Singular limits and properties of solutions of some degenerate elliptic and parabolic equations

Kin Ming Hui

Institute of Mathematics, Academia Sinica, Taipei 10617, Taiwan, Republic of Korea (kmhui@gate.sinica.edu.tw)

Sunghoon Kim*

Department of Mathematics, School of Natural Sciences, The Catholic University of Korea, 43 Jibong-ro, Wonmi-gu, Bucheon-si, Gyeonggi-do 14662, Republic of Korea (math.s.kim@catholic.ac.kr)

Let $n \geqslant 3$, $0 \leqslant m < n - 2/n$, $\rho_1 > 0$, $\beta > \beta_0^{(m)} = ((m\rho_1)/(n - 2 - nm))$, $\alpha_m = ((2\beta + \rho_1)/(1 - m))$ and $\alpha = 2\beta + \rho_1$. For any $\lambda > 0$, we prove the uniqueness of radially symmetric solution $v^{(m)}$ of $\triangle(v^m/m) + \alpha_m v + \beta x \cdot \nabla v = 0$, $v > 0$, in $\mathbb{R}^n \backslash \{0\}$ which satisfies $\lim_{|x| \to 0} |x|^{\alpha_m/\beta} v^{(m)}(x) = \lambda^{-((\rho_1)/((1-m)\beta))}$ and obtain higher order estimates of $v^{(m)}$ near the blow-up point $x = 0$. We prove that as $m \to 0^+$, $v^{(m)}$ converges uniformly in $C^2(K)$ for any compact subset K of $\mathbb{R}^n \backslash \{0\}$ to the solution v of $\triangle \log v + \alpha v + \beta x \cdot \nabla v = 0$, $v > 0$, in $\mathbb{R}^n \backslash \{0\}$, which satisfies $\lim_{|x| \to 0} |x|^{\alpha/\beta} v(x) = \lambda^{-\rho_1/\beta}$. We also prove that if the solution $u^{(m)}$ of $u_t = \Delta(u^m/m)$, $u > 0$, in $(\mathbb{R}^n \backslash \{0\}) \times (0, T)$ which blows up near $\{0\} \times (0, T)$ at the rate $|x|^{-\alpha_m/\beta}$ satisfies some mild growth condition on $(\mathbb{R}^n \backslash \{0\}) \times (0, T)$, then as $m \to 0^+$, $u^{(m)}$ converges uniformly in $C^{2+\theta, 1+\theta/2}(K)$ for some constant $\theta \in (0, 1)$ and any compact subset K of $(\mathbb{R}^n \backslash \{0\}) \times (0, T)$ to the solution of $u_t = \triangle \log u$, $u > 0$, in $(\mathbb{R}^n \backslash \{0\}) \times (0, T)$. As a consequence of the proof, we obtain existence of a unique radially symmetric solution $v^{(0)}$ of $\triangle \log v + \alpha v + \beta x \cdot \nabla v = 0$, $v > 0$, in $\mathbb{R}^n \backslash \{0\}$, which satisfies $\lim_{|x| \to 0} |x|^{\alpha/\beta} v(x) = \lambda^{-\rho_1/\beta}$.

1. Introduction

Recently there is a lot of study on the equation [**4**, **7**–**13**, **18**, **25**, **31**, **32**],

$$u_t = \triangle \phi_m(u), \quad u > 0, \tag{1.1}$$

where

$$\phi_m(u) = \begin{cases} u^m/m & \text{if } m \neq 0 \\ \log u & \text{if } m = 0 \end{cases} \tag{1.2}$$

* Corresponding author.

and the associated elliptic equation [**6**, **17**, **19**, **23**],

$$\triangle\phi_m(v) + \alpha_m v + \beta x \cdot \nabla v = 0, \quad v > 0, \tag{1.3}$$

where α_m and β are some constants. Recently, P. Daskalopoulos, M. del Pino, M. Fila, S. Y. Hsu, K. M. Hui, S. Kim, J. King, Ki-Ahm Lee, N. Sesum, M. Sáez, J. L. Vazquez, M. Winkler, E. Yanagida, E. Di Benedetto, U. Gianazza and N. Liao, etc. have many results on (1.1) and (1.3). The equation (1.1) appears in many physical models [**1**, **3**, **29**] and in the study of Ricci and Yamabe flow on manifolds [**5**, **14**, **28**, **33**]. When $m > 1$, it appears in modelling the evolution of various diffusion processes such as the flow of a gas through a porous medium [**1**]. When $m = 1$, (1.1) is the heat equation. When $0 < m < 1$, (1.1) is the fast diffusion equation. When $n \geqslant 3$ and $g = u^{4/n+2}dx^2$ is a metric on $\mathbb{R}^n$ which evolves by the Yamabe flow

$$\frac{\partial g}{\partial t} = -Rg \quad \text{on } (0, T)$$

where $R(\cdot, t)$ is the scalar curvature of the metric $g(\cdot, t)$, then u satisfies [**5**, **7**, **34**],

$$u_t = \frac{n-1}{m}\Delta u^m \quad \text{in } \mathbb{R}^n \times (0, T), \quad m = \frac{n-2}{n+2}$$

which after rescaling is equivalent to (1.1). Note that if $n \geqslant 3$, $0 \leqslant m < n - 2/n$, $\beta > 0$, $\alpha_m = ((2\beta + \rho_1)/(1-m))$ and $v^{(m)}$ is a solution of (1.3) in $\mathbb{R}^n$ (or $\mathbb{R}^n \backslash \{0\}$), then with $\rho_1 = 1$ and $T > 0$ the rescaled function

$$V^{(m)}(x, t) = (T - t)^{\alpha_m} v^{(m)} \left((T - t)^{\beta} x\right) \tag{1.4}$$

is a self-similar solution of (1.1) in $\mathbb{R}^n \times (0, T)$ (($\mathbb{R}^n \backslash \{0\}) \times (0, T)$, respectively) which vanishes at time T. Since solutions of (1.1) which vanishes at a finite time usually behaves like self-similar solutions of the form (1.4), in order to understand the behaviour of the solutions of (1.1), it is important to study the properties of solutions of (1.3).

For $m > (n-2)_+/n$, there are lots of studies on the solutions of (1.1) ([**3**, **29**]). However, there are not many studies on the equations (1.1) and (1.3) for the case $n \geqslant 3$ and $0 \leqslant m < n - 2/n$ until recently. This is because there is a big difference in the behaviour of solutions of (1.1) for the case $(n-2)_+/n < m < 1$ and the case $n \geqslant 3$, $0 \leqslant m < n - 2/n$ [**3**, **15**, **27**]. For example for any $0 \leqslant u_0 \in L^1_{\text{loc}}(\mathbb{R}^n)$, $u_0 \not\equiv 0$, when $(n-2)_+/n < m < 1$, there exists [**15**] a unique global positive smooth solution of (1.1) in $\mathbb{R}^n \times (0, \infty)$ with initial data u_0 on $\mathbb{R}^n$. However, for $n \geqslant 3$ and $0 \leqslant m < n - 2/n$ the Barenblatt solutions [**4**]

$$B_k(x, t) = \left(\frac{C_*}{k + (T-t)^{((2)/(n-2-nm))}|x|^2}\right)^{1/1-m} (T-t)^{n/n-2-nm},$$

$$C_* = \frac{2(n-2-mn)}{1-m}, \quad k > 0,$$

satisfy (1.1) in $\mathbb{R}^n \times (0, T)$ and vanishes identically at time T.

For the subcritical case $m < (n-2)_+/n$, M. Fila and M. Winkler [9–12] have obtained a lot of subtle phenomena for the solutions of (1.1). In [9, 10] they proved the sharp rate of convergence of solutions of (1.1) in $\mathbb{R}^n$ with $n > 4$ and $0 < m \leqslant n - 4/n - 2$ to the Barenblatt solutions as the extinction time is approached. In [11] they also proved the rate of convergence of solutions of (1.1) in $\mathbb{R}^n$ to separable solutions of (1.1) when $n > 10$ and $0 < m < (((n-2)(n-10))/((n-2)^2 - 4n + 8\sqrt{n-1}))$. In [12] they found an explicit dependence of the slow temporal growth rate of solutions of (1.1) in $\mathbb{R}^n$ on the initial spatial growth rate.

Properties of singular solutions of (1.1) are studied by E. Chasseige, J. L. Vazquez and M. Winkler in the papers [2, 30–32]. The existence of a singular solution of (1.1) for the case $n - 2/n < m < 1$ with initial value a nonnegative Borel measure on $\mathbb{R}^n$ which blows up at a singular set of $\mathbb{R}^n$ is proved by E. Chasseige and J. L. Vazquez in [2]. Finite blow-down or delay regularization behaviour for the solutions of the 2-dimensional logarithmic diffusion equation (1.1) (with $m = 0$) was studied in [30]. The asymptotic oscillating behaviour of singular solutions of (1.1) in bounded domains of $\mathbb{R}^n$ with $0 < m < n - 2/n$ and $n \geqslant 3$ was studied in [31] and the evolution of singularities of solutions of (1.1) in bounded domains of $\mathbb{R}^n$ with $0 < m < 1$ and $n \geqslant 3$ was studied in [32].

Another way to study the solutions of (1.1) and (1.3) is to study the singular limit of the solutions of (1.1) and (1.3) as $m \to 0$. Singular limit of solutions of (1.1) in $\mathbb{R}^2 \times (0, T)$ as $m \to 0^+$ and in $\Omega \times (0, \infty)$ for any bounded domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, as $m \to 0$ are proved by K. M. Hui in [20, 22]. Singular limit of solutions of (1.1) in $\mathbb{R}^n \times (0, \infty)$, $n \geqslant 2$, as $m \to 0^-$ is also proved by K. M. Hui in [22]. Singular limit of weak local solutions of (1.1) in $O \times (0, \infty)$ as $m \to 0$ for any open set $O \subset \mathbb{R}^n$ is proved by E. Di Benedetto, U. Gianazza and N. Liao in [8]. For $n \geqslant 3$, $0 < m \leqslant n - 2/n$ and either $\beta > 0$ or $\alpha = 0$, singular limit of solutions of

$$\triangle(v^m/m) + \alpha v + \beta x \cdot \nabla v = 0, \quad v > 0, \quad \text{in } \mathbb{R}^n$$

as $m \to 0^+$ is proved by S. Y. Hsu in [17].

In [23] K. M. Hui proved for any $n \geqslant 3$, $0 < m < n - 2/n$, $\rho_1 > 0$, $\lambda > 0$, $\beta \geqslant \beta_0^{(m)}$ and

$$\alpha_m = \frac{2\beta + \rho_1}{1 - m} \tag{1.5}$$

where

$$\beta_0^{(m)} = \frac{m\rho_1}{n - 2 - nm} \tag{1.6}$$

there exists a radially symmetric solution $v := v^{(m)}$ of (1.3) in $\mathbb{R}^n \backslash \{0\}$ which satisfies

$$\lim_{|x| \to 0} |x|^{\alpha_m/\beta} v(x) = \lambda^{-((\rho_1)/((1-m)\beta))}. \tag{1.7}$$

In this paper, we will prove that as $m \to 0^+$, the radially symmetric solution $v^{(m)}$ of (1.3) in $\mathbb{R}^n \backslash \{0\}$ with $\beta > 0$ and α_m given by (1.5) converges uniformly in $C^2(K)$

for any compact subset K of $\mathbb{R}^n \backslash \{0\}$ to the solution $v = v^{(0)}$ of

$$\triangle \log v + \alpha v + \beta x \cdot \nabla v = 0, \quad v > 0, \quad \text{in } \mathbb{R}^n \backslash \{0\}, \tag{1.8}$$

which satisfies

$$\lim_{|x| \to 0} |x|^{\alpha/\beta} v^{(0)}(x) = \lambda^{-\rho_1/\beta} \tag{1.9}$$

where $\alpha = \alpha_0 = 2\beta + \rho_1$. We will also prove that if $u^{(m)}$ is the solution of (1.1) in $(\mathbb{R}^n \backslash \{0\}) \times (0, T)$ with $\beta > 0$ and α_m given by (1.5) which blows up near $\{0\} \times (0, T)$ at the rate $|x|^{-\alpha_m/\beta}$, then as $m \to 0^+$, $u^{(m)}$ converges uniformly in $C^{2,1}(K)$ for any compact subset K of $(\mathbb{R}^n \backslash \{0\}) \times (0, T)$ to the solution u of

$$u_t = \triangle \log u, \quad u > 0, \quad \text{in } (\mathbb{R}^n \backslash \{0\}) \times (0, T). \tag{1.10}$$

For any $n \geqslant 3$, $0 \leqslant m < n - 2/n$, $\rho_1 > 0$, $\beta > \beta_0^{(m)}$, $\alpha_m = ((2\beta + \rho_1)/(1 - m))$ and $\lambda > 0$, we also prove the uniqueness of radially symmetric solution $v^{(m)}$ of (1.3) in $\mathbb{R}^n \backslash \{0\}$ which satisfies (1.7) and obtain higher order estimates of $v^{(m)}$ near the blow-up point $x = 0$.

Unless stated otherwise, we will now assume that $n \geqslant 3$, $0 \leqslant m < n - 2/n$, $\rho_1 > 0$, $\lambda > 0$, $\beta \geqslant \beta_0^{(m)}$ and ϕ_m, α_m, $\beta_0^{(m)}$, are given by (1.2), (1.5) and (1.6) respectively and $v = v^{(m)}$ is a radially symmetric solution of (1.3) in $\mathbb{R}^n \backslash \{0\}$ which satisfies (1.7) for the rest of the paper. We now recall a result of [23].

THEOREM 1.1 (Theorem 1.1 of [23]). *Let* $n \geqslant 3$, $0 < m < n - 2/n$, $\rho_1 > 0$, $\lambda > 0$ *and* $\beta \geqslant \beta_0^{(m)}$. *Then there exists a radially symmetric solution* $v = v^{(m)}$ *of* (1.3) *in* $\mathbb{R}^n \backslash \{0\}$ *which satisfies* (1.7) *and*

$$(v^{(m)})'(r) \leqslant 0 \quad \forall r = |x| > 0. \tag{1.11}$$

In this paper, we will prove the following main results.

THEOREM 1.2. *Let* $n \geqslant 3$, $0 \leqslant m < n - 2/n$, $\rho_1 > 0$, $\lambda > 0$, $\beta > \beta_0^{(m)}$ *and* ϕ_m, α_m, $\beta_0^{(m)}$, *be given by* (1.2), (1.5) *and* (1.6) *respectively and let* $v = v^{(m)}$ *be a radially symmetric solution of* (1.3) *in* $\mathbb{R}^n \backslash \{0\}$ *which satisfies* (1.7). *Let* $\tilde{w}(r) = r^{\alpha_m/\beta} v(r)$, $\rho = r^{\rho_1/\beta}$ *and* $\overline{w}(\rho) = \tilde{w}(r)$. *Then* $\overline{w}$ *can be extended to a function in* $C^2([0, \infty))$ *by setting*

$$\overline{w}(0) = \lambda^{-((\rho_1)/((1-m)\beta))},$$

$$\overline{w}_\rho(0) = A_1 \lambda^{-((m\rho_1)/((1-m)\beta))} \quad \text{and} \quad \overline{w}_{\rho\rho}(0) = A_2 \lambda^{-(((2m-1)\rho_1)/((1-m)\beta))} \tag{1.12}$$

where

$$A_1 = \frac{a_3}{a_2}, \quad A_2 = \frac{a_3(ma_3 - a_1)}{a_2^2},$$

and

$$a_1 = \frac{(n-2)\beta - 2m\alpha_m + \rho_1}{\rho_1}, \quad a_2 = \frac{\beta^2}{\rho_1}, \quad a_3 = \frac{\alpha_m \beta(n-2) - m\alpha_m^2}{\rho_1^2}. \tag{1.13}$$

Hence

$$\begin{cases} \overline{w}(\rho) = \lambda^{-((\rho_1)/((1-m)\beta))} + A_1 \lambda^{-((m\rho_1)/((1-m)\beta))}\rho \\ \quad + \dfrac{A_2}{2}\lambda^{-(((2m-1)\rho_1)/((1-m)\beta))}\rho^2 + o\left(\rho^2\right) \quad \text{as } \rho \to 0^+ \\ \overline{w}_\rho(\rho) = A_1 \lambda^{-((m\rho_1)/((1-m)\beta))} + A_2 \lambda^{-(((2m-1)\rho_1)/((1-m)\beta))}\rho + o\left(\rho\right) \quad \text{as } \rho \to 0^+ \end{cases}$$

or equivalently

$$\begin{cases} v^{(m)}(r) = r^{-\alpha_m/\beta}\left[\lambda^{-((\rho_1)/((1-m)\beta))} + A_1 \lambda^{-((m\rho_1)/((1-m)\beta))}r^{\rho_1/\beta}\right. \\ \quad \left. + \dfrac{A_2}{2}\lambda^{-(((2m-1)\rho_1)/((1-m)\beta))}r^{2\rho_1/\beta} + o\left(r^{2\rho_1/\beta}\right)\right] \quad \text{as } r \to 0^+ \\ (v^{(m)})'(r) = r^{-(\alpha_m/\beta)-1}\left[-\dfrac{\alpha_m}{\beta}\lambda^{-((\rho_1)/((1-m)\beta))}\right. \\ \quad \left. -\dfrac{(2\beta + m\rho_1)}{(1-m)\beta}A_1 \lambda^{-((m\rho_1)/((1-m)\beta))}r^{\rho_1/\beta} + o\left(r^{\rho_1/\beta}\right)\right] \quad \text{as } r \to 0^+. \end{cases} \tag{1.14}$$

THEOREM 1.3. *Let* $n \geqslant 3$, $0 \leqslant m < n - 2/n$, $\rho_1 > 0$, $\lambda > 0$, $\beta > \beta_0^{(m)}$ *and* α_m *be given by* (1.5). *Let* v_1, v_2 *be radially symmetric solutions of* (1.3) *in* $\mathbb{R}^n \backslash \{0\}$ *which satisfies* (1.7). *Then*

$$v_1(r) = v_2(r) \quad \forall r > 0. \tag{1.15}$$

THEOREM 1.4. *Let* $n \geqslant 3$, $\rho_1 > 0$, $\beta > 0$ *and* $\alpha = 2\beta + \rho_1$. *Suppose that* $v = v^{(0)}$ *is a radially symmetric solution of* (1.8) *in* $\mathbb{R}^n \backslash B_1$. *Then*

$$\lim_{r \to \infty} r^2 v(r) = \frac{2(n-2)}{\alpha - 2\beta}. \tag{1.16}$$

THEOREM 1.5. *Let* $n \geqslant 3$, $\rho_1 > 0$, $\lambda > 0$, $\beta > 0$ *and* $\alpha = 2\beta + \rho_1$. *Let* $\overline{m}_0 \in (0, n - 2/n)$ *satisfy* $\beta \geqslant \beta_0^{(\overline{m}_0)}$. *For any* $0 < m < \overline{m}_0$, *let* α_m *be given by* (1.5) *and let* $v^{(m)}$ *be the unique radially symmetric solution of* (1.3) *in* $\mathbb{R}^n \backslash \{0\}$ *which satisfies* (1.7) *given by theorems* 1.1 *and* 1.3. *Then as* $m \to 0^+$, $v^{(m)}$ *converges uniformly in* $C^2(K)$ *for any compact subset* K *of* $\mathbb{R}^n \backslash \{0\}$ *to the unique radially symmetric solution* v *of* (1.8) *which satisfies* (1.9).

THEOREM 1.6. *Let* $n \geqslant 3$, $\beta > 0$, $\lambda_1 \geqslant \lambda_2 > 0$ *and* $\alpha = 2\beta + 1$. *Suppose that* $0 \leqslant u_{0,1} \leqslant u_{0,2} \in L_{\text{loc}}^\infty(\mathbb{R}^n)$. *If* u_1, $u_2 \in C((\mathbb{R}^n \backslash \{0\}) \times (0, T)) \cap L_{\text{loc}}^\infty((\mathbb{R}^n \backslash \{0\}) \times [0, T))$ *are subsolution and supersolution of*

$$u_t = \triangle \log u, \quad u > 0, \quad \text{in } (\mathbb{R}^n \backslash \{0\}) \times (0, T) \tag{1.17}$$

which satisfies

$$u_i(x, 0) = u_{0,i}(x) \quad \text{in } \mathbb{R}^n \quad \forall i = 1, 2$$

and

$$V_{\lambda_1}(x, t) \leqslant u_i(x, t) \leqslant V_{\lambda_2}(x, t) \quad \text{in } (\mathbb{R}^n \backslash \{0\}) \times (0, T) \quad \forall i = 1, 2 \tag{1.18}$$

where

$$V_{\lambda_i}(x,t) = (T-t)^\alpha v_{\lambda_i}\left((T-t)^\beta |x|\right) \quad \forall i = 1, 2$$

and v_{λ_i} is the radially symmetric solution of (1.8) *which satisfies* (1.9) *with $\lambda = \lambda_1$, λ_2, respectively, then*

$$u_1 \leqslant u_2 \quad \text{in } (\mathbb{R}^n \backslash \{0\}) \times (0, T). \tag{1.19}$$

Hence if $u_{0,1} = u_{0,2}$, then $u_1 = u_2$ in $(\mathbb{R}^n \backslash \{0\}) \times (0, T)$.

THEOREM 1.7. *Let $n \geqslant 3$, $0 < \overline{m}_0 < n - 2/n$, $\lambda_1 > \lambda_2 > 0$, $\beta \geqslant \beta_0^{(\overline{m}_0)}$, $\alpha = 2\beta + 1$ and $T > 0$. For any $0 < m < \overline{m}_0$, let α_m be given by* (1.5) *with $\rho_1 = 1$ and*

$$V_{\lambda_i}^{(m)}(x,t) = (T-t)^{\alpha_m} v_{\lambda_i}^{(m)}\left((T-t)^\beta x\right) \quad \forall i = 1, 2 \tag{1.20}$$

where $v_{\lambda_i}^{(m)}$ is the radially symmetric solution of (1.3) *in $\mathbb{R}^n \backslash \{0\}$ which satisfies* (1.7) *with $\lambda = \lambda_1$, λ_2, respectively. Let $\{u_{0,m}\}_{0 < m < \overline{m}_0} \subset L_{\text{loc}}^\infty(\mathbb{R}^n \backslash \{0\})$, $u_{0,m} \geqslant 0$ for all $0 < m < \overline{m}_0$, be a family of functions satisfying*

$$V_{\lambda_1}^{(m)}(x,0) \leqslant u_{0,m}(x) \leqslant V_{\lambda_2}^{(m)}(x,0) \quad \text{in } \mathbb{R}^n \backslash \{0\} \tag{1.21}$$

and

$$u_{0,m} \to u_0 \quad \text{in } L_{\text{loc}}^1(\mathbb{R}^n \backslash \{0\}) \text{ as } m \to 0^+.$$

For any $0 < m < \overline{m}_0$, let $u^{(m)}$ be a solution of

$$\begin{cases} u_t = \triangle(u^m/m), \quad u > 0, \quad \text{in } (\mathbb{R}^n \backslash \{0\}) \times (0, T) \\ u(x,0) = u_{0,m} \quad \text{in } \mathbb{R}^n \backslash \{0\} \end{cases} \tag{1.22}$$

given by theorem 1.7 of [**23**] *which satisfies*

$$V_{\lambda_1}^{(m)}(x,t) \leqslant u^{(m)}(x,t) \leqslant V_{\lambda_2}^{(m)}(x,t) \quad \text{in } (\mathbb{R}^n \backslash \{0\}) \times (0, T). \tag{1.23}$$

Then $u^{(m)}$ converges uniformly in $C^{2+\theta, 1+\theta/2}(K)$ for some constant $\theta \in (0,1)$ and any compact subset K of $(\mathbb{R}^n \backslash \{0\}) \times (0, T)$ to the solution u of

$$\begin{cases} u_t = \triangle \log u, u > 0, \quad \text{in } (\mathbb{R}^n \backslash \{0\}) \times (0, T) \\ u(x,0) = u_0 \quad \text{in} \mathbb{R}^n \backslash \{0\} \end{cases} \tag{1.24}$$

as $m \to 0^+$ and u satisfies

$$V_1(x,t) \leqslant u(x,t) \leqslant V_2(x,t) \quad \text{in } (\mathbb{R}^n \backslash \{0\}) \times (0, T) \tag{1.25}$$

where

$$V_i(x,t) = (T-t)^\alpha v_{\lambda_i}\left((T-t)^\beta |x|\right) = \lim_{m \to 0} V_{\lambda_i}^{(m)}(x,t) \quad \forall i = 1, 2$$

and v_{λ_i} is the radially symmetric solution of (1.8) *given by theorem* 1.1 *which satisfies* (1.9) *with $\lambda = \lambda_i$, $i = 1, 2$, respectively.*

REMARK 1.8. By lemma 5.1 of [**23**] for any $n \geqslant 3$, $0 < m < n - 2/n$, and $\lambda_1 > \lambda_2 > 0$, $T > 0$, $\beta > \beta_0^m$, $\alpha_m = ((2\beta + 1)/(1 - m))$, $0 \leqslant u_0 \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\})$, if u_1 and u_2 are two solutions of

$$\begin{cases} u_t = \triangle(u^m/m), u > 0, & \text{in } (\mathbb{R}^n \backslash \{0\}) \times (0, T) \\ u(x, 0) = u_0 & \text{in } \mathbb{R}^n \backslash \{0\} \end{cases}$$

which satisfies (1.23), then $u_1 = u_2$ in $(\mathbb{R}^n \backslash \{0\}) \times (0, T)$.

The plan of the paper is as follows. We will prove theorems 1.2 and 1.3 in § 2. We will prove theorem 1.4 in § 3 and theorems 1.5, 1.6, and 1.7 in § 4.

We start with some definitions. We say that u is a solution of (1.1) in $(\mathbb{R}^n \backslash \{0\}) \times (0, T)$ if $u \in C^{2,1}((\mathbb{R}^n \backslash \{0\}) \times (0, T)) \cap L^\infty_{\mathrm{loc}}((\mathbb{R}^n \backslash \{0\}) \times (0, T))$ is positive in $(\mathbb{R}^n \backslash \{0\}) \times (0, T)$ and satisfies (1.1) in the classical sense in $(\mathbb{R}^n \backslash \{0\}) \times (0, T)$. We say that u is a subsolution (supersolution, respectively) of (1.1) in $(\mathbb{R}^n \backslash \{0\}) \times (0, T)$ if $u \in C((\mathbb{R}^n \backslash \{0\}) \times (0, T)) \cap L^\infty_{\mathrm{loc}}((\mathbb{R}^n \backslash \{0\}) \times (0, T))$ is positive in $(\mathbb{R}^n \backslash \{0\}) \times (0, T)$ and satisfies

$$\int_{\mathbb{R}^n} u(x, t_2)\eta(x, t_2) \, \mathrm{d}x \leqslant \int_{t_1}^{t_2} \int_{\mathbb{R}^n} (u\eta_t + \phi_m(u)\Delta\eta) \, \mathrm{d}x \, \mathrm{d}t$$
$$+ \int_{\mathbb{R}^n} u(x, t_1)\eta(x, t_1) \, \mathrm{d}x \quad \forall T > t_2 > t_1 > 0 \qquad (1.26)$$

($\geqslant$, respectively) for any $\eta \in C_0^{2,1}((\mathbb{R}^n \backslash \{0\}) \times (0, T))$. For any $0 \leqslant u_0 \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\})$, we say that a solution (or subsolution or supersolution) u of (1.1) in $(\mathbb{R}^n \backslash \{0\}) \times (0, T)$ has initial value u_0 if $u(\cdot, t) \to u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\})$ as $t \to 0$.

We say that v is a solution of (1.3) in $\mathbb{R}^n \backslash \{0\}$ if $u \in C^{2,1}(\mathbb{R}^n \backslash \{0\})$ is positive in $\mathbb{R}^n \backslash \{0\}$ and satisfies (1.3) in the classical sense in $R^n \backslash \{0\}$. For any $R > 0$, let $B_R = \{x \in \mathbb{R}^n : |x| < R\}$.

2. Uniqueness of radially symmetric solutions and higher order estimates at the origin

In this section, we will prove the uniqueness of radially symmetric solution $v^{(m)}$ of (1.3) in $\mathbb{R}^n \backslash \{0\}$ which satisfies (1.7) and obtain higher order estimates of $v^{(m)}$ near the blow-up point $x = 0$.

Let $\tilde{w}(r) = r^{\alpha_m/\beta} v^{(m)}(r)$, $\rho = r^{\rho_1/\beta}$ and $\overline{w}(\rho) = \tilde{w}(r)$. Then by the proof of Theorem 1.1 of [**23**], $\tilde{w}$ satisfies

$$\left(\frac{\tilde{w}_r}{\tilde{w}}\right)_r + \frac{n - 1 - ((2m\alpha_m)/(\beta))}{r} \cdot \frac{\tilde{w}_r}{\tilde{w}} + m\left(\frac{\tilde{w}_r}{\tilde{w}}\right)^2 + \frac{\beta r^{-1-\rho_1/\beta}\tilde{w}_r}{\tilde{w}^m}$$

$$= \frac{\alpha_m}{\beta} \cdot \frac{n - 2 - ((m\alpha_m)/(\beta))}{r^2} \quad \forall r > 0$$

$$\Rightarrow \left(\frac{\overline{w}_\rho}{\overline{w}}\right)_\rho + m\left(\frac{\overline{w}_\rho}{\overline{w}}\right)^2 + \frac{a_1}{\rho} \cdot \frac{\overline{w}_\rho}{\overline{w}} + \frac{a_2}{\rho^2} \cdot \frac{\overline{w}_\rho}{\overline{w}^m} = \frac{a_3}{\rho^2} \quad \forall \, \rho > 0 \qquad (2.1)$$

where a_1, a_2 and a_3 are constants given by (1.13). Note that by (1.7),

$$\lim_{\rho \to 0+} \overline{w}(\rho) = \lambda^{-((\rho_1)/((1-m)\beta))}. \tag{2.2}$$

Hence $\overline{w}(\rho)$ can be extended to a continuous function on $[0, \infty)$ by letting $\overline{w}(0) = \lambda^{-((\rho_1)/((1-m)\beta))}$.

LEMMA 2.1. *Let $n \geqslant 3$, $0 \leqslant m < n - 2/n$, $\rho_1 > 0$, $\lambda > 0$ and $\beta > \beta_0^{(m)}$. Then*

$$\overline{w}_\rho(\rho) > 0 \quad \forall \rho > 0 \tag{2.3}$$

or equivalently,

$$v^{(m)}(r) + \frac{\beta}{\alpha_m} r(v^{(m)})'(r) > 0 \quad \forall r > 0.$$

Hence

$$v^{(m)}(r) \geqslant \lambda^{-((\rho_1)/((1-m)\beta))} r^{-\alpha_m/\beta} \quad \forall r = |x| > 0. \tag{2.4}$$

Proof. Suppose that (2.3) does not hold. Then there exists a constant $\rho_2 > 0$ such that

$$\overline{w}_\rho(\rho_2) \leqslant 0. \tag{2.5}$$

Since $\beta > \beta_0^{(m)}$, $a_3 > 0$. Then by (2.1) and (2.5),

$$\left(\rho^{a_1} \cdot \overline{w}^m \cdot \frac{\overline{w}_\rho}{\overline{w}} \right)_\rho (\rho_2) = -a_2 \rho_2^{a_1-2} \overline{w}_\rho(\rho_2) + a_3 \rho_2^{a_1-2} \overline{w}(\rho_2)^m \geqslant a_3 \rho_2^{a_1-2} \overline{w}(\rho_2)^m > 0. \tag{2.6}$$

Hence by (2.5) and (2.6) there exists a constant $b \in (0, \rho_2)$ such that

$$\overline{w}_\rho(\rho) < 0 \quad \text{in } (\rho_2 - b, \rho_2).$$

Let (ρ_3, ρ_2), $\rho_3 \in [0, \rho_2)$, be the maximal interval such that

$$\overline{w}_\rho(\rho) < 0 \quad \forall \rho \in (\rho_3, \rho_2). \tag{2.7}$$

If $a_1 \geqslant 0$, by (2.1) and (2.7),

$$(\phi_m(\overline{w}))_{\rho\rho}(\rho) = \left(\overline{w}^m \cdot \frac{\overline{w}_\rho}{\overline{w}} \right)_\rho (\rho) \geqslant a_3 \frac{\overline{w}(\rho)^m}{\rho^2}$$

$$\geqslant a_3 \frac{\overline{w}(\rho_2)^m}{\rho^2} > 0 \quad \forall \rho \in (\rho_3, \rho_2) \tag{2.8}$$

$$\Rightarrow (\phi_m(\overline{w}))_\rho(\rho_2) \geqslant (\phi_m(\overline{w}))_\rho(\rho) + a_3 \overline{w}^m(\rho_2) \left(\frac{1}{\rho} - \frac{1}{\rho_2} \right) \quad \forall \rho \in (\rho_3, \rho_2)$$

$$\Rightarrow \phi_m(\overline{w}(\rho)) \geqslant \phi_m(\overline{w}(\rho_2)) + \left(\phi_m(\overline{w})_\rho(\rho_2) + a_3 \rho_2^{-1} \overline{w}^m(\rho_2) \right) (\rho - \rho_2)$$

$$+ a_3 \overline{w}^m(\rho_2) \log \left(\frac{\rho_2}{\rho} \right) \quad \forall \rho_3 < \rho < \rho_2. \tag{2.9}$$

If $a_1 < 0$, then by (2.1) and (2.7),

$$(\rho^{a_1}(\phi_m(\overline{w}))_\rho)_\rho = \left(\rho^{a_1} \cdot \overline{w}^m \cdot \frac{\overline{w}_\rho}{\overline{w}}\right)_\rho (\rho) \geqslant a_3 \rho^{a_1-2} \overline{w}(\rho)^m$$

$$\geqslant a_3 \overline{w}(\rho_2)^m \rho^{a_1-2} > 0 \qquad \forall \rho \in (\rho_3, \rho_2) \qquad (2.10)$$

$$\Rightarrow \rho_2^{a_1}(\phi_m(\overline{w}))_\rho(\rho_2) \geqslant \rho^{a_1}(\phi_m(\overline{w}))_\rho(\rho)$$
$$+ \frac{a_3 \overline{w}(\rho_2)^m}{1 - a_1}\left(\rho^{a_1-1} - \rho_2^{a_1-1}\right) \quad \forall \rho_3 < \rho < \rho_2$$

$$\Rightarrow \rho_2^{a_1}(\phi_m(\overline{w}))_\rho(\rho_2)\rho^{-a_1} \geqslant (\phi_m(\overline{w}))_\rho(\rho)$$
$$+ \frac{a_3 \overline{w}(\rho_2)^m}{1 - a_1}\left(\rho^{-1} - \rho_2^{a_1-1}\rho^{-a_1}\right) \quad \forall \rho_3 < \rho < \rho_2$$

$$\Rightarrow \phi_m(\overline{w}(\rho)) \geqslant \phi_m(\overline{w}(\rho_2)) + C_1(\rho^{1-a_1} - \rho_2^{1-a_1})$$
$$+ \frac{a_3 \overline{w}(\rho_2)^m}{1 - a_1}\log(\rho_2/\rho) \quad \forall \rho_3 < \rho < \rho_2. \qquad (2.11)$$

where

$$C_1 = \frac{1}{1 - a_1}\left(\rho_2^{a_1}(\phi_m(\overline{w}))_\rho(\rho_2) + \frac{a_3 \rho_2^{a_1-1}\overline{w}(\rho_2)^m}{1 - a_1}\right).$$

If $\rho_3 = 0$, then by (2.9) and (2.11),

$$\overline{w}(\rho) \to \infty \quad \text{as } \rho \to 0^+$$

which contradicts (2.2). Hence $\rho_3 > 0$ and

$$\overline{w}_\rho(\rho_3) = 0. \qquad (2.12)$$

By (2.8), (2.10) and (2.12),

$$\overline{w}_\rho(\rho) > 0 \quad \forall \rho \in (\rho_3, \rho_2)$$

which contradicts (2.7). Hence no such $\rho_2 > 0$ exists and (2.3) follows. By (2.2) and (2.3), (2.4) follows. □

LEMMA 2.2. *Let $n \geqslant 3$, $0 \leqslant m < n - 2/n$, $\rho_1 > 0$, $\lambda > 0$ and $\beta > \beta_0^{(m)}$. Then*

$$\lim_{\rho \to 0^+} \overline{w}_\rho(\rho) = \frac{a_3}{a_2}\lambda^{-((m\rho_1)/((1-m)\beta))} = \frac{\alpha_m\beta(n-2) - m\alpha_m^2}{\beta^2\rho_1} \cdot \lambda^{-((m\rho_1)/((1-m)\beta))} \qquad (2.13)$$

and

$$\lim_{r \to 0^+} r^{\alpha_m/\beta+1}(v^{(m)})'(r) = -\frac{\alpha_m}{\beta}\lambda^{-((\rho_1)/((1-m)\beta))} \qquad (2.14)$$

where a_1, a_2 and a_3 are constants given by (1.13). Hence $\overline{w}$ can be extended to a function in $C^1([0,\infty))$ by letting $\overline{w}(0) = \lambda^{-((\rho_1)/((1-m)\beta))}$ and $\overline{w}_\rho(0) = a_3/a_2\lambda^{-((m\rho_1)/((1-m)\beta))}$.

Proof. Let

$$q(\rho) = \frac{1}{\overline{w}_\rho(\rho)}.$$

By (2.1) $q(\rho)$ satisfies

$$q_\rho(\rho) = -\frac{(1-m)}{\overline{w}(\rho)} + \frac{q(\rho)}{\rho}\left[a_1 + \frac{\overline{w}(\rho)}{\rho}\left(\frac{a_2}{\overline{w}(\rho)^m} - a_3 q(\rho)\right)\right] \quad \forall \rho > 0. \qquad (2.15)$$

By lemma 2.1,

$$q(\rho) > 0 \quad \forall \rho > 0. \qquad (2.16)$$

By (2.2) and (2.3) there exists a constant $\rho_2 > 0$ such that

$$\lambda^{-((\rho_1)/((1-m)\beta))} < \overline{w}(\rho) < 2\lambda^{-((\rho_1)/((1-m)\beta))} \quad \forall\, 0 < \rho < \rho_2. \qquad (2.17)$$

We now claim that there exists a constant $\rho_0 > 0$ such that

$$\frac{a_2}{8a_3}\lambda^{((m\rho_1)/((1-m)\beta))} \leqslant q(\rho) \leqslant \frac{2^{1+m}a_2}{a_3}\lambda^{((m\rho_1)/((1-m)\beta))} \quad \forall 0 < \rho < \rho_0. \qquad (2.18)$$

To prove the inequality on the right-hand side of (2.18), we first suppose that the inequality on the right-hand side of (2.18) does not hold for any $\rho_0 > 0$. Then there exists a constant

$$0 < \rho_3 < \min\left\{\rho_2, \frac{a_2\lambda^{-\rho_1/\beta}}{8|a_1| + 1}\right\} \qquad (2.19)$$

such that

$$q(\rho_3) > \frac{2^{1+m}a_2}{a_3}\lambda^{((m\rho_1)/((1-m)\beta))}.$$

Then by continuity of $q(\rho)$ on $(0, \infty)$, there exists a maximal interval (ρ_4, ρ_3), $(0 \leqslant \rho_4 < \rho_3)$ such that

$$q(\rho) > \frac{2^{1+m}a_2}{a_3}\lambda^{((m\rho_1)/((1-m)\beta))} \quad \forall\, \rho \in (\rho_4, \rho_3). \qquad (2.20)$$

By (2.15), (2.16), (2.17), (2.19) and (2.20), $q(\rho)$ satisfies

$$q_\rho(\rho) \leqslant \frac{q(\rho)}{\rho}\left[\left(a_1 - \frac{a_3\overline{w}(\rho)}{4\rho}q(\rho)\right) + \frac{\overline{w}(\rho)}{\rho}\left(\frac{a_2}{\overline{w}(\rho)^m} - \frac{a_3}{2}q(\rho)\right) - \frac{a_3\overline{w}(\rho)}{4\rho}q(\rho)\right]$$

$$\leqslant \frac{q(\rho)}{\rho}\left[\left(a_1 - \frac{a_2\lambda^{-\rho_1/\beta}}{4\rho}\right) + \frac{\overline{w}(\rho)}{\rho}\left(a_2\lambda^{((m\rho_1)/((1-m)\beta))} - \frac{a_3}{2}q(\rho)\right)\right.$$

$$\left. - \frac{a_3\lambda^{-((\rho_1)/((1-m)\beta))}}{4\rho}q(\rho)\right] \qquad (2.21)$$

$$\leqslant -\frac{a_3\lambda^{-((\rho_1)/((1-m)\beta))}}{4\rho^2}q(\rho)^2 < 0 \qquad (2.22)$$

in (ρ_4, ρ_3). Dividing (2.21) by $q(\rho)^2$ and integrating over (ρ, ρ_3), $\rho_4 < \rho < \rho_3$,

$$
\overline{w}_\rho(\rho) = \frac{1}{q(\rho)} \leqslant \left(\frac{1}{q(\rho_3)} + \frac{a_3 \lambda^{-((\rho_1)/((1-m)\beta))}}{4\rho_3} \right)
$$
$$
- \frac{a_3 \lambda^{-((\rho_1)/((1-m)\beta))}}{4\rho} \quad \forall \rho \in (\rho_4, \rho_3)
$$
$$
\Rightarrow \overline{w}(\rho) \geqslant \overline{w}(\rho_3) + \left(\frac{1}{q(\rho_3)} + \frac{a_3 \lambda^{-((\rho_1)/((1-m)\beta))}}{4\rho_3} \right) (\rho - \rho_3)
$$
$$
+ \frac{a_3 \lambda^{-((\rho_1)/((1-m)\beta))}}{4} \log(\rho_3/\rho) \quad \forall \rho \in (\rho_4, \rho_3). \tag{2.23}
$$

If $\rho_4 = 0$, then by (2.23),

$$
\lim_{\rho \to 0^+} \overline{w}(\rho) = \infty
$$

which contradicts (2.2). Hence $\rho_4 > 0$ and

$$
q(\rho_4) = \frac{2^{1+m} a_2}{a_3} \lambda^{((m\rho_1)/((1-m)\beta))}. \tag{2.24}
$$

By (2.21) and (2.24),

$$
q(\rho) < q(\rho_4) = \frac{2^{1+m} a_2}{a_3} \lambda^{((m\rho_1)/((1-m)\beta))} \quad \forall \rho_4 < \rho < \rho_3
$$

which contradicts (2.20). Hence no such $\rho_3 > 0$ exists and

$$
q(\rho) \leqslant \frac{2^{1+m} a_2}{a_3} \lambda^{((m\rho_1)/((1-m)\beta))} \quad \forall 0 < \rho < \min\left(\rho_2, \frac{a_2 \lambda^{-\rho_1/\beta}}{8|a_1| + 1} \right). \tag{2.25}
$$

Now suppose the first inequality of (2.18) does not hold for any $\rho_0 > 0$. Then there exists a constant

$$
0 < \rho_5 < \min\left\{ \rho_2, \frac{a_2 \lambda^{-\rho_1/\beta}}{8|a_1| + 1} \right\} \tag{2.26}
$$

such that

$$
q(\rho_5) < \frac{a_2}{8a_3} \lambda^{((m\rho_1)/((1-m)\beta))}. \tag{2.27}
$$

By (2.17) and (2.27),

$$
\overline{w}(\rho_5) q(\rho_5) < \frac{a_2 \lambda^{-\rho_1/\beta}}{4a_3}. \tag{2.28}
$$

Then by (2.28) and continuity of $\overline{w}(\rho) q(\rho)$ on $(0, \infty)$ there exists a maximal interval (ρ_6, ρ_5) $(0 \leqslant \rho_6 < \rho_5)$ such that

$$
\overline{w}(\rho) q(\rho) < \frac{a_2 \lambda^{-\rho_1/\beta}}{4a_3} \quad \forall \rho \in (\rho_6, \rho_5). \tag{2.29}
$$

By (2.15), (2.16), (2.17), (2.26) and (2.29),

$$
\begin{aligned}
(\overline{w}(\rho)q(\rho))_\rho &= \overline{w}(\rho)q_\rho(\rho) + 1 \\
&\geqslant \frac{\overline{w}(\rho)q(\rho)}{\rho} \left[\left(a_1 + \frac{a_2\overline{w}(\rho)^{1-m}}{4\rho} \right) + \frac{3}{4\rho} \left(a_2\overline{w}(\rho)^{1-m} - 2a_3 q(\rho)\overline{w}(\rho) \right) \right. \\
&\quad \left. + \frac{a_3}{2\rho} \overline{w}(\rho)q(\rho) \right] \\
&\geqslant \frac{\overline{w}(\rho)q(\rho)}{\rho} \left[\left(a_1 + \frac{a_2\lambda^{-\rho_1/\beta}}{\rho} \right) + \frac{3}{4\rho} \left(a_2\lambda^{-\rho_1/\beta} - \frac{a_2\lambda^{-\rho_1/\beta}}{2} \right) + \frac{a_3}{2\rho} \overline{w}(\rho)q(\rho) \right] \\
&\geqslant \frac{a_3}{2\rho^2} \left(\overline{w}(\rho)q(\rho) \right)^2 \quad \text{on } (\rho_6, \rho_5).
\end{aligned}
\tag{2.30}
$$

Dividing (2.30) by $(\overline{w}(\rho)q(\rho))^2$ and integrating over (ρ, ρ_5), $\rho_6 < \rho < \rho_5$, we get

$$
\frac{\overline{w}_\rho(\rho)}{\overline{w}(\rho)} = \frac{1}{\overline{w}(\rho)q(\rho)} \geqslant \left(\frac{\overline{w}_\rho(\rho_5)}{\overline{w}(\rho_5)} - \frac{a_3}{2\rho_5} \right) + \frac{a_3}{2\rho} \quad \forall \rho \in (\rho_6, \rho_5)
$$

$$
\Rightarrow \log \overline{w}(\rho) \leqslant \log \overline{w}(\rho_5) + \left(\frac{\overline{w}_\rho(\rho_5)}{\overline{w}(\rho_5)} - \frac{a_3}{2\rho_5} \right)(\rho - \rho_5) + \frac{a_3}{2} \log \left(\frac{\rho}{\rho_5} \right) \quad \forall \rho \in (\rho_6, \rho_5).
\tag{2.31}
$$

If $\rho_6 = 0$, then by (2.31),

$$
\lim_{\rho \to 0^+} \overline{w}(\rho) = 0
$$

which contradicts (2.2). Hence $\rho_6 > 0$ and

$$
\overline{w}(\rho_6)q(\rho_6) = \frac{a_2\lambda^{-\rho_1/\beta}}{4a_3}.
\tag{2.32}
$$

By (2.30) and (2.32),

$$
\overline{w}(\rho)q(\rho) > \frac{a_2\lambda^{-\rho_1/\beta}}{4a_3} \quad \forall \rho_6 < \rho < \rho_5
$$

which contradicts (2.29). Hence no such $\rho_5 > 0$ exists and

$$
q(\rho) \geqslant \frac{a_2}{8a_3} \lambda^{((m\rho_1)/((1-m)\beta))} \quad \forall 0 < \rho < \min \left\{ \rho_2, \frac{a_2\lambda^{-\rho_1/\beta}}{8|a_1| + 1} \right\}.
\tag{2.33}
$$

By (2.25) and (2.33), (2.18) holds for

$$
\rho_0 = \min \left\{ \rho_2, \frac{a_2\lambda^{-\rho_1/\beta}}{8|a_1| + 1} \right\}.
$$

Let $\{\rho_i\} \subset \mathbb{R}^+$ be a sequence such that $\rho_i \to 0$ as $i \to \infty$. Then, by (2.18), the sequence $\{\rho_i\}$ has a subsequence which we may assume without loss of generality

to be the sequence $\{\rho_i\}$ itself such that

$$q_\infty := \lim_{i \to \infty} q(\rho_i) \quad \text{exists}$$

and

$$q_\infty \in \left[\frac{a_2}{8a_3} \lambda^{((m\rho_1)/((1-m)\beta))}, \frac{2^{1+m}a_2}{a_3} \lambda^{((m\rho_1)/((1-m)\beta))} \right]. \tag{2.34}$$

By (2.15),

$$\left(\rho^{-a_1} e^{a_2 \int_\rho^1 s^{-2} \overline{w}(s)^{1-m} \, ds} q(\rho) \right)_\rho = -\rho^{-a_1} \left[(1-m)\overline{w}(\rho)^{-1} + a_3 \rho^{-2} \overline{w}(\rho) q(\rho)^2 \right]$$

$$\times e^{a_2 \int_\rho^1 s^{-2} \overline{w}(s)^{1-m} \, ds} \quad \forall \rho > 0$$

$$\Rightarrow q(\rho) = \frac{q(1) + \int_\rho^1 s^{-a_1} [(1-m)\overline{w}(s)^{-1} + a_3 s^{-2} \overline{w}(s) q(s)^2] e^{a_2 \int_s^1 \sigma^{-2} \overline{w}(\sigma)^{1-m} \, d\sigma} \, ds}{\rho^{-a_1} e^{a_2 \int_\rho^1 s^{-2} \overline{w}(s)^{1-m} \, ds}}$$

$$\forall \rho > 0. \tag{2.35}$$

Since $\lim_{s \to 0^+} s^l e^{1/s} = \infty$ for any $l \in \mathbb{R}$, by (2.2) and (2.18),

$$\int_{\rho_i}^1 s^{-a_1} \left[(1-m)\overline{w}(s)^{-1} + a_3 s^{-2} \overline{w}(s) q(s)^2 \right]$$

$$\times e^{a_2 \int_s^1 \sigma^{-2} \overline{w}(\sigma)^{1-m} \, d\sigma} \, ds \to \infty \quad \text{as } i \to \infty$$

and

$$\rho_i^{-a_1} e^{a_2 \int_{\rho_i}^1 s^{-2} \overline{w}(s)^{1-m} \, ds} \to \infty \quad \text{as } i \to \infty.$$

Hence by (2.2), (2.34), (2.35) and l'Hospital rule,

$$q_\infty = \lim_{i \to \infty} q(\rho_i)$$

$$= \lim_{i \to \infty} \frac{-\rho_i^{-a_1} \left[(1-m)\overline{w}(\rho_i)^{-1} + a_3 \rho_i^{-2} \overline{w}(\rho_i) q(\rho_i)^2 \right] e^{a_2 \int_{\rho_i}^1 s^{-2} \overline{w}(s)^{1-m} \, ds}}{-a_1 \rho_i^{-(a_1+1)} e^{a_2 \int_{\rho_i}^1 s^{-2} \overline{w}(s)^{1-m} \, ds} - a_2 \rho_i^{-(a_1+2)} \overline{w}(\rho_i)^{1-m} e^{a_2 \int_{\rho_i}^1 s^{-2} \overline{w}(s)^{1-m} \, ds}}$$

$$= \lim_{i \to \infty} \frac{(1-m)\rho_i^2 \overline{w}(\rho_i)^{-1} + a_3 \overline{w}(\rho_i) q(\rho_i)^2}{a_1 \rho_i + a_2 \overline{w}(\rho_i)^{1-m}} = \frac{a_3}{a_2} \lambda^{-((m\rho_1)/((1-m)\beta))} q_\infty^2. \tag{2.36}$$

Hence by (2.34) and (2.36),

$$q_\infty = \frac{a_2}{a_3} \lambda^{((m\rho_1)/((1-m)\beta))}.$$

Since the sequence $\{\rho_i\}$ is arbitrary,

$$\lim_{\rho \to 0^+} q(\rho) = \frac{a_2}{a_3} \lambda^{((m\rho_1)/((1-m)\beta))}$$

and (2.13) follows. Since

$$r^{\alpha_m/\beta + 1} (v^{(m)})'(r) = \frac{\rho_1}{\beta} \rho \overline{w}_\rho(\rho) - \frac{\alpha_m}{\beta} r^{\alpha_m/\beta} v^{(m)}(r) \quad \forall \rho = r^{\rho_1/\beta} > 0, \tag{2.37}$$

by (1.7) and (2.13),

$$\lim_{r \to 0^+} r^{\alpha_m/\beta+1}(v^{(m)})'(r) = \frac{\rho_1}{\beta} \lim_{\rho \to 0^+} \rho\overline{w}_\rho(\rho) - \frac{\alpha_m}{\beta} \lim_{r \to 0^+} r^{\alpha_m/\beta} v^{(m)}(r)$$

$$= -\frac{\alpha_m}{\beta} \lambda^{-((\rho_1)/((1-m)\beta))}$$

and (2.14) follows. □

LEMMA 2.3. *Let* $n \geqslant 3$, $0 \leqslant m < n - 2/n$, $\rho_1 > 0$, $\lambda > 0$ *and* $\beta > \beta_0^{(m)}$. *Then*

$$\lim_{\rho \to 0^+} \overline{w}_{\rho\rho}(\rho) = \frac{a_3(ma_3 - a_1)}{a_2^2} \lambda^{-(((2m-1)\rho_1)/((1-m)\beta))} \tag{2.38}$$

where a_1, a_2 *and* a_3 *are constants given by* (1.13). *Hence* $\overline{w}$ *can be extended to a function in* $C^2([0,\infty))$ *by defining* $\overline{w}_\rho(0)$, $\overline{w}_\rho(0)$ *and* $\overline{w}_{\rho\rho}(0)$ *by* (1.12).

Proof. Let $\tilde{v}(\rho) = \overline{w}_\rho(\rho)$. Then by (2.1),

$$\begin{aligned}
\tilde{v}' &= (1-m)\frac{\tilde{v}^2}{\overline{w}} - \frac{a_1}{\rho}\tilde{v} - \frac{a_2}{\rho^2}\overline{w}^{1-m}\tilde{v} + \frac{a_3}{\rho^2}\overline{w} \\
&= (1-m)\frac{\tilde{v}^2}{\overline{w}} - \frac{a_1}{\rho}\tilde{v} - \frac{a_2\overline{w}^{1-m}}{\rho^2}\left(\tilde{v} - \frac{a_3}{a_2}\cdot\overline{w}^m\right) \quad \forall \rho > 0.
\end{aligned} \tag{2.39}$$

Let $v_1(\rho)$ be given by

$$\tilde{v}(\rho) = v_0 + v_1(\rho)\rho \tag{2.40}$$

where

$$v_0 = \frac{a_3}{a_2}\overline{w}^m(0).$$

Then by (2.39) for any $\rho > 0$,

$$\begin{aligned}
v_1'(\rho)\rho + v_1(\rho) = \tilde{v}'(\rho) &= \frac{(1-m)}{\overline{w}(\rho)}\tilde{v}^2(\rho) - a_1 v_1(\rho) - \frac{a_1 v_0}{\rho} \\
&\quad - \frac{a_2\overline{w}(\rho)^{1-m}}{\rho^2}\left[\frac{a_3}{a_2}\left(\overline{w}^m(0) - \overline{w}^m(\rho)\right) + v_1(\rho)\rho\right].
\end{aligned} \tag{2.41}$$

By the mean value theorem, for any $\rho > 0$ there exists a constant $\xi = \xi(\rho) \in (0, \rho)$ such that

$$\overline{w}^m(\rho) - \overline{w}^m(0) = m\overline{w}(\xi)^{m-1}\overline{w}_\rho(\xi)\rho. \tag{2.42}$$

By (2.41) and (2.42),

$$\begin{aligned}
v_1'(\rho) &= \frac{1}{\rho}\left[\frac{(1-m)}{\overline{w}(\rho)}\tilde{v}^2(\rho) - (1 + a_1)v_1(\rho)\right] \\
&\quad + \frac{a_2\overline{w}(\rho)^{1-m}}{\rho^2}\left[\frac{ma_3}{a_2}\overline{w}(\xi)^{m-1}\overline{w}_\rho(\xi) - v_1(\rho) - \frac{a_1 v_0}{a_2\overline{w}(\rho)^{1-m}}\right] \\
&= \frac{a_2\overline{w}(\rho)^{1-m}}{\rho}\left[\frac{1-m}{a_2\overline{w}(\rho)^{2-m}}\tilde{v}^2(\rho) - \frac{f_1(\rho)}{\rho}\right]
\end{aligned} \tag{2.43}$$

where

$$f_1(\rho) = \frac{(1 + a_1)}{a_2} \rho v_1(\rho) \overline{w}(\rho)^{m-1} + v_1(\rho) - f_2(\rho) \tag{2.44}$$

and

$$f_2(\rho) = \frac{ma_3}{a_2} \overline{w}(\xi(\rho))^{m-1} \overline{w}_\rho(\xi(\rho)) - \frac{a_1 a_3}{a_2^2} \overline{w}(\rho)^{m-1} \lambda^{-((m\rho_1)/((1-m)\beta))}.$$

Let

$$a_4 = \frac{a_3(ma_3 - a_1)}{a_2^2}.$$

Without loss of generality, we may assume that $a_4 > 0$. Then by (2.2) and (2.13),

$$\lim_{\rho \to 0^+} f_2(\rho) = a_4 \lambda^{-(((2m-1)\rho_1)/((1-m)\beta))} \tag{2.45}$$

and

$$\lim_{\rho \to 0^+} \rho v_1(\rho) = 0. \tag{2.46}$$

Let $0 < \varepsilon < 1/5$. By (2.45) and (2.46) there exists a constant $\rho_2 > 0$ such that

$$(1 - \varepsilon)a_4 \lambda^{-(((2m-1)\rho_1)/((1-m)\beta))} \leqslant f_2(\rho) \leqslant (1 + \varepsilon)a_4 \lambda^{-(((2m-1)\rho_1)/((1-m)\beta))}$$
$$\forall 0 < \rho \leqslant \rho_2 \tag{2.47}$$

and

$$\frac{(1 + a_1)}{a_2} \rho |v_1(\rho)| \overline{w}(\rho)^{m-1} \leqslant \varepsilon a_4 \lambda^{-(((2m-1)\rho_1)/((1-m)\beta))} \quad \forall 0 < \rho \leqslant \rho_2. \tag{2.48}$$

and (2.17) hold. Let

$$\rho_\varepsilon = \min \left(\rho_2, \frac{\varepsilon a_2 a_4 \lambda^{-(((2m-1)\rho_1)/((1-m)\beta))}}{16(1-m)} \cdot \inf_{0 < \rho \leqslant 1} \frac{\overline{w}(\rho)^{2-m}}{\overline{w}_\rho(\rho)^2} \right)$$

We claim that

$$(1 - 3\varepsilon)a_4 \lambda^{-(((2m-1)\rho_1)/((1-m)\beta))}$$
$$\leqslant v_1(\rho) \leqslant (1 + 3\varepsilon)a_4 \lambda^{-(((2m-1)\rho_1)/((1-m)\beta))} \quad \forall 0 < \rho < \rho_\varepsilon. \tag{2.49}$$

Suppose that the second inequality in (2.49) does not hold. Then there exists a constant $\rho_1' \in (0, \rho_\varepsilon)$ such that

$$v_1(\rho_1') > (1 + 3\varepsilon)a_4 \lambda^{-(((2m-1)\rho_1)/((1-m)\beta))}.$$

By continuity of $v_1(\rho)$ on $(0, \infty)$, there exists a maximal interval (ρ_3, ρ_4) containing ρ_1', $0 \leqslant \rho_3 < \rho_1' < \rho_4 \leqslant \rho_\varepsilon$, such that

$$v_1(\rho) > (1 + 3\varepsilon)a_4 \lambda^{-(((2m-1)\rho_1)/((1-m)\beta))} \quad \forall \rho \in (\rho_3, \rho_4). \tag{2.50}$$

Then by (2.44), (2.47), (2.48) and (2.50),

$$f_1(\rho) > \varepsilon a_4 \lambda^{-(((2m-1)\rho_1)/((1-m)\beta))} \quad \forall \rho \in (\rho_3, \rho_4). \tag{2.51}$$

Hence by (2.17), (2.43) and (2.51),

$$
\begin{aligned}
v_1'(\rho) &\leqslant \frac{a_2 \overline{w}(\rho)^{1-m}}{\rho} \left[\frac{1-m}{a_2 \overline{w}(\rho)^{2-m}} \tilde{v}^2(\rho) - \frac{\varepsilon a_4 \lambda^{-(((2m-1)\rho_1)/((1-m)\beta))}}{\rho} \right] \\
&\leqslant -\frac{\delta_0}{\rho^2} < 0 \quad \forall \rho \in (\rho_3, \rho_4)
\end{aligned}
\tag{2.52}
$$

for some constant $\delta_0 > 0$. Integrating (2.52) over (ρ, ρ_4),

$$v_1(\rho_4) - v_1(\rho) \leqslant \delta_0 \left(\frac{1}{\rho_4} - \frac{1}{\rho} \right) \quad \forall \rho \in (\rho_3, \rho_4)$$

$$\Rightarrow \overline{w}_\rho(\rho) = \tilde{v}(\rho) = v_0 + \rho v_1(\rho) \geqslant v_0 + \rho v_1(\rho_4) + \delta_0 - \frac{\delta_0 \rho}{\rho_4} \quad \forall \rho \in (\rho_3, \rho_4). \tag{2.53}$$

If $\rho_3 = 0$, then by (2.53) and lemma 2.2,

$$v_0 = \lim_{\rho \to 0^+} \overline{w}_\rho(\rho) \geqslant v_0 + \delta_0 > v_0$$

and contradiction arises. Hence $\rho_3 > 0$. Thus

$$v_1(\rho_3) = (1 + 3\varepsilon) a_4 \lambda^{-(((2m-1)\rho_1)/((1-m)\beta))}.$$

Then by (2.52),

$$v_1(\rho) < v_1(\rho_3) = (1 + 3\varepsilon) a_4 \lambda^{-(((2m-1)\rho_1)/((1-m)\beta))} \quad \forall \rho \in (\rho_3, \rho_4)$$

which contradicts (2.50). Hence no such $\rho_1' > 0$ exists and the second inequality in (2.49) follows. By a similar argument the first inequality in (2.49) also holds. Hence (2.49) holds. Since $\varepsilon \in (0, 1/5)$ is arbitrary, by (2.49),

$$\overline{w}_{\rho\rho}(0) = \lim_{\rho \to 0^+} \frac{v(\rho) - v_0}{\rho} = \lim_{\rho \to 0^+} v_1(\rho) = \frac{a_3(m a_3 - a_1)}{a_2^2} \cdot \lambda^{-(((2m-1)\rho_1)/((1-m)\beta))}$$

and the lemma follows. $\qquad \square$

By lemmas 2.2, 2.3, (2.37) and Taylor's expansions for $\overline{w}$ and $\overline{w}_\rho$, theorem 1.2 follows.

COROLLARY 2.4. *Let $n \geqslant 3$, $0 \leqslant m < n - 2/n$, $\rho_1 > 0$, $\lambda > 0$, $\beta > \beta_0^{(m)}$ and ϕ_m, α_m, $\beta_0^{(m)}$, be given by (1.2), (1.5) and (1.6) respectively and $v = v^{(m)}$ is a radially symmetric solution of (1.3) in $\mathbb{R}^n \backslash \{0\}$ which satisfies (1.7). Then*

$$(v^{(m)})'(r) < 0 \quad \forall r > 0. \tag{2.54}$$

Proof. By (1.3) and lemma 2.1,

$$(r^{n-1}v(r)^{m-1}v'(r))' = -\alpha_m \left(v(r) + \frac{\beta}{\alpha_m} r v'(r) \right) < 0 \quad \forall r > 0 \tag{2.55}$$

By theorem 1.2 there exists $\xi_0 > 0$ such that

$$v'(r) < 0 \quad \forall 0 < r \leqslant \xi_0. \tag{2.56}$$

By (2.55) and (2.56),

$$r^{n-1}v(r)^{m-1}v'(r) < \xi_0^{n-1}v(\xi_0)^{m-1}v'(\xi_0) < 0 \quad \forall r > \xi_0$$
$$\Rightarrow v'(r) < 0 \quad \forall r > \xi_0. \tag{2.57}$$

By (2.56) and (2.57), we get (2.54) the lemma follows. $\qquad\square$

We are now ready for the proof of Theorem 1.3.

Proof of Theorem 1.3. Note that the case $0 < m < n - 2/n$ and $\beta \geqslant ((\rho_1)/(n-2-nm))$ is already proved in [**23**]. We will give a new proof which includes all cases of the theorem. By (1.3), (1.7) and integration by parts,

$$r^{n-1}(v_1(r)^{m-1}v_1'(r) - v_2(r)^{m-1}v_2'(r)) + \beta r^n(v_1(r) - v_2(r))$$

$$= \sum_{i=1}^{2} (-1)^{i-1} \xi^{n-2-((m\alpha_m)/(\beta))}(\xi^{\alpha_m/\beta}v_i(\xi))^{m-1}\xi^{\alpha_m/\beta+1}v_i'(\xi)$$

$$+ \beta \xi^n(v_1(\xi) - v_2(\xi)) + (n\beta - \alpha_m) \int_{\xi}^{r} (v_1(\rho) - v_2(\rho))\rho^{n-1}\,\mathrm{d}\rho, \quad \forall r > \xi > 0. \tag{2.58}$$

By theorem 1.2, there exist constants $\xi_0 > 0$ and $C_0 > 0$ such that

$$\left| (\xi^{\alpha_m/\beta}v_i(\xi))^{m-1}\xi^{\alpha_m/\beta+1}v_i'(\xi) \right| \leqslant C_0 \quad \forall 0 < \xi < \xi_0, \quad i = 1, 2. \tag{2.59}$$

Since $\beta > \beta_0^{(m)}$, $n - 2 - ((m\alpha_m)/(\beta)) > 0$. Hence by (2.59),

$$\lim_{\xi \to 0} \sum_{i=1}^{2} \left| \xi^{n-2-((m\alpha_m)/(\beta))}(\xi^{\alpha_m/\beta}v_i(\xi))^{m-1}\xi^{\alpha_m/\beta+1}v_i'(\xi) \right| = 0. \tag{2.60}$$

By (1.14) of theorem 1.2 there exist constants $C > 0$ and $r_0 > 0$ such that

$$r^{\alpha_m/\beta}|v_1(r) - v_2(r)| \leqslant Cr^{2\rho_1/\beta} \quad \forall 0 < r < r_0. \tag{2.61}$$

Hence

$$|\xi^n(v_1(\xi) - v_2(\xi))| \leqslant C\xi^{n-((\alpha_m)/(\beta))+((2\rho_1)/(\beta))}$$

$$= C\xi^{(((n-2-nm))/((1-m)\beta))(\beta-\beta_0^{(m)})} \cdot \xi^{\rho_1/\beta} \to 0 \quad \text{as } \xi \to 0. \tag{2.62}$$

Letting $\xi \to 0$ in (2.58), by (2.60) and (2.62),

$$r^{n-1}(v_1(r)^{m-1}v_1'(r) - v_2(r)^{m-1}v_2'(r)) + \beta r^n(v_1(r) - v_2(r))$$

$$= (n\beta - \alpha_m) \int_0^r (v_1(\rho) - v_2(\rho))\rho^{n-1}\,\mathrm{d}\rho \quad \forall r > 0. \tag{2.63}$$

By corollary 2.4,

$$v_i'(r) < 0 \quad \forall r > 0, \quad i = 1, 2. \tag{2.64}$$

Since $n + ((2\rho_1 - \alpha_m)/(\beta)) = (((n-2-nm))/((1-m)\beta))(\beta - \beta_0^{(m)}) + \rho_1/\beta > 0$, by (2.61),

$$\left| (n\beta - \alpha_m) \int_0^r (v_1 - v_2)(\rho)\rho^{n-1}\,\mathrm{d}\rho \right|$$

$$\leqslant C \int_0^r \rho^{n-\alpha_m/\beta+((2\rho_1)/(\beta))-1}\,\mathrm{d}\rho = Cr^{n+((2\rho_1-\alpha_m)/(\beta))} \quad \forall 0 < r < r_0 \tag{2.65}$$

for some constant $C > 0$. Hence by (2.63) and (2.65),

$$r^{n-1}(v_1(r)^{m-1}v_1'(r) - v_2(r)^{m-1}v_2'(r)) + \beta r^n(v_1(r) - v_2(r))$$

$$\leqslant Cr^{n+((2\rho_1-\alpha_m)/(\beta))} \quad \forall 0 < r < r_0. \tag{2.66}$$

Let

$$\mathcal{D} = \{0 < r < r_0 : v_1(r) \geqslant v_2(r)\}.$$

By (2.64) and (2.66) for any $r \in \mathcal{D}$,

$$v_1(r)^{m-1}v_1'(r) + \beta r v_1(r)$$

$$\leqslant v_2(r)^{m-1}v_2'(r) + \beta r v_2(r) + Cr^{1+((2\rho_1-\alpha_m)/(\beta))}$$

$$\leqslant v_1(r)^{m-1}v_2'(r) + \beta r v_2(r) + Cr^{1+((2\rho_1-\alpha_m)/(\beta))}$$

$$\Rightarrow (v_1 - v_2)'(r) + \beta r v_1(r)^{1-m}(v_1 - v_2)(r) \leqslant Cr^{1+((2\rho_1-\alpha_m)/(\beta))}v_1(r)^{1-m}.$$

Hence

$$((v_1 - v_2)_+(r)\mathrm{e}^{\beta \int_{r_1}^r \rho v_1(\rho)^{1-m}\,\mathrm{d}\rho})'$$

$$\leqslant Cr^{1+((2\rho_1-\alpha_m)/(\beta))}v_1(r)^{1-m}\mathrm{e}^{\beta \int_{r_1}^r \rho v_1(\rho)^{1-m}\,\mathrm{d}\rho} \quad \forall 0 < r_1 < r < r_0$$

$$\Rightarrow (v_1 - v_2)_+(r_2) \leqslant (v_1 - v_2)_+(r_1)\mathrm{e}^{-\beta \int_{r_1}^{r_2} \rho v_1(\rho)^{1-m}\,\mathrm{d}\rho}$$

$$+ C\frac{\int_{r_1}^{r_2} \rho^{1+((2\rho_1-\alpha_m)/(\beta))}v_1(\rho)^{1-m}(\mathrm{e}^{\beta \int_{r_1}^\rho sv_1(s)^{1-m}\,\mathrm{d}s})\,\mathrm{d}\rho}{\mathrm{e}^{\beta \int_{r_1}^{r_2} \rho v_1(\rho)^{1-m}\,\mathrm{d}\rho}} \quad \forall 0 < r_1 < r_2 < r_0. \tag{2.67}$$

Since $rv_1(r)^{1-m} \approx r^{1-(1-m)\alpha_m/\beta} = r^{-1-\rho_1/\beta}$ near $r = 0$, both the numerator and denominator of the last term of (2.67) goes to infinity as $r_1 \to 0$. Hence by the

l'Hospital rule,

$$\lim_{r_1 \to 0} \frac{\int_{r_1}^{r_2} \rho^{1+((2\rho_1 - \alpha_m)/(\beta))} v_1(\rho)^{1-m} (e^{\beta \int_{r_1}^{\rho} s v_1(s)^{1-m} \, ds}) \, d\rho}{e^{\beta \int_{r_1}^{r_2} \rho v_1(\rho)^{1-m} \, d\rho}}$$

$$= \lim_{r_1 \to 0} \frac{r_1^{((2\rho_1 - \alpha_m)/(\beta))}}{\beta e^{\beta \int_{r_1}^{r_2} \rho v_1(\rho)^{1-m} \, d\rho}}. \tag{2.68}$$

Since

$$\lim_{r_1 \to 0} \frac{\int_{r_1}^{r_2} \rho \cdot v_1(\rho)^{1-m} \, d\rho}{r_1^{-\rho_1/\beta}} = \lim_{r_1 \to 0} \frac{r_1 v_1(r_1)^{1-m}}{\rho_1/\beta \, r_1^{-\rho_1/\beta - 1}} = \frac{\beta}{\rho_1} \lambda^{-\rho_1/\beta} \quad \forall 0 < r_2 < r_0,$$

for any $0 < r_2 < r_0$ there exists a constant $r_3 \in (0, r_2)$ such that

$$\int_{r_1}^{r_2} \rho v_1(\rho)^{1-m} \, d\rho \geqslant \frac{\beta}{2\rho_1} \lambda^{-\rho_1/\beta} r_1^{-\rho_1/\beta} \quad 0 < r_1 < r_3$$

$$\Rightarrow \left| \frac{r_1^{\theta}}{e^{\beta \int_{r_1}^{r_2} \rho v_1^{1-m}(\rho) \, d\rho}} \right| \leqslant \frac{r_1^{\theta}}{e^{\beta^2/2\rho_1 (\lambda r_1)^{-\rho_1/\beta}}} \to 0$$

$$\text{as } r_1 \to 0^+ \quad \forall 0 < r_2 < r_0, \theta \in \mathbb{R}. \tag{2.69}$$

By (1.7) and corollary 2.4

$$v_i(r) < \lambda^{-((\rho_1)/((1-m)\beta))} r^{-\alpha_m/\beta} \quad \forall r > 0, \quad i = 1, 2. \tag{2.70}$$

Hence by (2.68), (2.69) and (2.70),

$$\lim_{r_1 \to 0} \frac{(v_1 - v_2)_+(r_1)}{e^{\beta \int_{r_1}^{r_2} \rho v_1^{1-m} \, d\rho}} = 0 \quad \text{and}$$

$$\lim_{r_1 \to 0} \frac{\int_{r_1}^{r_2} \rho^{1+((2\rho_1 - \alpha_m)/(\beta))} v_1(\rho)^{1-m} (e^{\beta \int_{r_1}^{\rho} s v_1(s)^{1-m} \, ds}) \, d\rho}{e^{\beta \int_{r_1}^{r_2} \rho v_1(\rho)^{1-m} \, d\rho}} = 0. \tag{2.71}$$

By (2.67) and (2.71),

$$(v_1 - v_2)_+(r) \leqslant 0 \quad \forall 0 \leqslant r < r_0. \tag{2.72}$$

Similarly

$$(v_1 - v_2)_-(r) \leqslant 0 \quad \forall 0 \leqslant r < r_0. \tag{2.73}$$

By (2.72) and (2.73),

$$v_1(r) = v_2(r) \quad \forall 0 \leqslant r < r_0. \tag{2.74}$$

Then by (2.74) and standard O.D.E. theory, (1.15) holds. $\square$

3. Decay estimates of solutions of the elliptic logarithmic equation

In this section, we will prove the decay rate of solutions of the elliptic logarithmic equation (1.8).

LEMMA 3.1. *Let* $n \geqslant 3$, $\beta \in \mathbb{R}$, $\rho_1 > 0$ *and* $\alpha = 2\beta + \rho_1$. *Let* $v = v^{(0)}$ *be a radially symmetric solution of* (1.8) *in* $\mathbb{R}^n \backslash B_1$ *and* $w(r) = r^2 v(r)$. *Suppose that there exists a constant* $C_0 > 0$ *such that*

$$w(r) \leqslant C_0 \quad \forall r \geqslant 1. \tag{3.1}$$

Then, any sequence $\{w(r_i)\}_{i=1}^{\infty}$, $r_i \to \infty$ *as* $i \to \infty$, *has a subsequence* $\{w(r_i')\}_{i=1}^{\infty}$ *such that*

$$\lim_{i \to \infty} w(r_i') = \begin{cases} 0 \quad \text{or} \quad w_{\infty} & \text{if } v \notin L^1(\mathbb{R}^n \backslash B_1) \\ 0 \quad \text{or} \quad w_1 & \text{if } v \in L^1(\mathbb{R}^n \backslash B_1) \quad \text{and} \quad \beta > 0 \\ 0 & \text{if } v \in L^1(\mathbb{R}^n \backslash B_1) \quad \text{and} \quad \beta \leqslant 0 \end{cases} \tag{3.2}$$

where

$$w_{\infty} = \frac{2(n-2)}{\alpha - 2\beta} \quad \text{and} \quad w_1 = \frac{2}{\beta}.$$

Proof. We will use a modification of the proof of Lemma 2.1 of [19] to prove the lemma. Let $\{r_i\}_{i=1}^{\infty}$ be a sequence such that $r_i \to \infty$ as $i \to \infty$. By (3.1) the sequence $\{w(r_i)\}_{i=1}^{\infty}$ has a subsequence which we may assume without loss of generality to be the sequence itself that converges to some constant $a_0 \in [0, C_0]$ as $i \to \infty$. Multiplying (1.8) by r^{n-1} and integrating over $(1, r)$,

$$v'(r) = a_5 \frac{v(r)}{r^{n-1}} - \beta r v^2(r) + \frac{(n\beta - \alpha)}{r^{n-1}} v(r) \int_1^r \rho^{n-1} v(\rho) \, \mathrm{d}\rho \quad \forall r \geqslant 1. \tag{3.3}$$

where

$$a_5 = v(1)^{-1} v'(1) + \beta v(1). \tag{3.4}$$

Integrating (3.3) over (r, ∞), by (3.1), we have

$$v(r) = -a_5 \int_r^{\infty} s^{1-n} v(s) \, \mathrm{d}s + \beta \int_r^{\infty} s v^2(s) \, \mathrm{d}s$$

$$+ (\alpha - n\beta) \int_r^{\infty} s^{1-n} v(s) \left(\int_1^s \rho^{n-1} v(\rho) \, \mathrm{d}\rho \right) \mathrm{d}s \quad \forall r > 1. \tag{3.5}$$

By (3.1), (3.5) and l'Hospital rule,

$$
\begin{aligned}
a_0 &= \lim_{i \to \infty} r_i^2 v(r_i) \\
&= -a_5 \lim_{i \to \infty} \frac{\int_{r_i}^{\infty} s^{1-n} v(s)\, \mathrm{d}s}{r_i^{-2}} + \beta \lim_{i \to \infty} \frac{\int_{r_i}^{\infty} s v^2(s)\, \mathrm{d}s}{r_i^{-2}} \\
&\quad + (\alpha - n\beta) \lim_{i \to \infty} \frac{\int_{r_i}^{\infty} s^{1-n} v(s)(\int_1^s \rho^{n-1} v(\rho)\, \mathrm{d}\rho)\, \mathrm{d}s}{r_i^{-2}} \\
&= \frac{1}{2}\left(-a_5 \lim_{i \to \infty} \frac{v(r_i)}{r_i^{n-4}} + \beta \lim_{i \to \infty} \frac{r_i v^2(r_i)}{r_i^{-3}} \right. \\
&\quad \left. + (\alpha - n\beta) \lim_{i \to \infty} \frac{r_i^{1-n} v(r_i) \int_1^{r_i} \rho^{n-1} v(\rho)\, \mathrm{d}\rho}{r_i^{-3}} \right) \\
&= \frac{1}{2}\left(\beta a_0^2 + (\alpha - n\beta) \lim_{i \to \infty} \frac{r_i^2 v(r_i) \int_1^{r_i} \rho^{n-1} v(\rho)\, \mathrm{d}\rho}{r_i^{n-2}} \right) \\
&= \frac{1}{2}\left(\beta a_0^2 + (\alpha - n\beta) a_0 \lim_{i \to \infty} \frac{\int_1^{r_i} \rho^{n-1} v(\rho)\, \mathrm{d}\rho}{r_i^{n-2}} \right). \tag{3.6}
\end{aligned}
$$

If $v \notin L^1(\mathbb{R}^n \backslash B_1)$, then by (3.6) and the l'Hospital rule,

$$
a_0 = \frac{1}{2}\left(\beta a_0^2 + \frac{(\alpha - n\beta)}{n-2} a_0 \lim_{i \to \infty} r_i^2 v(r_i) \right) = \frac{\alpha - 2\beta}{2(n-2)} a_0^2
$$

$$
\Rightarrow a_0 = 0 \quad \text{or} \quad a_0 = \frac{2(n-2)}{\alpha - 2\beta} = w_\infty. \tag{3.7}
$$

If $v \in L^1(\mathbb{R}^n \backslash B_1)$, then by (3.6),

$$
a_0 = \frac{\beta}{2} a_0^2 \Rightarrow \begin{cases} a_0 = 0 \quad \text{or} \quad a_0 = \frac{2}{\beta} = w_1 & \text{if } \beta > 0 \\ a_0 = 0 & \text{if } \beta \leqslant 0. \end{cases} \tag{3.8}
$$

By (3.7) and (3.8), the lemma follows. $\qquad\square$

COROLLARY 3.2. *Let $n \geqslant 3$, $\beta \in \mathbb{R}$, $\rho_1 > 0$ and $\alpha = 2\beta + \rho_1$. Let $v = v^{(0)}$ be a radially symmetric solution of* (1.8) *in $\mathbb{R}^n \backslash B_1$ and $w(r) = r^2 v(r)$. Suppose that there exist constants $C_0 > C_1 > 0$ such that*

$$
C_1 \leqslant w(r) \leqslant C_0 \quad \forall r \geqslant 1.
$$

Then (1.16) *holds.*

LEMMA 3.3. *Let $n \geqslant 3$, $\rho_1 > 0$, $\beta > \beta_1^{(0)} := \rho_1/n - 2$ and $\alpha = 2\beta + \rho_1$. Let $v = v^{(0)}$ be a radially symmetric solution of* (1.8) *in $\mathbb{R}^n \backslash B_1$ and $w(r) = r^2 v(r)$. Then there exists a constant $C_1 > 0$ such that*

$$
w(r) \geqslant C_1 \quad \forall r \geqslant 1. \tag{3.9}
$$

Proof. By (3.3),

$$v'(r) + \beta r v^2(r) + |a_5| r^{1-n} v(r) \geqslant 0 \quad \forall r \geqslant 1 \tag{3.10}$$

where a_5 is given by (3.4). Let $H(r) = e^{-((|a_5|)/(n-2))r^{2-n}} v(r)$. Then by (3.10),

$$H'(r) \geqslant -\beta e^{((|a_5|)/(n-2))r^{2-n}} r H(r)^2 \geqslant -\beta e^{((|a_5|)/(n-2))} r H(r)^2 \quad \forall r \geqslant 1 \tag{3.11}$$

$$\Rightarrow -H(r)^{-2} H'(r) \leqslant \beta e^{((|a_5|)/(n-2))} r \qquad \forall r \geqslant 1$$

$$\Rightarrow v(r) \geqslant H(r) \geqslant \left(\frac{\beta e^{((|a_5|)/(n-2))}}{2} r^2 + H(1)^{-1} \right)^{-1} \qquad \forall r \geqslant 1. \tag{3.12}$$

By (3.12) there exists a constant $C_1 > 0$ such that (3.9) holds and the lemma follows. $\qquad \square$

LEMMA 3.4. *Let* $n \geqslant 3$, $\rho_1 > 0$, $\beta \leqslant \beta_1^{(0)} := \rho_1/n - 2$ *and* $\alpha = 2\beta + \rho_1$. *Let* $v = v^{(0)}$ *be a radially symmetric solution of* (1.8) *in* $\mathbb{R}^n \backslash B_1$ *and* $w(r) = r^2 v(r)$. *Then there exists a constant* $C_1 > 0$ *such that* (3.9) *holds.*

Proof. As observed in [**16**], w satisfies

$$\left(\frac{w'}{w} \right)' + \frac{n-1}{r} \cdot \frac{w'}{w} + \frac{\beta}{r} w' + \frac{(\alpha - 2\beta)w - 2(n-2)}{r^2} = 0 \quad \forall r \geqslant 1. \tag{3.13}$$

Multiplying (3.13) by r^{n-1} and integrating over $(1, r)$, $r > 1 > 0$,

$$\frac{r^{n-1} w'(r)}{w(r)} + \beta r^{n-2} w(r)$$

$$= \frac{w'(1)}{w(1)} + \beta w(1) + (n\beta - \alpha) \int_1^r \rho^{n-3} w(\rho) \, d\rho + 2(r^{n-2} - 1) \quad \forall r > 1$$

$$\Rightarrow \frac{r w'(r)}{w(r)} = 2 - \beta w(r) + \frac{(n\beta - \alpha)}{r^{n-2}} \int_0^r \rho^{n-3} w(\rho) \, d\rho + \frac{a_6}{r^{n-2}} \quad \forall r > 1 \tag{3.14}$$

where $a_6 = w(1)^{-1} w'(1) + \beta w(1) - 2$. Let

$$\mathcal{B} = \left\{ r > 1 : w(r) \leqslant \frac{1}{\rho_1} \right\}.$$

If there is a constant $R_0 > 1$ such that $\mathcal{B} \cap [R_0, \infty) = \emptyset$, then

$$w(r) \geqslant \frac{1}{\rho_1} \quad \forall r \geqslant R_0$$

and (3.9) follows. Hence we may assume that

$$\mathcal{B} \cap [R_1, \infty) \neq \emptyset \quad \forall R_1 > 1. \tag{3.15}$$

If $\int_1^\infty \rho^{n-3} w(\rho) \, d\rho = \infty$ holds, then by (3.14) and the l'Hospital rule,

$$\liminf_{\substack{r \to \infty \\ r \in \mathcal{B}}} \frac{r w'(r)}{w(r)} \geqslant 2 - \frac{\beta}{\rho_1} - \frac{\alpha - n\beta}{n-2} \cdot \frac{1}{\rho_1} = 2 - \frac{1}{n-2} > 0.$$

If $\int_1^\infty \rho^{n-3} w(\rho)\,\mathrm{d}\rho < \infty$ holds, then by (3.14),

$$\liminf_{\substack{r\to\infty \\ r\in\mathcal{B}}} \frac{rw'(r)}{w(r)} \geqslant 2 - \frac{\beta}{\rho_1} > 2 - \frac{1}{n-2} > 0.$$

Hence in both cases there exists a constant $R_2 \in \mathcal{B}$ such that

$$w'(r) > 0 \quad \forall r \in \mathcal{B} \cap [R_2, \infty). \tag{3.16}$$

Suppose that there exists a constant $R_3 > R_2$ such that $R_3 \notin \mathcal{B}$. Let

$$R_4 = \sup\left\{ r_1 > R_3 : w(r) > \frac{1}{\rho_1} \quad \forall R_3 \leqslant r < r_1 \right\}.$$

By (3.15),

$$R_4 < \infty \Rightarrow w(R_4) = \frac{1}{\rho_1}, \quad R_4 \in \mathcal{B} \quad \text{and} \quad w'(R_4) \leqslant 0$$

which contradicts (3.16). Thus no such point R_3 exists. Hence

$$[R_2, \infty) \subset \mathcal{B}. \tag{3.17}$$

By (3.16) and (3.17),

$$w(r) \geqslant w(R_2) \quad \forall r \geqslant R_2$$

and the lemma follows. $\qquad\square$

LEMMA 3.5. *Let $n \geqslant 3$, $\rho_1 > 0$, $\beta \leqslant \beta_1^{(0)} := \rho_1/n - 2$ and $\alpha = 2\beta + \rho_1$. Let $v = v^{(0)}$ be a radially symmetric solution of (1.8) in $\mathbb{R}^n \backslash B_1$ and $w(r) = r^2 v(r)$. Then there exists a constant $C_0 > 0$ such that (3.1) holds.*

Proof. By corollary 2.4 v satisfies (2.54). Since $\alpha \geqslant n\beta$, by (1.8), (1.9), (2.54) and lemma 3.4,

$$r^{n-1}\frac{v'(r)}{v(r)} + \beta r^n v(r) = \frac{v'(1)}{v(1)} + \beta v(1) - (\alpha - n\beta)\int_1^r \rho^{n-1} v(\rho)\,\mathrm{d}\rho \quad \forall r > 1$$

$$\leqslant \frac{v'(1)}{v(1)} + \beta v(1) - (\alpha - n\beta)\int_1^r \rho^{n-1} v(r)\,\mathrm{d}\rho \quad \forall r > 1$$

$$\leqslant a_5 - \frac{(\alpha - n\beta)}{n} r^n v(r) \quad \forall r > 1$$

$$\Rightarrow r^{n-1}\frac{v'(r)}{v(r)} + \frac{\alpha}{n} r^n v(r) \leqslant a_5 \quad \forall r > 1$$

$$\Rightarrow \frac{v'(r)}{v(r)^2} + \frac{\alpha}{n} r \leqslant \frac{|a_5|}{r^{n-1} v(r)} \leqslant \frac{C_2}{r^{n-3}} \leqslant C_2 \quad \forall r > 1 \tag{3.18}$$

for some constant $C_2 > 0$ where a_5 is given by (3.4). Integrating (3.18) over $(1, r)$,

$$\frac{1}{v(r)} \geqslant \frac{\alpha r^2}{2n} - C_2(r-1) - \frac{\alpha}{2n} + \frac{1}{v(1)} \geqslant \frac{\alpha r^2}{4n} + C_3 \quad \forall r > \max\left(1, \frac{4nC_2}{\alpha}\right) \tag{3.19}$$

where $C_3 = C_2 - \alpha/2n + 1/v(1)$. Then by (2.54) and (3.19), (3.1) holds for some constant $C_0 > 0$ and the lemma follows. $\qquad\square$

LEMMA 3.6 (cf. lemma 2.6 of [**19**]). *Let $n \geqslant 3$, $\rho_1 > 0$, $\beta > \beta_1^{(0)} = \rho_1/n - 2$ and $\alpha = 2\beta + \rho_1$. Let $v = v^{(0)}$ be a radially symmetric solution of (1.8) in $\mathbb{R}^n \backslash B_1$ and $w(r) = r^2 v(r)$. Then there exists a constant $C_0 > 0$ such that (3.1) holds.*

Proof. The proof of the lemma is similar to the proof of Lemma 2.6 of [**19**]. For the sake of completeness, we will give a sketch of the proof here. Let $A = \{r \in [1, \infty) : w'(r) \geqslant 0\}$. If there exists a constant $R_0 > 1$ such that $A \cap [R_0, \infty) = \emptyset$. Then $w'(r) < 0$ for all $r \geqslant R_0$ and (3.1) holds with $C_0 = \max_{1 \geqslant r \geqslant R_0} w(r)$.

We next suppose that $A \cap [R_0, \infty) \neq \emptyset$ for any $R_0 > 1$. By lemma 3.3 and the l'Hospital rule,

$$\limsup_{r \in A, \, r \to \infty} \frac{\int_1^r z^{n-1} v(z) \, \mathrm{d}z}{r^n v(r)} = \limsup_{r \in A, \, r \to \infty} \frac{\int_1^r z^{n-1} v(z) \, \mathrm{d}z}{r^{n-2} w(r)}$$

$$\leqslant \limsup_{r \in A, \, r \to \infty} \frac{r^{n-1} v(r)}{(n-2) r^{n-3} w(r) + r^{n-2} w'(r)} \leqslant \frac{1}{n-2}.$$

Hence there exists a constant $R_1 > 1$ such that

$$\int_1^r z^{n-1} v(z) \, \mathrm{d}z \leqslant \left(\frac{1}{n-2} + \frac{\rho_1}{2(n-2)(n\beta - \alpha)} \right) r^n v(r) \quad \forall r \geqslant R_1, r \in A. \quad (3.20)$$

By (3.3) and (3.20) for any $r \geqslant R_1, r \in A$,

$$\frac{r v'(r)}{v(r)} \leqslant \frac{a_5}{r^{n-2}} - \beta r^2 v(r) + (n\beta - \alpha) \left(\frac{1}{n-2} + \frac{\rho_1}{2(n-2)(n\beta - \alpha)} \right) r^2 v(r)$$

$$\leqslant \frac{a_5}{R_1^{n-2}} - \frac{\rho_1}{2(n-2)} w(r). \quad (3.21)$$

where a_5 is given by (3.4). Hence by (3.21),

$$0 \leqslant w'(r) = \frac{2w(r)}{r} \left(1 + \frac{1}{2} \frac{r v'(r)}{v(r)} \right)$$

$$\leqslant \frac{2w(r)}{r} \left(1 + \frac{a_5}{2R_1^{n-2}} - \frac{\rho_1}{4(n-2)} w(r) \right) \quad \forall r \geqslant R_1, r \in A$$

$$\Rightarrow w(r) \leqslant C_3 \quad \forall r \geqslant R_1, r \in A \quad (3.22)$$

for some constant $C_3 > 0$. Since $w'(r) < 0$ for any $r \in [R_1, \infty) \backslash A$, by (3.22) and the same argument as the proof of Lemma 2.6 of [**19**] (3.1) follows. $\square$

Proof of Theorem 1.4. If $n\beta > \alpha$, by corollary 3.2, lemmas 3.3 and 3.6, (1.16) follows. If $\alpha \geqslant n\beta$, by lemmas 3.4, 3.5 and corollary 3.2, (1.16) follows. $\square$

4. Singular limits of solutions

In this section, we will prove the singular limits of solutions of (1.1) and (1.3) as $m \to 0^+$. We first start with a lemma.

LEMMA 4.1. *Let $n \geqslant 3$, $0 < \overline{m}_0 < n - 2/n$, $\rho_1 > 0$, $\lambda > 0$, $\beta \geqslant \beta_0^{(\overline{m}_0)}$ and $\alpha_m = ((2\beta + \rho_1)/(1 - m))$. For any $0 < m < \overline{m}_0$, let $v^{(m)}$ be the radially symmetric solution of* (1.3) *in $\mathbb{R}^n \backslash \{0\}$ which satisfies* (1.7) *given by theorem* 1.1. *Then there exists a constant $m_0 \in (0, \overline{m}_0)$ such that*

$$\lambda^{-((\rho_1)/((1-m)\beta))} \leqslant r^{\alpha_m/\beta} v^{(m)}(r) \leqslant \lambda^{-((\rho_1)/((1-m)\beta))} \exp(C_m \lambda^{\rho_1/\beta} r^{\rho_1/\beta})$$

$$\forall |x| = r > 0, 0 < m < m_0 \tag{4.1}$$

holds where

$$C_m = \frac{\alpha_m}{\rho_1 \beta} \left(n - 2 - \frac{m \alpha_m}{\beta} \right). \tag{4.2}$$

Proof. We will use a modification of the technique of [**23**] to prove the theorem. Note that

$$\beta > \frac{m \rho_1}{n - 2 - nm} \quad \forall 0 < m < \overline{m}_0.$$

By the proof of Theorem 1.1 of [**23**], for any $i \in \mathbb{Z}^+$, $0 < m < \overline{m}_0$, there exists a radially symmetric solution v_i of

$$\begin{cases} \triangle \phi_m(v) + \alpha_m v + \beta x \cdot \nabla v = 0, \quad v > 0, \quad \text{in } \mathbb{R}^n \backslash B_{1/i} \\ v_i(1/i) = \lambda^{-((\rho_1)/((1-m)\beta))} i^{\alpha_m/\beta}, \\ v_i'(1/i) = -\frac{\alpha_m}{\beta} \lambda^{-((\rho_1)/((1-m)\beta))} i^{\alpha_m/\beta + 1} \end{cases}$$

which satisfies

$$v_i'(r) < 0 \quad \forall r \geqslant \tfrac{1}{i} \quad \text{and} \quad v_i(r) \geqslant \lambda^{-((\rho_1)/((1-m)\beta))} r^{-\alpha_m/\beta} \quad \forall r \geqslant \tfrac{1}{i}. \tag{4.3}$$

Moreover the sequence v_i has a subsequence which we may assume without loss of generality to be the sequence v_i itself that converges uniformly in $C^2(K)$ for any compact subset K of $\mathbb{R}^n \backslash \{0\}$ to $v = v^{(m)}$ as $i \to \infty$. Let $w_i(r) = r^{\alpha_m/\beta} v_i(r)$, $s = \log r$ and $z_i(s) = w_i^{-1}((\partial w_i)/(\partial s))$. Then by the proof of Theorem 1.1 of [**23**] (cf. [**17**]) and (4.3),

$$\left(\frac{w_{i,r}}{w_i} \right)_r + \frac{n - 1 - \frac{2m \alpha_m}{\beta}}{r} \cdot \frac{w_{i,r}}{w_i} + m \left(\frac{w_{i,r}}{w_i} \right)^2$$

$$+ \frac{\beta r^{-1-\rho_1/\beta} w_{i,r}}{w_i^m} = \frac{\alpha_m}{\beta} \cdot \frac{n - 2 - m \alpha_m/\beta}{r^2} \quad \forall r > 1/i, i \in \mathbb{N}$$

$$\Rightarrow z_{i,s} + \left(n - 2 - \frac{2m \alpha_m}{\beta} \right) z_i + m z_i^2 + \beta e^{-\rho_1/\beta s} w_i^{1-m} z_i$$

$$= \rho_1 C_m \quad \forall s > -\log i, i \in \mathbb{N}. \tag{4.4}$$

We now choose $m_0 \in (0, \overline{m}_0)$ such that

$$n - 2 - \frac{2m \alpha_m}{\beta} > 0 \quad \forall 0 < m < m_0.$$

Since by the proof of Theorem 1.1. of [**23**] $z_i(s) = w_i^{-1}((\partial w_i)/(\partial s)) \geqslant 0$ for all $s > -\log i$, by (4.3) and (4.4),

$$z_{i,s} + \beta \lambda^{-\rho_1/\beta} e^{-\rho_1/\beta s} z_i \leqslant \rho_1 C_m \quad \forall s > -\log i, \, i \in \mathbb{Z}^+, 0 < m < m_0 \qquad (4.5)$$

By (4.5) and an argument similar to the proof of Theorem 1.1 in [**23**],

$$z_i(s) \leqslant \frac{\rho_1 C_m}{\beta} \lambda^{\rho_1/\beta} e^{\rho_1/\beta s} \qquad\qquad \forall s > -\log i, \, i \in \mathbb{Z}^+, 0 < m < m_0$$

$$\Rightarrow w_i(r) \leqslant \lambda^{-((\rho_1)/((1-m)\beta))} \exp\{C_m \lambda^{\rho_1/\beta} r^{\rho_1/\beta}\} \quad \forall r > 1/i, \, i \in Z^+, 0 < m < m_0$$

$$\Rightarrow v_i(r) \leqslant \lambda^{-((\rho_1)/((1-m)\beta))} r^{-\alpha_m/\beta} \exp\{C_m \lambda^{\rho_1/\beta} r^{\rho_1/\beta}\}$$

$$\forall r > 1/i, \, i \in Z^+, 0 < m < m_0. \qquad (4.6)$$

Letting $i \to \infty$ in (4.3) and (4.6), we get (4.1) and the lemma follows $\qquad\square$

Proof of Theorem 1.5. Let $m_0 \in (0, \overline{m}_0)$ be given by lemma 4.1. Let $\{m_i\}_{i=1}^{\infty}$, $0 < m_i < m_0$ for all $i \in \mathbb{Z}^+$, be a sequence such that $m_i \to 0$ as $i \to \infty$. Let $R_2 > R_1 > 0$. By (4.1),

$$M_1(R_2) \leqslant v^{(m)}(x) \leqslant M_2(R_1, R_2) \quad \forall R_1 \leqslant |x| \leqslant R_2 \qquad (4.7)$$

where

$$\begin{cases} M_1(R_2) = \min(\lambda^{-n\rho_1/2\beta}, \lambda^{-\rho_1/\beta}) \min(R_2^{-n/2(2+\rho_1/\beta)}, R_2^{-(2+\rho_1/\beta)}) \\ M_2(R_1, R_2) = \max(\lambda^{-n\rho_1/2\beta}, \lambda^{-\rho_1/\beta}) \max(R_1^{-n/2(2+\rho_1/\beta)}, R_1^{-(2+\rho_1/\beta)}) \\ \quad \exp\left(\dfrac{n(n-2)(2\beta+\rho_1)}{2\rho_1\beta} \lambda^{\rho_1/\beta} R_2^{\rho_1/\beta}\right). \end{cases}$$

By (4.7) and the mean value theorem, for any $0 < m < m_0$ there exists $r_m \in (1,2)$ such that

$$|(v^{(m)})'(r_m)| = |v^{(m)}(2) - v^{(m)}(1)| \leqslant 2M_2(1,2). \qquad (4.8)$$

Multiplying (1.3) by r^{n-1} and integrating over (r_m, r), $R_1 \leqslant r \leqslant R_2$,

$$r^{n-1}(v^{(m)}(r))^{m-1}(v^{(m)})'(r)$$

$$= r_m^{n-1}(v^{(m)}(r_m))^{m-1}(v^{(m)})'(r_m)$$

$$+ \beta r_m^n v^{(m)}(r_m) - \beta r^n v^{(m)}(r) + (n\beta - \alpha_m) \int_{r_m}^{r} v^{(m)}(\rho)\rho^{n-1} \, d\rho. \qquad (4.9)$$

By (4.7), (4.8) and (4.9), for any $R_2 > R_1 > 0$ there exists a constant $M_3(R_1, R_2) > 0$ such that

$$|(v^{(m)})'(r)| \leqslant M_3(R_1, R_2) \quad \forall R_1 \leqslant r \leqslant R_2, 0 < m < m_0 \qquad (4.10)$$

$$\Rightarrow |v^{(m)}(r_1) - v^{(m)}(r_2)| \leqslant M_3(R_1, R_2)|r_1 - r_2| \quad \forall r_1, r_2 \in [R_1, R_2], 0 < m < m_0. \qquad (4.11)$$

By (1.3), (4.7) and (4.10), for any $R_1 \leqslant r \leqslant R_2$ and $0 < m_0 < \overline{m}_0$,

$$(v^{(m)})''(r) = (1-m)(v^{(m)}(r))^{-1}(v^{(m)})'(r)^2 - \alpha v^{(m)}(r)^{2-m}$$
$$- \beta r v^{(m)}(r)^{1-m}(v^{(m)})'(r) - (n-1)r^{-1}(v^{(m)})'(r) \qquad (4.12)$$

$$\Rightarrow |(v^{(m)})''(r)| \leqslant M_4(R_1, R_2) \quad \forall R_1 \leqslant r \leqslant R_2, 0 < m < m_0$$

$$\Rightarrow |(v^{(m)})'(r_1) - (v^{(m)})'(r_2)| \leqslant M_4(R_1, R_2)|r_1 - r_2|$$
$$\forall r_1, r_2 \in [R_1, R_2], 0 < m < m_0. \qquad (4.13)$$

for some constant $M_4(R_1, R_2) > 0$. By differentiating (1.3) with respect to $r > 0$ and repeating the above argument, there exists a constant $M_5(R_1, R_2) > 0$ such that

$$|(v^{(m)})'''(r)| \leqslant M_5(R_1, R_2) \quad \text{and} \quad |(v^{(m)})''(r_1) - (v^{(m)})''(r_2)|$$
$$\leqslant M_5(R_1, R_2)|r_1 - r_2| \quad \forall r, r_1, r_2 \in [R_1, R_2] \qquad (4.14)$$

holds for any $0 < m < m_0$. By (4.7), (4.10), (4.11), (4.13) and (4.14), the sequence $\{v^{(m_i)}\}_{i=1}^{\infty}$ is equi-Holder continuous in $C^2(K)$ for any compact subset K of $\mathbb{R}^n \setminus \{0\}$. By the Ascoli Theorem and a diagonalization argument the sequence $\{v^{(m_i)}\}_{i=1}^{\infty}$ has a subsequence which we may assume without loss of generality to be the sequence itself that converges uniformly in $C^2(K)$ for any compact subset K of $\mathbb{R}^n \setminus \{0\}$ to some positive function $v \in C^2(\mathbb{R}^n \setminus \{0\})$ as $i \to \infty$.

Putting $m = m_i$ in (4.12) and letting $i \to \infty$,

$$v''(r) = (v(r))^{-1}v'(r)^2 - \alpha v(r)^{2-1} - \beta v(r)v'(r), \quad v > 0, \quad \text{in} \mathbb{R}^n \setminus \{0\}$$

and hence v satisfies (1.8). Letting $m = m_i \to 0$ in (4.1),

$$\lambda^{-\rho_1/\beta} \leqslant |x|^{\alpha/\beta} v(x) \leqslant \lambda^{-\rho_1/\beta} \exp(C_0 \lambda^{\rho_1/\beta}|x|^{\rho_1/\beta}) \quad \forall x \in \mathbb{R}^n \setminus \{0\}$$

$$\Rightarrow \lim_{|x| \to 0} |x|^{\alpha/\beta} v(x) = \lambda^{-\rho_1/\beta}$$

where $C_0 = (((2\beta + \rho_1)(n-2))/(\rho_1 \beta))$. Then by theorem 1.3 v is the unique solution of (1.8) which satisfies (1.9). Since the sequence $\{m_i\}_{i=1}^{\infty}$ is arbitrary, $v^{(m)}$ converges uniformly in $C^2(K)$ for any compact subset of $\mathbb{R}^n \setminus \{0\}$ to the unique solution v of (1.8) which satisfies (1.9) as $m \to 0^+$ and the theorem follows. $\square$

Proof of Theorem 1.6. We will use a modification of the proof of Lemma 2.5 of [21] to prove the theorem. Let $h \in C_0^{\infty}(\mathbb{R}^n)$, $0 \leqslant h \leqslant 1$, $h(x) = 1$ for $|x| \leqslant 1$ and $h(x) = 0$ for $|x| \geqslant 2$. Let $\eta(x) = h(x)^4$ and $\eta_R(x) = \eta(x/R)$ for any $R > 0$. For any $R > 3\epsilon > 0$, let

$$\eta_{\epsilon,R}(x) = (1 - \eta(x/\epsilon))\eta_R(x).$$

Then

$$\begin{cases} \eta_{\epsilon,R} = 0 & \forall |x| \leqslant \epsilon \text{ or } |x| \geqslant 2R \\ \eta_{\epsilon,R}(x) = 1 & \forall 2\epsilon \leqslant |x| \leqslant R \end{cases}$$

and

$$|\triangle\eta_{\epsilon,R}(x)| \leqslant \frac{C_1}{\epsilon^2} \quad \forall \epsilon \leqslant |x| \leqslant 2\epsilon, \quad |\triangle\eta_{\epsilon,R}(x)| \leqslant \frac{C_1}{R^2} \quad \forall R \leqslant |x| \leqslant 2R \quad (4.15)$$

for some constant $C_1 > 0$. By Kato's inequality [24],

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^n} (u_1 - u_2)_+(x,t)\eta_{\epsilon,R}(x)\,\mathrm{d}x$$

$$\leqslant \int_{\mathbb{R}^n} (\log u_1 - \log u_2)_+(x,t)\triangle\eta_{\epsilon,R}(x)\,\mathrm{d}x$$

$$\leqslant \frac{C_1}{\epsilon^2} \int_{\epsilon\leqslant|x|\leqslant 2\epsilon} (\log u_1 - \log u_2)_+(x,t)\,\mathrm{d}x$$

$$+ \int_{R\leqslant|x|\leqslant 2R} (\log u_1 - \log u_2)_+(x,t)\,|\triangle\eta_R(x)|\,\mathrm{d}x \quad (4.16)$$

By (1.9) and lemma 2.1 there exists a constant $\epsilon_1 > 0$ such that

$$\lambda_i^{-\rho_1/\beta} \leqslant |x|^{\alpha/\beta}v_{\lambda_i}(x) \leqslant 2\lambda_i^{-\rho_1/\beta}, \quad \forall |x| \leqslant \epsilon_1, i = 1, 2. \quad (4.17)$$

Then by (1.18) and (4.17),

$$(\log u_1 - \log u_2)_+(x,t) \leqslant \log(2\lambda_2^{-\rho_1/\beta}((T-t)|x|)^{-\alpha/\beta})$$

$$- \log(\lambda_1^{-\rho_1/\beta}((T-t)|x|)^{-\alpha/\beta})$$

$$\leqslant \frac{\rho_1}{\beta}\log\left(\frac{\lambda_1}{\lambda_2}\right) + \log 2 \quad \forall |x| \leqslant \epsilon_1/T^\beta, 0 < t < T$$

$$\Rightarrow \left|\frac{1}{\epsilon^2}\int_{\epsilon\leqslant|x|\leqslant 2\epsilon} (\log u_1 - \log u_2)_+(x,t)\,\mathrm{d}x\right|$$

$$\leqslant 2^n\left(\frac{\rho_1}{\beta}\log\left(\frac{\lambda_1}{\lambda_2}\right) + \log 2\right)\omega_n\epsilon^{n-2} \quad \forall 0 < \epsilon \leqslant \frac{\epsilon_1}{2T^\beta}, 0 < t < T$$

$$\to 0 \quad \forall 0 < t < T \quad \text{as } \epsilon \to 0 \quad (4.18)$$

where ω_n is the surface area of the unit sphere S^{n-1} in $\mathbb{R}^n$. Letting $\epsilon \to 0$ in (4.16), by (4.18) we get

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^n} (u_1 - u_2)_+(x,t)\eta_R(x)\,\mathrm{d}x$$

$$\leqslant \int_{\mathbb{R}^n} (\log u_1 - \log u_2)_+(x,t)\,|\triangle\eta_R(x)|\,\mathrm{d}x \quad \forall 0 < t < T. \quad (4.19)$$

By theorem 1.4 there exists a constant $C_3 > 0$ such that

$$v_{\lambda_i}(x) \geqslant C_3|x|^{-2} \quad \forall |x| \geqslant 1, i = 1, 2. \quad (4.20)$$

By (1.18) and (4.20),

$$u_i(x,t) \geqslant (T-t)^\alpha \cdot C_3((T-t)^\beta |x|)^{-2} = C_3(T-T_1)|x|^{-2}$$

$$\forall |x| \geqslant (T-T_1)^{-\beta}, 0 < t \leqslant T_1 < T. \tag{4.21}$$

By (4.19), (4.21) and the same argument as the proof of Lemma 2.5 of [**21**] for any $T_1 \in (0,T)$ we get $u_1 \leqslant u_2$ in $(\mathbb{R}^n \backslash \{0\}) \times (0,T_1)$. Hence (1.19) holds.

If $u_{0,1} = u_{0,2}$ and both u_1, u_2 are solutions of (1.17) in $(\mathbb{R}^n \backslash \{0\}) \times (0,T)$ which satisfy (1.18), then we also have $u_2 \leqslant u_1$ in $(\mathbb{R}^n \backslash \{0\}) \times (0,T)$. Hence $u_1 = u_2$ in $(\mathbb{R}^n \backslash \{0\}) \times (0,T)$ and the theorem follows. $\qquad\square$

Proof of Theorem 1.7. Let $m_0 \in (0,\overline{m}_0)$ by given by lemma 4.1. Then by (1.20) and lemma 4.1, for any $x \in \mathbb{R}^n \backslash \{0\}$, $0 < t < T$, $0 < m < m_0$, $i = 1, 2$,

$$\lambda_i^{-((1)/((1-m)\beta))}|x|^{-\alpha_m/\beta}$$

$$\leqslant V_{\lambda_i}^{(m)}(x,t) \leqslant \lambda_i^{-((1)/((1-m)\beta))}|x|^{-\alpha_m/\beta} \exp(C_m \lambda_i^{1/\beta} T |x|^{1/\beta}) \tag{4.22}$$

$$\Rightarrow \underline{\lambda}_i |x|^{-\alpha_m/\beta} \leqslant V_{\lambda_i}^{(m)}(x,t) \leqslant \overline{\lambda}_i |x|^{-\alpha_m/\beta} \exp(\overline{C}_0 \lambda_i^{1/\beta} T |x|^{1/\beta}) \tag{4.23}$$

where C_m is given by (4.2) and

$$\overline{\lambda}_i = \max(\lambda_i^{-n/2\beta}, \lambda_i^{-1/\beta}), \quad \underline{\lambda}_i = \min(\lambda_i^{-n/2\beta}, \lambda_i^{-1/\beta}) \quad \text{and}$$

$$\overline{C}_0 = ((n(n-2)(2\beta+1))/(2\beta)).$$

By (1.23) and (4.23),

$$\underline{\lambda}_1 \min(|x|^{-n/2(2+1/\beta)}, |x|^{-(2+1/\beta)}) \leqslant u^{(m)}(x,t)$$

$$\leqslant \overline{\lambda}_2 \max(|x|^{-n/2(2+1/\beta)}, |x|^{-(2+1/\beta)})$$

$$\times \exp(\overline{C}_0 \lambda_2^{1/\beta} T |x|^{1/\beta}) \tag{4.24}$$

holds for any $x \in \mathbb{R}^n \backslash \{0\}$, $0 < t < T$ and $0 < m < m_0$. Let $\{m_i\}_{i=1}^\infty \subset (0,m_0)$ be a sequence of positive numbers such that $m_i \to 0$ as $i \to \infty$. By (4.24) the equation (1.1) for the sequence $\{u^{(m_i)}\}_{i=1}^\infty$ is uniformly parabolic on every compact subset of $(\mathbb{R}^n \backslash \{0\}) \times (0,T)$. By the Schauder estimates for parabolic equations [**26**], the sequence $u^{(m_i)}(x,t)$ is equi-bounded in $C^{2+\theta,1+\theta/2}(K)$ for some $\theta \in (0,1)$ for any compact subset K of $(\mathbb{R}^n \backslash \{0\}) \times (0,T)$. Hence by the Ascoli theorem and a diagonalization argument the sequence $u^{(m_i)}(x,t)$ has a subsequence which we may assume without loss of generality to be the sequence itself that converges uniformly in $C^{2+\theta,1+\theta/2}(K)$ for any compact subset K of $(\mathbb{R}^n \backslash \{0\}) \times (0,T)$ as $i \to \infty$ to a positive function $u(x,t) \in C^2(\mathbb{R}^n \backslash \{0\})$ which by (1.23) and Theorem 1.5 satisfies (1.25).

Putting $m = m_i$ in (4.22) and letting $i \to \infty$, by theorem 1.5,

$$\lambda_i^{-1/\beta}|x|^{-\alpha/\beta} \leqslant V_i(x, t)$$
$$\leqslant \lambda_i^{-1/\beta}|x|^{-\alpha/\beta} \exp(\overline{C}_0 \lambda_i^{1/\beta} T|x|^{1/\beta})$$
$$\forall x \in \mathbb{R}^n \backslash \{0\}, 0 < t < T, i = 1, 2. \tag{4.25}$$

By (4.24) and the mean value theorem for any $(x, t) \in (\mathbb{R}^n \backslash \{0\}) \times (0, T)$ there exists $\xi_i(x, t) \in (0, m_i]$ such that

$$\left| \frac{u^{(m_i)}(x, t)^{m_i} - 1}{m_i} - \log u(x, t) \right|$$

$$= \left| e^{\xi_i(x,t) \log u^{(m_i)}(x,t)} \log u^{(m_i)}(x, t) - \log u(x, t) \right|$$

$$\leqslant e^{\xi_i(x,t) \log u^{(m_i)}(x,t)} \left| \log u^{(m_i)}(x, t) - \log u(x, t) \right|$$

$$+ \left| e^{\xi_i(x,t) \log u^{(m_i)}(x,t)} - 1 \right| \cdot |\log u(x, t)|$$

$$\to 0 \quad \text{uniformly on every compact subset of } (\mathbb{R}^n \backslash \{0\}) \times (0, T) \text{ as } i \to \infty. \tag{4.26}$$

Hence putting $m = m_i$ in (1.1) and letting $i \to \infty$, by (4.26) u satisfies (1.10). It remains to prove that u has initial value u_0. For any $\psi \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$, we choose constants $R_2 > R_1 > 0$ such that $\operatorname{supp} \psi \subset B_{R_2} \backslash B_{R_1}$. Then

$$\left| \int_{\mathbb{R}^n \backslash \{0\}} u^{(m_i)}(x, t)\psi(x)\,\mathrm{d}x - \int_{\mathbb{R}^n \backslash \{0\}} u_{0,m}(x)\psi(x)\,\mathrm{d}x \right|$$

$$= \left| \int_0^t \int_{\mathbb{R}^n \backslash \{0\}} u_t^{(m_i)}(x, s)\psi(x)\,\mathrm{d}x\mathrm{d}s \right|$$

$$= \left| \int_0^t \int_{\mathbb{R}^n \backslash \{0\}} \left(\frac{u^{(m_i)}(x, s)^{m_i} - (T - s)^{m_i \alpha_{m_i}}}{m_i} \right) \cdot \triangle\psi(x)\,\mathrm{d}x\mathrm{d}s \right|$$

$$\leqslant \|\triangle\psi\|_{L^\infty} \int_0^t \int_{B_{R_2} \backslash B_{R_1}} (E_1 + E_2)\,\mathrm{d}x\mathrm{d}s \quad \forall 0 < t < T \tag{4.27}$$

where

$$E_k = \left| \frac{V_{\lambda_k}^{(m)}(x, s)^{m_i} - (T - s)^{m_i \alpha_{m_i}}}{m_i} \right| \quad \forall k = 1, 2.$$

Since

$$E_k = |(T - s)^{m_i \alpha_{m_i}}| \left| \frac{v_{\lambda_k}^{(m)}((T - s)^\beta |x|)^{m_i} - 1}{m_i} \right|$$

$$\to \left| \log v_{\lambda_k}((T - s)^\beta x) \right| \quad \text{uniformly on} (B_{R_2} \backslash B_{R_1}) \times (0, T_1)$$

for any $0 < T_1 < T$ as $i \to \infty$, letting $i \to \infty$ in (4.27),

$$\left| \int_{\mathbb{R}^n \setminus \{0\}} u(x,t)\psi(x)\,\mathrm{d}x - \int_{\mathbb{R}^n \setminus \{0\}} u_0(x)\psi(x)\,\mathrm{d}x \right|$$

$$\leqslant \|\triangle \psi\|_{L^\infty} \int_0^t \int_{B_{R_2} \setminus B_{R_1}} \left(\left| \log v_{\lambda_1}((T-s)^\beta x) \right| + \left| \log v_{\lambda_2}((T-s)^\beta x) \right| \right) \mathrm{d}x\mathrm{d}s$$

$$\leqslant C_1 \|\triangle \psi\|_{L^\infty} t \quad \forall 0 < t \leqslant T/2 \tag{4.28}$$

where

$$C_1 = \max_{(T/2)^\beta R_1 \leqslant |y| \leqslant T^\beta R_2} |\log v_{\lambda_1}(y)| + \max_{(T/2)^\beta R_1 \leqslant |y| \leqslant T^\beta R_2} |\log v_{\lambda_2}(y)|.$$

Letting $t \to 0$ in (4.28),

$$\lim_{t \to 0} \int_{\mathbb{R}^n \setminus \{0\}} u(x,t)\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n \setminus \{0\}} u_0(x)\psi(x)\,\mathrm{d}x \quad \forall \psi \in C_0^\infty(\mathbb{R}^n \setminus \{0\}). \tag{4.29}$$

By (4.29), any sequence $\{t_k\}_{k=1}^\infty$ converging to 0 as $k \to \infty$ will have a subsequence $\{t_{k_l}\}_{l=1}^\infty$ such that $u(x, t_{k_l})$ converges to $u_0(x)$ for a.e. $x \in \mathbb{R}^n \setminus \{0\}$ as $l \to \infty$. Then by the Lebesgue Dominated Convergence Theorem,

$$\lim_{l \to \infty} \int_{R_1 \leqslant |x| \leqslant R_2} |u(x, t_{k_l}) - u_0(x)|\,\mathrm{d}x = 0 \quad \forall R_2 > R_1 > 0.$$

Since the sequence $\{t_k\}_{k=1}^\infty$ is arbitrary, $u(\cdot, t)$ converges to u_0 in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $t \to 0$. Hence u has initial value u_0. Thus by theorem 1.6 u is the unique solution of (1.24). Hence $u^{(m)}$ converges uniformly in $C^{2+\theta, 1+\theta/2}(K)$ for some constant $\theta \in (0, 1)$ and any compact subset K of $(\mathbb{R}^n \setminus \{0\}) \times (0, T)$ to the solution u of (1.24) as $m \to 0^+$ and the theorem follows. $\qquad \square$

Acknowledgements

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