u(w) + \gamma(x - w), & \text{if } x < w \end{cases}$$

for some $0 < \beta \leq \gamma$ depending on D. When u is linear or piecewise linear with only one non-differentiable point at $x = w$, it is readily checked that $v_{u,D,w}$ is sufficiently and necessarily in this form.
ii. By the statement 4e in Section 5.4 in Gerber (1979), $H(X)$ is independent additive if and only if $v_{u,D,w}$ is linear or exponential. However, the latter statement is clearly impossible if $D < 1$. Therefore, the characterization requires further that $D = 1$.
iii. The comonotonic additivity of $H(X)$ implies that $H(X)$ is positively homogeneous. Indeed, if $H(X)$ is comonotonic additive,

$$H(rX) = rH(X) \quad \text{for any} \quad r \in \mathbb{Q}^{+}.$$

Since any risk measure H is Lipschitz continuous with respect to the supremum norm of $X \in L^{\infty}_{+}(\Omega, \mathcal{F}, \mathbb{P})$ (see Lemma 4.3 in Föllmer and Schied (2011)),

$$H(\alpha X) = \alpha H(X) \quad \text{for any} \quad \alpha \geq 0.$$

Therefore, by comonotonic additivity, positive homogeneity and the concavity of $v_{u,D,w}$, if X, Y are comonotonic and $\alpha \in [0, 1]$,

$$\begin{aligned} v_{u,D,w}(w) &= \mathbb{E}^{\mathbb{P}}\left[v_{u,D,w}\left(w + H(\alpha X + (1-\alpha)Y) - (\alpha X + (1-\alpha)Y)\right)\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[v_{u,D,w}\left(w + \alpha H(X) + (1-\alpha)H(Y) - (\alpha X + (1-\alpha)Y)\right)\right] \\ &\geq \alpha\mathbb{E}^{\mathbb{P}}\left[v_{u,D,w}(w + H(X) - X)\right] + (1-\alpha)\mathbb{E}^{\mathbb{P}}\left[v_{u,D,w}(w + H(Y) - Y)\right] \\ &= v_{u,D,w}(w). \end{aligned}$$

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Since (X, Y) is an arbitrary comonotonic pair, the inequality above is in fact an equality which holds true only when $v_{u,D,w}$ is linear. Finally, $v_{u,D,w}$ being linear is equivalent to that $D = 1$ and u is linear.

iv. Consider $D_1, D_2 \in (0, 1]$ first. Assume, to the contrary, that $H_{D_1}(X) < H_{D_2}(X)$. Since $D_1 \leq D_2$, we have $v_{u,D_1,w}(\cdot) \leq v_{u,D_2,w}(\cdot)$ and thus

$$\begin{aligned} u(w) &= v_{u,D_1,w}(w) \\ &= \mathbb{E}^{\mathbb{P}}\left[v_{u,D_1,w}\left(w + H_{D_1}(X) - X\right)\right] \\ &\leq \mathbb{E}^{\mathbb{P}}\left[v_{u,D_2,w}\left(w + H_{D_1}(X) - X\right)\right] \\ &< \mathbb{E}^{\mathbb{P}}\left[v_{u,D_2,w}\left(w + H_{D_2}(X) - X\right)\right] \\ &= v_{u,D_2,w}(w) \\ &= u(w), \end{aligned}$$

which leads to a contradiction. Finally, by Property (iv) in Theorem 3.5, $H_0(X) \geq H_D(X)$ for any $D \in [0, 1]$.

■

Notice that as the DA premium principle H is defined through indifference arguments, it possesses nice properties, for instance, Properties (iii)–(vi) in Theorem 3.5. However, it is not independent nor comonotonic additive unless $D = 1$ and the underlying utility u takes some specific forms. The monotonicity structure of H with respect to D depicts that as the insurer feels more disappointment averse, a greater premium $H(X)$ is charged to compensate the feeling of any potential disappointment.

Next, we generalize the classical Arrow–Pratt approximation from the EUT framework to the DA framework. In other words, we establish the first and second-order approximations of the DA premium for small-deviated and absolutely continuous loss X.

Proposition 3.7. *Assume that u is twice differentiable and ϵ is a bounded continuous random variable with zero mean and variance σ_ϵ^2. Suppose that $X = \mathbb{E}^{\mathbb{P}}[X] + k\epsilon$, where k is a positive constant such that X is positive. Let $h(k)$ be its associated DA premium which is also twice differentiable at* 0*. Then, when k is close to 0,*

$$h(k) \approx \mathbb{E}^{\mathbb{P}}[X] + h'(0)k + \frac{1}{2}h''(0)k^2,$$

where $h'(0)$ and $h''(0)$ are determined by the following equations:

$$h'(0) = \left(\frac{1}{D} - 1\right)\mathbb{E}^{\mathbb{P}}\left[\left(\epsilon - h'(0)\right)_+\right], \tag{12}$$

$$h''(0) = -\frac{u''(w)}{u'(w)} \frac{h'(0)^2 + \sigma_\epsilon^2 + \left(\frac{1}{D} - 1\right)\mathbb{E}^{\mathbb{P}}\left[\left(\epsilon - h'(0)\right)_+^2\right]}{1 + \left(\frac{1}{D} - 1\right)\mathbb{P}\left(\epsilon > h'(0)\right)}. \tag{13}$$

Proof. Let $v'_\pm$ be the right hand and left hand derivatives of v respectively. The right-hand side of (9) can be approximated, through Taylor series expansion on $v_{u,D,w}$, by

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[\left(v(w) + v'_-(w)\left(h(k) - \mathbb{E}^{\mathbb{P}}[X] - k\epsilon\right) + \frac{1}{2}v''_-(w)\left(h(k) - \mathbb{E}^{\mathbb{P}}[X] - k\epsilon\right)^2\right)\mathbb{1}_{\{h(k)-\mathbb{E}^{\mathbb{P}}[X]<k\epsilon\}}\right] \\
&\quad + \mathbb{E}^{\mathbb{P}}\left[\left(v(w) + v'_+(w)\left(h(k) - \mathbb{E}^{\mathbb{P}}[X] - k\epsilon\right) + \frac{1}{2}v''_+(w)\left(h(k) - \mathbb{E}^{\mathbb{P}}[X] - k\epsilon\right)^2\right)\mathbb{1}_{\{h(k)-\mathbb{E}^{\mathbb{P}}[X]>k\epsilon\}}\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\left(u(w) + \frac{1}{D}u'(w)\left(h(k) - \mathbb{E}^{\mathbb{P}}[X] - k\epsilon\right) + \frac{1}{2D}u''(w)\left(h(k) - \mathbb{E}^{\mathbb{P}}[X] - k\epsilon\right)^2\right)\mathbb{1}_{\{h(k)-\mathbb{E}^{\mathbb{P}}[X]<k\epsilon\}}\right] \\
&\quad + \mathbb{E}^{\mathbb{P}}\left[\left(u(w) + u'(w)\left(h(k) - \mathbb{E}^{\mathbb{P}}[X] - k\epsilon\right) + \frac{1}{2}u''(w)\left(h(k) - \mathbb{E}^{\mathbb{P}}[X] - k\epsilon\right)^2\right)\mathbb{1}_{\{h(k)-\mathbb{E}^{\mathbb{P}}[X]>k\epsilon\}}\right] \\
&= u(w) + u'(w)\left(h(k) - \mathbb{E}^{\mathbb{P}}[X]\right) - u'(w)\left(\frac{1}{D} - 1\right)\mathbb{E}^{\mathbb{P}}\left[\left(k\epsilon + \mathbb{E}^{\mathbb{P}}[X] - h(k)\right)_+\right] \\
&\quad + \frac{1}{2}u''(w)\left[\left(h(k) - \mathbb{E}^{\mathbb{P}}[X]\right)^2 + k^2\sigma_\epsilon^2\right] + \frac{1}{2}u''(w)\left(\frac{1}{D} - 1\right)\mathbb{E}^{\mathbb{P}}\left[\left(k\epsilon + \mathbb{E}^{\mathbb{P}}[X] - h(k)\right)_+^2\right].
\end{aligned}$$

Therefore, by (9), the DA premium satisfies an approximated equation

$$\begin{aligned}
&u'(w)\left(h(k) - \mathbb{E}^{\mathbb{P}}[X]\right) - u'(w)\left(\frac{1}{D} - 1\right)\mathbb{E}^{\mathbb{P}}\left[\left(k\epsilon + \mathbb{E}^{\mathbb{P}}[X] - h(k)\right)_+\right] \\
&\quad + \frac{1}{2}u''(w)\left[\left(h(k) - \mathbb{E}^{\mathbb{P}}[X]\right)^2 + k^2\sigma_\epsilon^2\right] \\
&\quad + \frac{1}{2}u''(w)\left(\frac{1}{D} - 1\right)\mathbb{E}^{\mathbb{P}}\left[\left(k\epsilon + \mathbb{E}^{\mathbb{P}}[X] - h(k)\right)_+^2\right] = 0.
\end{aligned}$$

Substituting $\mathbb{E}^{\mathbb{P}}[X] + h'(0)k + \frac{1}{2}h''(0)k^2$ in place of $h(k)$ and neglecting $O(k^3)$ terms yield

$$\begin{aligned}
&\left[u'(w)h'(0) - u'(w)\left(\frac{1}{D} - 1\right)\mathbb{E}^{\mathbb{P}}\left[\left(\epsilon - h'(0)\right)_+\right]\right]k \\
&\quad + \frac{1}{2}\left[u'(w)h''(0)\left(1 + \left(\frac{1}{D} - 1\right)\mathbb{P}\left(\epsilon > h'(0)\right)\right)\right. \\
&\quad \left. + u''(w)\left(h'(0)^2 + \sigma_\epsilon^2 + \left(\frac{1}{D} - 1\right)\mathbb{E}^{\mathbb{P}}\left[\left(\epsilon - h'(0)\right)_+^2\right]\right)\right]k^2 = 0.
\end{aligned}$$

By comparing the coefficients of k and k^2, we obtain Equations (12) and (13). ■

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Note that $h'(0)$ in (12) exists by Intermediate Value Theorem. Once $h'(0)$ is determined, $h''(0)$ can then be obtained from (13). In addition, $h'(0)$ is independent of the underlying utility, or in other words, the risk aversion of the insurer. On the other hand, $h'(0)$ decreases in D, which means that as the insurer is more disappointment averse, the first-order term of the DA premium increases. For $h''(0)$, it is influenced by the DA coefficient via the factor in (13). Using the integration by-parts and standard limiting arguments, one can show that $h''(0)$ tends to zero as D approaches zero. Therefore, if the insurer is extremely disappointment averse, the second-order term vanishes. Note also that when $D = 1$, we have $h'(0) = 0$ and

$$h''(0) = -\frac{u''(w)}{u'(w)}\sigma_\epsilon^2 =: h''_{\mathrm{EUT}}(0),$$

which is the classical Arrow–Pratt coefficient under the EUT.

Example 3.8. *Suppose that* ϵ *is uniformly distributed on* $[-\delta, \delta]$*, for some* $\delta > 0$ *close to 0, one readily obtains:*

$$h'(0) = \delta \times \frac{1-\sqrt{D}}{1+\sqrt{D}},$$

and

$$h''(0) = -\frac{u''(w)}{u'(w)}\sigma_\epsilon^2 \times \frac{4\sqrt{D}}{\left(1+\sqrt{D}\right)^2} = h''_{EUT}(0) \times \frac{4\sqrt{D}}{\left(1+\sqrt{D}\right)^2}.$$

Finally, we remark that, for any given u, D and w, since the DA premium principle is a convex risk measure satisfying the Fatou property, by Föllmer and Schied (2011), Theorem 4.33, $H(X)$ can be represented as

$$H(X) = \sup_{\mathbb{Q}\in\mathcal{M}(\mathbb{P})} \left(\mathbb{E}^{\mathbb{Q}}[X] - \alpha_{\min}(\mathbb{Q})\right), \tag{14}$$

where $\mathcal{M}(\mathbb{P})$ is the class of all probability measures on $(\Omega, \mathcal{F})$ which are absolutely continuous with respect to $\mathbb{P}$, and $\alpha_{\min}$ is the minimal penalty function defined on $\mathcal{M}(\mathbb{P})$. In addition, since $H(X)$ is law-invariant, $H(X)$ can be further represented as

$$H(X) = \sup_{\mu\in\mathcal{M}((0,1])} \left(\int_{(0,1]} AV@R_\lambda(X)\mu(d\lambda) - \beta_{\min}(\mu)\right)^3 \tag{15}$$

where $\mathcal{M}((0,1])$ is the family of all probability measures defined on (0, 1], and the minimal penalty function $\beta_{\min}$ defined on $\mathcal{M}((0,1])$ satisfies

$$\begin{aligned}\beta_{\min}(\mu) &= \sup_{X\in\mathcal{A}_H}\int_{(0,1]} AV@R_\lambda(X)\mu(d\lambda)\\ &= \sup_{X\in L_+^\infty(\Omega,\mathcal{F},\mathbb{P})}\int_{(0,1]}(AV@R_\lambda(X)\mu(d\lambda)-H(X)),\end{aligned}$$

for any $\mu\in\mathcal{M}((0,1])$, where $\mathcal{A}_H$ is an acceptable class of losses induced by H:

$$\{X\in L_+^\infty(\Omega,\mathcal{F},\mathbb{P})\,|\,H(X)\leq 0\}.$$

Although the maximizers in (14) and (15) should depend on the utility, disappointment extent, and initial fortune, the representations are too general and they give no clue on how they are connected with u, D and w. In the next section, further properties and more meaningful representations will be demonstrated when u is linear (Section 4.1) and when u is exponential (Section 4.2). In particular, the representations justify our earlier claim that the equivalent utility premium principle under the DA model is more realistic than that under the EUT.

4. Explicit examples of the DA premium principle H

In this section, we focus on two commonly used utilities, namely, linear and exponential. Our goals are to derive general explicit representation for the corresponding DA premium principle, and study other pertinent qualitative properties with respect to the disappointment coefficient D.

4.1. When u is linear

By exploiting the linearity of the underlying utility function, the family of Radon–Nikodym derivatives in (14) can be identified concisely and parameterized by the DA coefficient. We shall see that the DA premium principle adopted by a risk-neutral insurer depends solely on his/her degree of aversion toward disappointment but not the marginal utility. For simplicity, we let $u(x)=Ax$ for some $A>0$.

Theorem 4.1. *For any* $X\in L_+^\infty(\Omega,\mathcal{F},\mathbb{P})$,

$$H(X)=\sup_{\mathbb{Q}_\theta\in\mathcal{Q}_1}\mathbb{E}^{\mathbb{Q}_\theta}[X]=\sup_{\theta\in supp\,X}\frac{D\mathbb{E}^{\mathbb{P}}\left[X\mathbb{1}_{\{X\leq\theta\}}\right]+\mathbb{E}^{\mathbb{P}}\left[X\mathbb{1}_{\{X>\theta\}}\right]}{D\mathbb{P}(X\leq\theta)+\mathbb{P}(X>\theta)},\tag{16}$$

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where

$$\mathcal{Q}_1 := \left\{ \mathbb{Q}_\theta : \mathcal{F} \to [0,1] \middle| \mathbb{Q}_\theta \ll \mathbb{P} \quad \textit{and} \quad \exists \theta \in \textit{supp}\, X \textit{ such that } \frac{d\mathbb{Q}_\theta}{d\mathbb{P}} \right.$$

$$\left. = \frac{D\mathbb{1}_{\{X \le \theta\}} + \mathbb{1}_{\{X > \theta\}}}{D\mathbb{P}(X \le \theta) + \mathbb{P}(X > \theta)} \right\}.$$

Proof. By Equation (10),

$$H(X) = \mathbb{E}^{\mathbb{Q}}[X], \tag{17}$$

where $\mathbb{Q} : \mathcal{F} \to [0, 1]$ is defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{D\mathbb{1}_{\{X \le H(X)\}} + \mathbb{1}_{\{X > H(X)\}}}{D\mathbb{P}(X \le H(X)) + \mathbb{P}(X > H(X))}.$$

Note that $\mathbb{Q}$ defined here is absolutely continuous with respect to $\mathbb{P}$, i.e. $\mathbb{Q} \in \mathcal{M}(\mathbb{P})$. Also, Equation (23) coincides with representation (14). Therefore, we obtain representation (16). ■

The $\mathbb{P}$-equivalent probability measure $\mathbb{Q}_\theta$ above partitions the sample space Ω into the elation part $\{X \le \theta\}$ and the disappointment part $\{X > \theta\}$. Theorem 4.1 depicts that when the insurer is risk neutral, H can be written as the maximum of the expectation of X among all such scenarios $\mathbb{Q}_\theta$. Most importantly, by denoting

$$\beta_{D,\theta} := \frac{D}{D + (1 - D) S_X(\theta)},$$

the DA premium (16) can be rewritten as

$$\sup_{\theta \in \operatorname{supp} X} \beta_{D,\theta} \mathbb{E}^{\mathbb{P}}[X] + (1 - \beta_{D,\theta}) \mathrm{AV@R}_{S_X(\theta)}(X). \tag{18}$$

Therefore, the DA premium principle is indeed a weighted average of the expectation under the real-world measure $\mathbb{P}$ and the Average Value-at-Risk (AV@R) of X, where both the weighting and the risk level for the AV@R depend on the value of D. Alternatively, the DA premium principle (16) can be further rewritten into a risk-loading form:

$$(1 + \Lambda_D)\, \mathbb{E}^{\mathbb{P}}[X], \tag{19}$$

where the risk loading Λ_D is given by

$$\Lambda_D := \sup_{\theta \in \operatorname{supp} X} (1 - \beta_{D,\theta}) \frac{\mathrm{AV@R}_{S_X(\theta)}(X) - \mathbb{E}^{\mathbb{P}}[X]}{\mathbb{E}^{\mathbb{P}}[X]}. \tag{20}$$

This generalizes the classical expected value premium principle with a constant risk loading. The risk loading Λ_D here naturally takes into account the tail risk exposure of the loss and the DA of the insurer. Note that for any $\theta \in \operatorname{supp} X$,

$1 - \beta_{D,\theta}$ is decreasing in D, and hence, the risk loading Λ_D is also decreasing in D. Therefore, the more disappointment averse the insurer is, the larger the risk loading and hence the premium will be because the premium leans toward the tail risk measure AV@R more. In particular, when $D = 0$, the DA premium becomes $\operatorname{esssup} X$, which coincides with the one in Section 3. We also remark that the maximizer in (20) is not necessarily unique, and based on the proof above, $H(X)$ is indeed a maximizer.

Recall that with the zero-utility premium principle, which is $D = 1$ in our setting, the premium charged by a risk-neutral insurer is simply the net premium $\mathbb{E}^{\mathbb{P}}[X]$. However, this will lead to a definite ruin in the long run. Therefore, a risk loading is justifiable in practice, and it is natural that it takes the tail risk exposure into consideration. On the other hand, the DA premium principle derived from the DA framework, even when the insurer is risk neutral, automatically calibrates the risk loading in accordance with the tail risk exposure as measured by AV@R and the extent of the DA of the insurer. The idea is that the DA behavior toward huge loss that could be potentially incurred is similar to the central idea of the capital reserve in the insurance regulation. Under the disappointment framework, the risk level of AV@R used depends on the DA of the insurer rather than being specified by regulators.

Corollary 4.2. *Let $n \geq 2$ and $0 \leq a_1 < \cdots < a_n$. Suppose that $\mathbb{P}(X = a_1) = p_1, \ldots, \mathbb{P}(X = a_n) = p_n$, where $0 < p_1, \ldots, p_n < 1$ and $p_1 + \cdots + p_n = 1$. Then*

$$H(X) = \max_{i=1,\ldots,n-1} \frac{D\sum_{r=1}^{i} a_r p_r + \sum_{r=i+1}^{n} a_r p_r}{D\sum_{k=1}^{i} p_r + \sum_{k=i+1}^{n} p_r} = \mathbb{E}^{\mathbb{Q}^*}[X], \tag{21}$$

where $\mathbb{Q}^* : \sigma\{X\} \to [0, 1]$ is defined via

$$\mathbb{Q}^*(X = a_r) := \begin{cases} \frac{Dp_r}{\sum_{k=1}^{i^*}(Dp_k) + \sum_{k=i^*+1}^{n} p_k} & \text{for } r = 1, \ldots, i^*, \\ \frac{p_r}{\sum_{k=1}^{i^*}(Dp_k) + \sum_{k=i^*+1}^{n} p_k} & \text{for } r = i^*+1, \ldots, n, \end{cases}$$

where $i^* \in \{1, \ldots, n-1\}$ is the unique index such that $g(i^*+1) < D \leq g(i^*)$. Here $g(1) := 1$, $g(i) := \frac{\sum_{k=i+1}^{n}(a_k - a_i)p_k}{\sum_{k=1}^{i-1}(a_i - a_k)p_k}$ for $i = 2, \ldots, n-1$, and $g(n) := 0$.

Proof. By the expression (16),

$$\begin{aligned} H(X) &= \sup_{\theta \in \operatorname{supp} X} \frac{D\mathbb{E}^{\mathbb{P}}\left[X\mathbb{1}_{\{X\leq\theta\}}\right] + \mathbb{E}^{\mathbb{P}}\left[X\mathbb{1}_{\{X>\theta\}}\right]}{D\mathbb{P}(X \leq \theta) + \mathbb{P}(X > \theta)} \\ &= \max_{i=1,\ldots,n-1} \frac{D\sum_{r=1}^{i} a_r p_r + \sum_{r=i+1}^{n} a_r p_r}{D\sum_{k=1}^{i} p_r + \sum_{k=i+1}^{n} p_r}. \end{aligned} \tag{22}$$

Denote by i^* the maximizer of the rightmost expression of (22). It is readily checked that it satisfies $a_{i^*} \leq H(X) < a_{i^*+1}$. Then substituting the right-hand side of (22) into the inequality yields $g(i^*+1) < D \leq g(i^*)$. ■

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In fact, the expression of H in Corollary 4.2 can be obtained from Equation (10) directly by considering the cases when H lies in different portions between a_1 and a_n. Expression (21) provides a numerical method for calculating the DA premium, see Example 4.4 below.

Corollary 4.3. *For any $X \in L_+^\infty(\Omega, \mathcal{F}, \mathbb{P})$, the DA premium $H_D(X)$ for linear utility u is convex in D.*

Proof. For any $\theta \in \operatorname{supp} X$,

$$\frac{D\mathbb{E}^{\mathbb{P}}\left[X\mathbb{1}_{\{X\leq\theta\}}\right]+\mathbb{E}^{\mathbb{P}}\left[X\mathbb{1}_{\{X>\theta\}}\right]}{D\mathbb{P}(X\leq\theta)+\mathbb{P}(X>\theta)},$$

is convex in D; indeed, its derivative with respect to D equals

$$\frac{\mathbb{E}^{\mathbb{P}}\left[X\mathbb{1}_{\{X\leq\theta\}}\right]\mathbb{P}(X>\theta)-\mathbb{E}^{\mathbb{P}}\left[X\mathbb{1}_{\{X>\theta\}}\right]\mathbb{P}(X\leq\theta)}{\left(D\mathbb{P}(X\leq\theta)+\mathbb{P}(X>\theta)\right)^2},$$

in which the numerator is negative because

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\left[X\mathbb{1}_{\{X\leq\theta\}}\right]\mathbb{P}(X>\theta)-\mathbb{E}^{\mathbb{P}}\left[X\mathbb{1}_{\{X>\theta\}}\right]\mathbb{P}(X\leq\theta)\\&\quad\leq\ \theta\mathbb{P}(X\leq\theta)\mathbb{P}(X>\theta)-\theta\mathbb{P}(X>\theta)\mathbb{P}(X\leq\theta)=0.\end{aligned}$$

Therefore, its derivative is an increasing function in D. Finally, we use the fact that the supremum of convex functions is also convex. ■

This corollary suggests that the marginal premium charged by the insurer increases with his/her degree of DA.

As opposed to the EUT where H reduces to the net premium principle when the underlying utility u is linear, the DA premium principle H cannot be solved explicitly unless X is discrete because $v_{u,D,w}$, being distorted from the linear u, is now nonlinear instead. In the following, we demonstrate how $H(X)$ can be approximated through discretization arguments when X is continuous.

Example 4.4. *To approximate the DA premium for a bounded absolutely continuous loss X, we bound its distribution function F_X from above and below respectively by stepwise distribution functions $\underline{F}_n$ and $\overline{F}_n$ with constant jump sizes $1/n$. Define $\underline{X}_n$ and $\overline{X}_n$ to be discrete random variables corresponding to distribution functions $\underline{F}_n$ and $\overline{F}_n$ respectively, so that $\underline{X}_n \leq_{st} X \leq_{st} \overline{X}_n$. In general, for any unbounded absolutely continuous loss X, we replace $F_X(\cdot)$ by $\frac{F_X(\cdot)\wedge\alpha}{\alpha}$ for some α close to 1.*

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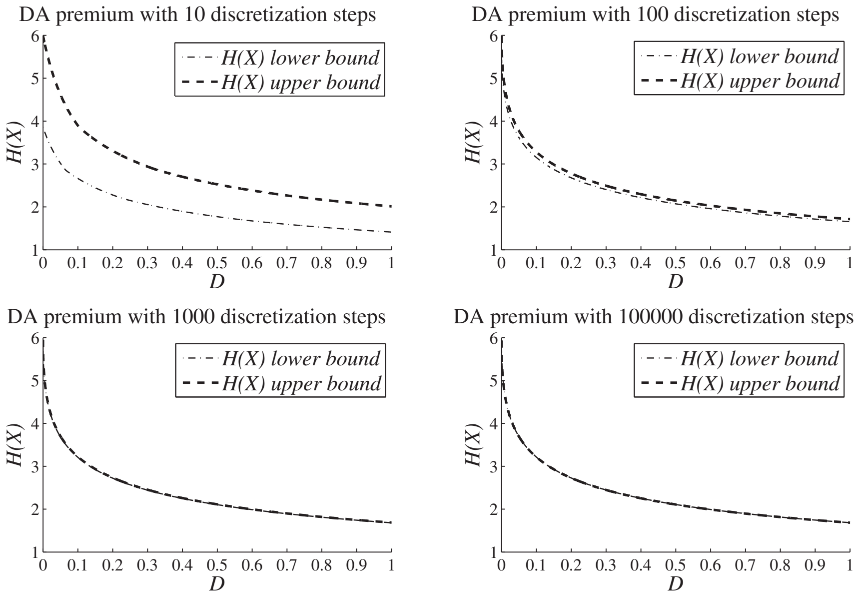

FIGURE 2: DA premium versus DA coefficient with exponential loss, mean $\lambda = 2$, and $\alpha = 0.95$, with various discretization steps n.

Let X be exponentially distributed with mean 2. *Assume that the DA coefficient D is* 0.4. *The following table shows the values of $H(\underline{X}_n)$ and $H(\overline{X}_n)$ for $\alpha = 0.95$ corrected up to 4 decimal places.*

n	$H(\underline{X}_n)$	$H(\overline{X}_n)$
10	1.8954	2.7022
100	2.2146	2.2955
1, 000	2.2506	2.2587
10, 000	2.2542	2.2550
100, 000	2.2546	2.2546

Therefore, the DA premium $H(X)$ approximately equals 2.2546. *Figure 2 shows the relation between the DA premium $H_D(X)$ and D.*

4.2. When u is exponential: H as an entropic risk measure

In this section, we assume that the insurer is risk averse modeled by the exponential utility: $u(x) = \frac{1}{A}(1 - e^{-Ax})$ for some $A > 0$. The results could be established in a similar manner as in Section 4.1, and hence we briefly outline and omit the details of the proofs in this section.

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Theorem 4.5. *For any* $X \in L^{\infty}_{+}(\Omega, \mathcal{F}, \mathbb{P})$,

$$H(X) = \sup_{\mathbb{Q}_\theta \in \mathcal{Q}_2} \frac{1}{A} \ln\left(\mathbb{E}^{\mathbb{Q}_\theta}\left[e^{AX}\right]\right) = \sup_{\theta \in supp\, X} \frac{1}{A} \ln\left(\mathbb{E}^{\mathbb{Q}_\theta}\left[e^{AX}\right]\right)$$
$$= \sup_{\mathbb{Q}_\theta \in \mathcal{Q}_2} \sup_{\mathbb{S} \in \mathcal{M}(\mathbb{Q}_\theta)} \left(\mathbb{E}^{\mathbb{S}}[X] - \frac{1}{A} J\left(\mathbb{S}|\mathbb{Q}_\theta\right)\right)$$
$$= \sup_{\theta \in supp\, X} \sup_{\mathbb{S} \in \mathcal{M}(\mathbb{Q}_\theta)} \left(\mathbb{E}^{\mathbb{S}}[X] - \frac{1}{A} J\left(\mathbb{S}|\mathbb{Q}_\theta\right)\right),$$

where

$$\mathcal{Q}_2 := \left\{\mathbb{Q}_\theta : \mathcal{F} \to [0, 1] \middle| \mathbb{Q}_\theta \ll \mathbb{P} \text{ and } \exists \theta \in \mathbb{R} \text{ such that } \frac{d\mathbb{Q}_\theta}{d\mathbb{P}} = \frac{D\mathbb{1}_{\{X \le \theta\}} + \mathbb{1}_{\{X > \theta\}}}{D\mathbb{P}(X \le \theta) + \mathbb{P}(X > \theta)}\right\},$$

$$\mathcal{M}(\mathbb{Q}_\theta) := \left\{\mathbb{S} : \mathcal{F} \to [0, 1] \middle| \mathbb{S} \ll \mathbb{Q}_\theta\right\},$$

and

$$J\left(\mathbb{S}|\mathbb{Q}_\theta\right) := \begin{cases} \mathbb{E}^{\mathbb{Q}_\theta}\left[\frac{d\mathbb{S}}{d\mathbb{Q}_\theta} \ln\left(\frac{d\mathbb{S}}{d\mathbb{Q}_\theta}\right)\right], & \text{if } \mathbb{S} \ll \mathbb{Q}_\theta, \\ +\infty, & \text{otherwise}, \end{cases}$$

where $J\left(\mathbb{S}|\mathbb{Q}_\theta\right)$ *is the relative entropy of* $\mathbb{S}$ *with respect to* $\mathbb{Q}_\theta$.

Corollary 4.6. *Let* $n \ge 2$ *and* $0 \le a_1 < \cdots < a_n$. *Suppose that* $\mathbb{P}(X = a_1) = p_1, \ldots, \mathbb{P}(X = a_n) = p_n$, *where* $0 < p_1, \ldots, p_n < 1$ *and* $p_1 + \cdots + p_n = 1$. *Then*

$$H(X) = \max_{i=1,\ldots,n-1} \frac{1}{A} \ln\left(\frac{D\sum_{r=1}^{i} e^{Aa_r} p_r + \sum_{r=i+1}^{n} e^{Aa_r} p_r}{D\sum_{k=1}^{i} p_r + \sum_{k=i+1}^{n} p_r}\right) = \frac{1}{A} \ln\left(\mathbb{E}^{\mathbb{Q}^*}\left[e^{AX}\right]\right),$$

where $\mathbb{Q}^* : \sigma\{X\} \to [0, 1]$ *is defined via*

$$\mathbb{Q}^*(X = a_r) := \begin{cases} \frac{Dp_r}{\sum_{k=1}^{i^*}(Dp_k) + \sum_{k=i^*+1}^{n} p_k} & \text{for } r = 1, \ldots, i^*, \\ \frac{p_r}{\sum_{k=1}^{i^*}(Dp_k) + \sum_{k=i^*+1}^{n} p_k} & \text{for } r = i^* + 1, \ldots, n, \end{cases}$$

where $i^* \in \{1, \ldots, n-1\}$ *is the unique index such that* $h(i^* + 1) < D \le h(i^*)$; *here* $h(1) := 1$, $h(i) := \frac{\sum_{k=i+1}^{n}(e^{Aa_k} - e^{Aa_i})p_k}{\sum_{k=1}^{i-1}(e^{Aa_i} - e^{Aa_k})p_k}$ *for* $i = 2, \ldots, n-1$, *and* $h(n) := 0$.

The expression in Corollary 4.6 will be exploited as a numerical method for finding the DA premium in Example 4.8 below. We remark that although the expression in Theorem 4.5 and Corollary 4.6 are similar to the distortion-exponential premium principle proposed in Tsanakas and Desli (2003), the probability measures in the family of the DA premium principle have different

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distortion functions depending on θ while the probability distortion in Tsanakas and Desli (2003) is fixed a prior. The expressions in Theorem 4.5 and Corollary 4.6 can also be rewritten into the form:

$$\frac{1}{A}\ln\left\{\left[1+\sup_{\theta\in\operatorname{supp}X}(1-\beta_{D,\theta})\frac{\mathrm{AV@R}_{S_X(\theta)}\left(e^{AX}\right)-\mathbb{E}^{\mathbb{P}}\left[e^{AX}\right]}{\mathbb{E}^{\mathbb{P}}\left[e^{AX}\right]}\right]\mathbb{E}^{\mathbb{P}}\left[e^{AX}\right]\right\}.$$

Using the expression in Theorem 4.5 or Corollary 4.6, we can also prove

Corollary 4.7. *For any $X\in L_+^\infty(\Omega,\mathcal{F},\mathbb{P})$, the DA premium $H_D(X)$ for exponential utility u is convex in D.*

Proof. For any $\theta\in\operatorname{supp}X$, the derivative of $H_D(X)$ with respect to D is

$$\frac{1}{A}\frac{\mathbb{E}^{\mathbb{P}}\left[X\mathbb{1}_{\{X\le\theta\}}\right]\mathbb{P}(X>\theta)-\mathbb{E}^{\mathbb{P}}\left[X\mathbb{1}_{\{X>\theta\}}\right]\mathbb{P}(X\le\theta)}{\left(D\mathbb{E}^{\mathbb{P}}\left[X\mathbb{1}_{\{X\le\theta\}}\right]+\mathbb{E}^{\mathbb{P}}\left[X\mathbb{1}_{\{X>\theta\}}\right]\right)\left(D\mathbb{P}(X\le\theta)+\mathbb{P}(X>\theta)\right)}\ (\le 0)$$

which is an increasing function in D. ■

Example 4.8. *Consider the same setting as in Example 4.4. We take the risk aversion parameter $A=0.5$.*

n	$H(\underline{X}_n)$	$H(\overline{X}_n)$
10	2.3082	3.7311
100	2.8570	3.0095
1, 000	2.9244	2.9396
10, 000	2.9312	2.9327
100, 000	2.9319	2.9321

Therefore, the DA premium $H(X)$ approximately equals 2.9320. The joint effect of D and A on $H(X)$ is demonstrated in Figure 3. When the insurer is more risk averse, the disappointment effect on the DA premium $H(X)$ diminishes.

5. Conclusion

In our present article, we presented a generalized zero-utility pricing principle, the DA premium principle, for an insurance policy under the DA Theory developed by Gul (1991). We established its properties, such as translation invariance, monotonicity, convexity, positive (non-)homogeneity, comonotonic (non-)additivity, independent (non-)additivity and we further came up with its convex risk measure representation. We also generalized the Arrow–Pratt approximation under the DA Theory. Explicit representations and discrete approximations of the premium principle for common modeling examples were also obtained.

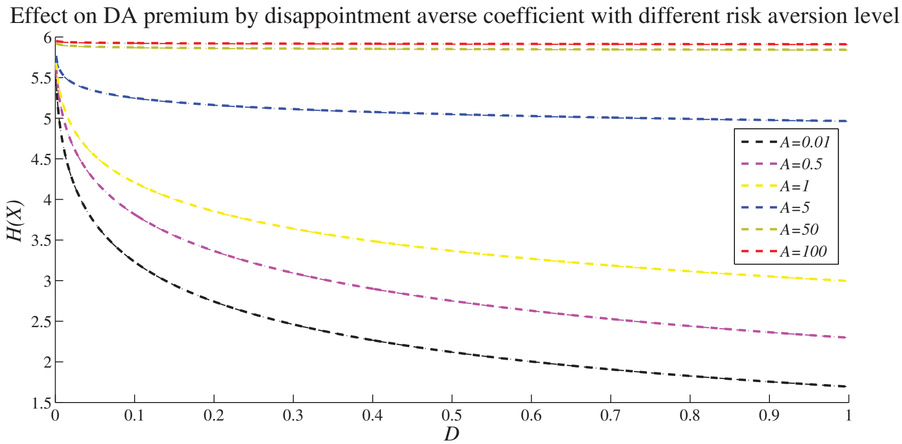

FIGURE 3: DA premium versus DA coefficient and risk aversion level with exponential loss, mean $\lambda = 2$, and $\alpha = 0.95$. (Color online)

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NOTES

1. In finance, a risk measure is usually defined in terms of the gain of a portfolio; while in this article, risk measures are considered from the loss perspective.

2. For any random variables Y and Z, Y is said to be stochastically smaller than Z, denoted as $Y \preceq_{\text{st}} Z$, if for all increasing functions $g : \mathbb{R} \to \mathbb{R}$, $\mathbb{E}[g(Y)] \le \mathbb{E}[g(Z)]$ provided that the expectations exist.

3. Since the α-level Value-at-Risk $\text{V@R}_\alpha(X)$ of any random variable X is defined as $\text{V@R}_\alpha(X) := \inf\{x : S(x) \le \alpha\} = q_X^-(1-\alpha)$ where $q_X^-(\cdot)$ is the lower quantile function of X, the α-level Average Value-at-Risk $\text{AV@R}_\alpha(X)$ is defined by $\text{AV@R}_\alpha(X) := \frac{1}{\alpha}\int_0^\alpha \text{V@R}_\gamma(X)d\gamma$.

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APPENDIX

A. THEOREM FOR POSITIVE HOMOGENEITY

We modify Theorem 4 in Section 3.5.7 in Goovaerts *et al.* (1984) and provide a proof as follows.

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Theorem A.1. *Let u be a function which is continuous, strictly increasing and $u(0) = 0$. Let $H(X)$ be the solution by*

$$\mathbb{E}[u(H(X) - X)] = 0.$$

Then $H(X)$ is positively homogeneous if and only if

$$u(x) = \begin{cases} \beta x^r & \text{for } x \geq 0 \\ -\gamma(-x)^r & \text{for } x < 0 \end{cases}$$

for some $r, \beta, \gamma > 0$.

Proof. It is easy to check that the given form of $u(x)$ yields the positive homogeneity of $H(X)$.

Assume that $H(X)$ is positively homogeneous, i.e. $H(kX) = kH(X)$ for $k \geq 0$. Consider a two-point random variable X_t such that $\mathbb{P}(X_t = 0) = t$ and $\mathbb{P}(X_t = 2x) = 1 - t$, where $x > 0$ and $t \in (0, 1)$. Then,

$$tu(H(X_t)) + (1 - t)u(H(X_t) - 2x) = 0.$$

By Lemma 1 in Section 2.10 in Goovaerts *et al.* (1984), we can choose $t \in (0, 1)$, denoted as t_0, such that $H(X_{t_0}) = x$. Therefore

$$t_0 u(x) + (1 - t_0)u(-x) = 0. \tag{23}$$

Consider another two-point random variable kX_{t_0}, for $k \geq 0$. By positive homogeneity,

$$t_0 u(kx) + (1 - t_0)u(-kx) = 0. \tag{24}$$

Hence, (23) and (24) imply

$$\frac{u(-x)}{u(x)} = \frac{-t_0}{1 - t_0} = \frac{u(-kx)}{u(kx)}.$$

Since $k \geq 0$ is arbitrary,

$$u(-x) = -Cu(x) \quad \text{for any} \quad x \geq 0, \tag{25}$$

where C is independent of x. Consider a three-point random variable Y such that $\mathbb{P}(Y = 0) = t_1$, $\mathbb{P}(Y = x - 1) = t_2$ and $\mathbb{P}(Y = x + 1) = 1 - t_1 - t_2$, where $x \geq 0$ and $t_1, t_2 \in (0, 1)$. Then

$$t_1 u(H(Y)) + t_2 u(H(Y) - (x - 1)) + (1 - t_1 - t_2)u(H(Y) - (x + 1)) = 0.$$

Using the similar argument, we can choose $t_1, t_2 \in (0, 1)$ such that $H(Y) = x$. Hence

$$t_1 u(x) + t_2 u(1) + (1 - t_1 - t_2)u(-1) = 0.$$

By positive homogeneity and (25), since $k \geq 0$,

$$t_1 u(kx) + t_2 u(k) + (1 - t_1 - t_2)(-Cu(k)) = 0,$$

or

$$u(kx) = \left(\frac{C(1 - t_1 - t_2) - t_2}{t_1}\right) u(k). \tag{26}$$

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Substituting $k = 1$ gives

$$u(x) = \left(\frac{C(1 - t_1 - t_2) - t_2}{t_1} \right) u(1)$$

which combines with (26) yields

$$u(kx) = \frac{u(k)u(x)}{u(1)} \quad \text{for any} \quad k \geq 0 \text{ and } x \geq 0.$$

Finally, with this relation between k and x, it is easy to deduce, together with (25), that

$$u(x) = \begin{cases} \beta x^r & \text{for } x \geq 0 \\ -\gamma(-x)^r & \text{for } x < 0 \end{cases}$$

for some $r, \beta, \gamma > 0$. ■

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