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RESEARCH ARTICLE

The Coble quadric

Vladimiro Benedetti [1], Michele Bolognesi [2], Daniele Faenzi [3] and Laurent Manivel [4]

[1]Laboratoire J.-A. Dieudonné, Université Côte d'Azur, CNRS, Parc Valrose, F-06108 Nice Cedex 2, France; E-mail: vladimiro.benedetti@univ-cotedazur.fr.
[2]Institut Montpelliérain Alexander Grothendieck, Université de Montpellier - CNRS, Place Eugène Bataillon, 34095 Montpellier Cedex 5, France; E-mail: michele.bolognesi@umontpellier.fr.
[3]Institut de Mathématiques de Bourgogne, Université de Bourgogne, 9 Avenue Alain Savary, 21078 Dijon Cedex, France; E-mail: daniele.faenzi@u-bourgogne.fr.
[4]Institut de Mathématiques de Toulouse, University of Toulouse III Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex 9, France; E-mail: manivel@math.cnrs.fr (corresponding author).

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Abstract
Given a smooth genus three curve C, the moduli space of rank two stable vector bundles on C with trivial determinant embeds in $\mathbb{P}^8$ as a hypersurface whose singular locus is the Kummer threefold of C; this hypersurface is the Coble quartic. Gruson, Sam and Weyman realized that this quartic could be constructed from a general skew-symmetric four-form in eight variables. Using the lines contained in the quartic, we prove that a similar construction allows to recover $\mathrm{SU}_C(2, L)$, the moduli space of rank two stable vector bundles on C with fixed determinant of odd degree L, as a subvariety of $G(2, 8)$. In fact, each point $p \in C$ defines a natural embedding of $\mathrm{SU}_C(2, \mathcal{O}(p))$ in $G(2, 8)$. We show that, for the generic such embedding, there exists a unique quadratic section of the Grassmannian which is singular exactly along the image of $\mathrm{SU}_C(2, \mathcal{O}(p))$ and thus deserves to be coined the Coble quadric of the pointed curve (C, p).

Contents

1. Introduction

A century ago, Arthur Coble proved that there exists a unique quartic hypersurface $\mathcal{C}$ in $\mathbb{P}^7$ that is singular exactly along the three-dimensional Kummer variety, image of the Jacobian of a nonhyperelliptic genus 3 curve C via the $|2\Theta|$-linear system ([Cob61]; see also [Bea03, Kol23]). This remarkable hypersurface is now named after him, and its many very special features have been studied by several algebraic geometers. For example, $\mathcal{C}$ is projectively self-dual [Pau02], it has close relationships with the Θ-geometry of the curve (e.g., a Schottky–Jung configuration of Kummer surfaces of Prym varieties [vGP92], etc.) and with moduli of configurations of points in the projective space [AB15].

Probably, the most striking property is, however, that $\mathcal{C}$ is the image, via the theta map, of the moduli space of semistable rank two vector bundles on C with trivial determinant. This was first remarked by Narasimhan and Ramanan in the seminal paper [NR87]. In particular, since the theta map is an embedding for rank two bundles with trivial determinant [Bea88], we can identify $\mathcal{C}$ with the moduli space $\mathrm{SU}_C(2)$ itself.

In rank two, there is, up to isomorphism, only one other moduli space $\mathrm{SU}_C(2, L)$ of rank two vector bundles on C, obtained by fixing the determinant to be any given line bundle L of odd degree (up to noncanonical isomorphisms, L is irrelevant). Contrary to $\mathcal{C}$, this moduli space is smooth and we can wonder what could be an analogue of the Coble quartic. The main results of this paper answer this natural question.

In order to achieve this, we will use the theory of theta representations [Vin76], in the way this was initiated in [GSW13] as a complex addition to arithmetic invariant theory. In our setting, the main point is that starting from the GL_8-module $\wedge^4\mathbb{C}^8$ one can easily construct the Coble quartics in terms of Pfaffian loci. From this point of view, the curve C defined by a general element of $\wedge^4\mathbb{C}^8$ is not immediately visible, but certain deep properties of the quartic $\mathcal{C}$ become easy to establish. For example, we give in Theorem 3.4 a short, self-contained proof of the self-duality of $\mathcal{C}$. Then we switch from $\mathbb{P}^7$ to the Grassmannian $G(2, 8)$ and observe that also in this Grassmannian, there exist natural Pfaffian loci corresponding to skew forms of rank at most 4 and 6, respectively of codimension 6 and 1:

$$D = D_{Z_6}(v) \subset Q = D_{Z_1}(v) \subset G(2, 8).$$

Here, v is a general element in $\wedge^4\mathbb{C}^8$ and Q is a quadric section of the Grassmannian that is singular exactly along the six-dimensional smooth locus D (the notation $D_{Z_i}(v)$ will be explained in Section 4.2). The connection with the Coble quartic comes from the fact that D parametrizes a family of lines on it, some of the so-called Hecke lines. We deduce (Theorem 4.8 later on):

Theorem 1.1. *$D \simeq \mathrm{SU}_C(2, L)$ for L of odd degree.*

Consequently, the moduli space, which is smooth, comes up with a natural hypersurface of which it is the singular locus, contrary to the even case for which the moduli space is singular and uniquely determined by its singular locus, which is the Kummer. We extend the unicity statement by proving (Theorem 5.1 later on):

Theorem 1.2. *Q is the only quadratic section of the Grassmannian that is singular along D.*

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Because of this property, Q really deserves to be called a *Coble quadric*. Moreover, exactly as the Coble quartic, we show this hypersurface is self-dual in a suitable sense (Theorem 5.15). As a matter of fact, for each point $p \in C$, there is an embedding

$$\varphi_p : \mathrm{SU}_C(2, \mathcal{O}_C(p)) \hookrightarrow G(2, 8),$$

(see [Bea91]), and we show that at least for the generic p, there exists a unique quadric section of the Grassmannian that is singular along the moduli space (Theorem 5.14).

Remarkably, we found other instances of this phenomenon: For example, an eightfold inside the flag variety $Fl(1, 7, 8)$ whose singular locus is an abelian threefold, essentially the Jacobian of the curve (see Remark 3.7).

The paper is organized as follows. In section 2, we recall a few classical results about lines on moduli spaces of vector bundles on curves and more specifically about lines in the Coble quartic. In section 3, we explain how the Coble quartic, the Kummer threefold and the associated Jacobian can be constructed from a skew-symmetric four-form in eight variables, and we give a short proof of the self-duality of the quartic. In section 4, we explain how this point of view allows to understand the lines in the Coble quartic in terms of orbital degeneracy loci [BFMT20a, BFMT20b], and we deduce Theorem 1 (see Theorem 21). The resulting description as a relative Pfaffian locus makes it clear that the odd moduli space is the singular locus of a special quadratic section of the Grassmannian $G(2, 8)$. In order to prove that this special quadric is unique, we need to study the square of the ideal of the Grassmannian $G(2, 6)$ in its Plücker embedding. Going back to the relative setting we deduce Theorem 2 (see Theorem 27). We finally complete the picture by explaining why and how the special quadric is also self-dual.

2. Lines in the Coble quartic

Throughout the text, we will denote by $\mathrm{U}_C(r, d)$ the moduli space of semistable vector bundles on a curve C of rank r and determinant of degree d. If L is a degree d line bundle on C, we will denote by $\mathrm{SU}_C(r, L)$ the subvariety of $U_C(r, d)$ parametrizing vector bundles of determinant L; moreover, $\mathrm{SU}_C(r) := \mathrm{SU}_C(r, \mathcal{O}_C)$. Since all the moduli spaces $\mathrm{SU}(r, L)$ are (noncanonically) isomorphic when the degree of L is fixed, we will also denote their isomorphism class by $\mathrm{SU}_C(r, d)$; it depends only on d modulo r. Finally, we will denote by $\mathrm{U}_C(r, d)^{\mathrm{eff}}$ the moduli space of vector bundles with effective determinant. When $d = 1$, this moduli space fibers over the curve C with fiber over c isomorphic to $\mathrm{SU}_C(2, \mathcal{O}_C(c))$.

2.1. Covering families of rational curves in $\mathrm{SU}_C(2)$

Rational curves in the moduli spaces $\mathrm{SU}_C(r, d)$ were extensively studied; see, for example, [NR75, OPP98, Hwa00, HR04, Sun05, MS09, Pal16]. Restricting to $g = 3$, $r = 2$ and $d = 0$, the results of [MTiB20] show that there exist two different families of covering lines, that is, families of rational curves of degree one with respect to the Theta embedding

$$\mathcal{C} := \mathrm{SU}_2(C) \hookrightarrow |2\Theta| = \mathbb{P}(V_8),$$

passing through a general point of the moduli space. We will denote these two families by $\mathcal{F}_H$ and $\mathcal{F}_R$ and consider them as subvarieties of the Grassmannain $G(2, V_8)$. In the sequel, we describe these two covering families in some detail. They are both of dimension six but behave very differently; we will illustrate this by showing how different are the corresponding variety of minimal rational tangents(VMRTs), which in our case, since we deal with lines, are just the spaces of lines through a fixed general point.

2.2. Hecke lines

Before going through the body of this section, we need to recall the following

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Definition 2.1. A vector bundle V on C is called (k,ℓ)-semistable (resp. (k,ℓ)-stable) if for any proper subbundle $W \subset V$ we have

$$\frac{\deg(W)+k}{\operatorname{rank}(W)} \leq (resp. <) \frac{\deg(V)+k-\ell}{\operatorname{rank}(V)}.$$

A generic Hecke line can be described by choosing a point $c \in C$ and a rank two vector bundle F on C with determinant $\det(F) = \mathcal{O}_C(c)$. Then the bundles E that fit into an exact sequence

$$0 \longrightarrow E \longrightarrow F \longrightarrow \mathcal{O}_c \longrightarrow 0$$

are parametrized by $\mathbb{P}(F_c^\vee) \simeq \mathbb{P}^1$. They have trivial determinant and are all stable when F is $(1,0)$-semistable in the sense of [MS09, Definition 2.5]. For vector bundles of rank two and degree one, this condition is equivalent to stability, hence also to semistability. The resulting curve in $\mathrm{SU}_C(2)$ is a line and such lines are called Hecke lines. Note that dualizing, we get an exact sequence

$$0 \longrightarrow F^\vee \longrightarrow E^\vee \longrightarrow \mathcal{O}_c \longrightarrow 0,$$

so a Hecke line parametrizes all the possible extensions of $\mathcal{O}_c$ by $F^\vee$.

By [Pal16, Remark 5.3], a general Hecke line defines a vector bundle $\mathcal{E}$ of rank 2 over $C \times \mathbb{P}^1$ fitting into an exact sequence

$$0 \longrightarrow p_1^* F^\vee \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{E}^\vee \longrightarrow p_1^* \mathcal{O}_c \longrightarrow 0,$$

where p_1 and p_2 are the projections of $C \times \mathbb{P}^1$ onto the two factors C and $\mathbb{P}^1$. An easy consequence is that, since $\mathcal{E}^\vee$ admits a *unique* jumping line at c, this point can be uniquely recovered from the Hecke line. (Beware this is only true for general Hecke lines.)

We will denote by $\mathcal{F}_H$ the family of Hecke lines in $\mathrm{SU}_C(2, \mathcal{O}_C)$, considered as a subvariety of the space $G(2, V_8)$ of lines in $\mathbb{P}(V_8)$.

Remark 2.2. Although a Hecke line does not always define a unique point in C, once we have fixed such a point c there is a well-defined morphism from $\mathrm{SU}_C(2, \mathcal{O}_C(c))$ to $\mathcal{F}_H$. By the previous observations, the resulting morphism from $\tilde{\mathcal{F}}_H := \mathrm{U}_C(2,1)^{\mathrm{eff}}$ to $\mathcal{F}_H$ is birational.

Conversely, Hecke lines passing through a general point $[E]$ of $\mathrm{SU}_C(2, K_C)$ (we make this choice of determinant just for convenience) are obtained by choosing a projection $E \to E_c \to \mathcal{O}_c$, where E_c denotes the fiber of the vector bundle E at the point $c \in C$. So they are parametrized by (the image in $\mathcal{F}_H$ of) the total space of the projective bundle $\mathbb{P}(E^\vee)$ over C. The tangent map of this morphism sends $\mathbb{P}(E^\vee)$ to the tangent space of the moduli space at $[E]$, which is the projectivization of

$$H^1(C, \mathcal{E}nd_0(E)) \simeq H^0(C, K_C \otimes \mathcal{E}nd_0(E))^\vee \simeq H^0(C, S^2E)^\vee$$

since $K_C \simeq \det(E)$. Here, $\mathcal{E}nd_0(E)$ denotes the vector bundle of traceless endomorphisms of E. This implies (see [Hwa00, HR04] for more general statements):

Proposition 2.3. *The VMRT of the family $\mathcal{F}_H$ of Hecke lines at a general point $[E]$ of the moduli space is the image of the ruled surface $\mathbb{P}(E^\vee)$ by the linear system $|\mathcal{O}_E(2)|$. In particular this surface contains no line.*

Equivalently, the latter claim means that a general Hecke line is not contained in any larger linear space contained in $\mathrm{SU}_C(2)$, although such larger linear spaces do exist.

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2.3. *Lines in the ruling*

For each line bundle $L \in \mathrm{Pic}^1(C)$, consider the rank two vector bundles E obtained as extensions of the form

$$0 \longrightarrow L \longrightarrow E \longrightarrow K_C \otimes L^\vee \longrightarrow 0.$$

Such extensions are parametrized by $\mathbb{P}_L := \mathbb{P}(\mathrm{Ext}^1(K_C \otimes L^\vee, L)) \simeq \mathbb{P}^3$. Hence, we obtain a ruling of $\mathrm{SU}_C(2)$ by a family of $\mathbb{P}^3$s parametrized by $\mathrm{Pic}^1(C)$, which we denote by $\mathbb{P}(\mathcal{R}) \to \mathrm{Pic}^1(C)$.

Note that $\mathbb{P}_L$ intersects the Kummer threefold along a copy C_L of C [OPP98, 1.1]. According to [OPP98, Theorem 1.3], a line in $\mathbb{P}_L$ is a Hecke line if and only if it meets C_L.

Moreover, by [OPP98, Proposition 1.2], two spaces $\mathbb{P}_L$ and $\mathbb{P}_M$ are always distinct for $L \neq M$ and, for sufficiently general choices of L and M, they are disjoint. When they meet, their intersection is a single point, or a line; the latter case happens exactly when $K_C - L - M$ is effective. In particular, if a line is contained in $\mathbb{P}_L \cap \mathbb{P}_M$, it must be a bisecant to both C_L and C_M.

Now, consider the family $\mathcal{F}_R$ of lines contained in the $\mathbb{P}^3$s of the ruling. By what we have just recalled, $\mathcal{F}_R$ is the birational image in $G(2, V_8)$ of the quadric bundle $G(2, \mathcal{R})$ over $\mathrm{Pic}^1(C)$.

Proposition 2.4. *The VMRT at a general point of* $\mathrm{SU}_C(2)$*, of the family* $\mathcal{F}_R$ *of lines in its ruling, is the disjoint union of eight planes in* $\mathbb{P}^5$.

Proof. It follows from [Pau02, Section 4.1] that the map $\mathbb{P}(\mathcal{R}) \longrightarrow \mathrm{SU}_C(2)$ is generically finite of degree 8. This means that eight $\mathbb{P}^3$s of the ruling pass through a general point $[E]$ of $\mathrm{SU}_C(2)$, and for each of them the lines passing through $[E]$ are parametrized by a projective plane. Finally, these projective planes are disjoint, again by [OPP98, Proposition 1.2]. □

For future use, we record the following easy consequence.

Corollary 2.5. *Any plane in* $\mathrm{SU}_C(2, K_C)$ *passing through a general point is contained in a unique* $\mathbb{P}^3$ *of the ruling.*

3. Four-forms and orbital degeneracy loci

In this section, we recall the definitions of some orbital degeneracy loci closely connected to the geometry of $\mathrm{SU}_C(2, \mathcal{O}_C)$, for C a general curve of genus 3. In particular, we recall how to recover the Coble quartic from a general four-form in eight variables. Using this description, we give a short proof of the self-duality statement of [Pau02]. Our references for orbital degeneracy loci (sometimes abbreviated as ODL) are [BFMT20a, BFMT20b]. All the results in this section are contained either in [Pau02] or in [GSW13]; our contribution consists in a new interpretation in terms of ODL and in the technique employed in the proof of the self-duality statement, which does not rely anymore on [Pau02].

Notation. We will denote by V_n and U_i complex vector spaces of dimension n and i, respectively (usually V_n will be fixed and U_i will be a variable subspace of V_n). We will also denote by $G(i, V_n)$ the Grassmannian of i-dimensional subspaces of V_n and by $Fl(i_1, \dots, i_k, V_n)$ the flag variety of flags of subspaces of V_n of dimensions $i_1 < \cdots < i_k$. Over the flag variety, we will denote by $\mathcal{U}_{i_j}$ the rank-i_j tautological bundle; over the Grassmannian, we will denote by $\mathcal{U}$ the tautological bundle and by $\mathcal{Q}$ the universal quotient bundle.

3.1. *A simple construction of the Coble quartic*

In this section, we recall some results from [GSW13]. The starting point is a general four-form in eight variables, $v \in \wedge^4 V_8 \simeq \wedge^4 V_8^\vee$, where V_8 denotes a complex eight-dimensional vector space. Recall that this is a *theta-representation*, being part of a $\mathbb{Z}_2$-grading of the exceptional Lie algebra

$$\mathfrak{e}_7 \simeq \mathfrak{sl}(V_8) \oplus \wedge^4 V_8.$$

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The action of the so-called theta-group, which here is $\mathrm{SL}(V_8)$, behaves very much as the action of the adjoint group on a simple complex Lie algebra. In particular, one has Jordan decompositions, and the GIT-quotient

$$\wedge^4 V_8 /\!/ \mathrm{SL}(V_8) \simeq \mathfrak{h}/W$$

for some finite complex reflection group W acting on what is called a Cartan subspace $\mathfrak{h}$ of the theta-representation. We will make this Cartan subspace explicit later on. For now, we just need to know that it coincides with the seven-dimensional representation of the Weyl group of E_7. As a consequence, the choice of v determines uniquely a nonhyperelliptic curve C of genus three (a plane quartic) with a marked flex point [GSW13, Remark 6.1].

We will construct from our general $v \in \wedge^4 V_8$ a collection of geometric objects defined as orbital degeneracy loci. The main point of this approach is that it allows us to reduce to simpler representations. Typically, the Borel–Weil theorem gives an isomorphism

$$\wedge^4 V_8 \simeq H^0(\mathbb{P}(V_8), \wedge^4 \mathcal{Q}) \simeq H^0(\mathbb{P}(V_8), \wedge^3 \mathcal{Q}^\vee(1)),$$

where $\mathcal{Q}$ denotes the rank seven quotient vector bundle on $\mathbb{P}(V_8)$. At the price of passing to a relative setting over $\mathbb{P}(V_8)$, this reduces the study of $\wedge^4 V_8$ to that of three-forms in seven variables.

But then the situation is much simpler because if V_7 is a seven-dimensional complex vector space, $\wedge^3 V_7^\vee \cong \wedge^4 V_7$ has finitely many orbits under the action of $\mathrm{GL}(V_7)$. Each orbit closure Y allows to associate to $v \in \wedge^4 V_8$ the locus $D_Y(v) \subset \mathbb{P}(V_8)$ of points x where the image of v lies in the corresponding $Y_x \subset \wedge^3 \mathcal{Q}^\vee(1)_x$ (this is exactly how orbital degeneracy loci are defined). By the general results of [BFMT20a], for v general the main properties of Y will be transferred to $D_Y(v)$, starting from its codimension. We can therefore focus on the orbit closures in $\wedge^3 V_7^\vee$ of codimension at most seven. Remarkably, there are only three such orbit closures (not counting the whole space), that we can index by their codimension: Y_1 is a hypersurface of degree 7, Y_4 is its singular locus, Y_7 is the singular locus of Y_4. The corresponding orbital degeneracy loci have been described in [GSW13, 6.1, 6.2].

Proposition 3.1. *For v general, the threefold $\mathrm{Kum}_C := D_{Y_4}(v)$ is the Kummer variety of a nonhyperelliptic genus three curve C. It is the singular locus of the quartic hypersurface $\mathcal{C} := D_{Y_1}(v)$. Its singular locus is the finite set $\mathrm{Kum}_C[2] := D_{Y_7}(v)$.*

Since the Coble quartic can be characterized as the unique quartic hypersurface that is singular along the Kummer threefold [Bea91, Proposition 3.1], we can immediately deduce that it coincides with $D_{Y_1}(v)$.

3.2. *Kempf collapsings*

A nice feature of our orbital degeneracy loci is the following. It turns out that the orbit closures they are associated to, although singular, admit nice resolutions of singularities by *Kempf collapsings*, which are birational contractions from total spaces of homogeneous vector bundles on flag manifolds. These homogeneous vector bundles are typically nonsemisimple, making them more difficult to handle. Nevertheless, these collapsings allow to describe the corresponding orbital degeneracy loci in terms of zero loci of sections of vector bundles.

In the cases we are interested in, we obtain the following descriptions, where U_k stands for a k-dimensional subspace of V_8. For A, B subspaces of a vector space V, we will denote by $(\wedge^p A) \wedge (\wedge^q B) \subset \wedge^{p+q} V$ the linear subspace spanned by the elements of the form $a_1 \wedge \cdots \wedge a_p \wedge b_1 \wedge \cdots \wedge b_q$ with $a_1, \ldots, a_p \in A$ and $b_1, \ldots, b_q \in B$. For vector subbundles $\mathcal{A}, \mathcal{B}$ of a the trivial bundle $V \otimes \mathcal{O}$, we use the same convention to define $(\wedge^p \mathcal{A}) \wedge (\wedge^q \mathcal{B})$ in $\wedge^{p+q} V \otimes \mathcal{O}$.

Proposition 3.2. *Let $v \in \wedge^4 V_8$ be a generic element. The Coble quartic $\mathcal{C}$ associated to v can be described as*

$$\left\{[U_1] \in \mathbb{P}(V_8) \mid \exists U_4 \supset U_1,\ v \in (\wedge^2 U_4) \wedge (\wedge^2 V_8) + \wedge^3 V_8 \wedge U_1\right\}.$$

The corresponding Kummer threefold Kum_C *is*

$$\left\{[U_1] \in \mathbb{P}(V_8) \mid \exists U_6 \supset U_2 \supset U_1,\ v \in \wedge^4 U_6 + \wedge^2 U_6 \wedge U_2 \wedge V_8 + \wedge^3 V_8 \wedge U_1\right\}.$$

The singular locus $\mathrm{Kum}_C[2]$ *of* Kum_C *is*

$$\left\{[U_1] \in \mathbb{P}(V_8) \mid \exists U_7 \supset U_4 \supset U_1,\ v \in \wedge^3 U_4 \wedge V_8 + (\wedge^2 U_4) \wedge (\wedge^2 U_7) + \wedge^3 V_8 \wedge U_1\right\}.$$

These results follow from a combination of [GSW13, Section 6] and [KW13, Section 3]. Let us clarify the statement for instance for Kum_C, the explanations for the other loci being similar. In [GSW13, Section 6], it is shown that $\mathrm{Kum}_C := D_{Y_4}(v)$ is the Kummer threefold. In [KW13, Section 3] it is proved that $Y_4 \subset \wedge^4 V_7$ is desingularized by a the total space of the vector bundle

$$\mathcal{W} := \wedge^4 \mathcal{U}_5 + \wedge^2 \mathcal{U}_5 \wedge \mathcal{U}_1 \wedge V_7$$

over the flag variety $Fl(1,5,V_7)$. Here, we denoted by $\mathcal{U}_1$ and $\mathcal{U}_5$, respectively, the rank one and rank five tautological vector bundles on $Fl(1,5,V_7)$. The projection from the total space of $\mathcal{W}$ to $Y_4 \subset \wedge^4 V_7$ is given by the composition of the inclusion of $\mathcal{W}$ inside $\wedge^4 V_7 \otimes \mathcal{O}_{Fl(1,5,V_7)}$ with the projection to $\wedge^4 V_7$. For v general, this desingularization $\mathrm{Tot}(\mathcal{W}) \to Y_4$ of Y_4 can be *relativized* to obtain a desingularization of $D_{Y_4}(v)$, as explained in [BFMT20a, Section 2]. For this, we simply consider the flag bundle $Fl(1,5,\mathcal{Q})$: By the previous discussion, any point of $x = [U_1] \in D_{Y_4}(v)$ must be the image of a flag $\bar{U}_1 \subset \bar{U}_5 \subset \mathcal{Q}_x = V_8/U_1$ such that v mod U_1 belongs to $\wedge^4 \bar{U}_5 + \wedge^2 \bar{U}_5 \wedge \bar{U}_1 \wedge \mathcal{Q}_x \subset \wedge^4 \mathcal{Q}_x$. This flag originates from a flag $(U_1 \subset U_2 \subset U_6 \subset V_8)$ (such that $\bar{U}_1 = U_2/U_1$, etc.), and we can rewrite the previous condition as asking that $x = [U_1]$ belongs to the projection of

$$Z(v) := \left\{(U_1 \subset U_2 \subset U_6) \in Fl(1,2,6,V_8),\ v \in \wedge^4 U_6 + \wedge^2 U_6 \wedge U_2 \wedge V_8 + \wedge^3 V_8 \wedge U_1\right\}.$$

This is the zero locus of a global section of a globally generated bundle, obtained as a quotient of the trivial bundle with fiber $\wedge^4 V_8$. For v general, this section is general, so $Z(v)$ is smooth. Moreover, the projection $Z(v) \to D_{Y_4}(v) \subset \mathbb{P}(V_8)$, obtained by just forgetting U_2 and U_6, is birational.

3.3. Self-duality of the Coble quartic

Because of the natural isomorphism $\wedge^4 V_8 \simeq \wedge^4 V_8^\vee$ (defined up to scalar or, more precisely, up to the choice of a volume form on V_8), the same constructions can be performed in the dual projective space $\mathbb{P}(V_8^\vee)$. This is related to the remarkable fact that the Coble quartic is projectively self-dual [Pau02]. Let us show how this duality statement easily follows from our approach in terms of orbital degeneracy loci.

First, consider a general point $[U_1]$ of $\mathcal{C} = D_{Y_1}(v)$. As we have seen in the previous section, there exists (a unique) $U_4 \supset U_1$ such that v belongs to $(\wedge^2 U_4) \wedge (\wedge^2 V_8) + \wedge^3 V_8 \wedge U_1$. Reducing modulo $(\wedge^2 U_4) \wedge (\wedge^2 V_8)$, we get

$$\bar{v} \in \wedge^3 (V_8/U_4) \otimes U_1 \simeq (V_8/U_4)^\vee.$$

In general, $\bar{v}$ is nonzero and defines a hyperplane in V_8/U_4, that is, a hyperplane U_7 of V_8, containing U_4. Note that this exactly means that

$$v \in (\wedge^2 U_4) \wedge (\wedge^2 V_8) + \wedge^3 U_7 \wedge U_1. \tag{3.1}$$

Lemma 3.3. $\mathbb{P}(U_7)$ *is the tangent hyperplane to* $\mathcal{C}$ *at* $[U_1]$.

Proof. Let $\tilde{\mathcal{C}}$ denote the variety of flags $(U_1 \subset U_4)$ such that v belongs to

$$\mathcal{F}(U_1, U_4) := (\wedge^2 U_4) \wedge (\wedge^2 V_8) + \wedge^3 V_8 \wedge U_1.$$

We know that the projection $\tilde{\mathcal{C}} \longrightarrow \mathcal{C}$ is birational. Moreover, as a subvariety of the flag manifold $Fl(1, 4, V_8)$, $\tilde{\mathcal{C}}$ is the zero-locus of the section of the vector bundle $\wedge^4 V_8/\mathcal{F}(U_1, U_4)$ defined by v since such a section vanishes exactly when $v \in \mathcal{F}(U_1, U_4) \subset \wedge^4 V_8$. Let $\mathfrak{p}(U_1, U_4)$ denote the stabilizer of the flag $(U_1 \subset U_4)$ inside $\mathfrak{gl}(V_8)$. The tangent space to $Fl(1, 4, V_8)$ at the corresponding point is the quotient $\mathfrak{gl}(V_8)/\mathfrak{p}(U_1, U_4)$, and the tangent space to $\tilde{\mathcal{C}}$ is the image, in this quotient, of the space of endomorphisms $X \in \mathfrak{gl}(V_8)$ such that $X(v)$ belongs to $\mathcal{F}(U_1, U_4)$, as follows from the normal exact sequence

$$0 \to T_{\tilde{\mathcal{C}}} \to T_{Fl(1,4,V_8)}|_{\tilde{\mathcal{C}}} \to (\wedge^4 V_8/\mathcal{F}(U_1, U_4))|_{\tilde{\mathcal{C}}} \to 0.$$

The tangent space to $\mathcal{C}$ is then the image of this space inside $\mathfrak{gl}(V_8)/\mathfrak{p}(U_1) \simeq \mathrm{Hom}(U_1, V_8/U_1)$, where $\mathfrak{p}(U_1)$ denotes the stabilizer of the line U_1.

So our claim will follow, if we can check that any $X \in \mathfrak{gl}(V_8)$ such that $X(v)$ belongs to $\mathcal{F}(U_1, U_4)$, must send U_1 into the hyperplane U_7. But (3.1) implies, once we apply X, that

$$X(v) \in U_4 \wedge (\wedge^3 V_8) + \wedge^3 U_7 \wedge X(U_1).$$

If $X(v)$ belongs to $\mathcal{F}(U_1, U_4)$, it has to vanish modulo U_4. So $\wedge^3 U_7 \wedge X(U_1)$ must also vanish modulo U_4, which is the case only if $X(U_1) \subset U_7$. □

Recall that once we fix a volume form on V_8, we get an isomorphism of $\wedge^4 V_8$ with $\wedge^4 V_8^\vee$. We will denote by $v^\vee$ the image of v. (Strictly speaking, it is uniquely defined only up to scalar, but this is irrelevant in our constructions.) To make things clearer, we will denote by $\mathcal{C}(v)$ the Coble quartic defined by v in $\mathbb{P}(V_8)$ and by $\mathcal{C}(v^\vee)$ the Coble quartic defined by $v^\vee$ in $\mathbb{P}(V_8^\vee)$.

Theorem 3.4. *The projective dual of $\mathcal{C}(v)$ is $\mathcal{C}(v^\vee)$.*

Proof. For $[U_1]$ a general point of $\mathcal{C}$, we have a flag $(U_1 \subset U_4 \subset U_7)$ such that v belongs to $(\wedge^2 U_4) \wedge (\wedge^2 V_8) + \wedge^3 U_7 \wedge U_1$. Choose an adapted basis $e_1, \ldots, e_8$ so that e_1 generates U_1, etc. The condition means that v is a linear combination of elementary tensors $e_i \wedge e_j \wedge e_k \wedge e_\ell$ with $i, j \leq 4$, and of $e_5 \wedge e_6 \wedge e_7 \wedge e_1$.

Now, recall that if the chosen volume form on V_8 is $e_1 \wedge \cdots \wedge e_8$, and $e_1^\vee, \ldots, e_8^\vee$ is the dual basis of $e_1, \ldots, e_8$, then the isomorphism of $\wedge^4 V_8$ with $\wedge^4 V_8^\vee$ sends the elementary tensor $e_i \wedge e_j \wedge e_k \wedge e_\ell$ to $\pm e_p^\vee \wedge e_q^\vee \wedge e_r^\vee \wedge e_s^\vee$, where $\{i, j, k, l\} \cap \{p, q, r, s\} = \emptyset$.

As a consequence, $v^\vee$ will be a linear combination of elementary tensors $e_p^\vee \wedge e_q^\vee \wedge e_r^\vee \wedge e_s^\vee$ with $p, q \geq 5$, and $e_2^\vee \wedge e_3^\vee \wedge e_4^\vee \wedge e_8^\vee$. In other words,

$$v^\vee \in (\wedge^2 U_4^\perp) \wedge (\wedge^2 V_8^\vee) + \wedge^3 U_1^\perp \wedge U_7^\perp.$$

This is exactly the condition that ensures that $[U_7^\perp]$ belongs to $\mathcal{C}(v^\vee)$. Thanks to the previous lemma, we deduce that $\mathcal{C}(v)^\vee \subset \mathcal{C}(v^\vee)$. Moreover, the symmetry between U_1 and $U_7^\perp$ implies that in general, U_1 can be recovered from U_7 exactly as U_7 is constructed from U_1, which means that $\mathcal{C}(v)^\vee$ is birationally equivalent to $\mathcal{C}(v)$. Finally, since $\mathcal{C}(v)^\vee$ and $\mathcal{C}(v^\vee)$ are both irreducible hypersurfaces, they must be equal. □

The previous discussion shows that it is natural to define the variety $\mathcal{C}(v, v^\vee) \subset Fl(1, 4, 7, V_8)$ parametrizing the flags $(U_1 \subset U_4 \subset U_7 \subset V_8)$ satisfying condition (3.1). This is a smooth variety dominating birationally both $\mathcal{C}(v)$ and $\mathcal{C}(v^\vee)$; there is a diagram

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$$
\begin{tikzcd}
 & & Fl(1,4,7,V_8) & & \\
Fl(1,4,V_8) & & \mathcal{C}(v,v^\vee) \arrow[u, hook] \arrow[dl] \arrow[dr] & & Fl(4,7,V_8) \\
 & \tilde{\mathcal{C}}(v) \arrow[ul, hook] \arrow[dl] & & \tilde{\mathcal{C}}(v^\vee) \arrow[ur, hook] \arrow[dr] & \\
\mathbb{P}(V_8) \supset \mathcal{C}(v) \arrow[rrrr, dashed, "d\mathcal{C}"] & & & & \mathcal{C}(v^\vee) \subset \mathbb{P}(V_8^\vee).
\end{tikzcd}
$$

One recovers that way the constructions explained is [Pau02, section 3.3]. We used the suggestive notation $d\mathcal{C}$ for the Gauss map, which sends a smooth point of $\mathcal{C}(v)$ to its tangent hyperplane, given by the differential of the cubic's equation.

3.4. *The Cartan subspace*

Recall that a Cartan subspace for the $\mathbb{Z}_2$-graded Lie algebra $\mathfrak{e}_7 = \mathfrak{sl}(V_8) \oplus \wedge^4 V_8$ is a maximal subspace of $\wedge^4 V_8$, made of elements of $\mathfrak{e}_6$ which are semisimple and commute [Vin76]. Among other nice properties, a general element of $\wedge^4 V_8$ is $\mathrm{SL}(V_8)$-conjugate to (finitely many) elements of any given Cartan subspace.

An explicit Cartan subspace of $\wedge^4 V_8$ is worked out in [Oed22, (3.1)]. It coincides with the space of Heisenberg invariants provided in [RSSS13, Remark 4.2]. Here is a list of seven generators, for a given basis $e_1, \dots, e_8$ of V_8:

$$
\begin{aligned}
h_1 &= e_1 \wedge e_2 \wedge e_3 \wedge e_4 + e_5 \wedge e_6 \wedge e_7 \wedge e_8, \\
h_2 &= e_1 \wedge e_3 \wedge e_5 \wedge e_7 + e_6 \wedge e_8 \wedge e_2 \wedge e_4, \\
h_3 &= e_1 \wedge e_5 \wedge e_6 \wedge e_2 + e_8 \wedge e_4 \wedge e_3 \wedge e_7, \\
h_4 &= e_1 \wedge e_6 \wedge e_8 \wedge e_3 + e_4 \wedge e_5 \wedge e_7 \wedge e_2, \\
h_5 &= e_1 \wedge e_8 \wedge e_4 \wedge e_5 + e_7 \wedge e_2 \wedge e_6 \wedge e_3, \\
h_6 &= e_1 \wedge e_4 \wedge e_7 \wedge e_6 + e_2 \wedge e_3 \wedge e_8 \wedge e_5, \\
h_7 &= e_1 \wedge e_7 \wedge e_2 \wedge e_8 + e_3 \wedge e_5 \wedge e_4 \wedge e_6.
\end{aligned}
$$

Combinatorially, each of these generators is given by a pair of complementary fourtuples of indices in $\{1, \dots, 8\}$. Each of these 14 fourtuples shares a pair of indices with any other distinct, not complementary fourtuple. This is the property that ensures the commutation in $\mathfrak{e}_6$, since the Lie bracket of $\mathfrak{e}_6$, restricted to $\wedge^4 V_8$, is given by the unique (up to scalar) $\mathfrak{sl}_8$-equivariant morphism

$$
\wedge^2(\wedge^4 V_8) \longrightarrow \wedge^4 V_8 \otimes \wedge^4 V_8 \longrightarrow S_{21111110} V_8 \simeq \mathfrak{sl}_8,
$$

where $S_{21111110}$ denotes the Schur functor corresponding to the partition $(2, 1, 1, 1, 1, 1, 1, 0)$. If we start with two elementary tensors given by fourtuples with a common pair of indices, we can include them into $\wedge^4 U_6$ for some codimension two subspace $U_6 \subset V_8$. But then the Lie bracket factors through $S_{21111110} U_6 = \{0\}$, so it has to vanish.

Each pair of indices in $\{1, \dots, 8\}$ belongs to three of the 14 fourtuples. For any triple (ijk) among

$$
(124),\ (137),\ (156),\ (235),\ (267),\ (346),\ (457),
$$

h_i, h_j and h_k share four disjoint pairs (for example h_1, h_2 and h_4 share $(13), (24), (57), (68)$). These seven triples always meet in exactly one index, so they are in correspondence with the lines in a Fano plane. One can find more on this in [Man06, Section 4].

A nice consequence of this description is the following

Proposition 3.5. *$\mathcal{C}(v)$ and $\mathcal{C}(v^\vee)$ are isomorphic.*

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Proof. Since our v is general, we may suppose up to the action of $\mathrm{SL}(V_8)$ that v belongs to our Cartan subspace above, given in terms of the basis $e_1, \dots, e_8$ of V_8. Denote the dual basis by $e_1^\vee, \dots, e_8^\vee$, and choose the volume form $e_1^\vee \wedge \cdots \wedge e_8^\vee$ on V_8. Then the induced isomorphism from $\wedge^4 V_8$ to $\wedge^4 V_8^\vee$ sends $e_I = e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge e_{i_4}$ to $\epsilon_{I,J} e_J$, where J is the complement of I in $\{1, \dots, 8\}$ and $\epsilon_{I,J}$ is the sign of the permutation $(i_1, \dots, i_4, j_1, \dots, j_4)$.

Now, observe that for each i, h_i is of the form $e_K + e_L$ for two complementary sets of indices K and L. Moreover, one can check that $\epsilon_{K,L}$ is always equal to 1. This implies that $h_i^\vee = e_K^\vee + e_L^\vee$ has exactly the same expression as h_i in terms of the dual basis. In other words the map $v \mapsto v^\vee$, when restricted to our Cartan subspace, is essentially the identity, and the claim follows. □

3.5. The abelian threefold

Remarkably, one can construct the abelian threefold whose Kummer variety is Kum_C by considering another orbital degeneracy locus. The idea is to use the flag variety $Fl(1, 7, V_8)$, the incidence correspondence in $\mathbb{P}(V_8) \times \mathbb{P}(V_8^\vee)$ parametrizing flags $(U_1 \subset U_7)$. The rank six quotient bundle $\mathcal{N} = \mathcal{U}_7/\mathcal{U}_1$ allows to realize the space of four-forms as

$$\wedge^4 V_8^\vee = H^0(Fl(1, 7, V_8), p_1^* \mathcal{O}(1) \otimes \wedge^3 \mathcal{N}^\vee).$$

Exactly as before, this allows to associate to any $\mathrm{GL}(V_6)$-orbit closure Y in $\wedge^3 V_6$ an orbital degeneracy locus $D_Y(v) \subset Fl(1, 7, V_8)$. Here, V_6 is a six-dimensional vector space. In particular, the cone Y_{10} over the Grassmannian $G(3, V_6)$ yields, for v generic, a smooth threefold $\mathrm{A}_C := D_{Y_{10}}(v)$. In similar terms as for the other orbital degeneracy loci, this threefold is

$$\mathrm{A}_C = \left\{ [U_1 \subset U_7] \in Fl(1, 7, V_8) \mid \exists U_1 \subset U_4 \subset U_7,\ v \in \wedge^3 U_4 \wedge V_8 + \wedge^4 U_7 + \wedge^3 V_8 \wedge U_1 \right\}. \tag{3.2}$$

Proposition 3.6. A_C *is a torsor over an abelian threefold, and the projection to* $\mathbb{P}(V_8)$ *is a double cover of* Kum_C.

Proof. This is [GSW13, Proposition 6.12]. □

Over a point of A_C, given by a flag $U_1 \subset U_7$, the four-form v defines a decomposable tensor in $\wedge^3(U_7/U_1)$. This tensor is never zero if v is general and therefore defines a four-dimensional space U_4 such that $U_1 \subset U_4 \subset U_7$. Hence, a rank-four vector bundle $\mathcal{U}_4$ on A_C, a subbundle of the trivial bundle $V_8 \otimes \mathcal{O}_{\mathrm{A}_C}$.

Remark 3.7. The proper orbit closures of the $\mathrm{GL}(V_6)$-action on $\wedge^3 V_6$ are, apart from the cone over the Grassmannian, a quartic hypersurface and the codimension five locus of partially decomposable tensors. In our relative setting, the quartic induces a hypersurface of bidegree $(2, 2)$ in $\mathbb{P}(V_8) \times \mathbb{P}(V_8^\vee)$, whose singular locus is an eightfold that is singular exactly along A_C. So once again we get a very interesting singular hypersurface. It would be very nice to find a modular interpretation of these loci.

4. Lines from alternating forms

In this section, we will identify the two covering families of lines in $\mathrm{SU}_C(2)$ in terms of orbital degeneracy loci; this will give a very explicit description of these families in terms of existence of special flags of vector spaces. As a consequence of this, we will obtain Theorem 4.8, in which we identify the moduli space $\mathrm{SU}_C(2, L)$, for L of odd degree, with an orbital degeneracy locus in $G(2, V_8)$ associated to $v \in \wedge^4 V_8$.

4.1. The ruling and its lines

Recall the definition of the abelian threefold A_C from Equation (3.2). Our next result relates it to the ruling described in section 2.3.

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Proposition 4.1. *The family* $\mathbb{P}(\mathcal{U}_4)$ *over* A_C *coincides with the ruling* $\mathbb{P}(\mathcal{R})$ *over* $\mathrm{Pic}^1(C)$ *of the moduli space* $\mathrm{SU}_C(2)$.

Proof. We need to prove that for any flag $(U_1 \subset U_7)$ in A_C, defining the four-plane U_4, the linear space $\mathbb{P}(U_4)$ is contained in $\mathcal{C}$. If we can show that $\mathcal{C}$ is even covered by this family of $\mathbb{P}^3$s, we will be done since the ruling is unique. So let us prove these two statements. □

Lemma 4.2. *The image of* $\mathbb{P}(\mathcal{U}_4)$ *in* $\mathbb{P}(V_8)$ *is contained in the Coble quartic.*

Proof. Consider a point of A_C and the associated flag $U_1 \subset U_4 \subset U_7$. By the very definition of A_C, this means we can write

$$v = e_1 \wedge w + v' + e_2 \wedge e_3 \wedge e_4 \wedge e_8$$

for some vectors $e_1 \in U_1$ and $e_2, e_3, e_4 \in U_4$, with $w \in \wedge^3 V_8$, $e_8 \in V_8$ and $v' \in \wedge^4 U_7$. Under the generality hypothesis we can suppose that $U_4 = \langle e_1, e_2, e_3, e_4 \rangle$, and it suffices to check that $U_1' = \mathbb{C}e_2$ defines a point of $\mathcal{C}$.

Modulo e_1 and e_2, the tensor w is a three-form in six variables. Since the secants of the Grassmannian $G(3,6)$ in its Plücker embedding fill up the ambient projective space, generically we can write $w = a \wedge b \wedge c + d \wedge e \wedge f$ modulo e_1 and e_2, for some vectors a, b, c, d, e, f. Modulo e_1 and e_2 again, v' is a four-form in only five variables, so it defines a hyperplane that will cut the three-dimensional space $\langle a, b, c \rangle$ in codimension one, say along $\langle a, b \rangle$, and similarly it will cut $\langle d, e, f \rangle$ in codimension one, say along $\langle d, e \rangle$. In other words, we may suppose that modulo e_1 and e_2, $v' = a \wedge b \wedge d \wedge e$. But then, modulo e_2 we get

$$v = e_1 \wedge (a \wedge b \wedge c + d \wedge e \wedge f) + a \wedge b \wedge d \wedge e.$$

So v belongs to $(\wedge^2 U_4') \wedge (\wedge^2 V_8) + \wedge^3 V_8 \wedge U_1'$ if $U_4' = \langle e_1, e_2, a, d \rangle$. The existence of such a space $U_4' \supset U_1'$ is precisely the required condition for U_1' to belong to $\mathcal{C}$, so we are done. □

Lemma 4.3. *The family* $\mathbb{P}(\mathcal{U}_4)$ *covers the Coble quartic.*

Proof. This can be done by a Chern class computation, being equivalent to the fact that the degree of $\mathbb{P}(\mathcal{U}_4)$ with respect to the relative hyperplane class does not vanish. Notice that by Equation (3.2), A_C can be considered as a subvariety of $Fl(1,4,7,V_8)$. Even more, it is the zero locus in the flag manifold of the section $\overline{v}$ of the rank 19 vector bundle

$$\mathcal{G} := \wedge^4 V_8 / (\wedge^3 \mathcal{U}_4 \wedge V_8 + \wedge^4 \mathcal{U}_7 + \wedge^3 V_8 \wedge \mathcal{U}_1)$$

over $Fl(1,4,7,V_8)$ defined by v. Since this section is general, the class of A_C in the Chow ring of the flag manifold is the top Chern class of $\mathcal{G}$. So the degree we are looking for is

$$\int_{\mathbb{P}(\mathcal{U}_4)} c_1(\mathcal{U}_1^\vee)^6 = \int_{\mathrm{A}_C} s_3(\mathcal{U}_4^\vee) = \int_{Fl(1,4,7,V_8)} c_{19}(\mathcal{G}) s_3(\mathcal{U}_4^\vee) = 32,$$

as can be computed using [GS, Schubert2 package]. This implies the claim. □

Remark. 32 is the expected number: Since the Coble hypersurface has degree 4, we recover the fact that exactly $8\mathbb{P}^3$s of the ruling pass through a general point of the quartic, as recalled in the proof of Proposition 2.4.

The previous statement allows to reconstruct the curve C purely in terms of the four-form and its associated orbital degeneracy loci. Indeed, we have recalled that a $\mathbb{P}^3$ of the ruling meets the Kummer threefold along a copy of the curve.

Corollary 4.4. *For any point of* A_C, *with associated flag* $(U_1 \subset U_4 \subset U_7)$, *the intersection of* $\mathbb{P}(U_4)$ *with* Kum_C *is a copy of the curve* C.

And of course we also recover the family of lines in the ruling as a quadric bundle. Indeed, the same arguments as in section 2.3 yield:

Corollary 4.5. *The total space of the fiber bundle $G(2, \mathcal{U}_4)$ over A_C maps birationally to the family $\mathcal{F}_R$ in $G(2, V_8)$.*

4.2. Hecke lines from alternating forms

In the previous section, we have defined some ODL $D_{Y_i}(v)$ from orbits inside the space of three-forms in seven variables (i.e., in the notation of the previous sections, inside $\wedge^3 V_7$). We will use a similar construction to obtain ODL inside the Grassmannian $G(2, V_8)$. The Borel–Weil theorem gives an isomorphism

$$\wedge^4 V_8 \simeq H^0(G(2, V_8), \wedge^4 \mathcal{Q}) = H^0(G(2, V_8), \wedge^2 \mathcal{Q}^\vee(1)),$$

where $\mathcal{Q}$ denotes the rank six quotient vector bundle on $G(2, V_8)$. Thus, in this case, we need to look at two-forms in six variables.

If V_6 is as before a six-dimensional complex vector space, $\wedge^4 V_6^\vee \simeq \wedge^2 V_6$ has only two proper $\mathrm{GL}(V_6)$-orbit closures, that we will index by their codimension: the Pfaffian cubic hypersurface Z_1 and its singular locus Z_6, that is the cone over the Grassmannian $G(2, V_6)$. These allow us to construct inside $G(2, V_8)$ the two orbital degeneracy loci $D_{Z_1}(v)$ and $D_{Z_6}(v)$.

Let us first consider $D_{Z_6}(v)$, which can also be defined by

$$D_{Z_6}(v) := \left\{ [U_2] \in G(2, V_8) \mid \exists U_6 \supset U_2, \quad v \in \wedge^3 V_8 \wedge U_2 + \wedge^4 U_6 \right\}.$$

Lemma 4.6. *$D_{Z_6}(v)$ is a smooth Fano sixfold of even index.*

Proof. By definition, $D_{Z_6}(v)$ is the projection in $G(2, V_8)$ of the locus $Z_6(v)$ in $Fl(2, 6, V_8)$ parametrizing flags $(U_2 \subset U_6 \subset V_8)$ such that v belongs to the 56-dimensional space $\wedge^3 V_8 \wedge U_2 + \wedge^4 U_6$. Taking the quotient of $\wedge^4 V_8$ by the latter, we get a rank 14 vector bundle $\mathcal{P}$ on $Fl(2, 6, V_8)$, generated by global sections since it is a quotient of a trivial bundle. Moreover, v defines a generic section of this bundle, and $Z_6(v)$ is the zero-locus of this section, hence it is smooth. Since $Fl(2, 6, V_8)$ has dimension 20, $Z_6(v)$ has dimension 6 and its canonical bundle is given by the adjunction formula. A straightforward computation yields

$$K_{Z_6(v)} = \det(\mathcal{U}_2)^{-3} \otimes \det(\mathcal{U}_6)^5.$$

On the other hand, for any $[U_2] \in G(2, V_8)$, the quotient of $\wedge^4 V_8$ by $\wedge^3 V_8 \wedge U_2$ is isomorphic to $\wedge^4 (V_8/U_2) \simeq \wedge^2 (V_8/U_2)^\vee \otimes \det(V_8/U_2)$. This is a space of skew-symmetric forms in six dimensions, and the existence of U_6 exactly means that v defines a skew-symmetric form in $\wedge^2 (V_8/U_2)^\vee \otimes \det(V_8/U_2)$ whose rank is at most two. In fact, the rank must be exactly two, since for v generic, a simple dimension count shows that the rank can never be zero. In particular, the projection of $Z_6(v)$ to $D_{Z_6}(v)$ is an isomorphism.

More than that, the kernel of our two-form on V_8/U_2 is U_6/U_2, so we get a nondegenerate skew-symmetric form on the quotient V_8/U_6, which is therefore identified with its dual. To be precise, since the skew-symmetric form has values in $\det(V_8/U_2)$, we get an isomorphism $V_8/U_6 \simeq (V_8/U_6)^\vee \otimes \det(V_8/U_2)$. Taking determinants, we deduce that $\det(U_2)^2 \simeq \det(U_6)^2$; in other words, the line bundle $\mathcal{L} = \det(\mathcal{U}_6) \otimes \det(\mathcal{U}_2)^\vee$ is 2-torsion on $Z_6(v)$.

But then we can rewrite the canonical bundle as

$$K_{Z_6(v)} = \det(\mathcal{U}_2) \otimes \det(\mathcal{U}_6) \otimes \mathcal{L}^{\otimes 4}.$$

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Note that $\det(\mathcal{U}_2)^\vee \otimes \det(\mathcal{U}_6)^\vee$ is very ample on $Fl(2,6,V_8)$ since it defines its canonical Plücker type embedding. Since $\mathcal{L}$ is torsion, we deduce that $Z_6(v)$ is Fano. But then its Picard group is torsion free, so $\mathcal{L}$ is actually trivial. So finally $K_{Z_6(v)} = \det(\mathcal{U}_2)^2$, hence the index is even. □

The previous discussion shows that $D_{Z_6}(v)$ is a Pfaffian locus defined by a skew-symmetric map $\psi_v : \mathcal{Q} \to \mathcal{Q}^\vee(1)$ associated with v. The restriction of ψ_v to $D_{Z_6}(v)$ has constant rank, hence its Kernel $\mathrm{Ker}(\psi_v)$ is a rank four vector bundle on $D_{Z_6}(v)$, that coincides with $(\mathcal{U}_6/\mathcal{U}_2)|_{D_{Z_6}(v)}$ and fits into an exact sequence

$$0 \to \mathrm{Ker}(\psi_v) \longrightarrow \mathcal{Q}|_{D_{Z_6}(v)} \longrightarrow \mathcal{Q}^\vee(1)|_{D_{Z_6}(v)} \longrightarrow \mathrm{Ker}(\psi_v)^\vee(1) \longrightarrow 0. \tag{4.1}$$

Let us set once and for all the more compact notation $D := D_{Z_6}(v)$ and $G := G(2,V_8)$. The exact sequence (4.1) allows to describe the normal bundle $\mathcal{N}_{D/G}$ as follows.

Lemma 4.7. *We have isomorphisms $\mathcal{N}_{D/G} \simeq \wedge^2 \mathrm{Ker}(\psi_v)^\vee(1)$ and $\mathcal{N}^\vee_{D/G} \simeq \mathcal{N}_{D/G}(-2)$.*

We will use this information later on. Our next goal is to prove the following theorem.

Theorem 4.8. *For a generic $v \in \wedge^4 V_8$, the orbital degeneracy locus $D_{Z_6}(v)$ is isomorphic to the moduli space $\mathrm{SU}_C(2, \mathcal{O}_C(c))$ of semistable rank two vector bundles on C with fixed determinant $\mathcal{O}_C(c)$, for a certain point $c \in C$.*

Remark 4.9. Such embeddings defined by Hecke lines are studied in [Bea91, section 3.4], and there is one, denoted φ_p in loc. cit., for each choice of a point p on the curve C. Here, we only get one of these embeddings, in agreement with the already mentioned fact that v does not only determine a genus three curve but a marked point on this curve.

Remark 4.10. An interesting consequence is that we know a minimal resolution of the structure sheaf of $\mathrm{SU}_C(2,\mathcal{O}_C(c))$ inside the Grassmannian $G(2,V_8)$. From this resolution, it is easy to check that the intersection with a general copy of $G(2,6)$ inside $G(2,V_8)$ is a K3 surface of genus 13. This kind of description is used in [KM23] to provide a new model for the general such K3 surface.

Let us begin by showing that $D_{Z_6}(v)$ defines a six-dimensional family of Hecke lines.

Proposition 4.11. *Let $[U_2] \in D_{Z_6}(v)$, then $\mathbb{P}(U_2) \subset \mathbb{P}(V_8)$ is a line in $\mathcal{C}$.*

Proof. Let $[U_1] \in \mathbb{P}(U_2)$ be a point in the line. By definition of $D_{Z_6}(v)$, one can write

$$(v \mod U_1) = u_2 \wedge v' + a \wedge b \wedge c \wedge d$$

for some $u_2 \in U_2$, some trivector v' and some vectors a, b, c, d. The trivector v' is a trivector in six variables; therefore, it can in general be written as $e \wedge f \wedge g + h \wedge i \wedge l$ for some vectors e, f, g, h, i, l since the secant variety of $G(3,6)$ is the whole Plücker space. Now, modulo U_2, $\dim(\langle a,b,c,d\rangle \cap \langle e,f,g\rangle) \geq 1$ and $\dim(\langle a,b,c,d\rangle \cap \langle h,i,l\rangle) \geq 1$. Thus, we can suppose that $a = e$ and $b = h$. But then if we let $U_4 = \langle U_2, a, b\rangle$, it is straightforward to check that $(v \bmod U_1) \in (\wedge^2 U_4) \wedge (\wedge^2 V_8)$. This ensures that $[U_1]$ belongs to $\mathcal{C}$. □

The point-line incidence variety of the family of lines parametrized by $D_{Z_6}(v)$ is given by the projective bundle $\mathbb{P}(\mathcal{U}_2) \to D_{Z_6}(v)$.

Proposition 4.12. *The family of lines parametrized by $D_{Z_6}(v)$ covers $\mathcal{C}$.*

Proof. This is again a Chern class computation. Indeed, by irreducibility of the varieties in play, it is sufficient to check that, if $\mathcal{U}_1^\vee$ denotes the relative dual tautological line bundle of $\mathbb{P}(\mathcal{U}_2) \to D_{Z_6}(v)$, then $c_1(\mathcal{U}_1^\vee)^6 \neq 0$; indeed $\mathcal{U}_1^\vee$ is the pullback to $\mathbb{P}(\mathcal{U}_2)$ of $\mathcal{O}_{\mathbb{P}(V_8)}(1)$. This implies that the image of $\mathbb{P}(\mathcal{U}_2)$ inside $\mathbb{P}(V_8)$ has dimension at least six and is thus the Coble quartic $\mathcal{C}$ by Proposition 4.11. Notice that

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one can work directly on $Z_6(v)$ since it is isomorphic to $D_{Z_6}(v)$. Since $Z_6(v)$ can be constructed as the zero locus of a section of a vector bundle inside the flag variety $Fl(2,4,V_8)$, we can verify that $c_1(\mathcal{U}_1^\vee)^6 \neq 0$ with [GS] by constructing the coordinate ring of the zero locus $Z_6(v)$ and of the projective bundle $\mathbb{P}(\mathcal{U}_2)$ over it, similarly to what we did in the proof of Lemma 4.3. □

Proposition 4.13. *The lines parametrized by $D_{Z_6}(v)$ are Hecke lines.*

Proof. Suppose by contradiction that the lines parametrized by $D_{Z_6}(v)$ are not Hecke. Since they form a covering family, they must be lines in the ruling, that is, $D_{Z_6}(v) \subset \mathcal{F}_R$. Now, recall that $\mathcal{F}_R$ is a birational image of the quadric bundle $G(2,\mathcal{U}_4)$ over A_C. The preimage of $D_{Z_6}(v)$ in $G(2,\mathcal{U}_4)$ is rationally connected, being birationally equivalent to the Fano manifold $D_{Z_6}(v)$. But then its projection to A_C must be constant. Since the fibers of this projection are only four-dimensional, while the dimension of $D_{Z_6}(v)$ is six, we get a contradiction. □

Proof of Theorem 4.8. Recall that the family $\mathcal{F}_H$ of Hecke lines has dimension seven, so $D_{Z_6}(v)$ cannot be the whole family. In fact, $\mathcal{F}_H$ has a rational map η to C, and by the same argument as above, the fact that $D_{Z_6}(v)$ is Fano ensures that its image in $\mathcal{F}_H$ is contained in a fiber of η, over some point $c \in C$. But then the morphism from $\mathrm{SU}_C(2,\mathcal{O}_C(c))$ to $\mathcal{F}_H$ is birational onto its image $D_{Z_6}(v)$. Since $\mathrm{SU}_C(2,\mathcal{O}_C(c))$ has Picard rank one [Ram73], this morphism must be an isomorphism. □

5. A Coble type quadric hypersurface

The aim of this section is to show that the Coble quadric hypersurface in $G(2,V_8)$ deserves its name in the sense that it is singular along the moduli space and it is uniquely determined by this property. So the section is mainly devoted to the proof of Theorem 5.1. In the last part, we also prove a self-duality statement concerning this hypersurface which is analogous to the self-duality of the Coble quartic in $\mathbb{P}(V_8)$.

5.1. The relative Pfaffian

As we have seen, the fact that $D_{Z_6}(v)$ is defined as a Pfaffian locus in $G(2,V_8)$ implies that it is the singular locus of a Pfaffian hypersurface, defined as the first degeneracy locus $D_{Z_1}(v)$ of the skew-symmetric morphism $\mathcal{Q} \longrightarrow \mathcal{Q}^\vee(1)$ defined by v.

Theorem 5.1. *The hypersurface $D_{Z_1}(v)$ of $G(2,V_8)$ is a quadratic section of the Grassmannian. It is the unique quadratic section that is singular along $D_{Z_6}(v)$.*

Remark 5.2. Starting from a genus three curve C and its Kummer threefold embedded in $\mathbb{P}^7$ by the linear system $|2\Theta|$, the original observation of Coble was that there exists a unique Heisenberg-invariant quartic $\mathcal{C}$ that is singular along the Kummer. Beauville proved much later that the Heisenberg-invariance hypothesis was actually not necessary [Bea03]. In our context, the curve and its Heisenberg group are not easily available (although there are connections between the latter and the Weyl group $W(E_7)$ of the theta-representation $\wedge^4 V_8$), so we do not use any Heisenberg-invariance hypothesis.

5.1.1. Structure of the proof of Theorem 5.1

Recall that D stands for $D_{Z_6}(v)$ and G for $G(2,V_8)$. That $D_{Z_1}(v)$ is a quadratic section of G follows from the fact that it is defined by a rank six Pfaffian, obtained as the image of v by the cubic morphism

$$S^3(\wedge^2 \mathcal{Q}^\vee(1)) \to \wedge^6 \mathcal{Q}^\vee(3) = \mathcal{O}_G(2).$$

In order to prove that this is the only quadratic section that is singular along D, recall that the conormal bundle of D in the Grassmannian G is the quotient of the ideal sheaf $\mathcal{I}_D$ by its square $\mathcal{I}_D^2$. Twisting by

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$\mathcal{O}_G(2)$ and taking cohomology, we get an exact sequence

$$0\longrightarrow H^0(G,\mathcal{I}_D^2(2))\longrightarrow H^0(G,\mathcal{I}_D(2))\longrightarrow H^0(D,\mathcal{N}_{D/G}^\vee(2))\longrightarrow H^1(G,\mathcal{I}_D^2(2)).$$

Observe that $H^0(G,\mathcal{I}_D(2))$ parametrizes quadratic sections of G (up to scalar) that contain D, while, since D is smooth, $H^0(G,\mathcal{I}_D^2(2))$ parametrizes quadratic sections that are singular along D. Our claim is that the latter space is one-dimensional. This will be proved in three steps: First, compute the dimension of the space of quadrics containing D; second, bound $H^0(D,\mathcal{N}_{D/G}^\vee(2))$ from below; third, prove that $H^1(G,\mathcal{I}_D^2(2))$ vanishes. These results are contained in Lemmas 5.3, 5.6, 5.11. From the fact that $H^1(\mathcal{I}_D^2(2))=0$, the exact sequence

$$0\longrightarrow H^0(\mathcal{I}_D^2(2))\longrightarrow H^0(\mathcal{I}_D(2))\longrightarrow H^0(\mathcal{N}_{D/G}^\vee(2))=H^0(\mathcal{N}_{D/G})\longrightarrow 0,$$

knowing that $h^0(\mathcal{I}_D(2))=71$ and $h^0(\mathcal{N}_{D/G})\geq 70$, will allow us to conclude that $h^0(\mathcal{I}_D^2(2))\leq 1$ and the proof will be complete.

5.1.2. Quadrics containing the moduli space

Let us count the quadric sections of $G=G(2,V_8)$ that contain the moduli space $D\simeq \mathrm{SU}_C(2,L)$.

Lemma 5.3. $h^0(G,\mathcal{I}_D(2))=71$.

Proof. Let us first recall the classical minimal resolution of the ideal I generated by submaximal Pfaffians of a generic skew-symmetric matrix of size 6; in other words, the ideal of the cone over the Grassmannian $G(2,V_6)$ inside $\wedge^2V_6$. As usual, we will use the notation S_λ, for λ a partition, in order to indicate the corresponding Schur functor. Letting $S=\mathbb{C}[\wedge^2V_6]$, this resolution is the following [Wey03, (6.4.6)]:

$$\begin{tikzcd}
 & 0 & \\
 & I \arrow[u] & \\
 & \wedge^4V_6^\vee\otimes S(-2) \arrow[u] & \\
 & S_{21111}V_6^\vee\otimes S(-3) \arrow[u] & \\
S_{311111}V_6^\vee\otimes S(-4) \arrow[ur] & \oplus & S_{22222}V_6^\vee\otimes S(-5) \arrow[ul] \\
 & S_{322221}V_6^\vee\otimes S(-6) \arrow[ul] \arrow[ur] & \\
 & S_{332222}V_6^\vee\otimes S(-7) \arrow[u] & \\
 & \det(V_6^\vee)^3\otimes S(-9) \arrow[u] & \\
 & 0 \arrow[u] &
\end{tikzcd}$$

Since D is a Pfaffian locus of the expected dimension, given by a skew-symmetric map $\mathcal{Q}\rightarrow\mathcal{Q}^\vee(1)$, we deduce the following free resolution of its twisted ideal sheaf (we used identifications like $S_{332222}\mathcal{Q}^\vee=\wedge^2\mathcal{Q}^\vee(-2)$):

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$$\begin{tikzcd}
 & 0 & \\
 & \mathcal{I}_D(2) \arrow[u] & \\
 & \wedge^4 \mathcal{Q} \arrow[u] & \\
 & \mathfrak{sl}(\mathcal{Q}) \arrow[u] & \\
S^2\mathcal{Q}^\vee(-1) \arrow[ur] & \oplus & S^2\mathcal{Q}(-1) \arrow[ul] \\
 & \mathfrak{sl}(\mathcal{Q})(-2) \arrow[ul] \arrow[ur] & \\
 & \wedge^2\mathcal{Q}(-3) \arrow[u] & \\
 & \mathcal{O}_G(-4) \arrow[u] & \\
 & 0 \arrow[u] &
\end{tikzcd}$$

Note that this resolution is self-dual, up to twist. Moreover, using the Bott–Borel–Weil theorem one can check that all the factors are acyclic homogeneous vector bundles, with two exceptions: $\wedge^4\mathcal{Q}$ has a nonzero space of sections, isomorphic to $\wedge^4 V_8$; and $S^2\mathcal{Q}^\vee(-1)$, which is one of the two irreducible factors of Ω^2_G, has a one dimensional cohomology group in degree two. We end up with a canonical exact sequence

$$0 \longrightarrow \wedge^4 V_8 \longrightarrow H^0(G, \mathcal{I}_D(2)) \longrightarrow \mathbb{C} \longrightarrow 0, \tag{5.1}$$

and our claim follows. □

Remark 5.4. Being defined by a cubic Pfaffian, the equation of the hypersurface $D_{Z_1}(v)$ must be a cubic $\mathrm{SL}(V_8)$-covariant of v in $\wedge^4 V_8$, taking values in $H^0(\mathcal{O}_G(2)) \simeq S_{22}V_8$. In fact, it is a $\mathrm{GL}(V_8)$-covariant, that by homogeneity with respect to V_8, must take its values in $S_{22}V_8 \otimes \det(V_8)$. One can check that the latter module has multiplicity one inside $S^3(\wedge^4 V_8)$, so this covariant is unique up to scalar. For example, it can be obtained as the composition

$$\begin{aligned} S^3(\wedge^4 V_8) &\hookrightarrow S^3(\wedge^2 V_8 \otimes \wedge^2 V_8) \to S^3(\wedge^2 V_8) \otimes S^3(\wedge^2 V_8) \to S^3(\wedge^2 V_8) \otimes \wedge^6 V_8 \to \\ &\to S^3(\wedge^2 V_8) \otimes \wedge^2 V_8^\vee \otimes \det(V_8) \to S^2(\wedge^2 V_8) \otimes \det(V_8) \to S_{22}V_8 \otimes \det(V_8). \end{aligned}$$

Following the natural morphisms involved in these arrows, this would allow to give an explicit formula for an equation of the quadratic hypersurface $D_{Z_1}(v)$ in terms of the coefficients of v (this was done in [RSSS13] for the Coble quartic itself). It would suffice to do this when v belongs to our preferred Cartan subspace; this is in principle a straightforward computation but the resulting formulas would be huge.

Remark 5.5. The embedding of $\wedge^4 V_8$ inside $H^0(G, \mathcal{I}_D(2))$ in Equation (5.1) is given by the derivatives of $D_{Z_1}(v)$ with respect to v, that is, can be obtained by polarizing the cubic morphism discussed in the previous remark. On the other hand, modulo these derivatives, (4) shows that there is a uniquely defined 'non-Pfaffian' quadric vanishing on D. This non-Pfaffian quadric comes from the contribution of $S^2\mathcal{Q}^\vee(-1)$ in the resolution of $\mathcal{I}_D(2)$. Since in this resolution, these two terms are connected one to the other through three morphisms having respective degree two, one and two with respect to v, the non-Pfaffian quadric must be given by a *quintic* covariant in v. And indeed, a computation with Lie [vLCL92] shows that

$$\mathrm{Hom}(S^5(\wedge^4 V_8), S_{22}V_8 \otimes (\det(V_8))^2)^{\mathrm{GL}(V_8)} \simeq \mathbb{C}^2.$$

A special line in this space of covariants is generated by the cubic covariant defining the Pfaffian quadric, twisted by the invariant quadratic form (defined by the wedge product). The quotient is our non-Pfaffian quadric. As before, we could in principle compute it explicitly by constructing a specific covariant. One way to construct such a covariant is to observe that

$$S^2(\wedge^4 V_8) \twoheadrightarrow S_{221111}V_8 \subset \wedge^2 V_8 \otimes \wedge^6 V_8 = \wedge^2 V_8 \otimes \wedge^2 V_8^\vee \otimes \det(V_8).$$

Taking the square of the resulting morphism, we can define a quartic covariant

$$S^4(\wedge^4 V_8) \longrightarrow S^2(\wedge^2 V_8) \otimes S^2(\wedge^2 V_8^\vee) \otimes \det(V_8)^2 \longrightarrow S_{22}V_8 \otimes \wedge^4 V_8^\vee \otimes \det(V_8)^2,$$

hence the desired quintic covariant.

5.1.3. The normal bundle of D in $G(2, V_8)$

Let us now bound from below the dimension of $H^0(D, \mathcal{N}_{D/G}^\vee(2))$. By Lemma 4.7, this space is isomorphic with $H^0(\mathcal{N}_{D/G})$, which parametrizes infinitesimal deformations of D inside G. Some of these deformations must be induced by the deformation of $[v]$ inside $\mathbb{P}(\wedge^4 V_8)$, which should provide 69 parameters. But recall that the family $\mathcal{F}_H$ of Hecke lines inside $SU_C(2)$ is a subvariety of $G(2, V_8)$, birationally fibered over the curve C, with one fiber isomorphic to $D \simeq \mathrm{SU}_C(2, \mathcal{O}_C(c))$ for some point $c \in C$. So we expect one extra deformation of D to be obtained by deforming c in the curve C. That these deformations are independent is essentially the content of

Lemma 5.6. $h^0(D, \mathcal{N}_{D/G}) \geq 70$.

Proof. The locus in $\wedge^4 V_6 \simeq \wedge^2 V_6^\vee$ corresponding to skew-symmetric forms of rank at most 2 is desingularized by the total space of $\wedge^4 \mathcal{U}_4$ over the Grassmannian $G(4, V_6)$ [Wey03, (6.4.2)]. As a consequence of this and of [BFMT20a, Proposition 2.3], the Pfaffian locus D is desingularized by the zero locus $Z := Z_6(v)$ inside $Fl(2, 6, V_8)$ of a (general) section of the bundle $\mathcal{V} = \wedge^4(V_8/\mathcal{U}_2)/\wedge^4(\mathcal{U}_6/\mathcal{U}_2)$. This bundle is an extension of irreducible bundles

$$0 \longrightarrow \wedge^3(\mathcal{U}_6/\mathcal{U}_2) \otimes (V_8/\mathcal{U}_6) \longrightarrow \mathcal{V} \longrightarrow \wedge^2(\mathcal{U}_6/\mathcal{U}_2) \otimes \det(V_8/\mathcal{U}_6) \longrightarrow 0.$$

By dimension count, Z is in fact isomorphic to D via the natural projection. Under this isomorphism and by Lemma 4.7, $\mathcal{N}_{D/G}$ can be identified with the restriction of $\mathcal{N} := \wedge^2(\mathcal{U}_6/\mathcal{U}_2) \otimes \det(V_8/\mathcal{U}_6)$ to Z. In order to compute the cohomology of this restriction, we can tensorize with $\mathcal{N}$ the Koszul complex $\wedge^\bullet \mathcal{V}^\vee$ of the global section of $\mathcal{V}$, whose zero locus is $Z \subset Fl(2, 6, V_8)$. This gives the following resolution of $\mathcal{N}_{D/G}$ by locally free sheaves on $Fl(2, 6, V_8)$

$$0 \longrightarrow \wedge^\bullet \mathcal{V}^\vee \otimes \mathcal{N} \longrightarrow \mathcal{N}_{D/G} \longrightarrow 0.$$

By applying the Bott–Borel–Weil theorem we can compute the cohomology groups of the bundles $\wedge^k \mathcal{V}^\vee \otimes \mathcal{N}$, for all $k \geq 0$. Those that do not vanish are the following:

$$H^0(\wedge^0 \mathcal{V}^\vee \otimes \mathcal{N}) = \wedge^4 V_8,$$
$$H^0(\wedge^1 \mathcal{V}^\vee \otimes \mathcal{N}) = \mathbb{C},$$

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Table 1. *Betti table of M.*

	0	1	2	3	4	5
0 :	·	·	·	·	·	·
1 :	·	·	·	·	·	·
2 :	·	·	·	·	·	·
3 :	·	·	·	·	·	·
4 :	105	399	595	405	105	·
5 :	·	·	·	21	35	15

$$H^2(\wedge^3\mathcal{V}^\vee\otimes\mathcal{N})=\mathbb{C},\quad H^3(\wedge^3\mathcal{V}^\vee\otimes\mathcal{N})=\mathbb{C}^2,$$
$$H^4(\wedge^4\mathcal{V}^\vee\otimes\mathcal{N})=H^5(\wedge^4\mathcal{V}^\vee\otimes\mathcal{N})=\wedge^4V_8,$$
$$H^4(\wedge^5\mathcal{V}^\vee\otimes\mathcal{N})=\mathfrak{sl}(V_8)\oplus\mathbb{C}^3,\quad H^5(\wedge^5\mathcal{V}^\vee\otimes\mathcal{N})=\mathfrak{sl}(V_8)\oplus\mathbb{C}^4,\quad H^6(\wedge^5\mathcal{V}^\vee\otimes\mathcal{N})=\mathbb{C},$$
$$H^6(\wedge^7\mathcal{V}^\vee\otimes\mathcal{N})=H^7(\wedge^7\mathcal{V}^\vee\otimes\mathcal{N})=\mathbb{C},$$
$$H^8(\wedge^9\mathcal{V}^\vee\otimes\mathcal{N})=H^9(\wedge^9\mathcal{V}^\vee\otimes\mathcal{N})=\mathbb{C},$$
$$H^{12}(\wedge^{13}\mathcal{V}^\vee\otimes\mathcal{N})=H^{13}(\wedge^{13}\mathcal{V}^\vee\otimes\mathcal{N})=\mathbb{C}.$$

A direct consequence is that $\chi(\mathcal{N}_{D/G})=70$. Moreover, observe that

$$H^q(\wedge^k\mathcal{V}^\vee\otimes\mathcal{N})=0\quad\text{for } q-k>1.$$

Since these groups give the first page of the spectral sequence in cohomology induced by the Koszul complex of $\mathcal{O}_Z$ twisted by $\mathcal{N}$, this implies that $H^i(\mathcal{N}_{D/G})=0$ for $i>1$. Therefore, $h^0(\mathcal{N}_{D/G})=\chi(\mathcal{N}_{D/G})+h^1(\mathcal{N}_{D/G})\geq 70$. □

5.1.4. An affine module *M*

As usual V_6 denotes a six-dimensional vector space. Let S be the coordinate ring of $\wedge^2V_6$. The ideal I of the cone over $G(2,V_6)$ is generated by the submaximal Pfaffians of the generic skew-symmetric matrix of size 6; the GL(6)-module generated by these submaximal Pfaffians is $\wedge^4V_6^\vee\subset S^2(\wedge^2V_6^\vee)$. The square of I is then generated by the symmetric square of this module. The module decomposes [vLCL92] as

$$S^2(\wedge^4V_6^\vee)=S_{221111}V_6^\vee\oplus S_{2222}V_6^\vee.$$

The first component is $\wedge^2V_6^\vee\otimes\det V_6^\vee$ and must be interpreted as parametrizing quartics that are multiples of linear forms by the Pfaffian cubic. The ideal they generate is S_+I_P, where $S_+\subset S$ is the irrelevant ideal, and I_P denotes the ideal of the Pfaffian hypersurface.

Consider the exact sequence

$$0\to S_+I_P\to I^2\to M:=I^2/S_+I_P\to 0.$$

The quotient module M is generated by $S_{2222}V_6^\vee$. According to [GS], the minimal resolution $R_\bullet$ of M has the Betti numbers of Table 1.

The minimal resolution is GL_6-equivariant, and it is not difficult to write it in terms of Schur functors. Indeed, we know that the quartic generators are parametrized by $S_{2222}V_6^\vee$, so the first syzygy module must be contained in $S_{2222}V_6^\vee\otimes\wedge^2V_6^\vee$, and it turns out that there is a unique GL_6-module of the correct dimension inside this tensor product. Proceeding inductively, we arrive at the following conclusion: The minimal GL_6-equivariant resolution of the S-module M has the following shape:

$$
\begin{tikzcd}
0 & \\
M \arrow[u] & \\
S_{2222}V_6^\vee \otimes S(-4) \arrow[u] & \\
(S_{32221}V_6^\vee \oplus \mathbf{S_{222211}}V_6^\vee) \otimes S(-5) \arrow[u] & \\
(S_{422211}V_6^\vee \oplus S_{33222}V_6^\vee \oplus \mathbf{S_{322221}}V_6^\vee) \otimes S(-6) \arrow[u] & \\
(S_{432221}V_6^\vee \oplus \mathbf{S_{422222}}V_6^\vee) \otimes S(-7) \arrow[u] & \mathbf{S_{333331}}V_6^\vee \otimes S(-8) \arrow[ul] \\
S_{442222}V_6^\vee \otimes S(-8) \arrow[u] & \mathbf{S_{433332}}V_6^\vee \otimes S(-9) \arrow[u] \arrow[ul] \\
 & \mathbf{S_{443333}}V_6^\vee \otimes S(-10) \arrow[u] \arrow[ul] \\
 & 0 \arrow[u]
\end{tikzcd}
$$

Here, vertical arrows have degree one and diagonal arrows have degree two. Notice that the complex in bold reproduces the resolution of the Pfaffian ideal I itself.

Remark 5.7. As J. Weyman observed, one could also obtain this resolution by considering the natural resolution of the Pfaffian hypersurface given by the total space of the vector bundle $\wedge^2\mathcal{U}$ over the Grassmannian $G(4, V_6)$. The morphism π from $\mathrm{Tot}(\wedge^2\mathcal{U})$ to $\wedge^2 V_6$ is a resolution of singularities, and one can check that M is the push-forward by π of the module given by the pull-back of the line bundle $\mathcal{O}(2)$ from the Grassmannian. Applying the geometric technique from [Wey03], one can extract the minimal resolution of M from the collection of $\mathrm{GL}(V_6)$-modules given by

$$F_i = \bigoplus_{j\geq 0} H^j(G(4, V_6), \mathcal{O}(2) \otimes \wedge^{i+j}(\wedge^2\mathcal{U})^\perp).$$

Here, $(\wedge^2\mathcal{U})^\perp$ is the kernel of the natural projection $\wedge^2 V_6^\vee \to \wedge^2\mathcal{U}^\vee$. The bundle $(\wedge^2\mathcal{U})^\perp$ is not semisimple but is an extension of $\mathcal{O}(-1)$ by $\mathcal{U}^\vee \otimes Q^\vee$. Remarkably, it is the contribution of $\mathcal{O}(-1)$ that reproduces the minimal resolution of I (twisted) inside that of M.

5.1.5. Relativizing M

Now, we want to use these results in the relative setting. Since $\wedge^4\mathcal{Q}$ is a vector bundle on $G(2, V_8)$ which is locally isomorphic to $\wedge^2 V_6$, we can relativize the construction of I and I_P and M. For convenience, let us restrict to the complement $\mathcal{X}$ of the zero section inside the total space of this vector bundle. We get sheaves of $\mathcal{O}_\mathcal{X}$-modules and ideals that we denote, respectively, by $\mathcal{I}', \mathcal{I}'_P, \mathcal{M}'$. Note that, since we avoid the zero section, we get an exact sequence

$$0 \to \mathcal{I}'_P \to \mathcal{I}'^2 \to \mathcal{M}' \to 0.$$

Then we consider $v \in \wedge^4 V_8$ as a general section of $\wedge^4\mathcal{Q}$, that we interpret as a morphism from $G = G(2, V_8)$ to the total space of $\wedge^4\mathcal{Q}$. By the definition of orbital degeneracy loci [BFMT20b, Definition 2.1], the ideal of $D_{Z_1}(v)$ is $\mathcal{I}_P := \mathcal{I}'_P \otimes \mathcal{O}_G$ and the ideal of $D = D_{Z_6}(v)$ is $\mathcal{I}_D := \mathcal{I}' \otimes \mathcal{O}_G$. Let us also denote $\mathcal{M} = \mathcal{M}' \otimes \mathcal{O}_G$. Of course, these tensor products are taken over $\mathcal{O}_\mathcal{X}$.

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Lemma 5.8. *There is an exact sequence*

$$0 \to \mathcal{I}_P \to \mathcal{I}_D^2 \to \mathcal{M} \to 0.$$

Proof. By the right exactness of tensor product, here by $\mathcal{O}_G$, we get an exact sequence

$$\mathcal{I}_P \to \mathcal{I}_D^2 \to \mathcal{M} \to 0.$$

But the map $\mathcal{I}_P \subset \mathcal{I}_D^2$ (which expresses the fact that D is contained in the singular locus of the Pfaffian hypersurface) clearly remains an injection, and we are done. □

In order to control $\mathcal{M}$, we will now consider the complex of vector bundles induced by the resolution we constructed for M. We can deduce a resolution of $\mathcal{M}'$ and then tensor out again by $\mathcal{O}_G$. In order to prove that we get a resolution of $\mathcal{M}$ (the resolution given just below), we need to check that the Tor-sheaves of $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{T}or_i(\mathcal{M}', \mathcal{O}_G)$ vanish for $i > 0$. All the Tor-sheaves we compute in the sequel will also be for $\mathcal{O}_{\mathcal{X}}$-modules.

$$\begin{tikzcd}
0 & \\
\mathcal{M} \arrow[u] & \\
S_{2222}\mathcal{Q}(-4) \arrow[u] & \\
S_{32221}\mathcal{Q}(-5) \oplus S_{222211}\mathcal{Q}(-5) \arrow[u] & \\
S_{422211}\mathcal{Q}(-6) \oplus S_{33222}\mathcal{Q}(-6) \oplus S_{322221}\mathcal{Q}(-6) \arrow[u] & \\
S_{432221}\mathcal{Q}(-7) \oplus S_{422222}\mathcal{Q}(-7) \arrow[u] & S_{333331}\mathcal{Q}(-8) \arrow[ul] \\
S_{442222}\mathcal{Q}(-8) \arrow[u] & S_{433332}\mathcal{Q}(-9) \arrow[u] \arrow[ul] \\
& S_{443333}\mathcal{Q}(-10) \arrow[u] \arrow[ul] \\
& 0 \arrow[u]
\end{tikzcd}$$

Lemma 5.9. *For any $i > 0$,*

1. $\mathcal{T}or_i(\mathcal{I}'_P, \mathcal{O}_G) = 0$,
2. $\mathcal{T}or_i(\mathcal{O}_{\mathcal{X}}/\mathcal{I}', \mathcal{O}_G) = 0$,
3. $\mathcal{T}or_i(\mathcal{I}'/\mathcal{I}'^2, \mathcal{O}_G) = 0$,
4. $\mathcal{T}or_i(\mathcal{M}', \mathcal{O}_G) = 0$.

Proof. (1) is obvious since $\mathcal{I}'_P$ is locally free. (2) is a consequence of the generic perfection theorem (see [EN67]), since I and therefore $\mathcal{I}'$ is perfect, and D has the expected dimension. (3) is a consequence of (2) because $\mathcal{I}'/\mathcal{I}'^2$ is a locally free $\mathcal{O}_{\mathcal{X}}/\mathcal{I}'$-module (recall that since we have a generality assumption the singular locus is avoided). Finally, to prove (4) observe first that, by the long exact sequence of Tor, $\mathcal{T}or_i(\mathcal{I}', \mathcal{O}_G) = \mathcal{T}or_{i+1}(\mathcal{O}_{\mathcal{X}}/\mathcal{I}', \mathcal{O}_G) = 0$ for any $i > 0$. Because of (3) this implies that

$\mathcal{T}or_i(\mathcal{I}'^2, \mathcal{O}_G) = 0$ for any $i > 0$. Then we can use the exact sequence of Lemma 5.8 to deduce that $\mathcal{T}or_i(\mathcal{M}', \mathcal{O}_G) = 0$ when $i > 1$ and that there is an exact sequence

$$0 \longrightarrow \mathcal{T}or_1(\mathcal{M}', \mathcal{O}_G) \longrightarrow \mathcal{I}_P \longrightarrow \mathcal{I}_D^2 \longrightarrow \mathcal{M} \longrightarrow 0.$$

By Lemma 5.8, $\mathcal{T}or_1(\mathcal{M}', \mathcal{O}_G)$ vanishes as well, and we are done. □

Lemma 5.10. $\mathcal{M}(2)$ *is acyclic.*

Proof. Twist the previous resolution of $\mathcal{M}$ by $\mathcal{O}(2)$, and deduce from the Bott–Borel–Weil theorem that all the bundles in the twisted resolution are acyclic. This implies the claim. □

Lemma 5.11. $H^i(\mathcal{I}_D^2(2)) = 0$ *for any* $i > 0$.

Proof. This follows immediately from Lemmas 5.8 and 5.10. □

This concludes the proof of Theorem 5.1. Note the following consequence: D has nonobstructed deformations.

Corollary 5.12. $h^0(\mathcal{N}_{D/G}) = 70$ *and* $h^i(\mathcal{N}_{D/G}) = 0$ *for any* $i > 0$.

5.2. *Deforming the Pfaffian hypersurface*

We already observed that, varying v in $\wedge^4 V_8$, we only get a codimension one family of deformations of D. The missing dimension is provided by the choice of the point on the curve C, but this is invisible in our constructions. We will nevertheless prove that the special quadric section of the Grassmannian deforms.

Lemma 5.13. *For a generic point* $p \in C$ *and the associated embedding* $\varphi_p : \mathrm{SU}_C(2, \mathcal{O}_C(p)) \hookrightarrow G(2, V_8)$ *(see Remark 4.9), there exists at most one quadric hypersurface* Q_p *in the Grassmannian that is singular along* $\mathrm{SU}_C(2, \mathcal{O}_C(p))$.

Proof. Such a quadric corresponds to a line in $H^0(G(2, V_8), \mathcal{I}^2_{\mathrm{SU}_C(2,\mathcal{O}_C(p))}(2))$ and we have computed in the proof of Theorem 5.1 that this space has dimension one for certain special points p. By semicontinuity, this dimension remains smaller or equal to one for p generic. □

Theorem 5.14. *For the generic embedding* φ_p*, there exists a unique quadric hypersurface of* $G(2, V_8)$ *that is singular along* $\mathrm{SU}_C(2, \mathcal{O}_C(p))$.

Proof. Let us consider the embedding $Q = D_{Z_1}(v) \hookrightarrow G$ from Theorem 5.1. Let $H'_{Q/G}$ be the so-called 'locally trivial Hilbert scheme' parametrizing locally trivial deformations of $Q \subset G$, as defined in [GK89, 2.2]. Remark that the construction of [GK89] is done for finite singularities, but their arguments, as the authors underline in the introduction, go through for arbitrary singularities because of [FK87]. Let

$$\mathcal{N}'_{Q/G} = \mathrm{Ker}(\mathcal{N}_{Q/G} \to \mathcal{T}^1_Q),$$

where $\mathcal{T}^1_Q$ denotes the first cotangent sheaf of Q (as defined, for instance, in [Ser06, Section 1.1.3]). In order that the locally trivial Hilbert scheme be smooth at Q, by [GK89, Prop. 2.3], we need that $H^1(Q, \mathcal{N}'_{Q/G}) = 0$. If this happens, then $h^0(Q, \mathcal{N}'_{Q/G}) = \dim(H'_{Q/G})$ and we will show that this equals 70. By [Ser06, Section 4.7.1], we have an exact sequence

$$0 \to T_Q \to T_G|_Q \to \mathcal{N}_{Q/G} \to \mathcal{T}^1_Q \to 0.$$

Hence, $\mathcal{N}'_{Q/G}$ coincides with the image of $T_{G/Q}$ inside $\mathcal{N}_{Q/G}$, which is exactly the (twisted) Jacobian ideal $\mathcal{J}_{Q/G}(2)$ restricted to Q. In turn, the Jacobian ideal of the Pfaffian locus of 6×6 matrices is exactly

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the ideal of 4×4 skew-symmetric minors. This implies that $\mathcal{J}_{Q/G}(2)$ is the twisted ideal $\mathcal{I}_D(2)/\mathcal{I}_Q(2)$ of D inside $\mathcal{O}_Q(2) = \mathcal{O}_Q(Q)$. Let us therefore consider the exact sequence

$$0 \to \mathcal{I}_Q(2) \to \mathcal{I}_D(2) \to \mathcal{J}_{Q/G}(2) \to 0. \tag{5.2}$$

By Lemma 5.3, we have $h^0(G, \mathcal{I}_D(2)) = 71$ and in the proof of the same lemma we showed that $h^i(G, \mathcal{I}_D(2)) = 0$, for $i > 0$. On the other hand, we have $\mathcal{I}_Q(2) = \mathcal{O}_G$. Via the long cohomology exact sequence associated to sequence (5.2), we deduce that $h^0(G, \mathcal{J}_{Q/G}(2)) = 70$ and $h^i(G, \mathcal{J}_{Q/G}(2)) = 0$ for $i > 0$. Hence $H'_{Q/G}$ is smooth of dimension 70 at $[Q]$. We have a natural map between Hilbert schemes

$$\sigma : H'_{Q/G} \to H_D,$$

where H_D is the component of the Hilbert scheme of $G(2, V_8)$ that contains the point $[D]$ defined by D. Both spaces have dimension 70 and are smooth, respectively, at $[Q]$ and $[D]$ by Corollary 5.12. In order to show that σ is dominant, it is enough to check that the induced morphism of tangent spaces is dominant. This is true because $H^0(G, \mathcal{J}_{Q/G}(2))$ and $H^0(\mathcal{N}_{D/G})$ are both dominated by $H^0(\mathcal{I}_D(2))$, and the morphism from $\mathcal{I}_D(2)$ to $\mathcal{N}_{D/G}$ factorizes through $\mathcal{J}_{Q/G}(2)$. This concludes the proof. □

5.3. Grassmannian self-duality

Exactly as we constructed the singular quadric hypersurface $D_{Z_1}(v) \subset G(2, V_8)$, there is another hypersurface $D_{Z_1}(v^\vee) \subset G(2, V_8^\vee) = G(6, V_8)$. Because of Proposition 3.5, these two hypersurfaces are projectively isomorphic since they are constructed from lines contained in isomorphic Coble quartics. But one should also expect some projective duality statement analogous to Theorem 3.4. Of course, we cannot refer to classical projective duality since we want to consider $D_{Z_1}(v)$ and $D_{Z_1}(v^\vee)$ really as hypersurfaces in Grassmannians, not as subvarieties of the ambient projective spaces. It turns out that a version of projective duality in this setting (and for certain other ambient varieties than Grassmannians) was once proposed in [Cha07] (that remained unpublished). We will refer to it as *Grassmannian duality*.

The idea is the following. Consider, say, a hypersurface H in $G(2, V_8)$ (or any Grassmannian, but let us restrict to the case we are interested in). At a smooth point $h = [U_2]$ of H, the tangent space to H is a hyperplane in $T_h G(2, V_8) = \mathrm{Hom}(U_2, V_8/U_2)$ or, equivalently, a line in the dual space $\mathrm{Hom}(V_8/U_2, U_2)$. If this line is generated by a surjective morphism, the kernel of this morphism is a four-dimensional subspace of V_8/U_2. Equivalently, this defines a six-dimensional space U_6 such that $U_2 \subset U_6 \subset V_8$. We get in this way a rational map from H to $G(6, V_8)$, and we can define the Grassmannian dual $H^\vee$ as the image of this rational map. For more details, see [Cha07, section 1.6]. Chaput has a remarkable biduality theorem generalizing the classical statement, according to which duality for subvarieties of Grassmannians is an involution [Cha07, Theorem 2.1].

So this Grassmannian duality is perfectly natural, and we have:

Theorem 5.15. *$D_{Z_1}(v) \simeq D_{Z_1}(v^\vee)$ is Grassmannian self-dual.*

Proof. Suppose that U_2 belongs to $D_{Z_1}(v)$. By definition, this means that there exists $U_4 \supset U_2$ (unique in general) such that

$$v \in U_2 \wedge (\wedge^3 V_8) + (\wedge^2 U_4) \wedge (\wedge^2 V_8).$$

If we mod out by $\wedge^2 U_4$, we get a tensor in $U_2 \otimes \wedge^3(V_8/U_4) \simeq U_2 \otimes (V_8/U_4)^\vee$, that is, a morphism from V_8/U_4 to U_2. Generically, this morphism has full rank, and its kernel defines some $U_6 \supset U_4$. So we get a flag $(U_2 \subset U_4 \subset U_6)$ such that

$$v \in U_2 \wedge (\wedge^2 U_6) \wedge V_8 + (\wedge^2 U_4) \wedge (\wedge^2 V_8). \tag{5.3}$$

□

Lemma 5.16. *U_6 defines a point of $D_{Z_1}(v^\vee)$.*

Proof. Using adapted basis, one checks that condition (5.3) implies that

$$v^\vee \in U_6^\perp \wedge (\wedge^2 U_2^\perp) \wedge V_8^\vee + (\wedge^2 U_4^\perp) \wedge (\wedge^2 V_8^\vee).$$

In particular, $v^\vee$ mod $U_6^\perp$ has rank at most four. □

Lemma 5.17. *U_6 defines a point of $D_{Z_1}(v)^\vee$.*

Proof. Using a basis of V_8 adapted to the flag $(U_2 \subset U_4 \subset U_6)$, we can rewrite relation (5.3) in the form

$$v = e_1 \wedge e_5 \wedge e_6 \wedge e_7 + e_2 \wedge e_5 \wedge e_6 \wedge e_8 + v', \quad v' \in (\wedge^2 U_4) \wedge (\wedge^2 V_8),$$

where $U_2 = \langle e_1, e_2 \rangle$ and $U_6 = \langle e_1, \dots, e_6 \rangle$. We can describe infinitesimal deformations of U_2 by some infinitesimal deformations of the vectors in the adapted basis, say $e_i \mapsto e_i + \epsilon\delta_i$, and we must keep a similar relation. Modding out by U_4, we only remain with the relation

$$\delta_1 \wedge e_5 \wedge e_6 \wedge e_7 + \delta_2 \wedge e_5 \wedge e_6 \wedge e_8 = 0 \qquad \text{mod } U_4,$$

which we can simply rewrite as $\delta_{18} = \delta_{27}$. This relation describes the tangent hyperplane to $D_{Z_1}(v)$ at U_2, as a hyperplane in $\mathrm{Hom}(U_2, V_8/U_2)$, orthogonal to the morphism $e_8^* \otimes e_1 - e_7^* \otimes e_2$. The kernel of this morphism is U_6/U_2, and we are done. □

These two lemmas together imply that $D_{Z_1}(v^\vee)$ coincides with the Grassmannian dual to $D_{Z_1}(v)$. The proof of the theorem is complete.

Note that we can resolve the singularities of $D_{Z_1}(v)$ by considering flags $(U_2 \subset U_4)$ as before, which gives a subvariety $\tilde{D}_{Z_1}(v) \subset Fl(2,4,V_8)$. By considering the flags $(U_2 \subset U_4 \subset U_6)$ as in the proof of the previous statement, we obtain a subvariety $D_{Z_1}(v, v^\vee) \subset Fl(2,4,6,V_8)$ that resolves simultaneously the singularities of $D_{Z_1}(v)$ and $D_{Z_1}(v^\vee)$. As for the Coble quartic, we get a diagram

$$\begin{tikzcd}
 & & Fl(2,4,6,V_8) & & \\
Fl(2,4,V_8) & & D_{Z_1}(v,v^\vee) \arrow[u, hook] \arrow[dl] \arrow[dr] & & Fl(4,6,V_8) \\
 & \tilde{D}_{Z_1}(v) \arrow[ul, hook] \arrow[dl] \arrow[rr, dashed] & & \tilde{D}_{Z_1}(v^\vee) \arrow[ur, hook] \arrow[dr] & \\
G(2,V_8) \supset D_{Z_1}(v) \arrow[rrrr, dashed, "dD"] & & & & D_{Z_1}(v^\vee) \subset G(6,V_8)
\end{tikzcd}$$

The birational map $\tilde{D}_{Z_1}(v) \dashrightarrow \tilde{D}_{Z_1}(v^\vee)$ must be a flop, resolved by two symmetric contractions.

Question. Is there a modular interpretation of $D_{Z_1}(v)$ as for the Coble quartic and of this diagram?

Remark 5.18. Our framework excludes the hyperelliptic genus three curves, but there should be a very similar story for these curves. In fact, consider a general pencil of quadrics in $\mathbb{P}^7 = \mathbb{P}(V_8)$. The eight singular members of the pencil define such a hyperelliptic curve C. It is a special case of the results of [DR77] that the moduli space $\mathrm{SU}_C(2, L)$, for L of odd degree, can be identified with the biorthogonal Grassmannian, that is the subvariety of $G(2, V_8)$ parametrizing subspaces that are isotropic with respect to any quadric in the pencil. On the other hand, the even moduli space $\mathrm{SU}_C(2)$ is a double cover of the six-dimensional quadric $\mathbb{Q}^6$, branched over a quartic section which is singular along a copy of the Kummer threefold of the curve. One expects this quartic to be of Coble type in the sense that it should be the unique quartic section of $\mathbb{Q}^6$ that is singular along the Kummer of C. It should also be self-dual in a suitable sense, and the whole story should be related to the representation theory of Spin_8. We plan to explore these topics in future work.

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