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DYNAMICS OF THE SAVING RATE IN THE NEOCLASSICAL GROWTH MODEL WITH CES PRODUCTION

MANUEL A. GÓMEZ
University of A Coruña

This paper characterizes the global dynamics of the saving rate in the neoclassical growth model with CES production. The study is based on qualitative phase-diagram analysis. The analytical conditions characterizing the cases that may arise theoretically depending on the parameters' configuration are obtained. It is well known that the saving rate behaves monotonically if technology is Cobb-Douglas. However, when the elasticity of substitution is nonunitary, the saving rate path may exhibit nonmonotonic behavior.

Keywords: Economic Growth, Saving, CES Production

1. INTRODUCTION

The Ramsey-Cass-Koopmans model is one of the most important theoretical frameworks in macroeconomics (Ramsey, 1928; Cass, 1965; Koopmans, 1965). It has become so familiar that one may presume that there is nothing new to say about it. However, the dynamics of the saving rate path is rather elusive from a theoretical standpoint, and general statements on its properties are hard to obtain. Cass and Koopmans dropped the fixed savings assumption of the Solow (1956) model, allowing dynamic optimizing savings behavior á la Ramsey. Recently, Barro and Sala-i-Martin (1995) characterized the global dynamics of the saving rate in the neoclassical growth model in the case of isoelastic utility and a Cobb-Douglas (CD) production function. They proved that, in this case, the saving rate is either monotonically increasing, monotonically decreasing, or constant throughout the entire transition path. However, a similar analysis assuming a constant-elasticity-of-substitution (CES) technology is lacking in the literature. The purpose of the present paper is to fill this gap.

This paper adds to a growing literature that analyzes the effect of the elasticity of substitution on a variety of issues in macroeconomics. The departure from the assumption of a unitary elasticity of substitution has important consequences in growth theory. If the elasticity of substitution is above one, long-run growth may

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arise even without any technological progress in the Solow and Ramsey-Cass-Koopmans growth models (e.g., Pitchford, 1960; Barro and Sala-i-Martin, 1995). If the elasticity of substitution is below one, the existence of multiple equilibria and poverty traps in the Diamond (1965) overlapping-generations growth model is possible (e.g., Azariadis, 1996). If the technology is CD, the factor income shares are constant and the direction of technical change is irrelevant for income distribution. However, the pronounced cycles in factor income distribution visible in many countries support the more general CES function and make possible biases of technical change an important issue (e.g., Blanchard, 1997; Bentolila and Saint-Paul, 2003; Acemoglu, 2003). Caselli (2005) shows that the relative roles assigned to differences in factor quantities and differences in the efficiency with which those factors of production are used in explaining cross-country income variations are sensitive to the elasticity of substitution. Klump and de La Grandville (2000) show a positive relationship between the elasticity of substitution and economic growth in the Solow growth model, a property the Diamond growth model does not exhibit (Miyagiwa and Papageorgiou, 2003). The elasticity of substitution also plays a key role in other issues such as the impact of corporate taxation on capital formation (Chirinko, 2002), the speed of convergence toward the balanced growth path (Turnovsky, 2002), the local determinacy properties of equilibria (Nishimura and Venditti, 2004), and the existence of scale effects (Zuleta, 2004). This renewed interest in the elasticity of substitution has been fostered by the empirical literature and, although the estimates obtained are wide-ranging [see, e.g., the reviews in Chirinko (2002) and Klump et al. (in press)], it appears that the CD specification must be rejected in favor of the CES specification (see also Duffy and Papageorgiou, 2000; Antràs, 2004; and Masanjala and Papageorgiou, 2004).

This paper analyzes the dynamics of the saving rate in the neoclassical growth model with CES production, which has as a particular case the CD technology. Our work hinges mainly on a phase-diagram analysis. This allows the qualitative characterization of the global dynamics of the saving rate. We obtain analytical conditions that characterize all the cases that may arise theoretically depending on the parameters' configuration. In particular, we identify the cases in which the saving rate exhibits globally nonmonotonic behavior. The monotonicity of the saving rate path that is obtained with CD technology does not fit in with the nonmonotonic saving patterns observed in many regions and countries around the world (see, e.g., Schmidt-Hebbel et al., 1996; Loayza et al., 2000). Hence, that theoretically the saving rate may exhibit nonmonotonic behavior when the elasticity of substitution is nonunitary suggests that the elasticity of substitution could also play a role in explaining the saving dynamics observed. For the sake of completeness, we characterize the dynamics of the saving rate for all configurations of the parameters, including those leading to endogenous growth, and take into account the irreversibility constraint on investment.

Recent related work has been done by Smetters (2003). He shows that for an elasticity of substitution below (above) one, the saving rate decreases (increases) along the transition path after the capital stock reaches a critical value identified analytically therein. Before reaching this value, the saving rate might increase (decrease) and so the entire saving rate path manifests overshooting (undershooting). He also devises a necessary and sufficient condition for the saving rate path to be nonmonotonic. However, checking this condition requires knowing the value of the saving rate (a jump variable) at the initial time, which is not available unless the transition path of the saving rate is (numerically) computed for specific parameter values. Hence, although Smetters's analytical results shed light on the behavior of the saving rate, these results do not allow us to characterize the global dynamics of the saving rate given the parameters' configuration, which is the goal of the present paper.

The remainder of this paper is organized as follows. Section 2 describes the model. Sections 3 and 4 analyze the dynamics of the saving rate for the cases of exogenous and endogenous growth, respectively. Section 5 concludes.

2. THE MODEL

Consider a closed economy populated by a fixed number of identical infinitely lived households that, for simplicity, is normalized to one. The household size, L, grows at the exogenous rate n, $L(t) = e^{nt}$, where $L(0)$ is normalized to one. Let $C(t)$ be aggregate consumption, and $\hat{C}(t) \equiv C(t)/L(t)$ be consumption per capita. Each household maximizes its dynastic utility

$$U = \int_0^\infty u[\hat{C}(t)]L(t)e^{-\rho t}dt = \int_0^\infty u[\hat{C}(t)]e^{-(\rho-n)t}dt, \qquad (1)$$

where ρ is the rate of time preference, $\rho > n$. The time argument is suppressed in all subsequent equations. The instantaneous utility function takes the isoelastic form

$$u(\hat{C}) = \begin{cases} \dfrac{\hat{C}^{1-\theta}}{1-\theta}, & \text{if } \theta \neq 1, \\ \log \hat{C}, & \text{if } \theta = 1, \end{cases} \qquad (2)$$

where $1/\theta > 0$ is the elasticity of intertemporal substitution.

Each individual supplies inelastically one unit of labor each period. Output, Y, is produced with the CES technology:

$$Y = F(K, T \cdot L) = A\,[\alpha K^{1-1/\sigma} + (1-\alpha)(T \cdot L)^{1-1/\sigma}]^{1/(1-1/\sigma)}$$
$$A > 0, \quad 0 < \alpha < 1, \quad \sigma > 0, \qquad (3)$$

where K denotes the capital stock and T the labor-augmenting technological progress. Labor productivity grows at the exogenous rate x, $T(t) = e^{xt}$, where $T(0)$ is normalized to 1. The term $T \cdot L$ is known as "effective labor." The parameter σ is the elasticity of substitution between capital and labor, and α is the capital weight in production. The CD technology is a particular case of the

CES technology where the elasticity of substitution is one and α equals the capital share of output.

The household's budget constraint is

$$F(K, T \cdot L) = I + C, \tag{4}$$

where I is gross investment. Throughout this paper, a dot over a variable denotes its time derivative, that is, $\dot{h} \equiv dh/dt$. The capital stock accumulates according to

$$\dot{K} = I - \delta K, \tag{5}$$

where δ is the depreciation rate. We assume that gross investment must be non-negative,

$$I \geq 0. \tag{6}$$

The household maximizes its dynastic utility (1) subject to constraints (4), (5), and (6). Let $y \equiv Y/(T \cdot L)$, $k \equiv K/(T \cdot L)$, $i \equiv I/(T \cdot L)$, and $c \equiv C/(T \cdot L)$ denote output, capital stock, gross investment and consumption per effective labor unit, respectively. The production function (3) can be expressed in intensive form as

$$y = f(k) = A\,[\alpha k^{1-1/\sigma} + (1-\alpha)]^{1/(1-1/\sigma)}. \tag{7}$$

The household's optimization problem can be equivalently expressed as

$$\max \int_0^\infty u(c)\, e^{-(\omega-\pi)t} dt, \tag{8a}$$

subject to

$$\dot{k} = i - \pi k, \tag{8b}$$

$$f(k) = i + c, \tag{8c}$$

$$i \geq 0, \tag{8d}$$

where

$$\omega = \rho + \theta x + \delta, \tag{9a}$$

$$\pi = x + n + \delta. \tag{9b}$$

Let J be the current-value Lagrangian of this problem,

$$J = u(c) + \lambda(i - \pi k) + \mu[f(k) - c - i].$$

The first-order necessary conditions for an interior optimum are[1]

$$\frac{\partial J}{\partial c} = c^{-\theta} - \mu = 0, \tag{10a}$$

$$\frac{\partial J}{\partial i} = \lambda - \mu \leq 0, \quad i \geq 0, \quad (\lambda - \mu) i = 0, \tag{10b}$$

$$\frac{\partial J}{\partial k} = -\pi\lambda + \mu f'(k) = (\omega - \pi)\lambda - \dot{\lambda}, \tag{10c}$$

plus the transversality condition:

$$\lim_{t\to\infty} e^{-(\omega-\pi)t}\lambda k = 0. \tag{10d}$$

When the nonnegativity constraint on gross investment is nonbinding, $i > 0$, equation (10b) yields $\lambda = \mu$. Log-differentiating (10a) and using (10c), we get

$$\frac{\dot{c}}{c} = \frac{1}{\theta}[f'(k) - \omega]. \tag{11a}$$

From (8b) and (8c), we get the overall resources constraint

$$\frac{\dot{k}}{k} = \frac{f(k)}{k} - \frac{c}{k} - \pi. \tag{11b}$$

The system (11a)–(11b) describes the dynamics of the economy in terms of c and k.

Because we want to focus on the behavior of the saving rate, we reformulate this system in terms of the marginal product of capital, $r \equiv f'(k)$, and the ratio of consumption to output, $z \equiv c\big/f(k)$. The saving rate, s, is then given by $s = 1 - z$. Henceforth, let

$$B = A^{\sigma-1}\alpha^{\sigma}, \tag{12a}$$

$$\bar{r} = B^{1/(\sigma-1)}. \tag{12b}$$

Noting that

$$r \equiv f'(k) = B^{1/\sigma}[f(k)\big/k]^{1/\sigma}, \tag{13}$$

the resource constraint (11b) can be rewritten as

$$\frac{\dot{k}}{k} = \frac{1}{B}(1 - z)\, r^{\sigma} - \pi. \tag{14}$$

Differentiating (13) with respect to time, we get

$$\dot{r} = f''(k)\,\dot{k} = \frac{1}{\sigma}B^{1/\sigma}[f(k)\big/k]^{-1+1/\sigma}[f'(k) - f(k)\big/k]\,\frac{\dot{k}}{k},$$

which after using (13), (14) and some algebra can be expressed in terms of r and z as

$$\dot{r} = -\frac{1}{\sigma}\left[\frac{1}{B}(1 - z)\, r^{\sigma} - \pi\right](1 - Br^{1-\sigma})\, r. \tag{15a}$$

Log-differentiating $z \equiv c/f(k)$ yields $\dot{z}/z = (\dot{c}/c) - f'(k)\,[k/f(k)]\,(\dot{k}/k)$, which after using (11a), (13), and (14) can be rewritten as

$$\dot{z} = \left[B\pi\, r^{1-\sigma} - \left(1 - z - \frac{1}{\theta}\right) r - \frac{\omega}{\theta}\right] z. \tag{15b}$$

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The system (15a)–(15b) has two nontrivial steady states if the elasticity of substitution is nonunitary, $\sigma \neq 1$, and only one if it is unitary, $\sigma = 1$. The first steady state is

$$r^* = f'(k^*) = \omega, \tag{16a}$$

$$z^* = 1 - B\pi\omega^{-\sigma} = 1 - (\pi/r^*)\,(r^*/\bar{r})^{1-\sigma}. \tag{16b}$$

The transversality condition (10d) is equivalent to

$$r^* = \omega > \pi. \tag{17}$$

We say that r^* is feasible if there exists $k^* \in (0, +\infty)$ such that condition (16a) is satisfied. If the elasticity of substitution is one, $\sigma = 1$, the fulfillment of the Inada conditions, $\lim_{k\to 0} f'(k) = \infty$ and $\lim_{k\to\infty} f'(k) = 0$, guarantees that r^* is feasible. However, if the elasticity of substitution is below one, $\sigma < 1$, one of the Inada conditions is violated, namely, $\lim_{k\to 0} f'(k) = \bar{r}$, and so, the condition

$$\omega = r^* < \bar{r} = B^{1/(\sigma-1)} \quad \text{if } \sigma < 1 \tag{18a}$$

is required to ensure that r^* is feasible. If the elasticity of substitution is above one, $\sigma > 1$, the other Inada condition is violated because $\lim_{k\to\infty} f'(k) = \bar{r}$, and so, the following condition is required for r^* to be feasible:

$$\omega = r^* > \bar{r} = B^{1/(\sigma-1)} \quad \text{if } \sigma > 1. \tag{18b}$$

We assume in this and in the following section that the steady state (r^*, z^*) is feasible, so that (18a) or (18b) is fulfilled if the elasticity of substitution is nonunitary, and that condition (17) is satisfied.[2] This steady state is one of exogenous growth, in which the variables expressed in per effective labor units are constant; that is, $\dot{y}^*/y^* = \dot{c}^*/c^* = \dot{k}^*/k^* = 0$.[3] The phase-diagram analysis performed in Section 3 shows that (r^*, z^*) is a saddle point.

If the elasticity of substitution is nonunitary, $\sigma \neq 1$, the second steady state of the system (15a)–(15b) is

$$\bar{r} = B^{1/(\sigma-1)}, \tag{19a}$$

$$\bar{z} = 1 - \frac{\bar{r} - \omega}{\theta\bar{r}} - \frac{\pi}{\bar{r}}. \tag{19b}$$

We say that the steady state $(\bar{r}, \bar{z})$ is feasible if $0 \leq \bar{z} \leq 1$. The phase-diagram analysis performed in Section 3 shows that the steady state $(\bar{r}, \bar{z})$ is unstable if (r^*, z^*) is feasible.

So far we have considered the case where the irreversibility constraint is not binding. When the irreversibility constraint, $i \geq 0$, is binding, the dynamics of the economy is driven by $c = f(k)$ and $\dot{k} = -\pi k$, or, equivalently, in terms of r and

z, by $z = 1$ and

$$\dot{r} = \frac{\pi}{\sigma}(1 - Br^{1-\sigma})\, r. \tag{20}$$

3. PHASE DIAGRAM ANALYSIS

In this section, we perform a phase-diagram analysis of the dynamics of the economy in (r, z)-plane. It should be noted that because $f''(k) < 0$ for all $k > 0$, there is a one-to-one monotonically decreasing relationship between the capital stock per effective labor unit, k, and the marginal product of capital, $r \equiv f'(k)$. Thus, the phase diagrams in (r, z)-plane may be easily interpreted in terms of the capital stock per effective labor unit, k, and the saving rate, $s = 1 - z$, by noting that k and s behave exactly opposite to how r and z, respectively, do. In the unitary elasticity case, $\sigma = 1$, the fulfillment of the Inada conditions entails that r take values in $(0, +\infty)$. In the low elasticity case, $\sigma < 1$, given that $\lim_{k\to 0} f'(k) = \bar{r}$, the relevant range of r is $(0, \bar{r})$. In the high elasticity case, $\sigma > 1$, given that $\lim_{k\to\infty} f'(k) = \bar{r}$, the relevant range of r is $(\bar{r}, +\infty)$. To facilitate the interpretation of the phase diagrams, the portions of the figures corresponding to irrelevant values of the marginal product of capital, r, are shaded.

Equation (15a) implies that there are three $\dot{r} = 0$–loci. The first two are vertical at $r = 0$ and $r = \bar{r}$, respectively. The third one is the curve:

$$z_r(r) = 1 - B\pi r^{-\sigma}, \tag{21}$$

which is upward sloping and concave, increases from minus infinity at $r = 0$, and goes to one as r goes to infinity, irrespective of the value of the elasticity of substitution. Given the configuration of the $\dot{r} = 0$–loci, the arrows point west (east) above (below) this curve for $r \in (0, \bar{r})$ if $\sigma < 1$, for $r \in (0, +\infty)$ if $\sigma = 1$, and for $r \in (\bar{r}, +\infty)$ if $\sigma > 1$, which are the relevant ranges of r in each case.

From (15b), the $\dot{z} = 0$–locus is (aside from $z = 0$) the curve

$$z_z(r) = 1 - B\pi r^{-\sigma} - \frac{1}{\theta} + \frac{\omega}{\theta r}. \tag{22}$$

It easily can be observed that the $\dot{z} = 0$–locus is above the $\dot{r} = 0$–locus given by (21) up to $r = r^*$ and is below it from $r = r^*$ onward. Given the configuration of the $\dot{z} = 0$–locus, for $z > 0$ the arrows point north (south) above (below) this curve irrespective of the value of the elasticity of substitution. Because the shape of the $\dot{z} = 0$–locus depends on the elasticity of substitution, it is analyzed for each particular case.

Whereas the nonnegativity constraint on gross investment is binding, z remains constant at 1, and (20) entails that r increases for $r \in (0, \bar{r})$ if $\sigma < 1$, for $r \in (0, +\infty)$ if $\sigma = 1$, and for $r \in (\bar{r}, +\infty)$ if $\sigma > 1$, which are the relevant ranges of r in each case.

The phase diagrams that follow have been depicted taking into account the results derived by Arrow and Kurz (1970) and summarized in the Appendix. We

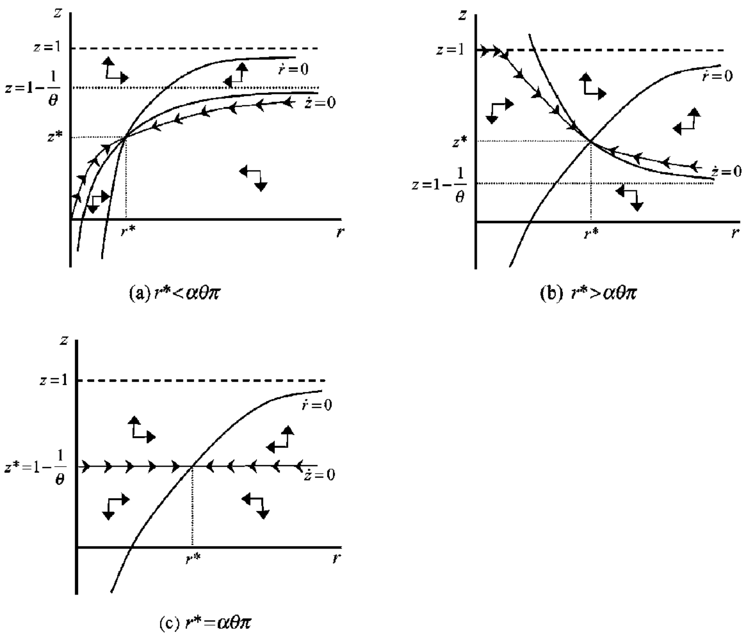

FIGURE 1. Transitional dynamics in the unitary elasticity case, $\sigma = 1$.

first consider the benchmark case of unitary elasticity of substitution, $\sigma = 1$, which has been analyzed by Barro and Sala-i-Martin (1995). Next, we analyze the cases of low elasticity, $\sigma < 1$, and high elasticity of substitution, $\sigma > 1$.

3.1. Unitary Elasticity of Substitution, $\sigma = 1$

In this case, the $\dot{z} = 0$–locus defined by (22) is the curve $z_z(r) = 1 - 1/\theta + (\omega - \theta\alpha\pi)/(\theta r)$, which is upward sloping and concave if $r^* = \omega < \alpha\theta\pi$; downward sloping and convex if $r^* > \alpha\theta\pi$, and horizontal if $r^* = \alpha\theta\pi$. In any case, as r goes to infinity, z goes to $1 - 1/\theta$.

Figure 1 illustrates the various phase diagrams in (r, z)-plane that may arise. In any case, the saving rate behaves in monotonic fashion. If $r^* < \theta\alpha\pi$, panel (a) shows that the saving rate increases monotonically toward its stationary value as the economy grows, starting at $k(0) < k^*$. If $r^* > \theta\alpha\pi$, panel (b) shows that the saving rate decreases monotonically during the transition toward its steady state if the economy starts at $k(0) < k^*$. If $r^* = \theta\alpha\pi$, panel (c) shows that the saving rate is constant during the transition.

3.2. Low Elasticity of Substitution, $0 < \sigma < 1$

In this case, equation (22) implies that the $\dot{z} = 0$–locus decreases from plus infinity at $r = 0$, reaches a minimum at

$$r_C = (B\pi\theta\sigma/\omega)^{1/(\sigma-1)}, \qquad \textbf{(23)}$$

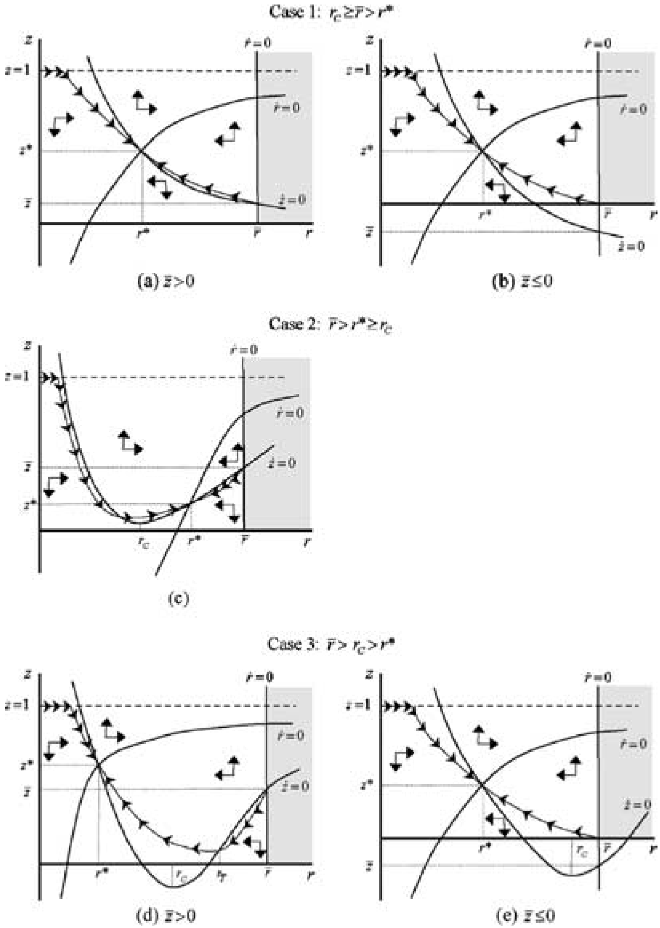

FIGURE 2. Transitional dynamics in the low-elasticity case, $\sigma < 1$.

and increases toward $1 - 1/\theta$ as r goes to infinity. Furthermore, it is convex up to the inflexion point $r_I = [(1+\sigma)/2]^{1/(\sigma-1)} r_C$, which is to the right of the minimum ($r_I > r_C$), and concave thereafter.

Figure 2 illustrates the various phase diagrams in (r, z)-plane that may arise. Depending on the relative position of r_C with respect to r^* and $\bar{r}$, we can distinguish three cases. In each case, the configuration of the two loci entails that the steady state (r^*, z^*) be a saddle point, whereas the steady state $(\bar{r}, \bar{z})$ is unstable when it is feasible.

In Case 1, $r_C \geq \bar{r} > r^*$, panels (a) and (b) show that convergence is monotonic, and that the saving rate decreases monotonically toward its stationary value as the economy grows, starting at $k(0) < k^*$.[4] In Case 2, $\bar{r} > r^* \geq r_C$, panel (c) shows that the saving rate increases monotonically toward its steady-state level as the economy develops, starting at $k(0) < k^*$. However, in the (unlikely) case that the economy starts with an initial capital stock sufficiently higher than its steady-state level, the saving rate would exhibit nonmonotonic behavior.[5] In Case 3, $\bar{r} > r_C > r^*$, panel (e) shows that convergence is monotonic if $\bar{z} \leq 0$, whereas panel (d) shows that convergence is globally nonmonotonic if $\bar{z} > 0$. In this case, if the economy starts with an initial capital stock sufficiently lower than its steady-state level, as the economy evolves the saving rate first increases, reaches a maximum at the turning point r_T where the saving rate path crosses the $\dot{z} = 0$–locus, and decreases thereafter toward its steady-state value. Because the turning point r_T must be to the right of r_C, an upper bound k_B on the turning value of k is implicitly defined by $f'(k_B) = r_C$.

The following Proposition summarizes the former findings, focusing on the (more interesting) case that the initial value of k is below its steady-state level, $k(0) < k^*$.

PROPOSITION 1. *Let* $0 < \sigma < 1$, *and assume that* $\bar{r} > r^*$ *and* $r^* > \pi$.

i) If a) $r_C \geq \bar{r}$ *or b)* $\bar{r} > r_C > r^*$ *and* $\bar{z} \leq 0$, *the saving rate path is monotonically decreasing for all* $k \in (0, k^*)$.

ii) If $r^* \geq r_C$, *the saving rate path is monotonically increasing for all* $k \in (0, k^*)$.

iii) If $\bar{r} > r_C > r^*$ *and* $\bar{z} > 0$, *then there exists* $k_T \in (0, k^*)$ *such that the saving rate path is monotonically increasing for all* $k \in (0, k_T)$, *and monotonically decreasing for all* $k \in (k_T, k^*)$. *Furthermore, an upper bound* k_B *on the turning point* k_T *is defined implicitly by*

$$f'(k_B) = r_C = (B\pi\theta\sigma/\omega)^{1/(\sigma-1)}. \quad \textbf{(24)}$$

Smetters (2003, Proposition 1) also provides an upper bound k_1 of the turning point k_T defined implicitly by $s^*[f'(k_1)/f'(k^*)]^{1-\sigma} = 1/\theta$, which can be equivalently expressed as

$$f'(k_1) = r^* \left[(1 - z^*)\theta\right]^{1/(\sigma-1)} = (B\pi\theta/\omega)^{1/(\sigma-1)}. \quad \textbf{(25)}$$

Because $\sigma^{1/(\sigma-1)} > 1$ for all $\sigma \in (0, 1)$, we have that $k_B < k_1$, and so, the upper bound defined implicitly by (24) is sharper than that derived by Smetters.

The condition $\bar{r} > r_C > r^*$ in case iii of Proposition 1 can be equivalently expressed as $B < B\pi\theta\sigma/\omega < \omega^{\sigma-1}$. Because $r^* = \omega > \pi$, a necessary condition for nonmonotonic behavior to arise when $k(0) < k^*$ is that $\theta\sigma > 1$. The next corollary summarizes this result.

COROLLARY 2. *Under the conditions of Proposition* 1, *a necessary condition for the saving rate path to exhibit nonmonotonic behavior in the interval* $(0, k^*)$ *is that* $\sigma > 1/\theta$.

3.3. High Elasticity of Substitution, $\sigma > 1$

In this case, equation (22) implies that the $\dot{z} = 0$–locus increases from minus infinity at $r = 0$, reaches a maximum at $r_C = (B\pi\theta\sigma/\omega)^{1/(\sigma-1)}$, and decreases toward $1 - 1/\theta$ as r goes to infinity. Furthermore, it is concave up to the inflexion point $r_I = [(1+\sigma)/2]^{1/(\sigma-1)} r_C$, which is to the right of the maximum ($r_I > r_C$), and convex thereafter.

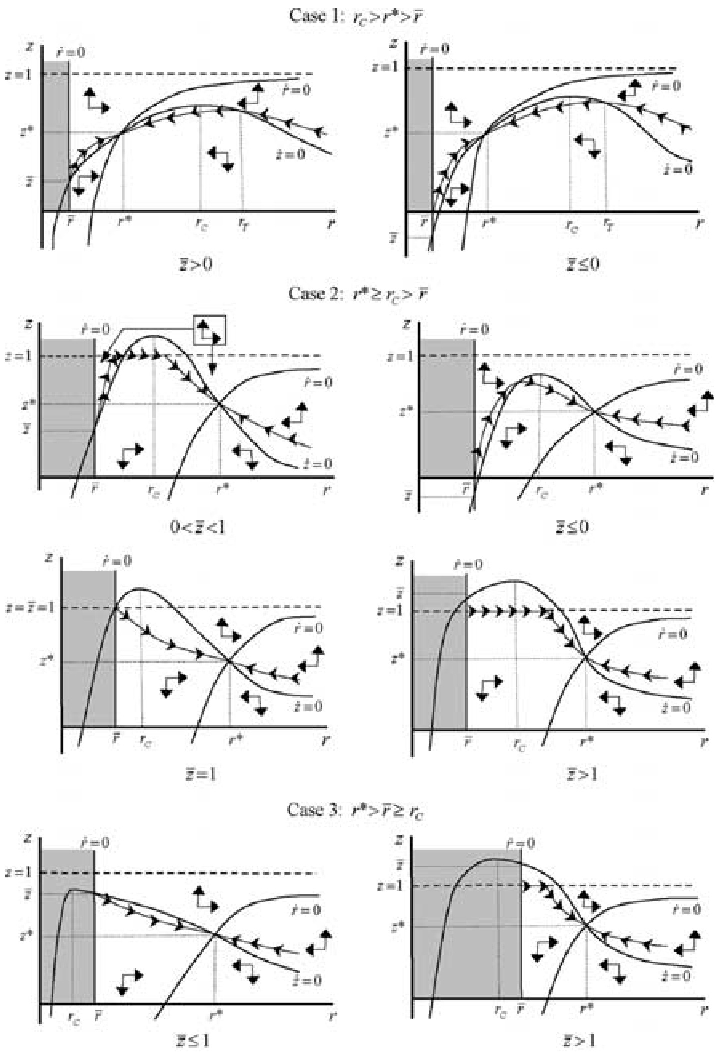

FIGURE 3. Transitional dynamics in the high-elasticity case, $\sigma > 1$.

Figure 3 illustrates the various phase diagrams in (r, z)-plane that may arise. Depending on the relative position of r_C with respect to r^* and $\bar{r}$, we can distinguish three cases. The configuration of the two loci entails that in each case the steady state (r^*, z^*) be a saddle point, whereas the steady state $(\bar{r}, \bar{z})$ is unstable when it is feasible.

In Case 1, $r_C > r^* > \bar{r}$, panels (a) and (b) show that convergence is globally nonmonotonic. If the economy starts with an initial capital stock sufficiently lower than its steady-state level, as the economy grows the saving rate first decreases, reaches a minimum at the turning point r_T where the saving rate path crosses the $\dot{z} = 0$–locus, and increases thereafter towards its stationary value. Because the turning point r_T must be to the right of r_C, an upper bound k_B on the corresponding turning value of k is implicitly defined by $f'(k_B) = r_C$. In Case 2, $r^* \geq r_C > \bar{r}$, panels (c) and (d) show that if $\bar{z} < 1$, the saving rate decreases monotonically toward its steady state as the economy grows, starting at $k(0) < k^*$. However, if the economy starts with an initial capital stock sufficiently higher than its steady-state level, the saving rate would exhibit nonmonotonic behavior.[6] Panels (e) and (f) show that if $\bar{z} \geq 1$, convergence is monotonic instead.[7] In Case 3, $r^* > \bar{r} \geq r_C$, panels (g) and (h) show that convergence is monotonic as well. Thus, we can state the following Proposition.

PROPOSITION 3. *Let $\sigma > 1$, and assume that $r^* > \bar{r}$ and $r^* > \pi$.*

i) If a) $r^* \geq r_C > \bar{r}$ or b) $\bar{r} \geq r_C$, the saving rate path is monotonically decreasing for all k in, $(0, k^*)$.

ii) If $r_C > r^*$, then there exists $k_T \in (0, k*)$ such that the saving rate path is monotonically decreasing for all $k \in (0, k_T)$, and monotonically increasing for all $k \in (k_T, k^*)$. Furthermore, an upper bound k_B on the turning point k_T is defined implicitly by

$$f'(k_B) = r_C = (B\pi\theta\sigma/\omega)^{1/(\sigma-1)}. \tag{26}$$

Smetters (2003, Proposition 1) also provides an upper bound k_1 of the turning point k_T defined implicitly by (21). Because $\sigma^{1/(\sigma-1)} > 1$ for all $\sigma \in (1, +\infty)$, we have that $k_B < k_1$, and so, the upper bound defined implicitly by (26) is sharper than that derived by Smetters.

Using (16a), (16b), and (23), the condition $r_C > r^*$ in case ii of Proposition 3 can be equivalently expressed as $(1 - z^*)\theta\sigma > 1$. Hence, a necessary condition for nonmonotonic behavior to arise when $k(0) < k^*$ is again that $\theta\sigma > 1$. We can state the following result.

COROLLARY 4. *Under the conditions of Proposition 3, a necessary condition for the saving rate path to exhibit nonmonotonic behavior in the interval $(0, k^*)$ is that $\sigma > 1/\theta$.*

4. THE CASE OF ENDOGENOUS GROWTH

Up to this point, we have assumed that r^* is feasible when the elasticity of substitution is nonunitary; that is, that there exists $k^* \in (0, +\infty)$ such that (16a) is satisfied or, equivalently, that condition (18a) or (18b) is fulfilled. It is well known (e.g., Barro and Sala-i-Martin, 1995) that when the elasticity of substitution is above one, $\sigma > 1$, endogenous growth may arise because the marginal and average product of capital approach a positive constant, $\bar{r}$, as k goes to infinity.[8] Actually, endogenous growth arises if r^* is infeasible, that is, if condition (18b) is not satisfied, so that

$$\omega = r^* \leq \bar{r} = B^{1/(\sigma-1)}. \tag{27}$$

Equation (19b) implies that the fulfillment of (27) ensures that $\bar{z} < 1$, whereas the transversality condition (10d) is equivalent to $\bar{z} > 0$. Thus, if parameter values are such that condition (27) is met and $\bar{z} > 0$, the steady state $(\bar{r}, \bar{z})$ is feasible, and

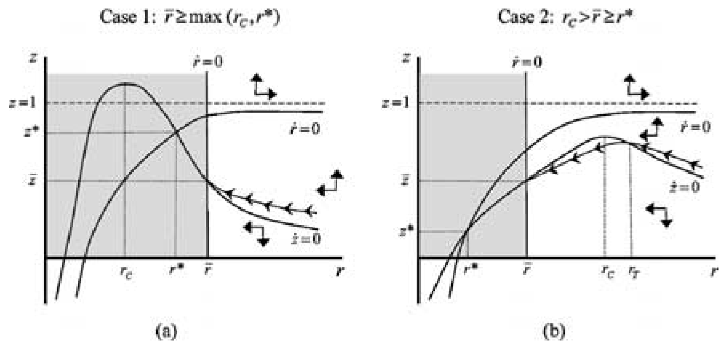

FIGURE 4. Transitional dynamics in the endogenous-growth case.

the phase diagram analysis (see Figure 4) shows that it is a saddle point. The arguments of Arrow and Kurz (1970) can be easily extended to deduce that the solution for the irreversible case coincides with the solution for the reversible case for all k, so that the irreversibility constraint is never binding. At the steady state $(\bar{r}, \bar{z})$, output, consumption and capital stock per effective labor units all grow at the common rate:[9]

$$\dot{\bar{y}}/\bar{y} = \dot{\bar{c}}/\bar{c} = \dot{\bar{k}}/\bar{k} = (\bar{r} - r^*)/\theta \geq 0. \tag{28}$$

Recalling the arguments given in Section 3, Figure 4 illustrates the various phase diagrams in (r,z)-plane that may arise. Depending on the relative position of r_C relative to r^* and $\bar{r}$, we can distinguish two cases. In Case 1, $\bar{r} \geq \max(r_C, r^*)$, panel (a) shows that the saving rate decreases monotonically toward its steady-state value as the economy develops. In Case 2, $r_C > \bar{r} \geq r^*$, panel (b) shows that the saving rate exhibits nonmonotonic behavior. The following proposition summarizes these findings.

PROPOSITION 5. *Let $\sigma > 1$, and assume that $\bar{r} \geq r^*$ and $\bar{z} > 0$.*

i) If $\bar{r} \geq r_C$, the saving rate path is monotonically decreasing for all $k \in (0, \infty)$.
ii) If $r_C > \bar{r}$, then there exists $k_T \in (0, \infty)$ such that the saving rate path is monotonically decreasing for all $k \in (0, k_T)$, and monotonically increasing for all $k \in (k_T, \infty)$. An upper bound k_B on the turning point k_T is defined implicitly by $f'(k_B) = r_C = (B\pi\theta\sigma/\omega)^{1/(\sigma-1)}$.

If case ii arises, then $r_C > \bar{r} \geq r^*$. Using (16a), (16b), and (23), the condition $r_C > r^*$ can be equivalently expressed as $(1 - z*)\theta\sigma > 1$. Hence, as in the exogenous growth case, a necessary condition for the saving rate to exhibit nonmonotonic behavior is that $\theta\sigma > 1$.

5. CONCLUSIONS

This paper has analyzed the dynamics of the saving rate in the neoclassical growth model with CES production. The global dynamics of the saving rate has been analyzed by means of a phase-diagram analysis. Analytical conditions have been obtained that characterize the different cases that may arise theoretically depending on the parameters' configuration. Differently from the Cobb-Douglas case, when the elasticity of substitution is nonunitary, the saving rate may exhibit globally nonmonotonic behavior.

Using panel data techniques with a data set for 82 countries over the period 1960 to 1987, Duffy and Papageorgiou (2000) found that the elasticity of substitution is lower than one in the poorest group of countries and greater than one in the richest group. This suggests that the elasticity of substitution may depend on the stage of development, being lower than one for developing economies and greater than one for developed economies. This paper has shown that when the elasticity of substitution is below one, as the economy grows, the saving rate path may be (i) monotonically decreasing, (ii) monotonically increasing, or (iii) nonmonotonic—first increasing and then decreasing—depending on the parameters' values. When the elasticity of substitution is above one, as the economy evolves, the saving rate path may be (i) monotonically decreasing, or (ii) nonmonotonic—first decreasing and then increasing. The richer dynamics that the saving rate may exhibit theoretically when the elasticity of substitution is nonunitary relative to that arising in the Cobb-Douglas case suggests that the elasticity of substitution may play a role in explaining the different saving patterns observed in the data. The results obtained by Duffy and Papageorgiou (2000) also indicate that the conditions for endogenous growth to arise when the elasticity of substitution is greater than one may not be in place. Nevertheless, we have characterized the dynamics of the saving rate for all configurations of the parameters, including those leading to endogenous growth.

NOTES

1. These conditions are also sufficient because, using (8c) to eliminate i from (8b) and (8d), the current-value Hamiltonian, $H = u(c) + \lambda(f(k) - c - \pi k)$, and the constraint are jointly concave in the states and the controls.

2. In this case, it can be readily shown that $0 < z^* < 1$. In particular, note that z^* can be rewritten as $z^* = 1 - (\pi/\omega)\ (r^*/\bar{r})^{1-\sigma}$ if the elasticity of substitution is nonunitary, $\sigma \neq 1$.

3. The case of endogenous growth is considered in Section 4.

4. Note that the minimum r_C, which is to the right of $\bar{r}$, is not depicted in Figure 2.

5. If $r_C = r^*$, the saving rate path crosses the $\dot{z} = 0$–locus at $r = r^*$, where the $\dot{z} = 0$–locus reaches a minimum and the saving rate path reaches a maximum. Thus, in this knife-edge case, the saving rate increases toward its steady-state value regardless of k approaching k^* from above or below.

6. If $\bar{z} < 1$, whether there exists an intermediate blocked interval [as illustrated in panel (c)] or not [as illustrated in panel (d)] depends on the parameters' configuration, and a general characterization may be intractable.

7. If $r_C = r^*$, the saving rate path crosses the $\dot{z} = 0$–locus at $r = r^*$, where the $\dot{z} = 0$–locus reaches a maximum and the saving rate path reaches a minimum. Thus, in this knife-edge case, the saving rate decreases toward its steady state regardless of k approaching k^* from above or below.

8. When the elasticity of substitution is below one, $\sigma < 1$, the model does not generate endogenous growth because the key Inada condition is satisfied, as the marginal and average products of capital approach zero as k goes to infinity. However, the violation of the Inada condition as k approaches zero entails that, if r^* is infeasible [i.e., condition (18a) is not satisfied], no steady state exists with a positive

value of k. The growth rate is always negative during the transition, the economy shrinks over time, and k, y, and c all approach zero (see, e.g., Barro and Sala-i-Martin, 1995).

9. We have included, somewhat arbitrarily, the borderline case $r^* = \bar{r}$ in the endogenous growth case, although the long-run growth rate is zero, because k does not approach a finite constant.

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APPENDIX

THE STRUCTURE OF THE POLICY FUNCTION: FREE AND BLOCKED INTERVALS

Following the terminology of Arrow and Kurz (1970), a free interval is a maximal interval in which the nonnegativity constraint on gross investment is nonbinding and the solution is equivalent to that obtained under the assumption that investment is reversible, and a blocked interval is a maximal interval in which the nonnegativity constraint is binding. If the elasticity of substitution is below 1 (and so, $\lim_{k\to\infty}[f'(k)\,k/f(k)] = 0$), or if the elasticity of substitution is unitary (and so, $\lim_{k\to\infty}[f'(k)\,k/f(k)] = \alpha > 0$) and $r^* > \alpha\theta\pi$, Arrow and Kurz (1970, Proposition 5) show that there exists $k_m > k^*$ such that for all $k \le k_m$ the solution coincides with that of the reversible case, while investment is zero for all $k \ge k_m$. When the elasticity of substitution is unitary and $r^* \le \alpha\theta\pi$, the solution coincides with that of the reversible case for all k, so that the nonnegativity constraint is never binding.

If the elasticity of substitution is constant and above 1, Arrow and Kurz (1970, Proposition 6) show that the solution has one of the following three forms: (i) it coincides with the solution for the reversible case for all k; (ii) it coincides with the solution for the reversible case for $k \le k_m$, and calls for zero investment for $k \ge k_m$; (iii) it coincides with the solution for the reversible case for $k \le k_m$, calls for zero investment for $k_m \le k \le k_M$, and has a free interval for $k > k_M$. Using Proposition 3a of Arrow and Kurz (1970), conveniently adapted to also consider the depreciation of the capital stock, we can distinguish between the cases of terminal free or blocked interval. Because $\lim_{k\to\infty} f'(k) = \lim_{r\to\bar{r}} r = \bar{r}$, we have that if $\bar{r} < r^* - \pi\theta$, or equivalently, using (15b), if $\bar{z} > 1$, then there is a final blocked interval; however, if $\bar{r} \ge r^* - \pi\theta$, i.e., if $\bar{z} \le 1$, then there is a terminal-free interval.